Nonlinear Analysis: Modelling and Control, Vol. 25, No. 1, 1–18
https://doi.org/10.15388/namc.2020.25.15717
ISSN: 1392-5113
eISSN: 2335-8963

Nonlinear dynamics of full-range CNNs with time-varying delays and variable coefficients*

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Received: July 10, 2018 / **Revised:** May 8, 2019 / **Published online:** January 10, 2020

Abstract. In the article, the dynamical behaviours of the full-range cellular neural networks (FRCNNs) with variable coefficients and time-varying delays are considered. Firstly, the improved model of the FRCNNs is proposed, and the existence and uniqueness of the solution are studied by means of differential inclusions and set-valued analysis. Secondly, by using the Hardy inequality, the matrix analysis, and the Lyapunov functional method, we get some criteria for achieving the globally exponential stability (GES). Finally, some examples are provided to verify the correctness of the theoretical results.

Keywords: cellular neural networks, globally exponential stability, differential inclusions, time-varying delays.

1 Introduction

Recently, the research of the cellular neural networks (CNNs) [6, 7] is a hot topic because of powerful parallel and nonlinear processing capabilities and its widespread applications, such as edge detection, image processing, optimization problems, and so on. These applications depend largely on the stability of CNNs. But not all CNNs are globally stable, some of them exhibit complex dynamic behaviors. And many studies have been done to understand the dynamical behaviors of CNNs. For example, chaos phenomenon was presented through a three-cell autonomous system in [43]. The local Hopf bifurcations of tri-neuron neural networks were studied, and the stability of Hopf bifurcations was determined by the Nyquist criterion and the graphical Hopf bifurcation theorem in [36, 37].

*This work was supported in part by the National Natural Science Foundation of China under grant U1703262.

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Because the GES takes a key pole in characterizing the dynamic behaviors of CNNs, the establishment of the criteria for GES is a significant problem in the study of CNNs. A lot of results were acquired on the GES of the CNNs with or without time delays. For instance, by the Hardy inequality, some sufficient conditions were derived to achieve GES of the CNNs with time-varying delays and variable coefficients in [4]. By constructing a new Lyapunov function, some criteria on the GES of the CNNs were acquired in [16,17]. Based on the new differential inequality, some conclusions were obtained for ensuring the GES of the CNNs in [40,41]. By utilizing a new generalized Halanay inequality, several delay-dependent criteria for the GES of the CNNs were established in [22] in which the boundedness of time-varying delays was no longer necessary. In [19], the GES of CNNs with multi-proportional delays was considered. The more theoretical results on the GES of the CNNs can be found in [1, 18, 21, 23, 33, 34, 38, 42].

Meanwhile, the GES of various generalized and improved CNNs is also widely studied, such as the switched CNNs [8], the shunting inhibitory CNNs [14], the memristor-based CNNs [35, 39], the inertial neural networks [13, 15], the quaternion-valued CNNs [20,30], etc. For the sake of getting the additional advantages in hardware implementation of the CNNs, the so-called FRCNNs, the improved CNNs model, which can overcome some disadvantages of CNNs implementation in the circuit, was introduced in [11, 27]. Practical experiments and extensive computer simulations about the stable networks show that the FRCNNs have larger cell densities, smaller power consumption, and higher speed than the CNNs. Furthermore, the dynamics of the FRCNNs is described via differential inclusions, which contain an unbounded set by term called a normal cone, so the theoretical research of the FRCNNs is more difficulty than the Filippov systems [5, 31] and polynomial systems [3,32], and the dynamic behaviors are more complicated than the discontinuous CNNs in [3,23,35,39]. Thus, it is imperative to investigate the dynamics of the FRCNNs. In recent years, the dynamic behaviors for the FRCNNs have been concerned by some scholars. In [10, 26], the authors compared the global dynamic behaviours of the CNNs and FRCNNs with the same parameters. The dynamics of the FRCNNs were rigorously analyzed according to set-valued analysis and differential inclusions in [24, 25, 28].

Note that the results of GES for FRCNNs in [26, 28] are restricted in constant coefficients and constant delays systems. In fact, the time-varying delays are common in the signal transmission among the neurons by reason of the finite speed of switching and transmitting signals. In addition, due to the environmental impact, the parameters of the FRCNNs will change during the dynamical processes. As far as we know, there are few works on the GES of the FRCNNs with time-varying delays and variable coefficients in the past ten years. Therefore, some novel results for the GES of the FRCNNs are eagerly anticipated.

The main contributions of the article are three points. Firstly, the improved model of the FRCNNs, which contains unbounded term called the normal cone, is proposed. Moreover, on account of the introduction of the normal cone terms, the existence of the solution for the improved FRCNNs is harder to prove than the discontinuous CNNs in [3, 23, 35, 39], we adopt a new method to overcome the difficulty. Finally, some novel results including the row diagonal dominant condition and the linear matrix inequality condition are acquired to guarantee the GES of FRCNNs.

This paper is organized as follows. Some notations, definitions, lemmas, and the model of FRCNNs are presented in Section 2. The existence and uniqueness of the solution are analyzed strictly by using differential inclusions and set-valued analysis, and several novel sufficient conditions are established for ensuring the GES of the FRCNNs in Section 3. An example is given to show the correctness of the results in Section 4. A short conclusion is given in Section 5.

2 Preliminaries and model formulation

In this section, some basic notations, preliminaries are provided. At the same time, the improved model of FRCNNs is proposed.

2.1 Notations

Throughout this paper, $\mathbb{R}_+$ denotes the set of positive real numbers. $|\cdot|$ means the absolute value. The n-dimensional Euclidean space is $\mathbb{R}^n$. $\|\cdot\|$ is the Euclidean norm. The symbol $(\cdot)^{\mathrm{T}}$ represents the transpose. $\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i$ is the scalar product of x and y. For a square matrix A, let A^{-1}, $\lambda_{\max}(A)$, and $\lambda_{\min}(A)$ be the inverse of A, the maximum and minimum eigenvalues of A, respectively. For a real matrix B, $B > 0$ ($B \geqslant 0$) if B is positive definite (semidefinite). $\operatorname{dist}(y, \Omega) = \inf_{x\in\Omega} \|y - x\|$ means the distance from y to Ω, where $y \in \mathbb{R}^n$, $\Omega \subset \mathbb{R}^n$. $m(\Omega)$ represents the element in Ω with the smallest Euclidean norm.

2.2 Tangent and normal cones

Firstly, some definitions concerning the set-valued map are given.

Definition 1. (See [2].) Let Y and Z be normed spaces.

(i) A set-valued map $\mathcal{G} : \Omega \subseteq Y \to Z$ is a map that for any $y \in \Omega$, there corresponds a nonempty set $\mathcal{G}(y) \subseteq Z$.

(ii) A set-valued map $\mathcal{G} : \Omega \subseteq Y \to Z$ is called upper semicontinuous at y if for any $\varepsilon > 0$, there exists $\eta > 0$ such that $\mathcal{G}(y + \eta\mathbb{B}(0,1)) \subseteq \mathcal{G}(y) + \varepsilon\mathbb{B}(0,1)$. $\mathcal{G}$ is upper semicontinuous if it is upper semicontinuous at each $y \in \Omega$.

