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# New criteria on global asymptotic synchronization of Duffing-type oscillator system*

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**Abstract.** In this paper, we are concerned with global asymptotic synchronization of Duffing-type oscillator system. Without using matrix measure theory, graph theory and LMI method, which are recently widely applied to investigating global exponential/asymptotic synchronization for dynamical systems and complex networks, four novel sufficient conditions on global asymptotic synchronization for above system are acquired on the basis of constant variation method, integral factor method and integral inequality skills.

**Keywords:** Duffing-type oscillator system, global asymptotic synchronization, constant variation method and integral factor method, integral inequality skills.

## 1 Introduction

The synchronization problem is an important issue of the complex networks. The complex networks exist everywhere in the real world, such as Internet, World Wide Web, electrical power grids, social community networks, global economic markets and ecosystems. The synchronization of complex networks has been widely investigated in the past few years, for example, see [4, 10, 11] and references therein.

Generally speaking, networks are formed by a large number of linked nodes coupled by edges. In practice, some coupled "nodes" will be more effective than a single "node". In a word, collective behaviors in networks or systems have attracted much attention. It is worth mentioning that synchronization in coupled oscillators or system has become an important research topic and can be an explanation of many natural phenomena and mechanical system [1,8,9]. In [9], the authors studied synchronization of coupled second-order linear harmonic oscillators with local interaction. By using graph theory and matrix theory, the synchronization conditions were obtained under mild network connectivity

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conditions. Duffing oscillator is an example of a periodically forced oscillator with a nonlinear elasticity, which is described by

$$x'' + cx' + g\big(x(t)\big) = p(t),$$

where $c > 0$ is the damping constant, $g(x)$ is smooth function. In physical sense, the Duffing oscillator can be interpreted as a forced oscillator with a spring whose restoring force is $g(x(t))$, $p(t)$ is a continuous function. Dynamical behaviors of Duffing oscillator have been widely investigated. For example, in [7, 14], the authors discussed the existence and stability of the periodic solutions of two kinds of Duffing-type equations. In addition, in [6], the authors established the synchronization conditions for a periodically forced Duffing oscillator with a chaotic pendulum. In [16], the synchronization chaos in two coupled Van der Pol–Duffing systems was investigated. In [3], the synchronization by the Ge–Yao–Chen partial region stability theory of chaotic Mathieu–Van der Pol and chaotic Duffing–Van der Pol system with fractional-order derivative was discussed.

In [15], by using Gegenbauer polynomial approximation, the authors addressed the chaos synchronization of two identical stochastic Duffing oscillators with bounded random parameters subject to harmonic excitations described by the following differential equations:

$$x''(t) + ax'(t) + b\big[x(t) + x^3(t)\big] = p(t), \quad a > 0.$$

In [12], the authors discussed the synchronization problem of the following Duffing oscillator network:

$$x_i'' + cx_i' + g\big(x_i(t)\big) = p(t), \quad i = 1, 2, \dots, N, \tag{1}$$

which was formed by $N$ coupled Duffing oscillators by local interaction. By using Lyapunov function method and graph theory method, which are different from those used in [9] and [6], the sufficient conditions on global asymptotic synchronization and global exponential synchronization for system (1) were obtained in [12].

In [13], the authors considered the more general Duffing oscillator dynamical networks without coupled delay

$$x_i''(t) + c_i x_i'(t) + g_i\big(t, x_i(t)\big) = p_i(t). \tag{2}$$

By transforming the problem of synchronization of nonidentical Duffing oscillators to the stability of the error systems, some sufficient criteria for impulsive global asymptotic synchronization of nonidentical networks both with and without coupling delays (2) were presented in [13] by using Lyapunov function method and inequality skills.

So far, the results on global exponential synchronization and global asymptotic synchronization for Duffing oscillator system have been acquired mainly by using Lyapunov functional method [13], graph theory method [9, 12] and matrix measure method [9, 12], numerical and experimental investigations [2] and chaos analysis [15, 16]. On the other hand, the restoring force of a spring of Duffing-type oscillator system in [12] was assumed to be increasing monotonously (see Remark 4).

On the other hand, in [12, 13], the linear controller was designed to achieve the global asymptotic synchronization and global exponential synchronization for the drive system and the response system discussed. The results of a control system with a plain linear control are well known and predictable, hence, the designs of some efficient nonlinear controllers are very necessary for Duffing oscillator system.

This motivates us to discuss global asymptotic synchronization of Duffing oscillator system by using some new study methods, getting rid of using graph theory, matrix measure theory, chaos analysis, numerical and experimental investigations, by setting new conditions on the restoring force of a spring instead of increasing restrict conditions on the restoring force of a spring in [12] and designing some nonlinear controllers instead of the linear controller designed in [12] and impulsive controllers.

In this paper, by setting Lipschitz conditions on the restoring force of a spring, we will establish concise and easily verified new sufficient conditions on global asymptotic synchronization for system (2) and its response system by using constant variation method, integral factor method and integrating inequality skills, which are different from those obtained in [2, 9, 12, 13, 15–18] and designing some nonlinear controllers, which are linear increasing in variable $x$ since our synchronization results will be derived from linear integrating inequalities. Thus, the contribution of the paper includes two aspects: (i) Four new study methods of global asymptotic synchronization: constant variation method, integral factor method and integral inequality skills are introduced in our papers; (ii) By using above study methods, four new sufficient conditions on global asymptotic synchronization for Duffing-type oscillator dynamical networks are established by setting Lipschitz conditions, which are less conservative than those obtained in the existing papers on the restoring force of a spring and designing more efficient nonlinear controllers with linear growth in variable $x$ than that designed in the existing papers for system (2) and its response system.

## 2 Preliminaries

We consider (2) as the drive system, and the response system is expressed by the following Duffing-type oscillator system:

$$y_i'' + c_i y_i' + g_i\big(t, y_i(t)\big) = p_i(t) + q_i(t), \quad i = 1, 2, \dots, N, \tag{3}$$

where $c_i > 0$, $N$ is a positive integer, $q_i(t)$ is the controller to be designed.

Throughout this paper, for systems (2) and (3), we always assume that:

(H1) There exists a positive constant $k_i$ such that $|g_i(t, x_i) - g_i(t, y_i)| \leqslant k_i |x_i - y_i|$ for all $x_i, y_i \in \mathbb{R}$, $i = 1, 2, \dots, N$, $|\cdot|$ is the norm of the Euclidean space $\mathbb{R}$.

(H2) There exists a constant $\eta$ with $\eta > c_i > 0$ such that

$$\frac{|e_i(t_0)| A_i G_i}{\eta} - \frac{\eta + p}{2\eta^2} |e_i(t_0)| A_i - \frac{|e_i(t_0)| A_i C_i}{\eta + c_i} + \frac{A_i D_i G_i}{c_i} \mathrm{e}^{-c_i t_0} - \frac{\eta + p}{\eta(\eta + c_i)} A_i D_i \mathrm{e}^{-c_i t_0} - \frac{A_i D_i C_i}{2c_i} \mathrm{e}^{-c_i t_0} < 0;$$

(H3) $c_i^2 < 4a$, $2(k_i - m + p)/l < c_i/2$, $a > 0$;
(H4) $b + \alpha < 0$, $b + \alpha + c < d_1 + \alpha < c + k/(c + b - d_1)$;
(H5) $b + \alpha < 0$, $d_1 + \alpha \leqslant \min\{c, b + c + \alpha\}$, where $k = \max_{1\leqslant i\leqslant N}\{k_i\}$, $l = \sqrt{4a - c_i^2}/2$, $c_i > 0$, $c = \min_{1\leqslant i\leqslant N}\{c_i\}$, $A_i = \exp\{[k_i/(c_i(\eta - c_i)) + (\eta + p)\eta][\mathrm{e}^{\eta t_0} - 1]\}$, $B_i = \eta + p + k_i/(\eta - c_i)$, $C_i = k_i/(c_i(\eta - c_i))$, $D_i = (p|e_i(t_0)| + |e_i'(t_0)|\mathrm{e}^{c_i t_0})/(\eta - c_i)$, $G_i = (\eta + p)/\eta + B_i + C_i$, $e_i(t_0) = y_i(t_0) - x_i(t_0)$, $e_i'(t_0) = y_i'(t_0) - x_i'(t_0)$, $t_0 \geqslant 0$ is initial time, $a$ is defined in controller (5), $\alpha$, $b$, $d_1$ are defined in controllers (37).

