A sufficient and necessary condition of existence of blow-up radial solutions for a k-Hessian equation with a nonlinear operator[*]

Xinguang Zhang[a,1]**, Lishan Liu**[b]**, Yonghong Wu**[c]**, Yujun Cui**[d]

[a]School of Mathematical and Informational Sciences, Yantai University,
Yantai 264005, Shandong, China
zxg123242@163.com

[b]School of Mathematical Sciences, Qufu Normal University,
Qufu, 273165 Shandong, China
mathlls@163.com

[c]Department of Mathematics and Statistics, Curtin University of Technology,
Perth, WA 6845, Australia
y.wu@curtin.edu.au

[d]Department of Mathematics, Shandong University of Science and Technology,
Qingdao, 266590, Shandong, China
cyj720201@163.com

Abstract. In this paper, we establish the results of nonexistence and existence of blow-up radial solutions for a k-Hessian equation with a nonlinear operator. Under some suitable growth conditions for nonlinearity, the result of nonexistence of blow-up solutions is established, a sufficient and necessary condition on existence of blow-up solutions is given, and some further results are obtained.

Keywords: k-Hessian equation, existence and nonexistence, sufficient and necessary condition, blow-up solutions.

1 Introduction

Let D^2u be a square matrix of second-order partial derivatives for the scalar-valued function u, which is called the Hessian matrix in mathematics. In general, the Hessian matrix can describe the local curvature of the function u with multiple variables. Suppose $\lambda_1, \lambda_2, \ldots, \lambda_N$ are the eigenvalues of the Hessian matrix D^2u, define the k-Hessian

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[1]Corresponding author.

operator $S_k(\lambda(D^2u))$ as follows:

$$S_k\big(\lambda\big(D^2u\big)\big) = \sum_{1\leqslant i_1<i_2<\cdots<i_k\leqslant N} \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k}, \quad k = 1, 2, \ldots, N,$$

where $\lambda(D^2u) = (\lambda_1, \lambda_2, \ldots, \lambda_N)$ is the vector of eigenvalues of D^2u, that is $S_k(\lambda(D^2u))$ is the sum of the kth principal minors of the Hessian matrix, and here we define

$$\Gamma_k := \big\{\lambda \in \mathbb{R}^N \colon S_l\big(\lambda\big(D^2u\big)\big) > 0, \ 1 \leqslant l \leqslant k\big\}.$$

On the other hand, the k-Hessian operator can also be expressed as the following form of divergence:

$$S_k\big(\lambda\big(D^2u\big)\big) = \frac{1}{k}\sum_{i,j=1}^{N}\big(S_k^{ij}u_i\big)_j,$$

where $S_k^{ij} = \partial S_k(D^2u)/\partial u_{ij}$, and for more details, see Trudinger et al. [50] and Gavitone [7].

For $k > 1$, $S_k(\lambda(D^2u))$ is a second-order fully nonlinear differential operator. When $k = N$, the k-Hessian operator reduces to the Monge–Ampère operator $\det D^2u$, and if $k = 1$, then $S_k(\lambda(D^2u))$ turns into a Laplace operator, which implies that the k-Hessian operator constructs a discrete collection of partial differential operators including the Monge–Ampère operator $\det D^2u$ and the Laplace operator Δu as special cases.

In this paper, we consider the existence and nonexistence of blow-up solutions for the following k-Hessian equation with a nonlinear operator:

$$\mathfrak{B}\big(S_k^{1/k}\big(\lambda\big(D^2u\big)\big)\big)S_k^{1/k}\big(\lambda\big(D^2u\big)\big) = f\big(|x|, u\big), \quad x \in \mathbb{R}^N \ (N > 2k), \qquad (1)$$

where $f : C((0, +\infty) \times \mathbb{R} \to [0, +\infty))$ is nondecreasing, and the nonlinear operator $\mathfrak{B} \in \mathcal{X}$ satisfies the following property:

$$\mathcal{X} = \big\{\mathfrak{B} \in C^2\big([0, +\infty), [0, +\infty)\big) \colon \text{there exists a constant } \alpha > 0$$
$$\text{such that, for any } 0 < b < 1, \ \mathfrak{B}(bs) \leqslant b^\alpha\mathfrak{B}(s)\big\}.$$

Many existing work for the k-Hessian equation is devoted to constructing mathematical theory rather than modeling or exploring new applications. Here we will briefly mention some potential implications in physics and applied mathematics. Escudero [3] described some phenomena of non-equilibrium phase transitions and statistical physics by using the k-Hessian equation. In addition, the k-Hessian equation also gives a class of important fully nonlinear elliptic operators related to geometric optics [51], study of quasilinear parabolic problems [40], optimal transportation, and isometric embedding [19].