Definition 2 [Tangent cones]. (See [2, 9].) Let $\Omega \subset \mathbb{R}^n$ is a nonempty closed convex set, the tangent cone to Ω at $y \in \Omega$ is given by

$$T_\Omega(y) = \left\{ h \in \mathbb{R}^n\colon \ \liminf_{s\to 0^+} \frac{\operatorname{dist}(y + sh,\, \Omega)}{s} = 0 \right\}.$$

Definition 3 [Normal cones]. (See [2, 9].) Let $\Omega \subset \mathbb{R}^n$ be a nonempty closed convex set, the normal cone to Ω at $y \in \Omega$ is defined as

$$N_\Omega(y) = \bigl\{ u \in \mathbb{R}^n\colon \ \langle u, h\rangle \leqslant 0 \ \forall h \in T_\Omega(y) \bigr\}.$$

In particular, let $\Omega = Q$, where Q is of the hypercube $[-1, 1]^n$, it can easily obtain that

$$N_Q(y) = \big(N_{[-1,1]}(y_1),\, N_{[-1,1]}(y_2),\, \dots,\, N_{[-1,1]}(y_n)\big)^{\mathrm{T}}$$

for all $y \in Q$, where

$$N_{[-1,1]}(y_i) = \begin{cases} [0, +\infty), & y_i = 1, \\ 0, & -1 < y_i < 1, \\ (-\infty, 0], & y_i = -1. \end{cases}$$

Now, the lemma about the properties of normal cone is presented as follows.

Lemma 1. (*See* [26].) *Let $\Omega \subset \mathbb{R}^n$ be a nonempty closed convex set.*

(i) *$T_\Omega(x)$ and $N_\Omega(x)$ are nonempty closed convex cones in $\mathbb{R}^n$.*

(ii) *$N_\Omega(x)$ is a monotone operator. That is, for any $x, y \in \Omega$, $n_x \in N_\Omega(x)$, $n_y \in N_\Omega(y)$, then*

$$\langle x - y,\, n_x - n_y \rangle \geqslant 0.$$

(iii) *If choose $\Omega = Q$ and $P = \operatorname{diag}(p_1, p_2, \dots, p_n) \geqslant 0$ for any $x, y \in Q$ and $n_x \in N_Q(x), n_y \in N_Q(y)$, we have*

$$\big\langle x - y,\, P(n_x - n_y)\big\rangle \geqslant 0.$$

2.3 The solution of differential inclusions

Consider the differential inclusion

$$\dot{x}(t) \in F\big(t, x(t)\big) - N_\Omega\big(x(t)\big), \qquad x(t_0) \in \Omega. \tag{1}$$

Definition 4. (See [2].) An absolutely continuous function $x(t)$ is called a solution of inclusion (1) on interval $[t_0, t_1]$ if for almost all $t \in [t_0, t_1]$, $x(t_0) \in \Omega$ and $\dot{x}(t) \in F(t, x(t)) - N_\Omega(x(t))$.

Lemma 2. (*See* [29].) *Let $F : [0, T] \times \Omega \to \Omega$ be a set-valued map with nonempty compact convex values, F is measurable with respect to (t, x) and upper semicontinuous with respect to x. Assume that there exists an integrable function $u(t)$ over $[0, T]$ such that $m(F(t, x(t))) \leqslant u(t)$, then given any $x(0) \in \Omega$, the differential inclusion* (1) *has at least one solution $x(t)$ for almost all $t \in [0, T]$.*

2.4 Some useful inequalities

Lemma 3 [Hardy inequality]. (*See* [12].) *There exist constants $a_k \geqslant 0$, $p_k > 0$, $k = 1, 2, \dots, m+1$, such that*

$$\left(\prod_{k=1}^{m+1} a_k^{p_k}\right)^{1/S_{m+1}} \leqslant \left(\sum_{k=1}^{m+1} p_k a_k^r\right)^{1/r} S_{m+1}^{-1/r}, \tag{2}$$

where $r > 0$ and $S_{m+1} = \sum_{k=1}^{m+1} p_k$.

Especially, if let $p_{m+1} = 1$, $r = S_{m+1} = \sum_{k=1}^{m} p_k + 1$, one has

$$a_{m+1} \prod_{k=1}^{m} a_k^{p_k} \leqslant \frac{1}{r} \sum_{k=1}^{m} p_k a_k^r + \frac{1}{r} a_{m+1}^r. \tag{3}$$

2.5 System formulation

Consider the following CNN with delay:

$$\dot{x}(t) = -x(t) + AG\bigl(x(t)\bigr) + BG\bigl(x(t-\tau)\bigr) + I, \tag{4}$$

where $x(t) = (x_1(t), x_2(t), \dots, x_n(t))^{\mathrm{T}}$ is the vector of neuron state variables. $A \in \mathbb{R}^{n\times n}$ is the neuron interconnection matrix. τ is a constant delay. $B \in \mathbb{R}^{n\times n}$ is the delayed neuron interconnection matrix. $I \in \mathbb{R}^n$ is the constant input. $G(x(t)) = (g(x_1(t)), g(x_2(t)), \dots, g(x_n(t)))^{\mathrm{T}}$ is an active function in which

$$g(x_i) = \frac{1}{2}\bigl(|x_i + 1| - |x_i - 1|\bigr) = \begin{cases} 1, & x_i > 1, \\ x_i, & -1 \leqslant x_i \leqslant 1, \\ -1, & x_i < -1. \end{cases}$$

The hardware realization of the CNNs chips is required to constrain the range of input states for original CNNs, so the following so-called signal FRCNNs is introduced:

$$\dot{x}(t) = -x(t) + AG\bigl(x(t)\bigr) + BG\bigl(x(t-\tau)\bigr) + I - mL\bigl(x(t)\bigr), \tag{5}$$

where $m \geqslant 0$ is a constant, and $L(x) = (l(x_1), l(x_2), \dots, l(x_n))^{\mathrm{T}}$ is the vector function satisfying

$$l(x_i) = \begin{cases} x_i - 1, & x_i > 1, \\ 0, & -1 \leqslant x_i \leqslant 1, \\ x_i + 1, & x_i < -1. \end{cases}$$

Remark 1. When the slope m is large enough, the role of the nonlinearity $mL(x(t))$ is to prevent the state variable x of system (5) exceedingly deviating the saturated region Q. Let $m \to +\infty$,

$$\lim_{m\to+\infty} ml(x_i) = N_{[-1,1]}(x_i) = \begin{cases} [0, +\infty), & x_i = 1, \\ 0, & -1 \leqslant x_i \leqslant 1, \\ (-\infty, 0], & x_i = -1. \end{cases}$$

The ideal hard-limiter $N_{[-1,1]}(x_i)$ constrains the state variables x_i to change in the interval $[-1, 1]$.