We cite the following notations:

$$F_i = e_i(t_0)\mathrm{e}^{ct_0/2}\sin lt_0 + \frac{1}{l}\mathrm{e}^{ct_0/2}\cos lt_0\left[e_i'(t_0) + \frac{ce_i(t_0)}{2}\right] - \frac{\sin 2lt_0}{l}\big(g\big(y(t_0)\big) - g\big(x(t_0)\big) - a\big[H\big(e_i(t_0)\big) - e_i(t_0)\big]\big),$$

$$L_1 = V_1(t_0) + \frac{V_2(t_0)}{b + c - d_1}, \qquad L_2 = \frac{V_2(t_0)}{d_1 - b - c},$$

$$M_i = \frac{e_i(t_0)\mathrm{e}^{ct_0/2} - b_i(t_0)\sin lt_0}{\cos lt_0}, \qquad U_i = V_i + b + c - d_1, \quad V_i = \frac{k_i}{c + b - d_1}.$$

**Definition 1.** Drive system (2) and response system (3) are said to be globally asymptotically synchronized if for arbitrary solutions of systems (2) and (3) denoted by $[x_1(t), x_2(t), \dots, x_N(t)]^{\mathrm{T}}$ and $[y_1(t), y_2(t), \dots, y_N(t)]^{\mathrm{T}}$, we have

$$\lim_{t\to\infty}\big|y_i(t) - x_i(t)\big| = 0, \quad i = 1, 2, \dots, N.$$

**Lemma 1.** (*See* [5].) *Assume that $u(t)$, $b(t)$ are continuous in $(\alpha, \beta)$, $a(t)$, $q(t) \in L[\alpha, \beta]$, $b(t)$, $q(t)$ are nonnegative functions. If $u(t) \leqslant a(t) + q(t)\int_{t_0}^t b(s)u(s)\,\mathrm{d}s$, $t \in (\alpha, \beta)$, then*

$$u(t) \leqslant a(t) + q(t)\int_{t_0}^{t} a(s)b(s)\exp\left\{\int_{s}^{t} q(r)b(r)\right\}\mathrm{d}s, \quad t \in (\alpha, \beta).$$

**Lemma 2.** (*See* [5].) *If $u(t) \leqslant a(t) + \int_{t_0}^t K(t, s)u(s)\,\mathrm{d}s$, then*

$$u(t) \leqslant a(t) + \frac{\int_{t_0}^t v(s)a(s)b(s)\,\mathrm{d}s}{1 - t_0 + v(t)},$$

*where*

$$v(t) = \exp\left\{-\int_{t_0}^{t} K(t, s)\,\mathrm{d}s\right\}, \quad b(t) = K(t, t) + \int_{t_0}^{t} K(t, s)\,\mathrm{d}s, \quad 0 \leqslant t_0 \leqslant s \leqslant t \leqslant b.$$

## 3 Global asymptotic synchronization by constant variation method and integrating factor method

In this section, we drive two sufficient conditions on global asymptotic synchronization for systems (2) and (3) by using constant variation method and integral factor method. Letting $e_i(t) = y_i(t) - x_i(t)$, $i = 1, 2, \dots, N$, the error system can be expressed as follows:

$$e_i''(t) + c_i e_i'(t) + \left[g_i\bigl(t, y_i(t)\bigr) - g_i\bigl(t, x_i(t)\bigr)\right] = q_i(t), \quad i = 1, 2, \dots, N, \tag{4}$$

where $c_i > 0$, the controllers are designed as

$$q_i(t) = -\mathrm{e}^{-c_i t} H'\bigl(e_i(t)\bigr) e_i'(t) \tag{5}$$

and

$$q_i(t) = -a e_i(t) - m\bigl|e_i(t)\bigr| - H\bigl(e_i(t)\bigr), \tag{6}$$

$H(x)$ is a continuous nonlinear function satisfying $|H(x)| \leqslant p|x|$, $p > 0$ is a positive constant, $a$, $m$ are two constants.

Hence, in order to prove that the drive system (2) and the response system (3) are globally asymptotically synchronized, we only need to prove that $\lim_{t\to\infty} |e_i(t)| = 0$.

**Theorem 1.** *Assume that* (H1) *and* (H2) *hold, then the drive system* (2) *and the response system* (3) *can acquire global asymptotical synchronization under controller* (5).

*Proof.* On the basis of systems (4) and (5), one has

$$e_i''(t) + c_i e_i'(t) + g_i\bigl(t, y_i(t)\bigr) - g_i\bigl(t, x_i(t)\bigr) + \mathrm{e}^{-c_i t} H'\bigl(e_i(t)\bigr) e_i'(t) = 0. \tag{7}$$

Multiplying (7) by $\mathrm{e}^{c_i t}$ gives

$$\frac{d[e_i'(t)\mathrm{e}^{c_i t}]}{dt} + \mathrm{e}^{c_i t}\bigl\{\mathrm{e}^{-c_i t} H'\bigl(e_i(t)\bigr) e_i'(t) + g_i\bigl(t, y_i(t)\bigr) - g_i\bigl(t, x_i(t)\bigr)\bigr\} = 0. \tag{8}$$

Integrating (8) over $[t_0, t]$ gives

$$\begin{aligned} &e_i'(t)\mathrm{e}^{c_i t} - e_i'(t_0)\mathrm{e}^{c_i t_0} \\ &\quad + \int_{t_0}^{t} \mathrm{e}^{c_i s}\bigl\{\mathrm{e}^{-c_i s} H'\bigl(e_i(s)\bigr) e_i'(s) + g_i\bigl(t, y_i(s)\bigr) - g_i\bigl(t, x_i(s)\bigr)\bigr\}\,\mathrm{d}s = 0, \end{aligned}$$

from which one has

$$\begin{aligned} &e_i'(t) - e_i'(t_0)\mathrm{e}^{c_i(t_0 - t)} + \mathrm{e}^{-c_i t}\bigl[H\bigl(e_i(t)\bigr) - H\bigl(e_i(t_0)\bigr)\bigr] \\ &\quad + \mathrm{e}^{-c_i t} \int_{t_0}^{t} \mathrm{e}^{c_i s}\bigl[g_i\bigl(t, y_i(s)\bigr) - g_i\bigl(t, x_i(s)\bigr)\bigr]\,\mathrm{d}s = 0. \end{aligned} \tag{9}$$

By (9) we have

$$e_i'(t) + \eta e_i(t) - e_i'(t_0)\mathrm{e}^{c_i(t_0-t)} + \mathrm{e}^{-c_i t}\big[H\big(e_i(t)\big) - H\big(e_i(t_0)\big)\big] + \mathrm{e}^{-c_i t}\int_{t_0}^{t} \mathrm{e}^{c_i s}\big[g_i\big(s, y_i(s)\big) - g_i\big(s, x_i(s)\big)\big]\,\mathrm{d}s = \eta e_i(t). \tag{10}$$

Multiplying (10) with $\mathrm{e}^{\eta t}$ gives

$$\frac{\mathrm{d}[e_i(t)\mathrm{e}^{\eta t}]}{\mathrm{d}t} = \mathrm{e}^{\eta t}\Big\{e_i'(t_0)\mathrm{e}^{c_i(t_0-t)} - \mathrm{e}^{-c_i t}\big[H\big(e_i(t)\big) - H\big(e_i(t_0)\big)\big] - \mathrm{e}^{-c_i t}\int_{t_0}^{t} \mathrm{e}^{c_i s}\big[g_i\big(s, y_i(s)\big) - g_i\big(s, x_i(s)\big)\big]\,\mathrm{d}s + \eta e_i(t)\Big\}. \tag{11}$$

Integrating (11) over $[t_0, t]$ gives

$$\begin{aligned} e_i(t)\mathrm{e}^{\eta t} - e_i(t_0)\mathrm{e}^{\eta t_0} &= e_i'(t_0)\mathrm{e}^{c_i t_0}\int_{t_0}^{t} \mathrm{e}^{(\eta - c_i)s}\,\mathrm{d}s + \eta\int_{t_0}^{t} \mathrm{e}^{\eta s} e_i(s)\,\mathrm{d}s \\ &\quad - \int_{t_0}^{t} \mathrm{e}^{(\eta - c_i)s}\,\mathrm{d}s \int_{t_0}^{s} \mathrm{e}^{c_i r}\big[g_i\big(r, y_i(r)\big) - g_i\big(r, x_i(r)\big)\big]\,\mathrm{d}r \\ &\quad + \int_{t_0}^{t} \mathrm{e}^{(\eta - c)s}\big[H\big(e_i(s)\big) - H\big(e_i(t_0)\big)\big]\,\mathrm{d}s \\ &= \frac{e_i'(t_0)\mathrm{e}^{c_i t_0}}{\eta - c_i}\big[\mathrm{e}^{(\eta - c_i)t} - \mathrm{e}^{(\eta - c_i)t_0}\big] + \eta\int_{t_0}^{t} \mathrm{e}^{\eta s} e_i(s)\,\mathrm{d}s \\ &\quad - \int_{t_0}^{t} \mathrm{e}^{(\eta - c_i)s}\,\mathrm{d}s \int_{t_0}^{s} \mathrm{e}^{c_i r}\big[g\big(r, y_i(r)\big) - g\big(r, x_i(r)\big)\big]\,\mathrm{d}r \\ &\quad + \int_{t_0}^{t} \mathrm{e}^{(\eta - c_i)s}\big[H\big(e_i(s)\big) - H\big(e_i(t_0)\big)\big]\,\mathrm{d}s, \end{aligned}$$

from which it follows that

$$\begin{aligned} \big|e_i(t)\big|\mathrm{e}^{\eta t} &\leqslant \big|e_i(t_0)\big|\mathrm{e}^{\eta t_0} + \left|\frac{e_i'(t_0)\mathrm{e}^{c_i t_0}}{\eta - c_i}\right|\big|\mathrm{e}^{(\eta - c_i)t} - \mathrm{e}^{(\eta - c_i)t_0}\big| + \eta\int_{t_0}^{t} \mathrm{e}^{\eta s}\big|e_i(s)\big|\,\mathrm{d}s \\ &\quad + \int_{t_0}^{t} \mathrm{e}^{(\eta - c_i)s}\,\mathrm{d}s \int_{t_0}^{s} \mathrm{e}^{c_i r}\big|g_i\big(r, y_i(r)\big) - g_i\big(r, x_i(r)\big)\big|\,\mathrm{d}r \end{aligned}$$

$$
\begin{aligned}
&+\int_{t_0}^{t} \mathrm{e}^{(\eta-c_i)s} p\big[\big|e_i(s)\big|+\big|e_i(t_0)\big|\big]\,\mathrm{d}s \\
&\leqslant \big|e_i(t_0)\big|\mathrm{e}^{\eta t_0}+\frac{p|e_i(t_0)|+|e_i'(t_0)\mathrm{e}^{c_i t_0}|}{|\eta-c_i|}\big|\mathrm{e}^{(\eta-c_i)t}-\mathrm{e}^{(\eta-c_i)t_0}\big|+\eta\int_{t_0}^{t}\mathrm{e}^{\eta s}\big|e_i(s)\big|\,\mathrm{d}s \\
&+k_i\int_{t_0}^{t}\mathrm{e}^{(\eta-c_i)s}\,\mathrm{d}s\int_{t_0}^{s}\mathrm{e}^{c_i r}\big|e_i(r)\big|\,\mathrm{d}r+p\int_{t_0}^{t}\mathrm{e}^{(\eta-c_i)s}\big|e_i(s)\big|\,\mathrm{d}s. \qquad (12)
\end{aligned}
$$