The k-Hessian equation of form (1) has attracted much research interest, and there are many rich literatures concerning the k-Hessian equation. If $\mathfrak{B}(x) = x^{k-1}$, then the k-Hessian equation (1) reduces to the standard Hessian equation $S_k(\lambda(D^2u)) = f$ and the

standard Monge–Ampère equation when $k = n$. Caffarelli et al. [1] studied the existence and a priori estimate of the smooth solutions for the k-Hessian equation

$$S_k\big(\lambda\big(D^2u\big)\big) = f \quad \text{in } \Omega \subset \mathbb{R}^N,$$

$$u = \varphi \quad \text{on } \partial\Omega,$$

and the work was then extended to more general equations in [25,49]. If $\mathfrak{B}(x) = \text{const} \neq 0$, then the k-Hessian equation becomes the Hessian equation

$$S_k^{1/k}\big(\lambda\big(D^2u\big)\big) = f, \quad x \in \mathbb{R}^N \ (N > 2k),$$

Covei [2] considered the existence of positive radial solutions for the above Hessian equation, provided that $f = p(|x|)h(u)$. If $\mathfrak{B}(x) = |x|^{p-2}$, $p \geqslant 2$, the k-Hessian equation (1) turns into the form

$$\varphi_p\big(S_k^{1/k}\big(\lambda\big(D^2u\big)\big)\big) = f\big(|x|, -u\big) \quad \text{in } \mathbb{R}^N,$$

which is a p-Poisson Hessian equation, and few work was reported. Thus, since the k-Hessian equation (1) involves a nonlinear operator $\mathfrak{B}$, it includes many interesting and important equations such as the Hessian equation, the Monge–Ampère equation, the p-Poisson Hessian equation as special cases. To the best of our knowledge, no result has been reported on the existence and nonexistence of blow-up solutions for the k-Hessian equation (1), and this is the first paper using the iterative method to study the k-Hessian equation involving a nonlinear operator.

The rest of the paper is organized as follows. In Section 2, we firstly study the property of the nonlinear operator $\mathfrak{B}$ and then transform the k-Hessian partial differential equation (1) to an ordinary differential equation. In Section 3, we establish the results of the nonexistence of blow-up solutions for the k-Hessian equation (1) by some estimations. In Section 4, a necessary and sufficient condition on existence of blow-up solutions for equation (1) is established and some further results are obtained.

2 Preliminary results on radial solutions

Before we give a detailed description of our main results, we firstly establish a property for the inverse operator of the operator $s\mathfrak{B}(s)$.

Lemma 1. *Let $\mathfrak{L}(s) = s\mathfrak{B}(s)$, $\mathfrak{B} \in \mathcal{X}$, then there exists a nonnegative increasing inverse mapping $\mathfrak{L}^{-1}(s)$, and for any $0 < b < 1$,*

$$\mathfrak{L}^{-1}(bs) \geqslant b^{1/(1+\alpha)}\mathfrak{L}^{-1}(s).$$

Proof. Firstly, we assert that $\mathfrak{B}$ is an increasing operator if $\mathfrak{B} \in \mathcal{X}$. In fact, for any $\mathfrak{B} \in \mathcal{X}$ and $s, t \in [0, +\infty)$, without loss of the generality, let $0 \leqslant s < t$. If $s = 0$, then, obviously, $\mathfrak{B}(s) \leqslant \mathfrak{B}(t)$ holds. If $s \neq 0$, let $c_0 = s/t$, then $0 < c_0 < 1$. It follows from the property of $\mathfrak{B}$ that

$$\mathfrak{B}(s) = \mathfrak{B}(c_0t) \leqslant c_0^\alpha \mathfrak{B}(t) \leqslant \mathfrak{B}(t),$$

which implies that $\mathfrak{B}$ is an increasing operator. Thus we have $\mathfrak{L}'(s) = (s\mathfrak{B}(s))' > 0$ for any $s > 0$, which implies that $\mathfrak{L}$ is a bijection on $(0, \infty)$ and has a nonnegative increasing inverse mapping $\mathfrak{L}^{-1}(s)$.