When $m \to +\infty$, from the analysis of Remark 1 and the definition of $G(x)$, system (5) can be expressed as

$$\dot{x}(t) \in -x(t) + Ax(t) + Bx(t-\tau) + I - N_Q\big(x(t)\big), \tag{6}$$

where the ideal hard-limiter $N_Q(x(t)$ constrains the state variables to evolve within the hypercube $Q = [-1, 1]^n$. Then system (6) displays a number of advantages in the very large-scale integration implementation compared with system (4), including smaller power consumption and simpler design process.

In this paper, the system of FRCNN with variable coefficients and time-varying delay is described by the following differential inclusion:

$$\dot{x}(t) \in -x(t) + A(t)x(t) + B(t)x(t-\tau(t)) + I - N_Q\big(x(t)\big), \tag{7}$$

where $A(t) = (a_{ij}(t))_{n\times n}$ is the neuron interconnection matrix and $B(t) = (b_{ij}(t))_{n\times n}$ is the delayed neuron interconnection matrix. $\tau(t)$ is a time-varying delay.

Remark 2. The continuous systems in [19, 21, 22] and the discontinuous systems in [23, 35, 39] about GES were studied, but the set-valued term $N_Q(x(t))$ is not introduced into their systems. In [28], the system $\dot{x}(t) \in Ax(t) + I - N_Q(x(t))$ was proposed and studied. It was further researched in [24]. However, they thought about the models with constant coefficients and constant delays. In fact, if we consider the long term dynamics of the system, it is more common for the systems with time-varying delays and variable coefficients because of the states-changing. Hence, we consider system (7) in this paper, which is more general than the existing ones.

Remark 3. If system (7) is written in terms of the form

$$\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} \in -x_i(t) + \sum_{j=1}^{n} a_{ij}(t)x_j(t) + \sum_{j=1}^{n} b_{ij}(t)x_j\big(t-\tau(t)\big) - N_{[-1,1]}\big(x_i(t)\big),$$

we can replace $x_j(t-\tau(t))$ by $x_j(t-\tau_{ij}(t))$, that is, time-varying delays can be related to index i, j. In the analysis of the model, we are going to use the upper bound of $\tau(t)$. If we replace $\tau(t)$ with $\tau_{ij}(t)$, we can take the maximum of upper bound for $\tau_{ij}(t)$ as the upper bound, and the analysis process is similar.

To achieve the main results, Assumption 1 and Definition 5 are needed.

Assumption 1. The variable coefficients $a_{ij}(t)$ and $b_{ij}(t)$ are continuous, $0 < \tau(t) \leqslant \tau$, $\inf_{t\in R^+}\{1 - \dot{\tau}(t)\} > 0$, and $\dot{\tau}(t)$ is continuous, where τ is a constant.

The initial condition of system (7) is given by $x(\theta) = \varphi(\theta)$, $\theta \in [-\tau, 0]$, where $\varphi(\theta) \in C([-\tau, 0], Q)$, $C([-\tau, 0], Q)$ is the Banach space of continuous function with $\|\varphi\| = \sup_{-\tau\leqslant\theta\leqslant0} |\phi(\theta)|_\infty$, $|\phi(\theta)|_\infty = (\max_{1\leqslant i\leqslant n} |\phi_i(\theta)|^r)^{1/r}$, $r \geqslant 1$.

Definition 5. System (7) is globally exponentially stable for any two solutions $x(t), y(t)$ with initial functions $\varphi(\theta), \phi(\theta)$ if there exist constants $\varepsilon > 0$ and $M \geqslant 1$ such that

$$\big\|x(t) - y(t)\big\| \leqslant M\|\varphi - \phi\|\mathrm{e}^{-\varepsilon t}, \quad t \geqslant 0.$$

3 Main results

In this section, the existence of the solution is discussed by using set-valued map and differential inclusions, and some globally exponentially stable conditions of system (7) are acquired.

Theorem 1. *Under Assumption 1, for any initial function* $\varphi(\theta)$ *of system* (7), $\theta \in [-\tau, 0]$, *there exists a unique solution* $x(t)$ *defined on* $[0, +\infty)$.

Proof. Firstly, we prove that there exists a solution $x(t)$ of system (7) on a right neighbourhood of zero with the initial function $\varphi(\theta)$. Denote $\widehat{Q} = Q \times [-\tau, 0]$ and $Z(t) = (z^{\mathrm{T}}(t), \theta(t))^{\mathrm{T}}$, where $z(t) = (z_1(t), z_2(t), \dots, z_n(t))^{\mathrm{T}} \in \mathbb{R}^n$ and $\theta(t) \in \mathbb{R}$. Consider the following system:

$$\begin{aligned} \dot{Z}(t) &= \begin{pmatrix} \dot{z}(t) \\ \dot{\theta}(t) \end{pmatrix} \\ &\in \begin{pmatrix} -z(t) + A(t)z(t) + I + B(t)\varphi(\theta(t)) \\ 1 - \dot{\tau}(t) \end{pmatrix} - \begin{pmatrix} N_Q(z(t)) \\ N_{[-\tau,0]}(\theta(t)) \end{pmatrix} \\ &\doteq \mathcal{F}\bigl(t, Z(t)\bigr) - N_{\widehat{Q}}\bigl(Z(t)\bigr), \\ Z(0) &= \begin{pmatrix} z(0) \\ \theta(0) \end{pmatrix} = \begin{pmatrix} \varphi(0) \\ -\tau(0) \end{pmatrix}. \end{aligned} \tag{8}$$

For any $T > 0$, $\mathcal{F}$ is continuous bounded on $[0, T] \times \widehat{Q}$ and $Z(0) \in \widehat{Q}$. From Lemma 2 there exists one solution $Z(t) = (z^{\mathrm{T}}(t), \theta(t))^{\mathrm{T}}$ of system (8) on a right neighbourhood of zero.

Secondly, we conclude that the solution $x(t)$ of system (7) exists on $[0, +\infty)$. Inductively, assume that $x(t)$ is a solution of (7) on $[-\tau, N\tau]$ for some integer $N \geqslant 0$. Let

$$x(t) = \begin{cases} \varphi(t), & t \in [-\tau, 0), \\ z(t), & t \in [0, N\tau], \end{cases}$$

where $x(t)$ is continuous and $x(t) \in Q$ for $t \in [-\tau, N\tau]$. It is obvious that $x(t)$ is absolutely continuous on $[0, N\tau]$. Moreover, by solving the inclusion

$$\begin{aligned} \dot{Z}(t) &= \begin{pmatrix} \dot{z}(t) \\ \dot{\theta}(t) \end{pmatrix} \\ &\in \begin{pmatrix} -z(t) + A(t)z(t) + I + B(t)x(N\tau + \theta(t)) \\ 1 - \dot{\tau}(t) \end{pmatrix} - \begin{pmatrix} N_Q(z(t)) \\ N_{[-\tau,0]}(\theta(t)) \end{pmatrix}, \\ Z(0) &= \begin{pmatrix} z(0) \\ \theta(0) \end{pmatrix} = \begin{pmatrix} x(N\tau) \\ -\tau(0) \end{pmatrix}, \end{aligned}$$

we can extend the solution $x(t)$ of system (7) on a right neighbourhood of $N\tau$. According to $x(t) \in Q$, we conclude that there exists a solution of (7) with the initial function $\varphi(\theta)$ for $t \geqslant 0$.