By changing the order of two time integrals, one has

$$
\begin{aligned}
k_i\int_{t_0}^{t}\mathrm{e}^{(\eta-c_i)s}\,\mathrm{d}s\int_{t_0}^{s}\mathrm{e}^{c_i r}\big|e_i(r)\big|\,\mathrm{d}r &= k_i\int_{t_0}^{t}\mathrm{e}^{c_i r}\big|e_i(r)\big|\,\mathrm{d}r\int_{r}^{t}\mathrm{e}^{(\eta-c_i)s}\,\mathrm{d}s \\
&= \int_{t_0}^{t}\frac{k_i}{\eta-c_i}\mathrm{e}^{c_i r}\big[\mathrm{e}^{(\eta-c_i)t}-\mathrm{e}^{(\eta-c_i)r}\big]\big|e_i(r)\big|\,\mathrm{d}r. \qquad (13)
\end{aligned}
$$

Substituting (13) into (12) gives

$$
\begin{aligned}
\big|e_i(t)\big|\mathrm{e}^{\eta t} \leqslant{}& \big|e_i(t_0)\big|\mathrm{e}^{\eta t_0}+\frac{p|e_i(t_0)|+|e_i'(t_0)|\mathrm{e}^{c_i t_0}}{|\eta-c_i|}\big|\mathrm{e}^{(\eta-c_i)t}-\mathrm{e}^{(\eta-c_i)t_0}\big| \\
&+\eta\int_{t_0}^{t}\mathrm{e}^{\eta s}\big|e_i(s)\big|\,\mathrm{d}s+\int_{t_0}^{t}\frac{k_i}{\eta-c_i}\big[\mathrm{e}^{(\eta-c_i)t+c_i r}-\mathrm{e}^{\eta r}\big]\big|e_i(r)\big|\,\mathrm{d}r \\
&+p\int_{t_0}^{t}\mathrm{e}^{(\eta-c_i)s}\big|e_i(s)\big|\,\mathrm{d}s. \qquad (14)
\end{aligned}
$$

On the basis of (14), one has since $\eta > c_i$,

$$
\begin{aligned}
\big|e_i(t)\big| \leqslant{}& \big|e_i(t_0)\big|\mathrm{e}^{\eta(t_0-t)}+\frac{p|e_i(t_0)|+|e_i'(t_0)|\mathrm{e}^{c_i t_0}}{\eta-c_i}\mathrm{e}^{-c_i t} \\
&+\int_{t_0}^{t}\bigg[(\eta+p)\mathrm{e}^{\eta(s-t)}+\frac{k_i}{\eta-c_i}\mathrm{e}^{c_i(s-t)}\bigg]\big|e_i(s)\big|\,\mathrm{d}s,
\end{aligned}
$$

from which, by using Lemma 2, it follows that

$$
\big|e_i(t)\big| \leqslant \big|e_i(t_0)\big|\mathrm{e}^{\eta(t_0-t)}+\frac{p|e_i(t_0)|+|e_i'(t_0)|\mathrm{e}^{c_i t_0}}{\eta-c_i}\mathrm{e}^{-c_i t}+\frac{\int_{t_0}^{t} v(s)a(s)b(s)\,\mathrm{d}s}{1-t_0+v(t)}, \qquad (15)
$$

where

$$
v(t)=\exp\bigg\{-\int_{t_0}^{t}K(t,s)\,\mathrm{d}s\bigg\}, \qquad b(t)=K(t,t)+\int_{t_0}^{t}K(t,s)\,\mathrm{d}s,
$$

$$a(t) = \left|e_i(t_0)\right|\mathrm{e}^{\eta(t_0-t)} + \frac{p|e_i(t_0)| + |e_i'(t_0)|\mathrm{e}^{c_i t_0}}{\eta - c_i}\mathrm{e}^{-c_i t} > 0,$$

$$K(t,s) = (\eta + p)\mathrm{e}^{\eta(s-t)} + \frac{k_i}{\eta - c_i}\mathrm{e}^{c_i(s-t)}.$$

Consequently,

$$\begin{aligned}
v(t) &= \exp\left\{-\int_{t_0}^{t}\left[(\eta+p)\mathrm{e}^{\eta(s-t)} + \frac{k_i}{\eta-c_i}\mathrm{e}^{c_i(s-t)}\right]\mathrm{d}s\right\} \\
&= \exp\left\{\frac{\eta+p}{\eta}\left[\mathrm{e}^{\eta(t_0-t)} - 1\right] + \frac{k_i}{c_i(\eta-c_i)}\left[\mathrm{e}^{c_i(t_0-t)} - 1\right]\right\} \\
&\leqslant \exp\left\{\left[\frac{k_i}{c_i(\eta-c_i)} + \frac{\eta+p}{\eta}\right]\left[\mathrm{e}^{\eta t_0} - 1\right]\right\} = A_i,
\end{aligned}$$

$$\begin{aligned}
b(t) &= \eta + p + \frac{k_i}{\eta-c_i} + \frac{\eta+p}{\eta}\left[1 - \mathrm{e}^{\eta(t_0-t)}\right] + \frac{k_i}{c_i(\eta-c_i)}\left[1 - \mathrm{e}^{c_i(t_0-t)}\right] \\
&= B_i + \frac{\eta+p}{\eta}\left[1 - \mathrm{e}^{\eta(t_0-t)}\right] + C_i\left[1 - \mathrm{e}^{c_i(t_0-t)}\right] > 0.
\end{aligned}$$

As a result

$$\begin{aligned}
0 &\leqslant \int_{t_0}^{t} v(s)a(s)b(s)\,\mathrm{d}s \\
&\leqslant A_i\int_{t_0}^{t}\left[B_i + \frac{\eta+p}{\eta}\left[1 - \mathrm{e}^{\eta(t_0-s)}\right] + C_i\left[1 - \mathrm{e}^{c_i(t_0-s)}\right]\right] \\
&\quad\times\left(\left|e_i(t_0)\right|\mathrm{e}^{\eta(t_0-s)} + D_i\mathrm{e}^{-c_i s}\right)\mathrm{d}s \\
&= A_i\int_{t_0}^{t}\left[G_i - \frac{\eta+p}{\eta}\mathrm{e}^{\eta(t_0-s)} - C_i\mathrm{e}^{c_i(t_0-s)}\right]\left(\left|e_i(t_0)\right|\mathrm{e}^{\eta(t_0-s)} + D_i\mathrm{e}^{-c_i s}\right)\mathrm{d}s \\
&= A_i\left[-\frac{|e_i(t_0)|G_i}{\eta}\mathrm{e}^{\eta(t_0-t)} + \frac{|e_i(t_0)|G_i}{\eta} + \frac{\eta+p}{2\eta^2}\left|e_i(t_0)\right|\mathrm{e}^{2\eta(t_0-t)}\right. \\
&\quad - \frac{\eta+p}{2\eta^2}\left|e_i(t_0)\right| + \frac{|e_i(t_0)|C_i}{\eta+c_i}\mathrm{e}^{(\eta+c_i)(t_0-t)} - \frac{|e_i(t_0)|C_i}{\eta+c_i} \\
&\quad - \frac{D_iG_i}{c_i}\mathrm{e}^{-c_i t} + \frac{D_iG_i}{c_i}\mathrm{e}^{-c_i t_0} + \frac{\eta+p}{\eta(\eta+c_i)}D_i\mathrm{e}^{-(\eta+c_i)t+\eta t_0} \\
&\quad \left.- \frac{\eta+p}{\eta(\eta+c_i)}D_i\mathrm{e}^{-c_i t_0} + \frac{D_iC_i}{2c_i}\mathrm{e}^{-2c_i t + c_i t_0} - \frac{D_iC_i}{2c_i}\mathrm{e}^{-c_i t_0}\right],
\end{aligned}$$

from which, together with (H2), it follows that

$$
\begin{aligned}
0 &\leqslant \int_{t_0}^{t} v(s)a(s)b(s)\,\mathrm{d}s \\
&\leqslant -\frac{|e_i(t_0)|A_iG_i}{\eta}\mathrm{e}^{\eta(t_0-t)} + \frac{\eta+p}{2\eta^2}\big|e_i(t_0)\big|A_i\mathrm{e}^{2\eta(t_0-t)} + \frac{|e_i(t_0)|A_iC_i}{\eta+c_i}\mathrm{e}^{(\eta+c_i)(t_0-t)} \\
&\quad - \frac{A_iD_iG_i}{c_i}\mathrm{e}^{-c_it} + \frac{\eta+p}{\eta(\eta+c_i)}A_iD_i\mathrm{e}^{-(\eta+c_i)t+\eta t_0} + \frac{A_iD_iC_i}{2c_i}\mathrm{e}^{-2c_it+c_it_0}.
\end{aligned} \tag{16}
$$