On the other hand, for any $0 < b < 1$, let $c = b^{1/(1+\alpha)}$, then $0 < c < 1$. Thus we have

$$\mathfrak{L}(cx) = cx\mathfrak{B}(cx) \leqslant c^{1+\alpha}x\mathfrak{B}(x) = c^{1+\alpha}\mathfrak{L}(x), \quad x > 0.$$

Consequently, let $s = \mathfrak{L}(x)$, then

$$c\mathfrak{L}^{-1}(s) = cx \leqslant \mathfrak{L}^{-1}\big(c^{1+\alpha}\mathfrak{L}(x)\big) = \mathfrak{L}^{-1}(bs),$$

that is

$$b^{1/(1+\alpha)}\mathfrak{L}^{-1}(s) \leqslant \mathfrak{L}^{-1}(bs).$$

The proof is completed. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 1. Clearly, if $a \geqslant 1$, then we have

$$\mathfrak{L}^{-1}(as) \leqslant a^{1/(1+\alpha)}\mathfrak{L}^{-1}(s).$$

Remark 2. The operator set $\mathcal{X}$ includes a large class of operators and the standard type of operators is $\mathfrak{B}(s) = \sum_{i=1}^{n} s^{\alpha_i}$, $\alpha_i > 0$. In fact, take $\alpha = \min\{\alpha_1, \dots, \alpha_n\} > 0$, then for any $0 < c < 1$, one has

$$\mathfrak{B}(cs) \leqslant c^{\alpha}\mathfrak{B}(s).$$

For $B_R := \{x \in \mathbb{R}^N \colon |x| < R\}$ and $r = \sqrt{x_1^2 + x_2^2 + \cdots + x_N^2}$, the following lemma has been proven.

Lemma 2. (*See* [18].) *Assume $y(r) \in C^2[0, R)$ is radially symmetric and $y('0) = 0$. Then the function $u(|x|) = y(r)$ with $r = |x| < R$ is $C^2(B_R)$, and*

$$\lambda(D^2 u) = \begin{cases} (y''(r), \frac{y'(r)}{r}, \dots, \frac{y'(r)}{r}), & r \in (0, R), \\ (y''(0), y''(0), \dots, y''(0)), & r = 0, \end{cases}$$

$$S_k\big(\lambda(D^2 u)\big) = \begin{cases} C_{N-1}^{k-1} y''(r)(\frac{y'(r)}{r})^{k-1} + C_{N-1}^{k}(\frac{y'(r)}{r})^{k}, & r \in (0, R), \\ C_N^{k}(y''(0))^{k}, & r = 0, \end{cases}$$

where $C_N^k = N!/(k!(N-k)!)$.

Now we consider the k-Hessian equation (1). By using Lemma 1, equation (1) can be transformed into the following form:

$$S_k\big(\lambda(D^2 u)\big) = \big[\mathfrak{L}^{-1}\big(f(|x|, u)\big)\big]^k, \quad x \in \mathbb{R}^N \; (N > 2k). \tag{2}$$

Make the radial transformation $u(|x|) = y(r)$, then from Lemma 2 and (2), $u(|x|) = y(r)$ is a radial solution of equation (1) if and only if $y(r)$ is a solution of the ODE

$$C_{N-1}^{k-1} y''(r) \left(\frac{y'(r)}{r} \right)^{k-1} + C_{N-1}^k \left(\frac{y'(r)}{r} \right)^k = \left[\mathfrak{L}^{-1} \big(f(r, -y(r)) \big) \right]^k, \quad r > 0. \quad (3)$$

Take any central value $y(0) = d > 0$ for equation (1). Since

$$y'(0) = \lim_{r \to 0} \frac{y(r) - d}{r - 0} = \lim_{\xi \to 0} y'(\xi)$$

$$= \lim_{\xi \to 0} \left(\frac{k}{\xi^{N-k}} \int_0^\xi \frac{s^{N-1}}{C_{N-1}^{k-1}} \left[\mathfrak{L}^{-1} \big(f(r, y(s)) \big) \right]^k \, \mathrm{d}s \right)^{1/k} = 0,$$

we have $y(r) \in C^1[0, R)$. Consequently,

$$y''(0) = \lim_{r \to 0} \frac{y'(r) - y'(0)}{r - 0} = \lim_{r \to 0} \left(\frac{k}{r^N} \int_0^r \frac{s^{N-1}}{C_{N-1}^{k-1}} \left[\mathfrak{L}^{-1} \big(f(r, y(s)) \big) \right]^k \, \mathrm{d}s \right)^{1/k}$$

$$= \lim_{r \to 0} \left(\frac{k r^{N-1} [\mathfrak{L}^{-1}(f(r, y(r)))]^k}{C_{N-1}^{k-1} r^{N-1}} \right)^{1/k} = \left(\frac{k [\mathfrak{L}^{-1}(f(0, d))]^k}{C_{N-1}^{k-1}} \right)^{1/k}, \quad (4)$$

which implies that $y(r) \in C^2[0, R)$. Thus it follows from (4) that the following lemma holds.