Finally, the uniqueness of the solution will be proven. By contradiction, let $x(t)$, $y(t)$ be any two solutions for system (7) with initial function $\varphi(\theta)$. Denote $\rho(t) = \|x(t) - y(t)\|^2/2$. Then $\rho(t) = 0$ will be proved by mathematical induction for $t \geqslant 0$.

For $t \in [-\tau, 0]$, it is easy to get $\rho(t) = 0$.

Let integer $m \geqslant 0$, $t \in [-\tau, m\tau]$, suppose that $\rho(t) = 0$. Next, we need to prove $\rho(t) = 0$ when $t \in [-\tau,\ (m+1)\tau]$. From the result of second step there exist $n_{x(t)}(t) \in N_Q(x(t))$ and $n_{y(t)}(t) \in N_Q(y(t))$ such that

$$\dot{x}(t) = -x(t) + A(t)x(t) + B(t)x\big(t - \tau(t)\big) + I - n_{x(t)}(t)$$

and

$$\dot{y}(t) = -y(t) + A(t)y(t) + B(t)y\big(t - \tau(t)\big) + I - n_{y(t)}(t).$$

It follows that

$$\begin{aligned}\dot{\rho}(t) &= \big\langle x(t) - y(t),\ \dot{x}(t) - \dot{y}(t)\big\rangle \\ &\leqslant \big\langle x(t) - y(t),\ A(t)(x(t) - y(t)\big\rangle \\ &\quad + \big\langle x(t) - y(t),\ B(t)\big(x\big(t - \tau(t)\big) - y\big(t - \tau(t)\big)\big)\big\rangle,\end{aligned}$$

the above inequality is obtained by $\langle x(t) - y(t),\ n_{x(t)}(t) - n_{y(t)}(t)\rangle \geqslant 0$. Due to the fact $\rho(t) = 0$, $t \in [-\tau, m\tau]$, we get $x(t - \tau(t)) - y(t - \tau(t)) = 0$ for $t \in [m\tau, (m+1)\tau]$. Hence, we have

$$\dot{\rho}(t) \leqslant \big\langle x(t) - y(t),\ A(t)\big(x(t) - y(t)\big)\big\rangle \leqslant 2\lambda_{\max}\big(A(t)\big)\rho(t).$$

By the Gronwall inequality and $\rho(m\tau) = 0$ we obtain

$$0 \leqslant \rho(t) \leqslant \rho(m\tau)\mathrm{e}^{2\lambda_{\max}(A(t))(t - m\tau)} = 0.$$

Therefore $\rho(t) = 0$ for $t \in [-\tau, (m+1)\tau]$. This completes the proof. □

Remark 4. The local existence of the solutions for the discontinuous systems in [23, 35, 39] was proved by using the classical existence theorem of differential inclusions. Since the normal cone $N_Q(x(t))$ is a set-valued map and not an upper semicontinuous, the above method can not apply to our system. Here, we transform system (7) into the form of differential inclusion (1). By using Lemma 2 we get the local existence of the solution for system (7).

Theorem 2. *Under Assumption 1, system* (7) *is globally exponentially stable if there exist constants* $v_k > 0$, $k = 1, 2, \dots, K_1$, $\mu_k > 0$, $k = 1, 2, \dots, K_2$, $\omega_i > 0, p_{ij}, p_{ij}^*, q_{ij}, q_{ij}^*$, $i, j = 1, 2, \dots, n$, *such that*

$$\begin{aligned}&\omega_i r - \sum_{j=1}^{n}\sum_{k=1}^{K_1}\omega_j v_k\big|a_{ij}(t)\big|^{rq_{ij}/v_k} - \sum_{j=1}^{n}\omega_j\big|a_{ij}(t)\big|^{rq_{ij}^*} \\ &\quad - \sum_{j=1}^{n}\omega_j\sum_{k=1}^{K_2}\mu_k\big|b_{ij}(t)\big|^{rp_{ij}/\mu_k} - \sum_{j=1}^{n}\omega_j\frac{|b_{ij}(\psi^{-1}(t))|^{rp_{ij}^*}}{1 - \dot{\tau}(\psi^{-1}(t))} > 0, \qquad (9)\end{aligned}$$

where $r = \sum_{k=1}^{K_1} \upsilon_k + 1 = \sum_{k=1}^{K_2} \mu_k + 1$, $K_1 q_{ij} + q_{ij}^* = 1$, $K_2 p_{ij} + p_{ij}^* = 1$, *and* $\psi^{-1}(t)$ *is the inverse function of* $\psi(t) = t - \tau(t)$.

Proof. Let $x(t)$ and $y(t)$ be any two solutions of system (7) with the initial functions $\varphi(\theta), \phi(\theta)$. Denote $\eta_i(t) = x_i(t) - y_i(t)$, $i = 1, 2, \ldots, n$. From Theorem 1 there exist the measurable functions $n_{x_i(t)}(t) \in N_{[-1,1]}(x_i(t))$ and $n_{y_i(t)}(t) \in N_{[-1,1]}(y_i(t))$ such that

$$\begin{aligned} \frac{\mathrm{d}\eta_i(t)}{\mathrm{d}t} &= -\eta_i(t) + \sum_{j=1}^{n} a_{ij}(t)\eta_j(t) + \sum_{j=1}^{n} b_{ij}(t)\eta_j\bigl(t - \tau(t)\bigr) \\ &\quad - \bigl(n_{x_i(t)}(t) - n_{y_i(t)}(t)\bigr). \end{aligned} \tag{10}$$

Define $\zeta_i(t) = \eta_i(t)/\omega_i$, then

$$\begin{aligned} \frac{\mathrm{d}|\zeta_i(t)|^r}{\mathrm{d}t} &= r\bigl|\zeta_i(t)\bigr|^{r-1} \operatorname{sign}\bigl(\zeta_i(t)\bigr)\frac{\mathrm{d}\zeta_i(t)}{\mathrm{d}t} \\ &= r\bigl|\zeta_i(t)\bigr|^{r-1} \operatorname{sign}\bigl(\zeta_i(t)\bigr)\frac{1}{\omega_i}\frac{\mathrm{d}\eta_i(t)}{\mathrm{d}t} \\ &= r\bigl|\zeta_i(t)\bigr|^{r-1} \operatorname{sign}\bigl(\zeta_i(t)\bigr) \\ &\quad \times \Biggl(-\zeta_i(t) + \sum_{j=1}^{n} \frac{\omega_j}{\omega_i} a_{ij}(t)\zeta_j(t) + \sum_{j=1}^{n} \frac{\omega_j}{\omega_i} b_{ij}(t)\zeta_j\bigl(t - \tau(t)\bigr) \\ &\quad - \frac{r}{\omega_i}\bigl|\zeta_i(t)\bigr|^{r-1} \operatorname{sign}\bigl(\zeta_i(t)\bigr)\bigl(n_{x_i(t)}(t) - n_{y_i(t)}(t)\bigr)\Biggr), \end{aligned} \tag{11}$$

where

$$\operatorname{sign}\bigl(\zeta_i(t)\bigr) = \begin{cases} 1, & \zeta_i(t) > 0, \\ 0, & \zeta_i(t) = 0, \\ -1, & \zeta_i(t) < 0. \end{cases}$$