Since $\eta > 0$, $c_i > 0$, by (16) and (H2), one has

$$\lim_{t\to\infty}\int_{t_0}^{t} a(s)b(s)v(s)\,\mathrm{d}s = 0. \tag{17}$$

It is clear that

$$\lim_{t\to\infty} v(t) = \exp\biggl\{-\frac{\eta+p}{\eta} + \frac{k_i}{c_i(c_i-\eta)}\biggr\}. \tag{18}$$

Substituting (17) and (18) into (15) gives $\lim_{t\to\infty}|e_i(t)| = 0$. The proof of Theorem 1 is accomplished. □

**Theorem 2.** *Assume that* (H1) *and* (H3) *hold, then the drive system* (2) *and the response system* (3) *can acquire global asymptotical synchronization under controller* (6).

*Proof.* Based on systems (4) and (6), one has

$$
\begin{aligned}
&e_i''(t) + c_ie_i'(t) + g_i\big(t,y_i(t)\big) - g_i\big(t,x_i(t)\big) + ae_i(t) + m\big|e_i(t)\big| \\
&\quad + H\big(e_i(t)\big) = 0.
\end{aligned} \tag{19}
$$

Letting $r_1 = -c_i/2 - (\sqrt{4a-c_i^2}/2)i$, $r_2 = -c_i/2 + (\sqrt{4a-c_i^2}/2)i$, by using constant variation formula, from (19) it follows

$$e_i(t) = a_i(t)\mathrm{e}^{-c_it/2}\cos\frac{\sqrt{4a-c_i^2}}{2}t + b_i(t)\mathrm{e}^{-c_it/2}\sin\frac{\sqrt{4a-c_i^2}}{2}t, \tag{20}$$

where $a_i(t)$ and $b_i(t)$ satisfy the following differential equations:

$$a_i'(t)\mathrm{e}^{-c_it/2}\cos\frac{\sqrt{4a-c_i^2}}{2}t + b_i'(t)\mathrm{e}^{-c_it/2}\sin\frac{\sqrt{4a-c_i^2}}{2}t = 0 \tag{21}$$

and

$$
\begin{aligned}
&-a_i'(t)\mathrm{e}^{-c_it/2}\biggl[\frac{c_i}{2}\cos\frac{\sqrt{4a-c_i^2}}{2}t + \frac{\sqrt{4a-c_i^2}}{2}\sin\frac{\sqrt{4a-c_i^2}}{2}t\biggr] \\
&\quad + b_i'(t)\mathrm{e}^{-c_it/2}\biggl[-\frac{c_i}{2}\sin\frac{\sqrt{4a-c_i^2}}{2}t + \frac{\sqrt{4a-c_i^2}}{2}\cos\frac{\sqrt{4a-c_i^2}}{2}t\biggr] \\
&= g\big(t,x_i(t)\big) - g\big(t,y_i(t)\big) - m\big|e_i(t)\big| - H\big(e_i(t)\big).
\end{aligned} \tag{22}
$$

Denoting $l = \sqrt{4a - c_i^2}/2$, by (21) and (22), it follows that

$$a_i'(t) = \frac{e^{c_i t/2}}{l}\big(g\big(t, y_i(t)\big) - g\big(t, x_i(t)\big) - m\big|e_i(t)\big| - H\big(e_i(t)\big)\big)\sin lt \tag{23}$$

and

$$b_i'(t) = \frac{e^{c_i t/2}}{l}\big(g_i\big(t, x_i(t)\big) - g_i\big(t, y_i(t)\big) - m\big|e_i(t)\big| - H\big(e_i(t)\big)\big)\cos lt. \tag{24}$$

Integrating (23) and (24) over $[t_0, t]$, respectively, it follows that

$$a_i(t) = a_i(t_0) + \frac{1}{l}\int_{t_0}^{t} e^{c_i s/2}\big(g_i\big(s, y_i(s)\big) - g_i\big(s, x_i(s)\big) - m\big|e_i(s)\big| - H\big(e_i(s)\big)\big) \times \sin ls\,ds \tag{25}$$

and

$$b_i(t) = b_i(t_0) + \frac{1}{l}\int_{t_0}^{t} e^{c_i s/2}\big(g_i\big(s, y_i(s)\big) - g_i\big(s, x_i(s)\big) - am\big|e_i(s)\big| - H\big(e_i(s)\big)\big) \times \cos ls\,ds. \tag{26}$$

Substituting (25) and (26) into (20) gives

$$\begin{aligned} e_i(t) = {} & \Bigg\{a_i(t_0) + \frac{1}{l}\int_{t_0}^{t} e^{c_i s/2}\big(g_i\big(s, y_i(s)\big) - g_i\big(s, x_i(s)\big) - m\big|e_i(s)\big| \\ & - H\big(e_i(s)\big)\big)\sin ls\,ds\Bigg\} e^{-c_i t/2}\cos lt \\ & + \Bigg\{b_i(t_0) + \frac{1}{l}\int_{t_0}^{t} e^{c_i s/2}\big(g_i\big(s, y_i(s)\big) - g_i\big(s, x_i(s)\big) - m\big|e_i(s)\big| \\ & - H\big(e_i(s)\big)\big)\cos ls\,ds\Bigg\} e^{-ct/2}\sin lt. \end{aligned} \tag{27}$$

Based on (27), one has

$$e_i(t_0)e^{c_i t_0/2} = a_i(t_0)\cos lt_0 + b_i(t_0)\sin lt_0 \tag{28}$$

and

$$\begin{aligned} & e_i'(t_0)e^{c_i t_0/2} + \frac{c_i}{2}e_i(t_0)e^{c_i t_0/2} \\ & \quad = \frac{1}{l}e^{c_i t_0/2}\sin 2lt_0\big(g\big(t_0, y(t_0)\big) - g\big(t_0, x(t_0)\big) \\ & \qquad - \big[H\big(e_i(t_0)\big) + m\big|e_i(t_0)\big|\big]\big) - a_i(t_0)l\sin lt_0 + b_i(t_0)l\cos lt_0. \end{aligned} \tag{29}$$

On the basis of (28) and (29), we get

$$\begin{aligned} b_i(t_0) &= e_i(t_0)\mathrm{e}^{c_i t_0/2}\sin lt_0 + \frac{1}{l}\mathrm{e}^{c_i t_0/2}\cos lt_0\left[e_i'(t_0) + \frac{c_i e_i(t_0)}{2}\right] \\ &\quad - \frac{\sin 2lt_0}{l}\bigl(g\bigl(t_0, y(t_0)\bigr) - g\bigl(t_0, x(t_0)\bigr) - \bigl[H\bigl(e_i(t_0)\bigr) + m\bigl|e_i(t_0)\bigr|\bigr]\bigr) \\ &= F_i \end{aligned} \tag{30}$$

and

$$a_i(t_0) = \frac{e_i(t_0)\mathrm{e}^{c_i t_0/2} - b_i(t_0)\sin lt_0}{coslt_0} = M_i. \tag{31}$$

Substituting (30) and (31) into (27) gives

$$\begin{aligned} e_i(t) &= \Biggl\{M_i + \frac{1}{l}\int_{t_0}^{t}\mathrm{e}^{c_i s/2}\bigl(g\bigl(s, y_i(s)\bigr) - g\bigl(s, x_i(s)\bigr) - m\bigl|e_i(s)\bigr| \\ &\qquad - H\bigl(e_i(s)\bigr)\bigr)\sin ls\,\mathrm{d}s\Biggr\}\mathrm{e}^{-c_i t/2}\cos lt \\ &\quad + \Biggl\{F_i + \frac{1}{l}\int_{t_0}^{t}\mathrm{e}^{c_i s/2}\bigl(g\bigl(s, y_i(s)\bigr) - g\bigl(s, x_i(s)\bigr) \\ &\qquad - m\bigl|e_i(s)\bigr| - H\bigl(e_i(s)\bigr)\bigr)\cos ls\,\mathrm{d}s\Biggr\}\mathrm{e}^{-c_i t/2}\sin lt, \end{aligned}$$

from which, it follows that

$$\begin{aligned} \bigl|e_i(t)\bigr| &\leqslant \Biggl(|M_i| + \frac{k_i - m + p}{l}\int_{t_0}^{t}\mathrm{e}^{c_i s/2}\bigl|e_i(s)\bigr|\,\mathrm{d}s\Biggr)\mathrm{e}^{-c_i t/2} \\ &\quad + \Biggl(|F_i| + \frac{k_i - m + p}{l}\int_{t_0}^{t}\mathrm{e}^{c_i s/2}\bigl|e_i(s)\bigr|\,\mathrm{d}s\Biggr)\mathrm{e}^{-c_i t/2} \\ &= \bigl(|M_i| + |F_i|\bigr)\mathrm{e}^{-c_i t/2} + \frac{2(k_i - m + p)}{l}\mathrm{e}^{-c_i t/2}\int_{t_0}^{t}\mathrm{e}^{c_i s/2}\bigl|e_i(s)\bigr|\,\mathrm{d}s. \end{aligned} \tag{32}$$