Lemma 3. *For any number $d > 0$, assume $y \in C[0, R) \cap C^1(0, R)$ is a solution to the Cauchy problem*

$$y'(r) = \left(\frac{k}{r^N} \int_0^r \frac{s^{N-1}}{C_{N-1}^{k-1}} \left[\mathfrak{L}^{-1} \big(f(r, y(s)) \big) \right]^k \, \mathrm{d}s \right)^{1/k}, \quad y(0) = d, \ r > 0.$$

Then $y(r) \in C^2(B_R)$ and satisfies equation (3) with $y'(0) = 0$, and

$$\lambda_r = \left(y''(r), \frac{y'(r)}{r}, \dots, \frac{y'(r)}{r} \right) \in \Gamma_k, \quad 0 \leqslant r < R.$$

Thus in this paper, we only focus on the classical blow-up solutions of the k-Hessian equation (1), namely, a function $u(r)$ in $C^2[0, +\infty)$ satisfies the k-Hessian equation (1) and $\lim_{r \to +\infty} u(r) = \infty$. Normally, a blow-up behaviour is often caused by singularity [12, 16, 20, 24, 26, 27, 59, 87] and impulse behavior of nonlinearities [4–6, 8, 10, 21, 30, 34, 44, 45, 52, 53, 56, 63–65, 90, 91] in various complex process, but here blow-up phenomena refers to the boundary blow of solution; for more details, see [75, 77, 79, 85].

Our main tool is analysis method and iterative technique as well as some suitable estimations. Here, in order to make the article more complete, we also recall some analysis methods and numerical techniques, which have been used to study various nonlinear ordinary differential equations and partial differential equations such as iterative methods [9, 11, 28, 29, 42, 43, 48, 54, 60, 61, 72, 76, 81, 84, 86], variational methods [17, 31–33, 36–39, 41, 46, 62, 67, 74, 78, 80, 83, 88, 89], fixed point theorems [13–15, 22, 23, 35, 47, 55, 57, 66, 70, 71, 73, 82], and upper and lower solution methods [58, 68, 69]. These theoretical methods and tools not only enrich the scientific field of differential equations, but also give some new scientific explanations and practical applications for the corresponding natural phenomena.

3 Nonexistence of blow-up solutions

Now let us list a growth assumption, which is to be used in the rest of this paper.

(A) $f \in C((0, +\infty) \times [0, +\infty), (0, +\infty))$ is nondecreasing on the second variable, and there exists a positive constant $\beta < 1 + \alpha$ such that

$$f(s, cu) \geqslant c^{\beta} f(s, u) \quad \forall (s, u) \in (0, +\infty) \times [0, +\infty), \ c \in (0, 1].$$

Remark 3. It follows from (A) that, for any $c \geqslant 1$, $(s, u) \in (0, +\infty) \times [0, +\infty)$,

$$f(s, cu) \leqslant c^{\beta} f(s, u).$$

Under the above growth condition, we have the following result of nonexistence of blow-up solution for the k-Hessian equation (1).

Theorem 1. *Suppose* (A) *holds, and the fast decay condition is satisfied*:

$$\int\limits_0^\infty \left(\frac{k}{t^{N-k}} \int\limits_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} \left[\mathfrak{L}^{-1}\big(f(s, 1)\big) \right]^k \mathrm{d}s \right)^{1/k} \mathrm{d}t < +\infty, \tag{5}$$

then the k-Hessian equation (1) *has no positive entire blow-up solutions*.

Proof. We shall establish the result of nonexistence by using the method of contradiction. Suppose that the k-Hessian equation (1) has a positive entire blow-up solution, say $u = y(r)$. Then by (5), there exists a sufficiently large $\widetilde{r} > 0$ such that

$$\int\limits_{\widetilde{r}}^\infty \left(\frac{k}{t^{N-k}} \int\limits_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} \left[\mathfrak{L}^{-1}\big(f(s, 1)\big) \right]^k \mathrm{d}s \right)^{1/k} \mathrm{d}t < 1.$$

Since y is radially symmetric, by using the standard integrating procedure for (3), we have

$$y(r) = y(0) + \int\limits_0^r \left(\frac{k}{t^{N-k}} \int\limits_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} \left[\mathfrak{L}^{-1}\big(f(s, y(s))\big) \right]^k \mathrm{d}s \right)^{1/k} \mathrm{d}t. \tag{6}$$

Consequently, for any $r > \widetilde{r} > 0$, (6) yields

$$y(r) = y(\widetilde{r}) + \int_{\widetilde{r}}^{r} \left(\frac{k}{t^{N-k}} \int_{0}^{t} \frac{s^{N-1}}{C_{N-1}^{k-1}} \left[\mathfrak{L}^{-1}\big(f\big(s, y(s)\big)\big) \right]^{k} \mathrm{d}s \right)^{1/k} \mathrm{d}t. \tag{7}$$