From (9) choose a constant $\varepsilon > 0$ such that

$$\begin{aligned} &(\varepsilon - r) + \sum_{j=1}^{n} \frac{\omega_j}{\omega_i} \sum_{k=1}^{K_1} \upsilon_k \bigl|a_{ij}(t)\bigr|^{r q_{ij}/\upsilon_k} + \sum_{j=1}^{n} \frac{\omega_j}{\omega_i} \bigl|a_{ij}(t)\bigr|^{r q_{ij}^*} \\ &\quad + \sum_{j=1}^{n} \frac{\omega_j}{\omega_i} \sum_{k=1}^{K_2} \mu_k \bigl|b_{ij}(t)\bigr|^{r p_{ij}/\mu_k} + \mathrm{e}^{\varepsilon\tau} \sum_{j=1}^{n} \frac{\omega_j}{\omega_i} \frac{|b_{ij}(\psi^{-1}(t))|^{r p_{ij}^*}}{1 - \dot{\tau}(\psi^{-1}(t))} < 0. \end{aligned} \tag{12}$$

Consider the following Lyapunov functional:

$$\begin{aligned} V(t) &= \bigl|\zeta_{i(t)}(t)\bigr|^r \mathrm{e}^{\varepsilon t} \\ &\quad + \sum_{j=1}^{n} \frac{\omega_j}{\omega_{i(t)}} \int\limits_{t-\tau(t)}^{t} \frac{|b_{i(t)j}(\psi^{-1}(s))|^{r p_{i(t)j}^*}}{1 - \dot{\tau}(\psi^{-1}(s))} \bigl|\zeta_j(t)\bigr|^r \mathrm{e}^{\varepsilon(s + \tau(\psi^{-1}(s)))}\,\mathrm{d}s, \end{aligned} \tag{13}$$

where $i(t) \in \{1, 2, \ldots, n\}$, $|\zeta_{i(t)}(t)| = |\zeta(t)|_\infty = (\max_{1 \leqslant i \leqslant n} |\zeta_i(t)|^r)^{1/r}$. According to (11) and based on the following result, which obtain by Lemma 1

$$\begin{aligned} &\frac{r}{\omega_i}\left|\zeta_i(t)\right|^{r-1} \operatorname{sign}\bigl(\zeta_i(t)\bigr)\bigl(n_{x_i(t)}(t) - n_{y_i(t)}(t)\bigr) \\ &\quad = \frac{r}{\omega_i}\left|\zeta_i(t)\right|^{r-1} \frac{\zeta_i(t)}{|\zeta_i(t)|}\bigl(n_{x_i(t)}(t) - n_{y_i(t)}(t)\bigr) \geqslant 0, \end{aligned}$$

we have

$$\begin{aligned} \dot{V}(t) = {}& \mathrm{e}^{\varepsilon t}\Biggl(\varepsilon\left|\zeta_{i(t)}(t)\right|^r + r\left|\zeta_{i(t)}(t)\right|^{r-1} \operatorname{sign}\bigl(\zeta_{i(t)}(t)\bigr)\Biggl\{\sum_{j=1}^n \frac{\omega_j}{\omega_{i(t)}} a_{i(t)j}(t)\zeta_j(t) \\ &+ \sum_{j=1}^n \frac{\omega_j}{\omega_{i(t)}} b_{i(t)j}(t)\zeta_j\bigl(t-\tau(t)\bigr) - \zeta_{i(t)}(t) \\ &- \frac{1}{\omega_{i(t)}}\bigl(n_{x_{i(t)}(t)}(t) - n_{y_{i(t)}(t)}(t)\bigr)\Biggr\} \\ &+ \mathrm{e}^{\varepsilon\tau(\psi^{-1}(t))}\sum_{j=1}^n \frac{\omega_j}{\omega_{i(t)}} \frac{|b_{i(t)j}(\psi^{-1}(t))|^{rp^*_{i(t)j}}}{1 - \dot{\tau}(\psi^{-1}(t))}\left|\zeta_j(t)\right|^r \\ &- \sum_{j=1}^n \frac{\omega_j}{\omega_{i(t)}}\left|b_{i(t)j}(t)\right|^{rp^*_{i(t)j}}\left|\zeta_j\bigl(t-\tau(t)\bigr)\right|^r\Biggr) \\ \leqslant {}& \mathrm{e}^{\varepsilon t}\Biggl((\varepsilon - r)\left|\zeta_{i(t)}(t)\right|^r + r\sum_{j=1}^n \frac{\omega_j}{\omega_{i(t)}}\left|\zeta_{i(t)}(t)\right|^{r-1}\left|a_{i(t)j}(t)\right|\left|\zeta_j(t)\right| \\ &+ r\sum_{j=1}^n \frac{\omega_j}{\omega_{i(t)}}\left|\zeta_{i(t)}(t)\right|^{r-1}\left|b_{i(t)j}(t)\right|\left|\zeta_j\bigl(t-\tau(t)\bigr)\right| \\ &+ \mathrm{e}^{\varepsilon\tau}\sum_{j=1}^n \frac{\omega_j}{\omega_{i(t)}} \frac{|b_{i(t)j}(\psi^{-1}(t))|^{rp^*_{i(t)j}}}{1 - \dot{\tau}(\psi^{-1}(t))}\left|\zeta_j(t)\right|^r \\ &- \sum_{j=1}^n \frac{\omega_j}{\omega_{i(t)}}\left|b_{i(t)j}(t)\right|^{rp^*_{i(t)j}}\left|\zeta_j\bigl(t-\tau(t)\bigr)\right|^r\Biggr). \end{aligned} \tag{14}$$