In view of (32), by using Lemma 1, we get

$$\begin{aligned} \bigl|e_i(t)\bigr| &\leqslant \bigl(|M_i| + |F_i|\bigr)\mathrm{e}^{-c_i t/2} \\ &\quad + \frac{2(k_i - m + p)(|M_i| + |F_i|)}{l}\mathrm{e}^{-c_i t/2}\int_{t_0}^{t}\mathrm{e}^{-c_i s/2}\mathrm{e}^{c_i s/2}\exp\Biggl\{\frac{2k_i}{l}\int_{s}^{t}\mathrm{e}^{\mathrm{d}r}\Biggr\}\,\mathrm{d}s \end{aligned}$$

$$
\begin{aligned}
&= \big(|M_i| + |F_i|\big)\mathrm{e}^{-c_i t/2} \\
&\quad + \frac{2(k_i - m + p)(|M_i| + |F_i|)}{l}\mathrm{e}^{-c_i t/2}\int_{t_0}^{t} \exp\biggl\{\frac{2(k_i - m + p)}{l}(t - s)\biggr\}\,\mathrm{d}s \\
&= \big(|M_i| + |F_i|\big)\mathrm{e}^{-c_i t/2} \\
&\quad + \frac{2(k_i - m + p)(|M_i| + |F_i|)}{l}\mathrm{e}^{(2(k_i - m + p)/l - c_i/2)t}\frac{l}{2(k_i - m + p)} \\
&\quad \times \big[\mathrm{e}^{-2(k_i - m + p)t_0/l} - \mathrm{e}^{-2(k_i - m + p)t/l}\big] \\
&= \big(|M_i| + |F_i|\big)\mathrm{e}^{-c_i t/2} \\
&\quad + \big(|M_i| + |F_i|\big)\big[\mathrm{e}^{-2(k_i - m + p)t_0/l + (2(k_i - m + p)/l - c_i/2)t} - \mathrm{e}^{-c_i t/2}\big] \\
&= \big(|M_i| + |F_i|\big)\mathrm{e}^{-2(k_i - m + p)t_0/l + (2(k_i - m + p)/l - c_i/2)t}. \qquad (33)
\end{aligned}
$$

Since $c_i > 0$, $2(k_i - m + p)/l < c_i/2$, by (33), it follows that $\lim_{t\to\infty} |e_i(t)| = 0$. This accomplishes the proof of Theorem 2. □

## 4 Global asymptotic synchronization by integral inequality skills

In this section, we will drive two sufficient conditions on global asymptotic synchronization for Duffing-type oscillator system (2) by using integral inequality skills.

Setting $x_i' = z_i$, $i = 1, 2, \dots, N$, we can acquire the following system from system (2):

$$x_i' = z_i, \quad z_i' = -c_i z_i - g_i\big(t, x_i(t)\big) + p_i(t), \quad i = 1, 2, \dots, N. \qquad (34)$$

Consequently, in order to study the synchronization of system (2) and its response system, we only need to study the synchronization of system (34) and its response system.

We consider system (34) as the drive system, the response system is expressed as the following equations:

$$u_i' = v_i + P_i(t), \qquad v_i' = -c_i v_i - g_i\big(t, u_i(t)\big) + p_i(t) + Q_i(t), \qquad (35)$$

where $P_i(t)$, $Q_i(t)$ are the controllers.

By setting $E_i(t) = u_i(t) - x_i(t)$, $R_i(t) = v_i(t) - z_i(t)$, the error system can be expressed as

$$E_i' = R_i + P_i(t), \qquad R_i' = -c_i R_i - \big[g_i\big(t, u_i(t)\big) - g_i\big(t, x_i(t)\big)\big] + Q_i(t). \qquad (36)$$

The controllers are designed as follows:

$$P_i(t) = bE_i(t) + h\big(E_i(t)\big), \qquad Q_i(t) = d_1 R_i(t) + h\big(R_i(t)\big), \qquad (37)$$

where $h(x)$ is a continuous function satisfying $|h(x)| \leqslant \alpha|x|$, $\alpha > 0$ is a constant.

**Definition 2.** The drive system (34) and the response system (35) are said to be globally asymptotically synchronized if

$$\lim_{t\to\infty}\bigl|u_i(t)-x_i(t)\bigr|=0,\quad \lim_{t\to\infty}\bigl|v_i(t)-z_i(t)\bigr|=0,\quad i=1,2,\dots,N.$$

**Theorem 3.** *Assume that* (H1) *and* (H4) *hold, then the drive system* (34) *and the response system* (35) *can acquire global asymptotical synchronization under controllers* (37).

*Proof.* Two Lyapunov functions are designed as follows:

$$F_1(t)=\sum_{i=1}^{N}\bigl|E_i(t)\bigr|,\qquad F_2(t)=\sum_{i=1}^{N}\bigl|R_i(t)\bigr|.$$

In view of (36) and (37), one has

$$\begin{aligned}F_1'(t)&=\sum_{i=1}^{N}\operatorname{sign}\bigl[E_i(t)\bigr]\bigl[R_i(t)+bE_i(t)+h\bigl(E_i(t)\bigr)\bigr]\\&\leqslant\sum_{i=1}^{N}\bigl[(b+\alpha)\bigl|E_i(t)\bigr|+\bigl|R_i(t)\bigr|\bigr]=(b+\alpha)F_1(t)+F_2(t)\end{aligned}\tag{38}$$

and

$$\begin{aligned}F_2'(t)&=\sum_{i=1}^{N}\operatorname{sign}\bigl[R_i(t)\bigr]\bigl(-c_iR_i(t)-\bigl[g_i\bigl(t,u_i(t)\bigr)-g_i\bigl(t,x_i(t)\bigr)\bigr]\\&\qquad+d_1R_i(t)+h\bigl(R_i(t)\bigr)\bigr)\\&=\sum_{i=1}^{N}\bigl[(\alpha-c_i)\bigl|R_i(t)\bigr|+k_i\bigl|E_i(t)\bigr|+d_1\bigl|R_i(t)\bigr|\bigr]\\&\leqslant(d_1-c+\alpha)F_2(t)+kF_1(t).\end{aligned}\tag{39}$$

Multiplying (38) and (39) with $\mathrm{e}^{-(b+\alpha)t}$ and $\mathrm{e}^{(c-d_1-\alpha)t}$, respectively, it follows that

$$\frac{\mathrm{d}[F_1(t)\mathrm{e}^{-(b+\alpha)t}]}{\mathrm{d}t}\leqslant F_2(t)\mathrm{e}^{-(b+\alpha)t}\tag{40}$$

and

$$\frac{d[F_2(t)\mathrm{e}^{(c-d_1-\alpha)t}]}{\mathrm{d}t}\leqslant kF_1(t)\mathrm{e}^{(c-d_1-\alpha)t}.\tag{41}$$

Integrating (40) and (41) over $[t_0,t]$, respectively, one has

$$F_1(t)\leqslant F_1(t_0)\mathrm{e}^{(b+\alpha)(t-t_0)}+\mathrm{e}^{(b+\alpha)t}\int_{t_0}^{t}F_2(s)\mathrm{e}^{-(b+\alpha)s}\,\mathrm{d}s\tag{42}$$

and

$$F_2(t)\leqslant F_2(t_0)\mathrm{e}^{(c-d_1-\alpha)(t_0-t)}+k\mathrm{e}^{-(c-d_1-\alpha)t}\int_{t_0}^{t}F_1(s)\mathrm{e}^{(c-d_1-\alpha)s}\,\mathrm{d}s.\tag{43}$$

Since $d_1 > b + c$, substituting (43) into (42) gives

$$\begin{aligned}
F_1(t) &\leqslant F_1(t_0)\mathrm{e}^{(b+\alpha)(t-t_0)} + \mathrm{e}^{(b+\alpha)t} \\
&\quad \times \int_{t_0}^{t} \mathrm{e}^{-(b+\alpha)s} \left\{ F_2(t_0)\mathrm{e}^{(c-d_1-\alpha)(t_0-s)} + k\mathrm{e}^{(d-c)s} \int_{t_0}^{s} F_1(r)\mathrm{e}^{(c-d_1-\alpha)r}\,\mathrm{d}r \right\} \mathrm{d}s \\
&= L_1\mathrm{e}^{(b+\alpha)(t-t_0)} + L_2\mathrm{e}^{(d_1-c+\alpha)(t-t_0)} \\
&\quad + \mathrm{e}^{(b+\alpha)t} k \int_{t_0}^{t} \mathrm{e}^{(d_1-c-b)s}\,\mathrm{d}s \int_{t_0}^{s} F_1(r)\mathrm{e}^{(c-d_1-\alpha)r}\,\mathrm{d}r \\
&= L_1\mathrm{e}^{(b+\alpha)(t-t_0)} + L_2\mathrm{e}^{(d_1-c+\alpha)(t-t_0)} \\
&\quad + \mathrm{e}^{(b+\alpha)t} k \int_{t_0}^{t} F_1(r)\mathrm{e}^{(c-d_1-\alpha)r}\,\mathrm{d}r \int_{r}^{t} \mathrm{e}^{(d_1-c-b)s}\,\mathrm{d}s \\
&= L_1\mathrm{e}^{(b+\alpha)(t-t_0)} + L_2\mathrm{e}^{(d_1-c+\alpha)(t-t_0)} \\
&\quad + \frac{k\mathrm{e}^{(b+\alpha)t}}{d_1-b-c} \int_{t_0}^{t} F_1(r)\mathrm{e}^{(c-d_1-\alpha)r} \left[\mathrm{e}^{(d_1-c-b)t} - \mathrm{e}^{(d_1-c-b)r}\right] \mathrm{d}r \\
&\leqslant L_1\mathrm{e}^{(b+\alpha)(t-t_0)} + L_2\mathrm{e}^{(d_1-c+\alpha)(t-t_0)} + \frac{k\mathrm{e}^{(d_1-c+\alpha)t}}{d_1-b-c} \int_{t_0}^{t} F_1(r)\mathrm{e}^{(c-d_1-\alpha)r}\mathrm{d}r. \qquad (44)
\end{aligned}$$