Thus it follows from (7), (A), and Remark 1 that

$$y(r) \leqslant y(\widetilde{r}) + \int_{\widetilde{r}}^{r} \left(\frac{k}{t^{N-k}} \int_{0}^{t} \frac{s^{N-1}}{C_{N-1}^{k-1}} \left[\mathfrak{L}^{-1}\big(\big(y(s) + 1\big)^{\beta} f(s, 1)\big) \right]^{k} \mathrm{d}s \right)^{1/k} \mathrm{d}t$$

$$\leqslant y(\widetilde{r}) + \big(y(r) + 1\big)^{\beta/(1+\alpha)} \int_{\widetilde{r}}^{r} \left(\frac{k}{t^{N-k}} \int_{0}^{t} \frac{s^{N-1}}{C_{N-1}^{k-1}} \left[\mathfrak{L}^{-1}\big(f(s, 1)\big) \right]^{k} \mathrm{d}s \right)^{1/k} \mathrm{d}t$$

$$\leqslant y(\widetilde{r}) + \big(y(r) + 1\big)^{\beta/(1+\alpha)} \quad \forall r > \widetilde{r},$$

which implies that

$$1 \leqslant \frac{y(\widetilde{r})}{y(r)} + \frac{(y(r) + 1)^{\beta/(1+\alpha)}}{y(r)} \to 0 \quad \text{as } r \to +\infty,$$

which contradicts with the assumption that y is the positive entire blow-up solution of equation (1) since $0 < \beta/(1 + \alpha) < 1$. Thus (1) has no positive entire radial blow-up solution, and the proof is completed. □

4 Existence of entire blow-up solutions

Theorem 2. *Assume that* (A) *holds. Then the k-Hessian equation* (1) *has infinitely many positive entire blow-up radial solutions if and only if the function f satisfies the following slow decay condition*:

$$\int_{0}^{\infty} \left(\frac{k}{t^{N-k}} \int_{0}^{t} \frac{s^{N-1}}{C_{N-1}^{k-1}} \left[\mathfrak{L}^{-1}\big(f(s, 1)\big) \right]^{k} \mathrm{d}s \right)^{1/k} \mathrm{d}t = \infty. \tag{8}$$

Proof. Sufficient condition. We firstly prove that the k-Hessian equation (1) has a positive radial solution. For this purpose, according to Lemma 3 and (3), we consider the equivalent integral equation

$$y(r) = y(0) + \int_{0}^{r} \left(\frac{k}{t^{N-k}} \int_{0}^{t} \frac{s^{N-1}}{C_{N-1}^{k-1}} \left[\mathfrak{L}^{-1}\big(f\big(s, y(s)\big)\big) \right]^{k} \mathrm{d}s \right)^{1/k} \mathrm{d}t. \tag{9}$$

In the following, we shall construct a positive increasing sequence $\{y^{(m)}(r)\}_{m \geqslant 1}$ by using (9), which is bounded above on $[0, R]$ for any fixed $R > 0$. For this purpose,

we take the central value $y(0) = d > 0$ for equation (3), and then define recursively a sequence $\{y^{(m)}(r)\}_{m \geqslant 1}$ on $[0, \infty)$ by

$$y^{(0)}(r) = d \quad \forall r \geqslant 0$$

and

$$y^{(m)}(r) = d + \int_0^r \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} \left[\mathfrak{L}^{-1} \big(f\big(s, y^{(m-1)}(s)\big) \big) \right]^k \mathrm{d}s \right)^{1/k} \mathrm{d}t. \quad (10)$$

Obviously, $\{y^{(m)}(r)\}_{m \geqslant 1}$ is an increasing sequence of nonnegative and increasing functions, and $y^{(m)}(r) \geqslant d$ for all $r \geqslant 0$ and $m \in N$. Thus it is sufficient to prove that the sequences $\{y^{(m)}(r)\}_{m \geqslant 1}$ are bounded from above on bounded subsets for obtaining our desired conclusion.

Fix $R > 0$, since f is increasing with respect to the second variable y on $[0, \infty)$, then it follows from (A) and Remarks 1 and 3 that

$$y^{(m)}(R) = d + \int_0^R \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} \left[\mathfrak{L}^{-1}\big(f\big(s, y^{(m-1)}(s)\big) \big) \right]^k \mathrm{d}s \right)^{1/k} \mathrm{d}t$$

$$\leqslant d + \int_0^R \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} \left[\mathfrak{L}^{-1}\big(f\big(s, y^{(m)}(s)\big) \big) \right]^k \mathrm{d}s \right)^{1/k} \mathrm{d}t$$

$$\leqslant d + \int_0^R \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} \left[\mathfrak{L}^{-1}\big(\big(y^{(m)}(s) + 1\big)^\beta f(s, 1)\big) \right]^k \mathrm{d}s \right)^{1/k} \mathrm{d}t$$

$$\leqslant d + \left[\big(y^{(m)}(R)\big) + 1 \right]^{\beta/(1+\alpha)}$$

$$\times \int_0^R \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} \left[\mathfrak{L}^{-1}\big(f(s, 1)\big) \right]^k \mathrm{d}s \right)^{1/k} \mathrm{d}t. \quad (11)$$