Moreover, by Hardy inequality (3) we have

$$\begin{aligned} \dot{V}(t) \leqslant {}& \mathrm{e}^{\varepsilon t}\Biggl((\varepsilon - r)\left|\zeta_{i(t)}(t)\right|^r + \sum_{j=1}^n \frac{\omega_j}{\omega_{i(t)}}\sum_{k=1}^{K_1} \upsilon_k\left|a_{i(t)j}(t)\right|^{rq_{i(t)j}/\upsilon_k}\left|\zeta_{i(t)}(t)\right|^r \\ &+ \sum_{j=1}^n \frac{\omega_j}{\omega_{i(t)}}\left|a_{i(t)j}(t)\right|^{rq^*_{i(t)j}}\left|\zeta_j(t)\right|^r \end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}^{n}\frac{\omega_j}{\omega_{i(t)}}\sum_{k=1}^{K_2}\mu_k\big|b_{i(t)j}(t)\big|^{rp_{i(t)j}/\mu_k}\big|\zeta_{i(t)}(t)\big|^r \\
&+\mathrm{e}^{\varepsilon\tau}\sum_{j=1}^{n}\frac{\omega_j}{\omega_{i(t)}}\frac{|b_{i(t)j}(\psi^{-1}(t))|^{rp^*_{i(t)j}}}{1-\dot{\tau}(\psi^{-1}(t))}\big|\zeta_j(t)\big|^r\Bigg) \\
\leqslant\ &\mathrm{e}^{\varepsilon t}\Bigg((\varepsilon-r)+\sum_{j=1}^{n}\frac{\omega_j}{\omega_{i(t)}}\sum_{k=1}^{K_1}\upsilon_k\big|a_{i(t)j}(t)\big|^{rq_{i(t)j}/\upsilon_k} \\
&+\sum_{j=1}^{n}\frac{\omega_j}{\omega_{i(t)}}\big|a_{i(t)j}(t)\big|^{rq^*_{i(t)j}}+\sum_{j=1}^{n}\frac{\omega_j}{\omega_{i(t)}}\sum_{k=1}^{K_2}\mu_k\big|b_{i(t)j}(t)\big|^{rp_{i(t)j}/\mu_k} \\
&+\mathrm{e}^{\varepsilon\tau}\sum_{j=1}^{n}\frac{\omega_j}{\omega_{i(t)}}\frac{|b_{i(t)j}(\psi^{-1}(t))|^{rp^*_{i(t)j}}}{1-\dot{\tau}(\psi^{-1}(t))}\Bigg)\big|\zeta_{i(t)}(t)\big|^r. \qquad (15)
\end{aligned}
$$

From (12) and (15) we have

$$\dot{V}(t)\leqslant 0.$$

Thus,

$$V(t)\leqslant V(0).$$

Due to (13), one has

$$V(t)\geqslant\big|\zeta_{i(t)}(t)\big|^r\mathrm{e}^{\varepsilon t},$$

and

$$
\begin{aligned}
V(0)&=\big|\zeta_{i(0)}(0)\big|^r \\
&\quad+\sum_{j=1}^{n}\frac{\omega_j}{\omega_{i(0)}}\int_{-\tau(0)}^{0}\frac{|b_{i(0)j}(\psi^{-1}(s))|^{rp^*_{i(0)j}}}{1-\dot{\tau}(\psi^{-1}(s))}\big|\zeta_j(s)\big|^r\mathrm{e}^{\varepsilon(s+\tau(\psi^{-1}(s)))}\,\mathrm{d}s \\
&\leqslant\Bigg(1+\sum_{j=1}^{n}\frac{\omega_j}{\omega_{i(0)}}\int_{-\tau(0)}^{0}\frac{|b_{i(0)j}(\psi^{-1}(s))|^{rp^*_{i(0)j}}}{1-\dot{\tau}(\psi^{-1}(s))}\mathrm{e}^{\varepsilon(s+\tau(\psi^{-1}(s)))}\,\mathrm{d}s\Bigg) \\
&\quad\times\sup_{s\in[-\tau,0]}\big|\zeta(s)\big|_\infty^r \\
&\leqslant(1+M)\sup_{s\in[-\tau,0]}\big|\zeta(s)\big|_\infty^r,
\end{aligned}
$$

where

$$\underline{\omega}=\min_{1\leqslant j\leqslant n}\omega_j$$

and

$$M=\sum_{j=1}^{n}\frac{\omega_j}{\underline{\omega}}\max_{1\leqslant i\leqslant n}\int_{-\tau(0)}^{0}\frac{|b_{ij}(\psi^{-1}(s))|^{rp^*_{ij}}}{1-\dot{\tau}(\psi^{-1}(s))}\mathrm{e}^{\varepsilon(s+\tau(\psi^{-1}(s)))}\,\mathrm{d}s.$$

Hence

$$\left|\zeta_i(t)\right|^r \leqslant \left|\zeta_{i(t)}(t)\right|^r \leqslant V(0)\mathrm{e}^{-\varepsilon t} \leqslant (1+M) \sup_{s\in[-\tau,0]} \left|\zeta(s)\right|_\infty^r \mathrm{e}^{-\varepsilon t},$$

i.e.,

$$\left|x_i(t) - y_i(t)\right| \leqslant M_0\|\varphi - \phi\|\mathrm{e}^{-\varepsilon t/r},$$

where $M_0 \geqslant 1$ is a constant. It shows that the solution of system (7) is global exponentially stable. □

Remark 5. When we calculate the time derivative of the Lyapunov function in the proof of Theorem 2, system (7) contains set-valued item $N_Q(x(t))$, which leads to the additional difficulty compared with the systems in [19,21–23]. We overcome this difficulty by using the monotonicity of the normal cone and the measurable selection theorem.

When $r = 1$, the following condition is established by directly estimating the right-hand side of (14), instead of employing the Hardy inequality.

Corollary 1. *Under Assumption 1, system* (7) *is global exponentially stable if there exist constants* $\omega_i > 0$, $i = 1, 2, \ldots, n$, *such that*

$$\omega_i - \sum_{j=1}^n \omega_j\left|a_{ij}(t)\right| - \sum_{j=1}^n \omega_j \frac{|b_{ij}(\psi^{-1}(t))|}{1 - \dot{\tau}(\psi^{-1}(t))} > 0.$$

When $r = 1$, let

$$V(t) = \sum_{i=1}^n \omega_i\left(\left|\zeta_i(t)\right|\mathrm{e}^{\varepsilon t} + \sum_{j=1}^n \int_{t-\tau(t)}^t \frac{|b_{ij}(\psi^{-1}(s))|}{1 - \dot{\tau}(\psi^{-1}(s))}\left|\zeta_j(t)\right|\mathrm{e}^{\varepsilon(s+\tau(\psi^{-1}(s)))}\,\mathrm{d}s\right).$$

The following column diagonal dominant condition can be derived.

Corollary 2. *Under Assumption 1, system* (7) *is global exponentially stable if there exist constants* $\omega_j > 0$, $j = 1, 2, \ldots, n$, *such that*

$$\omega_j - \sum_{i=1}^n \omega_i\left|a_{ij}(t)\right| - \sum_{i=1}^n \omega_i \frac{|b_{ij}(\psi^{-1}(t))|}{1 - \dot{\tau}(\psi^{-1}(t))} > 0.$$

In Theorem 2, choose $K_1 = K_2 = 1$, $v_k = \mu_k = r - 1$, $p_{ij} = q_{ij} = (r-1)/r$, $p_{ij}^* = q_{ij}^* = 1/r$, $i, j = 1, 2, \ldots, n$. We have the following corollary.