Setting $z(t) = \int_{t_0}^{t} F_1(r)\mathrm{e}^{(c-d_1-\alpha)r}\,\mathrm{d}r$, we have $z'(t) = F_1(t)\mathrm{e}^{(c-d_1-\alpha)t}$, then by (44),

$$z'(t) \leqslant L_1\mathrm{e}^{(b+c-d_1)t-(b+\alpha)t_0} + L_2\mathrm{e}^{(c-d_1-\alpha)t_0} + \frac{k}{d_1-c-b} z(t). \qquad (45)$$

Multiplying (45) with $\mathrm{e}^{kt/c+b-d_1}$ and integrating over $[t_0, t]$, one has

$$\begin{aligned}
z(t)\mathrm{e}^{kt/(c+b-d_1)} &\leqslant \int_{t_0}^{t} \left(L_1\mathrm{e}^{-(b+\alpha)t_0}\mathrm{e}^{U_i s} + L_2\mathrm{e}^{(c-d_1-\alpha)t_0}\mathrm{e}^{V_i s}\right) \mathrm{d}s \\
&= \frac{L_1\mathrm{e}^{-(b+\alpha)t_0}}{U_i}\left[\mathrm{e}^{U_i t} - \mathrm{e}^{U_i t_0}\right] + \frac{L_2\mathrm{e}^{(c-d_1-\alpha)t_0}}{V_i}\left[\mathrm{e}^{V_i t} - \mathrm{e}^{V_i t_0}\right],
\end{aligned}$$

from which it follows that

$$z(t) \leqslant \frac{L_1\mathrm{e}^{-(b+\alpha)t_0}}{U_i}\left[\mathrm{e}^{(b+c-d_1)t} - \mathrm{e}^{U_i t_0 - V_i t}\right] + \frac{L_2\mathrm{e}^{(c-d_1-\alpha)t_0}}{V_i}\left[1 - \mathrm{e}^{V_i(t_0-t)}\right]. \qquad (46)$$

Substituting (46) into (45) gives

$$0 \leqslant F_1(t) \leqslant \left(L_1 - \frac{V_i L_1}{U_i}\right)\mathrm{e}^{(b+\alpha)(t-t_0)} + \left(L_2 + \frac{V_i L_1}{U_i}\right)\mathrm{e}^{(d_1+\alpha-c-V_i)(t-t_0)}. \qquad (47)$$

Since $b + \alpha < 0$, $b + \alpha + c < d + \alpha < c + k/(c + b - d) < c$, then from (47) we have $\lim_{t\to\infty} F_1(t) = 0$. Hence,

$$\lim_{t\to\infty} E_i(t) = 0. \tag{48}$$

Substituting (48) into (43) gives

$$\begin{aligned}
0 &\leqslant F_2(t) \qquad (49)\\
&\leqslant V_F(t_0)\mathrm{e}^{(c-d_1-\alpha)(t_0-t)}\\
&\quad + k\mathrm{e}^{(d_1+\alpha-c)t}\int_{t_0}^{t}\bigg[\bigg(L_1 - \frac{V_iL_1}{U_i}\bigg)\mathrm{e}^{(b+\alpha)(s-t_0)}\mathrm{e}^{(c-d_1-\alpha)s}\\
&\qquad + \bigg(L_2 + \frac{V_iL_1}{U_i}\bigg)\mathrm{e}^{(d_1+\alpha-c-V_i)(s-t_0)}\mathrm{e}^{(c-d_1-\alpha)s}\bigg]\,\mathrm{d}s\\
&= \bigg[F_2(t_0) + \frac{k(L_2 + \frac{V_iL_1}{U_i})}{V_i} - \frac{kL_1 - \frac{kV_iL_1}{U_i}}{b + c - d_1}\bigg]\mathrm{e}^{(d_1+\alpha-c)(t-t_0)}\\
&\quad + \frac{kL_1 - \frac{kV_iL_1}{U_i}}{b + c - d_1}\mathrm{e}^{(b+\alpha)(t-t_0)} - \frac{k(L_2 + \frac{V_iL_1}{U_i})}{V_i}\mathrm{e}^{(d_1+\alpha-c-V_i)(t-t_0)}. \qquad (50)
\end{aligned}$$

Since $b + \alpha < 0$, $d_1 + \alpha < c$, $d_1 + \alpha - c + k/(d_1 - b - c) < 0$, then from (50) we have $\lim_{t\to\infty} F_2(t) = 0$. Hence,

$$\lim_{t\to\infty} R_i(t) = 0. \tag{51}$$

Combining (48) with (51) accomplishes the proof of Theorem 3. □

**Theorem 4.** *Assume that* (H1) *and* (H5) *hold, then the drive system* (34) *and the response system* (35) *can acquire global asymptotical synchronization under controllers* (37).

*Proof.* Two Lyapunov functions are designed as follows:

$$F_1(t) = \sum_{i=1}^{N}\big|E_i(t)\big|, \qquad F_2(t) = \sum_{i=1}^{N}\big|R_i(t)\big|.$$

In view of the proofs of (42) and (43), one has

$$F_1(t) \leqslant F_1(t_0)\mathrm{e}^{(b+\alpha)(t-t_0)} + \mathrm{e}^{(b+\alpha)t}\int_{t_0}^{t} F_2(s)\mathrm{e}^{-(b+\alpha)s}\,\mathrm{d}s \tag{52}$$

and

$$F_2(t) \leqslant F_2(t_0)\mathrm{e}^{(c-d_1-\alpha)(t_0-t)} + k\mathrm{e}^{-(c-d_1-\alpha)t}\int_{t_0}^{t} F_1(s)\mathrm{e}^{(c-d_1-\alpha)s}\,\mathrm{d}s. \tag{53}$$

Since $d_1 < b + c$, substituting (53) into (52), the similar proof to that of (44) gives

$$F_1(t) \leqslant L_1 \mathrm{e}^{(b+\alpha)(t-t_0)} + L_2 \mathrm{e}^{(d_1-c+\alpha)(t-t_0)} + \frac{k\mathrm{e}^{(b+\alpha)t}}{b+c-d_1} \int_{t_0}^{t} F_1(r)\mathrm{e}^{-(b+\alpha)r}\,\mathrm{d}r,$$

from which, we have

$$F_1(t)\mathrm{e}^{-(b+\alpha)t} \leqslant L_1\mathrm{e}^{-(b+\alpha)t_0} + L_2\mathrm{e}^{(d_1-c-b)t-(d_1-c+\alpha)t_0} + \frac{k}{b+c-d_1}\int_{t_0}^{t} F_1(r)\mathrm{e}^{-(b+\alpha)r}\,\mathrm{d}r. \tag{54}$$

Setting $Z(t) = \int_{t_0}^{t} F_1(s)\mathrm{e}^{-(b+\alpha)s}\,\mathrm{d}s$, we have $Z'(t) = F_1(t)\mathrm{e}^{-(b+\alpha)t}$. Consequently, by (54), we can get

$$Z'(t) \leqslant L_1\mathrm{e}^{-(b+\alpha)t_0} + L_2\mathrm{e}^{(d_1-c-b)t-(d_1-c+\alpha)t_0} + \frac{k}{b+c-d_1}Z(t). \tag{55}$$

Multiplying (55) by $\mathrm{e}^{kt/(d_1-b-c)}$ gives

$$\frac{\mathrm{d}[Z(t)\mathrm{e}^{kt/(d_1-b-c)}]}{\mathrm{d}t} \leqslant L_1\mathrm{e}^{kt/(d_1-b-c)-(b+\alpha)t_0} + L_2\mathrm{e}^{(d_1-c-b+k/(d_1-b-c))t-(d_1-c+\alpha)t_0}. \tag{56}$$

Integrating (56) over $[t_0, t]$, we have by noting $Z(t_0) = 0$

$$Z(t)\mathrm{e}^{kt/(d_1-b-c)} \leqslant L_1\mathrm{e}^{-(b+\alpha)t_0}\int_{t_0}^{t}\mathrm{e}^{ks/(d_1-b-c)}\,\mathrm{d}s + L_2\mathrm{e}^{-(d_1-c+\alpha)t_0}\int_{t_0}^{t}\mathrm{e}^{(d_1-c-b+k/(d_1-b-c))s}\,\mathrm{d}s.$$