Hence from the monotonicity of $\{y^{(m)}(R)\}_{m \geqslant 1}$ we have

$$L(R) := \lim_{m \to \infty} \big(y^{(m)}(R) \big) \leqslant +\infty.$$

We assert that $L(R)$ is finite. In fact, if not, by (11), (8), and $0 < \beta < 1 + \alpha$, one has

$$1 \leqslant \frac{d}{y^{(m)}(R)} + \frac{\left[\big(y^{(m)}(R)\big) + 1 \right]^{\beta/(1+\alpha)}}{y^{(m)}(R)}$$

$$\times \int_0^R \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} \left[\mathfrak{L}^{-1}\big(f(s, 1)\big) \right]^k \mathrm{d}s \right)^{1/k} \mathrm{d}t$$

$$\to 0 \quad \text{as } m \to \infty,$$

which is a contradiction. So $L(R)$ is finite, and the mapping $L : (0, \infty) \to (0, \infty)$ is increasing since $y^{(m)}(r)$ is an increasing function in $(0, \infty)$. Thus for any $r \in [0, R]$ and $m \geqslant 1$, we have

$$y^{(m)}(r) \leqslant y^{(m)}(R) \leqslant L(R),$$

which implies that the sequence $\{(y^{(m)}(r))\}_{m \geqslant 1}$ is bounded from above on bounded sets. Letting

$$y(r) := \lim_{m \to \infty} y^{(m)}(r) \quad \forall r \geqslant 0,$$

and taking the limit for (10), we get that y is a positive solution of (1).

Now to complete the proof, we only need to prove that y blows up. Indeed, by (9) and (8), we have

$$y(r) = d + \int_0^r \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} \left[\mathfrak{L}^{-1}\big(f(s, y(s))\big) \right]^k \mathrm{d}s \right)^{1/k} \mathrm{d}t$$

$$\geqslant d + \int_0^r \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} \left[\mathfrak{L}^{-1}\big(f(s, d)\big) \right]^k \mathrm{d}s \right)^{1/k} \mathrm{d}t$$

$$\geqslant d + \int_0^r \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} \left[\mathfrak{L}^{-1}\big(f(s, \min\{1, d\})\big) \right]^k \mathrm{d}s \right)^{1/k} \mathrm{d}t$$

$$\geqslant d + \big(\min\{1, d\}\big)^{\beta/(1+\alpha)}$$

$$\times \int_0^r \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} \left[\mathfrak{L}^{-1}\big(f(s, 1)\big) \right]^k \mathrm{d}s \right)^{1/k} \mathrm{d}t \to +\infty \quad \text{as } r \to +\infty,$$

which implies that y is a blow-up solution of the k-Hessian equation (1). Noticing that $d \in (0, \infty)$ is chosen arbitrarily, the k-Hessian equation (1) has infinitely many positive entire blow-up solutions.

Necessary condition. For the proof of necessity, we still adopt the method of contradiction. Suppose y is any positive entire blow-up solution of the k-Hessian equation (1) and (8) is not true, that is

$$\int_0^\infty \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} \left[\mathfrak{L}^{-1}\big(f(s, 1)\big) \right]^k \mathrm{d}s \right)^{1/k} \mathrm{d}t < \infty,$$

and

$$y(r) = y(0) + \int_0^r \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} \left[\mathfrak{L}^{-1}\big(f(s, y(s))\big) \right]^k \mathrm{d}s \right)^{1/k} \mathrm{d}t,$$

$$\lim_{r \to \infty} y(r) = +\infty.$$

Then it follows from the fact of y being a positive solution of the k-Hessian equation (1) that

$$y'(r) = \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} \left[\mathfrak{L}^{-1}\big(f\big(s, y(s)\big)\big) \right]^k \mathrm{d}s \right)^{1/k} \geqslant 0,$$

which implies that $y(r)$ is increasing on $[0, \infty)$. By (A), (9), and Remark 1, one has

$$y(r) \leqslant y(0) + \int_0^r \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} \left[\mathfrak{L}^{-1}\big((y(s) + 1\big)^\beta f(s, 1)\big) \right]^k \mathrm{d}s \right)^{1/k} \mathrm{d}t$$

$$\leqslant y(0) + \left[(y(r)) + 1 \right]^{\beta/(1+\alpha)} \int_0^r \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} \left[\mathfrak{L}^{-1}\big(f(s, 1)\big) \right]^k \mathrm{d}s \right)^{1/k} \mathrm{d}t.$$

Noticing that $\lim_{r \to +\infty} y(r) = +\infty$, we have

$$1 \leqslant \frac{y(0)}{y(r)} + \frac{\left[(y(r)) + 1 \right]^{\beta/(1+\alpha)}}{y(r)} \int_0^r \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} \left[\mathfrak{L}^{-1}\big(f(s, 1)\big) \right]^k \mathrm{d}s \right)^{1/k} \mathrm{d}t$$

$$\to 0 \quad \text{as } r \to \infty,$$

which is a contradiction. The necessity of Theorem 2 is proved. $\qquad\square$

5 Further results

In this section, we discuss some alternative conditions of (A) and give some further results.