Corollary 3. *Suppose that Assumption 1 holds. If there exist constants* $\omega_i > 0$, $i = 1, 2, \ldots, n$, *and* $r \geqslant 1$ *such that*

$$\omega_i r - r\sum_{j=1}^n \omega_j\left|a_{ij}(t)\right| - (r-1)\sum_{j=1}^n \omega_j\left|b_{ij}(t)\right| - \sum_{j=1}^n \omega_j \frac{|b_{ij}(\psi^{-1}(t))|}{1 - \dot{\tau}(\psi^{-1}(t))} > 0,$$

system (7) *is global exponentially stable.*

Next, the delay-dependent condition is established upon linear matrix inequality.

Theorem 3. *Under Assumption 1, system* (7) *is global exponentially stable if there exist a positive definite diagonal matrix* $P > 0$ *and symmetric matrix* $Q > 0$ *such that*

$$-2P + PA(t) + A^{\mathrm{T}}(t)P + \frac{1}{1-\dot{\tau}(\psi^{-1}(t))}Q + PB(t)Q^{-1}\big(PB(t)\big)^{\mathrm{T}} < 0, \tag{16}$$

where $\psi^{-1}(t)$ *is the inverse function of* $\psi(t) = t - \tau(t)$.

Proof. Let $x(t)$ and $y(t)$ be any two solutions of (7) with the initial functions $\varphi(\theta)$ and $\phi(\theta)$ for almost all $\theta \in [-\tau, 0)$, then there exist the measurable functions $n_{x(t)}(t) \in N_Q(x(t))$ and $n_{y(t)}(t) \in N_Q(y(t))$ such that

$$\dot{x}(t) = -x(t) + A(t)x(t) + B(t)x\big(t-\tau(t)\big) + I - n_{x(t)}(t)$$

and

$$\dot{y}(t) = -y(t) + A(t)y(t) + B(t)y\big(t-\tau(t)\big) + I - n_{y(t)}(t).$$

Denote $\zeta(t) = x(t) - y(t)$, it is easy to get

$$\dot{\zeta}(t) = -\zeta(t) + A(t)\zeta(t) + B(t)\zeta\big(t-\tau(t)\big) - \big(n_{x(t)}(t) - n_{y(t)}(t)\big). \tag{17}$$

According to (16), take $\varepsilon > 0$ such that

$$(\varepsilon-2)P + PA(t) + A^{\mathrm{T}}(t)P + \frac{\mathrm{e}^{\varepsilon\tau(\psi^{-1}(t))}}{1-\dot{\tau}(\psi^{-1}(t))}Q + PB(t)Q^{-1}\big(PB(t)\big)^{\mathrm{T}} < 0. \tag{18}$$

Define a Lyapunov functional as

$$V(t) = \zeta^{\mathrm{T}}(t)P\zeta(t)\mathrm{e}^{\varepsilon t} + \int_{t-\tau(t)}^{t} \frac{1}{1-\dot{\tau}(\psi^{-1}(s))}\zeta^{\mathrm{T}}(s)Q\zeta(s)\mathrm{e}^{\varepsilon(s+\tau(\psi^{-1}(s)))}\,\mathrm{d}s.$$

Thus

$$\begin{aligned}
\dot{V}(t) &= \varepsilon\zeta^{\mathrm{T}}(t)P\zeta(t)\mathrm{e}^{\varepsilon t} + 2\zeta^{\mathrm{T}}(t)P\dot{\zeta}(t)\mathrm{e}^{\varepsilon t} + \frac{1}{1-\dot{\tau}(\psi^{-1}(t))}\zeta^{\mathrm{T}}(t)Q\zeta(t)\mathrm{e}^{\varepsilon(t+\tau(\psi^{-1}(t)))} \\
&\quad - \zeta^{\mathrm{T}}\big(t-\tau(t)\big)Q\zeta\big(t-\tau(t)\big)\mathrm{e}^{\varepsilon t} \\
&= \mathrm{e}^{\varepsilon t}\bigg[\varepsilon\zeta^{\mathrm{T}}(t)P\zeta(t) + 2\zeta^{\mathrm{T}}(t)P\dot{\zeta}(t) + \frac{\mathrm{e}^{\varepsilon\tau(\psi^{-1}(t))}}{1-\dot{\tau}(\psi^{-1}(t))}\zeta^{\mathrm{T}}(t)Q\zeta(t) \\
&\quad - \zeta^{\mathrm{T}}\big(t-\tau(t)\big)Q\zeta\big(t-\tau(t)\big)\bigg].
\end{aligned}$$

From (17) we have

$$\dot V(t) = \mathrm{e}^{\varepsilon t}\Big[\varepsilon\zeta^{\mathrm T}(t)P\zeta(t) - 2\zeta^{\mathrm T}(t)P\zeta(t) + 2\zeta^{\mathrm T}(t)PA(t)\zeta(t) + 2\zeta^{\mathrm T}(t)PB(t)\zeta\big(t-\tau(t)\big) + \frac{\mathrm{e}^{\varepsilon\tau(\psi^{-1}(t))}}{1-\dot\tau(\psi^{-1}(t))}\zeta^{\mathrm T}(t)Q\zeta(t) - 2\zeta^{\mathrm T}(t)P\big(n_{x(t)}(t) - n_{y(t)}(t)\big) - \zeta^{\mathrm T}\big(t-\tau(t)\big)Q\zeta\big(t-\tau(t)\big)\Big].$$

By Lemma 1 it has $\zeta^{\mathrm T}(t)P(n_{x(t)}(t) - n_{y(t)}(t)) \geqslant 0$, then

$$\begin{aligned}\dot V(t) &\leqslant \mathrm{e}^{\varepsilon t}\Big[\zeta^{\mathrm T}(t)\Big(\varepsilon P - 2P + PA(t) + A^{\mathrm T}(t)P + \frac{\mathrm{e}^{\varepsilon\tau(\psi^{-1}(t))}}{1-\dot\tau(\psi^{-1}(t))}Q\Big)\zeta(t) + 2\zeta^{\mathrm T}(t)PB(t)\zeta\big(t-\tau(t)\big) - \zeta^{\mathrm T}\big(t-\tau(t)\big)Q\zeta\big(t-\tau(t)\big)\Big]\\ &\leqslant \mathrm{e}^{\varepsilon t}W^{\mathrm T}\begin{pmatrix}(\varepsilon-2)P + PA(t) + A^{\mathrm T}(t)P + \frac{\mathrm{e}^{\varepsilon\tau(\psi^{-1}(t))}}{1-\dot\tau(\psi^{-1}(t))}Q & PB(t)\\ (PB(t))^{\mathrm T} & -Q\end{pmatrix}W,\end{aligned}$$

where $W = (\zeta^{\mathrm T}(t), \zeta^{\mathrm T}(t-\tau(t)))^{\mathrm T}$. Based on the Schur complement lemma and condition (18), it has $\dot V(t) \leqslant 0$ for all $t \geqslant 0$. Then,

$$V(0) \geqslant V(t) \geqslant \zeta^{\mathrm T}(t)P\zeta(t)\mathrm{e}^{\varepsilon t} \geqslant \lambda_{\min}(P)\big\|\zeta(t)\big\|^2\mathrm{e}^{\varepsilon t}.$$