Consequently,

$$\begin{aligned} 0 &\leqslant F_1(t)\mathrm{e}^{-(b+\alpha)t} \\ &\leqslant \frac{L_1\mathrm{e}^{-(b+\alpha)t_0}}{V_i}\big[\mathrm{e}^{-V_i t_0} - \mathrm{e}^{-V_i t}\big] + L_2 U_i \mathrm{e}^{-(d_1-c+\alpha)t_0}\big(\mathrm{e}^{-U_i t_0} - \mathrm{e}^{-U_i t}\big) \\ &= \frac{L_1\mathrm{e}^{-(b+\alpha+V_i)t_0}}{V_i}\big[1 - \mathrm{e}^{-V_i(t-t_0)}\big] + L_2 U_i \mathrm{e}^{-(d_1-c+\alpha+U_i)t_0}\big[1 - \mathrm{e}^{-U_i(t-t_0)}\big], \end{aligned}$$

from which it follows that since $L_2 < 0$, $L_1 > 0$, $U_i > 0$, $V_i > 0$,

$$\begin{aligned} 0 &\leqslant F_1(t)\mathrm{e}^{-(b+\alpha)t} \\ &\leqslant \frac{L_1\mathrm{e}^{-(b+\alpha+V_i)t_0}}{V_i} + L_2 U_i \mathrm{e}^{-(d_1-c+\alpha+U_i)t_0}\big[1 - \mathrm{e}^{-U_i(t-t_0)}\big]. \end{aligned} \tag{57}$$

By (57), one has

$$\begin{aligned} 0 &\leqslant F_1(t) \\ &\leqslant \frac{L_1}{V_i}\mathrm{e}^{(b+\alpha)t-(b+\alpha+V_i)t_0} + L_2U_i\mathrm{e}^{(b+\alpha)t-(d_1-c+\alpha+V_i)t_0}\bigl[1-\mathrm{e}^{-V_i(t-t_0)}\bigr]. \end{aligned} \tag{58}$$

Since $b+\alpha<0$, $V_i>0$, from (58) it follows that

$$\lim_{t\to\infty} F_1(t) = 0.$$

Since $L_2<0$, substituting (58) into (53) gives

$$\begin{aligned} 0 &\leqslant F_2(t) \\ &\leqslant F_2(t_0)\mathrm{e}^{(c-d_1-\alpha)(t_0-t)} + k\mathrm{e}^{-(c-d_1-\alpha)t} \\ &\quad\times \int_{t_0}^{t} \mathrm{e}^{(c-d_1+b)s}\biggl\{\frac{L_1}{V_i}\mathrm{e}^{-(b+\alpha+V_i)t_0} + \mathrm{e}^{-(d_1-c+\alpha+V_i)t_0}L_2U_i\bigl[1-\mathrm{e}^{-V_i(s-t_0)}\bigr]\biggr\}\,\mathrm{d}s \\ &\leqslant F_2(t_0)\mathrm{e}^{(c-d_1-\alpha)(t_0-t)} + k\mathrm{e}^{-(c-d_1-\alpha)t}\int_{t_0}^{t}\mathrm{e}^{(c-d_1+b)s}\frac{L_1}{V_i}\mathrm{e}^{-(b+\alpha+V_i)t_0}\,\mathrm{d}s \\ &= V_2(t_0)\mathrm{e}^{(\alpha+d_1-c)(t-t_0)} \\ &\quad+ \frac{kL_1}{V_i(c-d_1+b)}\bigl[\mathrm{e}^{(b+\alpha)t-(b+\alpha+V_i)t_0} - \mathrm{e}^{(\alpha+d_1-c)t-(c-d_1+b)t_0}\bigr]. \end{aligned} \tag{59}$$

Since $b+\alpha<0$, $d_1+\alpha<c$, on the basis of (59), we have $\lim_{t\to\infty} F_2(t)=0$. This completes the proof of Theorem 4. □

**Remark 1.** In [9,12] and [13], global exponential synchronization and global asymptotic synchronization for system (2) were discussed by using graph theory, matrix theory and Lyapunov function method. While in our paper, getting rid of using above methods, we acquire four new sufficient conditions on global asymptotic synchronization for system (2) and its response system by using constant variation method, integral factor method and inequality skills. As a result, some new study methods are introduced in our paper.

**Remark 2.** In [12] and [13], a linear controller was respectively designed to achieve the global asymptotic synchronization and global exponential synchronization between the drive system and the response system discussed. In our paper, some more generalized nonlinear controllers, which are linearly increasing in variable $x$, are designed to acquire the global asymptotic synchronization between the drive system and the response system. Consequently, new nonlinear controllers are introduced in our paper.

**Remark 3.** In [12], Assumption 1 was assumed:

**Assumption 1.** There are two positive numbers $\theta_1$ and $\theta_2$ such that

$$\theta_1(x-y)^2 \leqslant (x-y)\bigl[g(x)-g(y)\bigr] \leqslant \theta_2(x-y)^2.$$

When $x \geqslant y$, from Assumption 1 we have

$$0 \leqslant \theta_1(x-y) \leqslant g(x) - g(y) \leqslant \theta_2(x-y).$$

When $x < y$, from Assumption 1 we have

$$\theta_2(x-y) \leqslant g(x) - g(y) \leqslant \theta_1(x-y) \leqslant 0.$$

Hence, $g(x)$ is monotonously increasing on $x$. Thus, in [12], the restoring force of a spring of oscillator dynamical networks was assumed to be increasing monotonously.

**Remark 4.** In [12], $g(x)$ is assumed to be monotonously increasing, while in our paper, without the assumption of monotonicity on $g(x)$, we assume that $g(t,x)$ only satisfies global Lipschitz conditions. On other hand, in [12, 13], linear controllers were designed, while in our paper, some generalized nonlinear controllers are designed. Consequently, our results on global asymptotic synchronization for Duffing oscillator dynamical networks are less conservative than those obtained in [12, 13]. In [13], the sufficient conditions on global impulsive synchronization for system (2) and its response system were acquired. In our paper, we acquire new sufficient conditions on global asymptotic synchronization for system (2) and its response system under new controllers.

## 5 Examples

In this section, we give four examples for showing our results.

*Example 1.* We consider the following dynamical network (the model was discussed in [12]):

$$x_i'' + cx_i' + g\bigl(x_i(t)\bigr) = p(t), \tag{60}$$

where $g(x_i) = 2(x_i + 1)$, $p(t) = \cos t$. The control input was designed as in [12]:

$$\mu_i = d\sum_{j=1}^{N} a_{ij}(u_j - u_i).$$

In [12], the coupled Duffing oscillator network and its response system can achieve the global synchronization with the control input $\mu_i$. We consider the drive system (60) and the response system

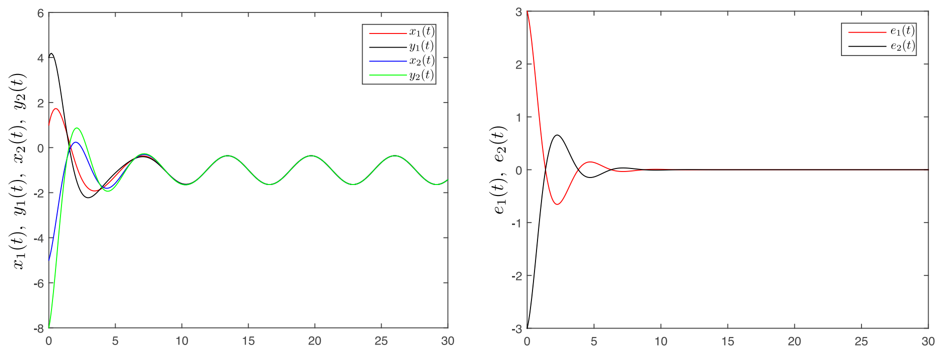
$$y_i'' + cy_i' + g\bigl(y_i(t)\bigr) = p(t) + q_i(t) \tag{61}$$

with the controller

$$q_i(t) = -\mathrm{e}^{-ct}H'\bigl(e_i(t)\bigr)e_i'(t), \tag{62}$$

where $g(x_i(t)) = x_i(t) + \sin x_i(t) + 2$, $p(t) = \cos t$, $k_i = 2$, $c = 1.2$, $\eta = 1.6$, $H(e_i(t)) = e_i(t)/(1+e_i^2(t))$ (linear growth $|e_i(t)/(1+e_i^2(t))| \leqslant |e_i(t)|$), $p = 1, i = 1, 2$. Then the error system can be described by as follows:

$$e_i''(t) + ce_i'(t) + \bigl[g\bigl(y_i(t)\bigr) - g\bigl(x_i(t)\bigr)\bigr] = -\mathrm{e}^{-ct}\frac{1 - e_i^2(t)}{(1 + e_i^2(t))^2}e_i'(t), \quad i = 1, 2.$$




(a) (b)

**Figure 1.** Example 1: (a) the trajectories of $x_1(t)$, $y_1(t)$ and $x_2(t)$, $y_2(t)$; (b) the error states $e_1(t)$ and $e_2(t)$.