(B) $f \in C((0, +\infty) \times [0, +\infty), (0, +\infty))$ is nondecreasing on the second variable u, and there exists a positive constant $\beta < 1 + \alpha$ such that, for all $s \in (0, +\infty)$, $f(s, u)/u^\beta$ is decreasing on the variable $u \in (0, +\infty)$.

Remark 4. Assumption (B) is easy to check, and there is a large class of functions to satisfy assumption (B). The typical example is $f(s, u) = \sum_{i=1}^m g_i(s) u^{\beta_i}$, where $\max\{\beta_1, \beta_2, \ldots, \beta_m\} < 1 + \alpha$, $g_i(s) > 0$, $s \in (0, \infty)$, $i = 1, 2, \ldots, m$.

In fact, clearly, $f(s, u)$ is nondecreasing on the second variable u on $[0, +\infty)$. Take $\beta = \max\{\beta_1, \beta_2, \ldots, \beta_m\}$, then $f(s, u)/u^\beta$ is decreasing on the variable $u \in (0, +\infty)$.

Now we show that assumption (B) implies assumption (A).

Proof. Notice that $f(s, u)/u^\beta$ is decreasing on the variable $u \in (0, +\infty)$, for any $0 < c \leqslant 1$ and $u \in (0, +\infty)$, we have

$$\frac{f(s, u)}{u^\beta} \leqslant \frac{f(s, cu)}{(cu)^\beta},$$

which implies that $f(s, cu) \geqslant c^\beta f(s, u)$. The proof is completed. $\qquad\square$

Thus we have the following corollaries:

Corollary 1. *Suppose* (B) *holds and* f *satisfies the fast decay condition* (5)*, then the* k-*Hessian equation* (1) *has no positive entire blow-up solutions.*

Corollary 2. *Suppose* (B) *holds, then the* k-*Hessian equation* (1) *has infinitely many positive entire blow-up radial solutions if and only if* f *satisfies the slow decay condition* (8)*.*

(C) $f : (0, +\infty) \times [0, +\infty) \to (0, +\infty)$ *is continuous and there exists a positive constant* $\beta < 1 + \alpha$ *such that* $\partial f / \partial u \geqslant 0$ *and*

$$u \frac{\partial f(s, u)}{\partial u} \leqslant \beta f(s, u) \quad \forall u \in (0, \infty). \tag{12}$$

Obviously, (12) implies that $f(s, u)/u^\beta$ is decreasing on the variable $u \in (0, +\infty)$, and thus Corollaries 1 and 2 can be rewritten as:

Corollary 3. *Suppose* (C) *holds and* f *satisfies the fast decay condition* (5)*, then the* k-*Hessian equation* (1) *has no positive entire blow-up solution.*

Corollary 4. *Suppose* (C) *holds, then the* k-*Hessian equation* (1) *has infinitely many positive entire blow-up radial solutions if and only if* f *satisfies the slow decay condition* (8)*.*

Now we consider the case where the nonlinearity includes a weigh function p, i.e, $f(x, u) = p(|x|)g(u)$.

(D) $f \in C([0, +\infty), (0, +\infty))$ *is nondecreasing and* $p(t) \geqslant 0$ *for* $t \in [0, +\infty)$*, and there exists a positive constant* $\beta < 1 + \alpha$ *such that*

$$0 < \sup_{u > 0} \frac{f(u)}{u^\beta} < \infty.$$

By checking the proof of Theorems 1 and 2, the conclusions can be rewritten as follows:

Theorem 3. *Suppose* (D) *holds, and the following fast decay condition holds*:

$$\int\limits_0^\infty \left(\frac{k}{t^{N-k}} \int\limits_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} \left[\mathfrak{L}^{-1}\big(p(s)\big) \right]^k \mathrm{d}s \right)^{1/k} \mathrm{d}t < +\infty,$$

then the k-*Hessian equation* (1) *has no positive entire blow-up solution.*

Theorem 4. *Assume that* (D) *holds, and the weigh function* p *satisfies the following slow decay condition*:

$$\int\limits_0^\infty \left(\frac{k}{t^{N-k}} \int\limits_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} \left[\mathfrak{L}^{-1}\big(p(s)\big) \right]^k \mathrm{d}s \right)^{1/k} \mathrm{d}t = \infty. \tag{13}$$