Furthermore,

$$\begin{aligned}V(0) &= \zeta^{\mathrm T}(0)P\zeta(0) + \int_{-\tau(0)}^{0}\frac{1}{1-\dot\tau(\psi^{-1}(s))}\zeta^{\mathrm T}(s)Q\zeta(s)\mathrm{e}^{\varepsilon(s+\tau(\psi^{-1}(s)))}\,\mathrm{d}s\\ &\leqslant \lambda_{\max}(P)\big\|\zeta(0)\big\|^2 + \lambda_{\max}(Q)L\int_{-\tau(0)}^{0}\big\|\zeta(s)\big\|^2\,\mathrm{d}s\\ &\leqslant \big(\lambda_{\max}(P) + \lambda_{\max}(Q)L\tau(0)\big)\sup_{s\in[-\tau,0]}\big\|\zeta(s)\big\|^2,\end{aligned}$$

where

$$L = \sup_{s\in[-\tau,0]}\frac{\mathrm{e}^{\varepsilon(s+\tau(\psi^{-1}(s)))}}{1-\dot\tau(\psi^{-1}(s))}.$$

Thus,

$$\begin{aligned}\big\|x(t)-y(t)\big\| = \big\|\zeta(t)\big\| &\leqslant \sqrt{\frac{\lambda_{\max}(P) + \lambda_{\max}(Q)L\tau(0)}{\lambda_{\min}(P)}}\sup_{s\in[-\tau,0]}\big\|\zeta(s)\big\|\mathrm{e}^{-\varepsilon t/2}\\ &\leqslant M_0\|\varphi-\phi\|\mathrm{e}^{-\varepsilon t/2},\end{aligned}$$

where $M_0 \geqslant 1$ is a constant. The proof is completed. □

4 Numerical simulation

For the sake of demonstrating the correctness of main results, a second-order FRCNN (7) with time-varying delays is shown as follows:

$$\begin{pmatrix} \dot{x_1}(t) \\ \dot{x_2}(t) \end{pmatrix} = \left(-I + A(t)\right) \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + B(t) \begin{pmatrix} x_1(t-\tau(t)) \\ x_2(t-\tau(t)) \end{pmatrix} - N_Q \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, \tag{19}$$

where

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad N_Q \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} N_{[-1,1]}(x_1(t)) \\ N_{[-1,1]}(x_2(t)) \end{pmatrix}.$$

We take

$$A(t) = \begin{pmatrix} \frac{2+\cos t}{16} & \frac{1+\sin t}{16} \\ \frac{1-\sin t}{13} & \frac{1+\cos t}{13} \end{pmatrix}, \qquad B(t) = \begin{pmatrix} \frac{2-\cos t}{16} & \frac{2+\cos t}{16} \\ \frac{1-\cos t}{13} & \frac{2+\sin t}{13} \end{pmatrix},$$

and $\tau(t) = 1 + \cos t/2$. It is easy to see that $\inf_{t\in\mathbb{R}_+}\{1-\dot{\tau}(t)\} > 0$. Choose $\omega_i = 1$, $i = 1, 2$, and $r = 2$, then

$$\begin{aligned} &2 - 2\sum_{j=1}^{2} \left|a_{1j}(t)\right| - \sum_{j=1}^{2} \left|b_{1j}(t)\right| - \sum_{j=1}^{2} \frac{|b_{1j}(\psi^{-1}(t))|}{1-\dot{\tau}(\psi^{-1}(t))} \\ &\quad = 2 - \frac{5+\cos t+\sin t}{8} - \frac{1}{8(1-\dot{\tau}(\psi^{-1}(t)))} > 0 \end{aligned}$$

and

$$\begin{aligned} &2 - 2\sum_{j=1}^{2} \left|a_{2j}(t)\right| - \sum_{j=1}^{2} \left|b_{2j}(t)\right| - \sum_{j=1}^{2} \frac{|b_{2j}(\psi^{-1}(t))|}{1-\dot{\tau}(\psi^{-1}(t))} \\ &\quad = 2 - \frac{7-\sin t+\cos t}{13} - \frac{3-\cos(\psi^{-1}(t))-\sin(\psi^{-1}(t))}{13(1-\dot{\tau}(\psi^{-1}(t)))} > 0. \end{aligned}$$

This means the condition of Theorem 2 holds. Hence, system (19) is globally exponentially stable. Figure 1 shows the dynamic behaviors of second-order FRCNN (19) with eight initial states. Figure 2 illustrates the 2-dimensional phase plots with eight initial states. The simulation results show that system (19) exponentially converges to zero.

Remark 6. From the above analysis, the exponential stability of system (19) is guaranteed by using Theorem 2. In fact, if choose $P = I$, $Q = I/2$, based on Theorem 3, we can also acquire that system (19) is global exponentially stable.

Remark 7. From the simulation results in Figs. 1 and 2 we can get that the variation range of the input states does not exceed that of the output states. In other words, the normal cone improves the variation range of the input states, which is consistent with our theoretical analysis. That is, the difference between our system and the classical CNNs system.

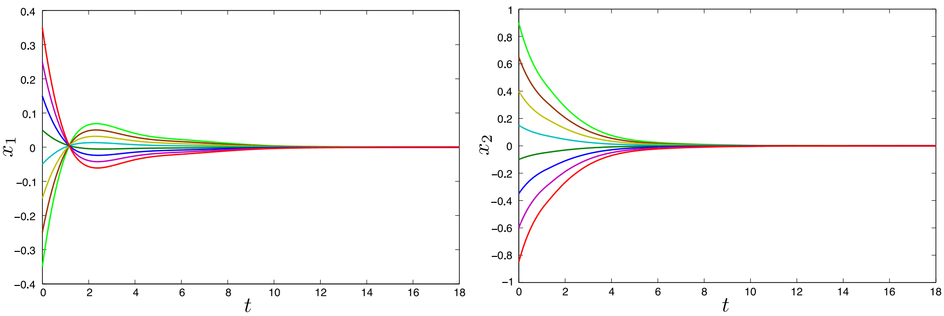

Figure 1. The states of variables $x_1(t)$ and $x_2(t)$.

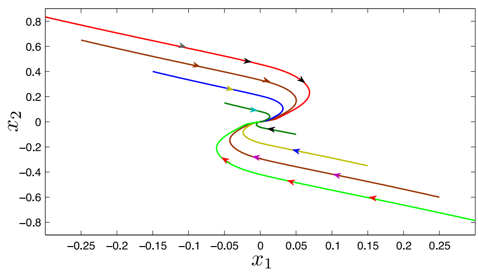

Figure 2. Phase plots of variable $x(t)$.

5 Conclusion

In this paper, the improved model of the FRCNNs is proposed and researched. The existence of the solution is proved by using set-valued analysis and differential inclusions. Based on the Hardy inequality and the matrix analysis method, some sufficient conditions upon the time-varying delays are obtained to achieve the GES. In this paper, the time-varying delay is bounded and continuously differentiable. However, the time-varying delay may be unbounded or discontinuous. Therefore, the relatively weak conditions about the time-varying delay will be considered in the future.

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