We take all the initial conditions as: $x_1(0) = 1$, $y_1(0) = 4$, $x_1'(0) = 3$, $y_1'(0) = 2$, $x_2(0) = -5$, $y_2(0) = -8$, $x_2'(0) = 2$, $y_2'(0) = 3$. It is easy to verify that (H1) and (H2) are satisfied. Hence, by Theorem 1 the drive system (60) and the response system (61) are globally asymptotically synchronized under controllers (62).

In [12], $g(x)$ is assumed to be monotonously increasing (see Remark 5), while $g(x) = x(t)+\sin x(t)$ is not monotonously increasing in our paper, in [12] and [13], the controller $\mu_i$ is linear, while our controller (62) is nonlinear. Consequently, our result of global asymptotic synchronization with nonlinear controller is less conservative than that in [12] and [13].

The curves of variables $x_1(t)$, $y_1(t)$, $x_2(t)$ and $y_2(t)$ are shown in Fig. 1(a), the error curves of the drive-response system $e_1(t)$ and $e_2(t)$ are shown in Fig. 1(b).

*Example 2.* In [12], the authors considered the Duffing-type oscillator dynamical network (60)

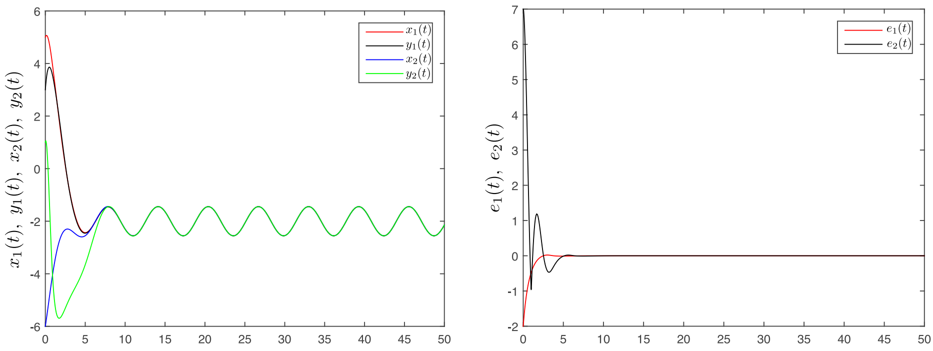
$$x_i'' + cx_i' + g\big(x_i(t)\big) = p(t),$$

where $g(x) = 0.5x - 0.5\sin x + 2$, $p(t) = \cos t$, the controller was designed as

$$\mu_i = d\sum_{j=1}^{N} a_{ij}\big[(u_j - u_i) + (v_j - v_i)\big].$$

The coupled Duffing oscillator network and its response system can achieve the global synchronization with the control input. We consider the drive system (60) and the response system (61) with the controller $q_i(t) = -ae_i(t) - m|e_i(t)| - e_i^3(t)/(1+e_i^2(t))$, $a = 2$, $c = 1.8$, $p = 1$, $m = 2.5$, $g(x_i(t)) = 0.5x_i(t) - 0.5\sin x_i(t) + 2$, $p(t) = \cos t$, $k_i = 1$, $i = 1, 2$, $|e_i^3(t)/(1 + e_i^2(t))| \leqslant |e_i(t)|$ (linear growth). The error system can be described by as follows:

$$\begin{aligned} &e_i''(t) + ce_i'(t) + \big[g\big(y_i(t)\big) - g\big(x_i(t)\big)\big] \\ &\quad = -ae_i(t) - m\big|e_i(t)\big| - \frac{e_i^3(t)}{1 + e_i^2(t)}, \quad i = 1, 2. \end{aligned}$$



(a) (b)

**Figure 2.** Example 2: (a) the trajectories of $x_1(t)$, $y_1(t)$ and $x_2(t)$, $y_2(t)$; (b) the error states $e_1(t)$ and $e_2(t)$.

We take all the initial conditions as follows: $x_1(0) = 5$, $y_1(0) = 1$, $x_1'(0) = 3$, $y_1'(0) = 4$, $x_2(0) = -6$, $y_2(0) = 2$, $x_2'(0) = 1$, $y_2'(0) = 2$.

It is easy to verify that (H1) and (H3) are satisfied. Hence, by Theorem 2, the drive system (60) and the response system (61) are globally asymptotically synchronized.

In [12, 13], the linear controller was designed, while in our paper, the efficient nonlinear controller is designed to achieve global synchronization; in [12], $g(x)$ is monotonously increasing, while in our paper, $g(x)$ is not monotonously increasing, so, our result on global asymptotic synchronization is less conservative than that in [12, 13].

The curves of variables $x_1(t)$, $y_1(t)$, $x_2(t)$ and $y_2(t)$ are shown in Fig. 2(a), the error curve of the drive-response system $e_1(t)$ and $e_2(t)$ is shown in Fig. 2(b).

*Example 3.* (See [13, Ex. 4.1].). We consider the drive system

$$x_i' = z_i, \quad z_i' = -c_i z_i - g_i\bigl(t, x_i(t)\bigr) + p_i(t), \quad i = 1, 2. \tag{63}$$

and the response system

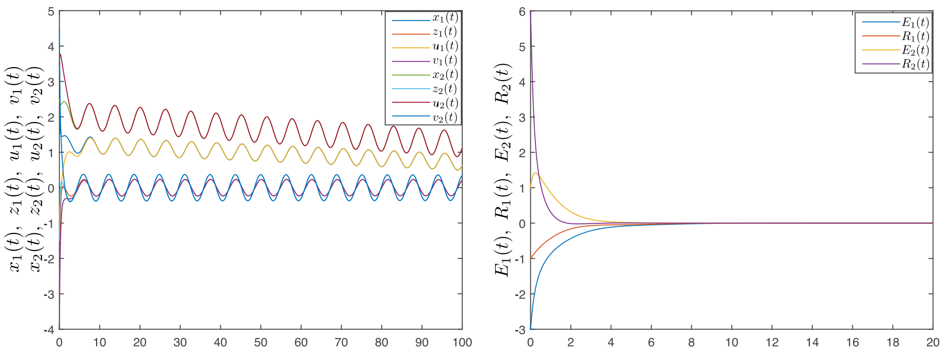
$$u_i' = v_i + P_i(t), \quad v_i' = -c_i v_i - g_i\bigl(t, u_i(t)\bigr) + p_i(t) + Q_i(t), \quad i = 1, 2, \tag{64}$$

with the controllers

$$P_i(t) = bE_i(t) + 2\sin\bigl(E_i(t)\bigr), \qquad Q_i(t) = d_1 R_i(t) + 2\sin\bigl(R_i(t)\bigr), \tag{65}$$

where $p_i(t) = i\cos t$, $g_i(t, x_i(t)) = \theta_i(i + x_i(t))\sin t$ with $\theta_i = 0.2\sqrt{i}$, $k_1 = 0.2$, $k_2 = 0.2\sqrt{2}$, $c_i = 5 + (-1)^i 0.5i$, $c_1 = 4.5$, $c_2 = 6$, $c = 4.5$, $b = -3$, $d_1 = 2$, $2|\sin x_i(t)| \leqslant 2|x_i(t)|$ (linear growth), $\alpha = 2$, $i = 1, 2$. Correspondingly, the error system can be described by as follows:

$$\begin{aligned} E_i' &= R_i + bE_i(t) + 2\sin\bigl(E_i(t)\bigr), \\ R_i' &= -c_i R_i - \bigl[g_i\bigl(t, u_i(t)\bigr) - g_i\bigl(t, x_i(t)\bigr)\bigr] \\ &\qquad + d_1 R_i(t) + 2\sin\bigl(R_i(t)\bigr), \quad i = 1, 2. \end{aligned}$$



(a) (b)

**Figure 3.** Example 3: (a) the trajectories of $x_1(t)$, $z_1(t)$, $u_1(t)$, $v_1(t)$ and $x_2(t)$, $z_2(t)$, $u_2(t)$, $v_2(t)$; (b) the error states $E_1(t)$ $R_1(t)$ and $E_2(t)$, $R_2(t)$.

In [13], the impulsive synchronization of system (63) was discussed. In this paper, we consider the global asymptotical synchronization of system (63) and system (64). We take all the initial conditions as follows: $x_1(0) = 2$, $z_1(0) = -3$, $u_1(0) = -1$, $v_1(0) = -4$, $x_2(0) = 2.5$, $z_2(0) = -1.5$, $u_2(0) = 3.5$, $v_2(0) = 4.5$. It is easy to verify that (H1) and (H4) are satisfied. Hence, by Theorem 4, the drive system (63) and the response system (64) with $i = 1, 2$ are globally asymptotically synchronized under the controllers (65). The curves of variables $x_1(t)$, $z_1(t)$, $u_1(t)$, $v_1(t)$, $x_2(t)$, $z_2(t)$, $u_2(t)$ and $v_2(t)$ are shown in Fig. 3(a), the error curves of the drive-response system $E_1(t)$, $R_1(t)$, $E_2(t)$ and $R_2(t)$ are shown in Fig. 3(b).

## 6 Conclusion

In this paper, the asymptotic synchronization of Duffing-type oscillator system is considered. Without using matrix measure theory, graphic theory and LMI method, four new sufficient conditions on global asymptotic synchronization for above system are presented by applying some new study methods: constant variation method, integral factor method and integrating inequality method. Furthermore, our results on global asymptotic synchronization of Duffing-type oscillator system are less conservative than those acquired in [12] and different from those acquired in [13].

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