Then the k-*Hessian equation* (1) *has infinitely many positive entire blow-up radial solutions if and only if the slow decay condition* (13) *is satisfied.*

In addition, noticing that

$$\int\limits_0^\infty \left(\frac{k}{t^{N-k}} \int\limits_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} \, \mathrm{d}s \right)^{1/k} \mathrm{d}t = \infty$$

always holds, if the weigh $p \equiv \mathrm{const} \neq 0$, then we only have the result of existence of blow-up positive solutions, that is

Corollary 5. *Assume that* (D) *holds, then the k-Hessian equation* (1) *has infinitely many positive entire blow-up radial solutions.*

6   Numerical example

In this section, we present an example to illustrate our main results.

Example 1. Consider the existence and nonexistence of radial blow-up solutions for the following 3-Hessian equation:

$$\left(S_3^{1/3}\left(\lambda(D^2 u)\right)\right)^2 S_3^{1/3}\left(\lambda(D^2 u)\right) = p(|x|)u^{1/4} \quad \text{in } \mathbb{R}^7, \tag{14}$$

where $p(s) > 0$, $s \in (0, +\infty)$. Then

- (i) If $p(r) = 4/(r^3(1 + r^4)^2)$, the k-Hessian equation (14) has no positive entire blow-up solutions.
- (ii) If $p(r) = e^r(4r^{-3} + r^{-2})$, the k-Hessian equation (14) has infinitely many positive entire blow-up radial solution.

In fact, here $k = 3$, $N = 7$, $f(x, u) = p(|x|)u^{1/4}$, and $\mathfrak{B}(x) = x^2$. Thus we have $\mathfrak{L}^{-1}(x) = x^{1/3}$ and

$$f(s, cu) = p(s)(cu)^{1/4} \geqslant p(s)c^{1/3}u^{1/2} = c^{1/3} f(s, u)$$

for all $(s, u) \in (0, +\infty) \times [0, +\infty)$, $c \in (0, 1]$, which implies that (A) holds.

(i) If $p(r) = 4/(r^3(1 + r^4)^2)$, then

$$\int\limits_0^\infty \left(\frac{k}{t^{N-k}} \int\limits_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} \left[\mathfrak{L}^{-1}\left(f(s, 1)\right)\right]^k \mathrm{d}s \right)^{1/k} \mathrm{d}t$$

$$= \int\limits_0^\infty \left(\frac{3}{t^4} \int\limits_0^t \frac{4s^3}{C_6^2(1 + s^4)^2} \, \mathrm{d}s \right)^{1/3} \mathrm{d}t = \frac{4}{5} \int\limits_0^\infty \frac{1}{\sqrt[3]{s^4 + 1}} \, \mathrm{d}t < \infty.$$

According to Theorem 1, the k-Hessian equation (14) has no positive entire blow-up solutions.

(ii) If $p(r) = \mathrm{e}^r(4r^{-3} + r^{-2})$, we have

$$
\int_0^\infty \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} \left[\mathfrak{L}^{-1}\big(f(s,1)\big) \right]^k \mathrm{d}s \right)^{1/k} \mathrm{d}t
$$

$$
= \int_0^\infty \left(\frac{3}{t^4} \int_0^t \frac{\mathrm{e}^s(4s^3 + s^4)}{C_6^2} \,\mathrm{d}s \right)^{1/3} \mathrm{d}t = \int_0^\infty \mathrm{e}^{t/3} \,\mathrm{d}t = \infty.
$$

According to Theorem 2, the k-Hessian equation (14) has infinitely many positive entire blow-up radial solutions.

Remark 5. From Example 1 we can see that assumption (A) permits $f(s,u)$ to have singularity at $s = 0$, which allows that the k-Hessian equation (1) can describe some singular phenomena such as those in [22, 35, 58, 68, 69, 71, 82].

7   Conclusion

The k-Hessian operator is an important fully nonlinear and non-uniform elliptic partial differential operator, which includes the Monge–Ampère operator and the Laplace operator as special cases, thus the study of the k-Hessian equation is interesting and challenging. In this paper, we introduce a new nonlinear operator $\mathfrak{B}$ with suitable growth condition, and the k-Hessian equation is extended to more generalized cases. In addition, we also put forward some suitable growth conditions for nonlinearity for obtaining the results of nonexistence and existence of blow-up radial solutions for the k-Hessian equation. The conditions are easy to be checked, and the corresponding results improve and generalize some recent results such as those in [1, 2, 18, 25, 49].

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