Nonlinear Analysis: Modelling and Control, Vol. 25, No. 4, 638–657
https://doi.org/10.15388/namc.2020.25.17437

ISSN: 1392-5113
eISSN: 2335-8963

Turing instability and spatially homogeneous Hopf bifurcation in a diffusive Brusselator system

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Received: March 9, 2019 / **Revised:** August 15, 2019 / **Published online:** July 1, 2020

Abstract. The present paper deals with a reaction–diffusion Brusselator system subject to the homogeneous Neumann boundary condition. When the effect of spatial diffusion is neglected, the local asymptotic stability and the detailed Hopf bifurcation of the unique positive equilibrium of the associated ODE system are analyzed. In the stable domain of the ODE system, the effect of spatial diffusion is explored, and local asymptotic stability, Turing instability and existence of Hopf bifurcation of the constant positive equilibrium are demonstrated. In addition, the direction of spatially homogeneous Hopf bifurcation and the stability of the spatially homogeneous bifurcating periodic solutions are also investigated. Finally, numerical simulations are also provided to check the obtained theoretical results.

Keywords: Brusselator reaction–diffusion system, local stability, Turing instability, spatially homogeneous Hopf bifurcation, normal form.

1 Introduction

In the chemical reaction process, assume that A_{in} and B_{in} represent the input chemicals, D and E denote the out chemicals, as well as U and V are intermediates, respectively. Then the well-known Brusselator system can be described by the chemical reaction process [12]

$$\begin{aligned} B_{\mathrm{in}} &\longrightarrow U, \\ A_{\mathrm{in}} + U &\Longleftrightarrow V + D, \\ 2U + V &\longrightarrow 3U, \\ U &\longrightarrow E. \end{aligned} \tag{1}$$

[1]Corresponding author.

[2]Supported by the National Natural Science Foundation of China (61763024) and Foundation of a Hundred Youth Talents Training Program of Lanzhou Jiaotong University (152022).

[3]Supported by the National Natural Science Foundation of China (61563026).

If the reactor for the above reaction process is bounded by a bounded open domain Ω with a smooth boundary $\partial\Omega$ in $\mathbb{R}^N$ $(N \geqslant 1)$ and we use $U(X,T)$ and $V(X,T)$ to denote respectively the concentrations of intermediates U and V at time T and space location $X \in \Omega$, then the chemical reaction process (1) can be described by the following reaction–diffusion system [1]:

$$\begin{aligned}&\frac{\partial U}{\partial T} = D_1\Delta U + A - (B+1)U + U^2V, \quad X \in \Omega,\ T > 0,\\ &\frac{\partial V}{\partial T} = D_2\Delta V + BU - U^2V, \quad X \in \Omega,\ T > 0,\\ &\frac{\partial U}{\partial n} = \frac{\partial V}{\partial n} = 0, \quad X \in \partial\Omega,\ T > 0,\\ &U(X,0) = V_0(X) \geqslant 0, \quad V(X,0) = V_0(X) \geqslant 0, \quad X \in \partial\Omega,\end{aligned} \tag{2}$$

where D_1 and D_2 are respectively the diffusion coefficients of the intermediates U and V, Δ is the Laplace operator in $\mathbb{R}^N$, and n is the outward unit normal vector on $\partial\Omega$.

Introduce new time and space variables t and x: $T = t$ and $X = \sqrt{D_1}x$, and let

$$U(T,X) = U(t,\sqrt{D_1}x) = Au(t,x), \qquad V(T,X) = V(t,\sqrt{D_1}x) = \frac{B}{A}v(t,x).$$

Use a and b to denote the parameters A and B, respectively, and let $\sigma = D_2/D_1$. Then system (2) is transformed into the following reaction–diffusion system:

$$\begin{aligned}&\frac{\partial u}{\partial t} = \Delta u + 1 - (b+1)u + bu^2v, \quad x \in \Omega,\ t > 0,\\ &\frac{\partial v}{\partial t} = \sigma\Delta v + a^2(u - u^2v), \quad x \in \Omega,\ t > 0,\\ &\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0, \quad x \in \partial\Omega,\ t > 0,\\ &u(x,0) = u_0(x) \geqslant 0, \quad v(x,0) = v_0(x) \geqslant 0, \quad x \in \partial\Omega,\end{aligned} \tag{3}$$

in which the scaled spatial domain is still represented by Ω, and σ measures the ratio of two diffusion coefficients D_2 and D_1. It is easy to see that system (3) has always the constant positive equilibrium $E^* = (1,1)$ for any positive parameters a and b, and the diffusion coefficient σ has no any effect on the existence, positivity and location of the constant equilibrium of system (3).

Reaction–diffusion systems similar to (2) and (3) have been concerned extensively by many researchers from the theoretical and numerical aspects; see [6, 7, 9, 10, 13, 16, 17]. For example, for system (3), Peng and Wang in [10] obtained the existence and non-existence of positive nonconstant steady states, and Zuo and Wei [16] investigated in detail Hopf bifurcations and global steady state bifurcations, which bifurcate from the unique positive constant equilibrium. Subsequently, Jia, Li and Wu [6] further considered the coexistence of nonconstant positive steady states for system (2). In addition, by taking b as the bifurcation parameter, Li and Wang [9] studied the local asymptotic stability of

E^* for system (3) when Ω is a general domain with a smooth boundary and the detailed spatially homogeneous Hopf bifurcation of (3) at E^* when $\Omega = (0, \pi)$. The stability and Hopf bifurcation analysis for system (3) with the delayed feedback control is considered by Zuo and Wei [17]. On the other hand, from the view points of numerical investigations, Siraj-ul-Islam, Ali and Haq [13], as well as Jiwari and Yuan [7], provided respectively the numerical difference methods solving system (2) when $D_1 = D_2$ and the spatial domain Ω is taken as a rectangle $(0, L_x) \times (0, L_y)$ in the planes. For the study of dynamical behaviors of the population reaction–diffusion systems with time periodic coefficients, see [3, 4].

As pointed out by Turing [11], different diffusion rates of two chemical reactants could sometimes lead to the so-called *Turing instability* or *diffusion-driven instability*. Although Turing's idea has attracted the attention of a great number of investigators and was also successfully developed on the theoretical backgrounds, the search for Turing patterns in real chemical or biological systems turned out to be difficult. Based on this case, one aim of the present article is to explore the Turing instability of system (3). In addition, under the case when a is fixed, the effect of the variation of b on the dynamics of system (3) is investigated and the detailed spatially homogeneous Hopf bifurcation also carried out when Ω is a general domain in $\mathbb{R}^N$ with a smooth boundary.

The remaining parts of this paper are arranged as follows. In the next section, the local asymptotic stability and the detailed Hopf bifurcation of the unique positive equilibrium E^* of the ODE system corresponding to the reaction–diffusion system (3) are provided according to the qualitative theory of ODE dynamical systems. In Section 3, by analyzing in detail the eigenvalue problem of the linearized system of (3) at the constant positive steady state E^*, the local asymptotic stability and diffusion-driven instability of the constant positive steady state E^* of (3) are analyzed. In addition, the existence and properties of Hopf bifurcation of (3) at E^* are obtained in Section 4 by employing the normal form method and the center manifold theorem for reaction–diffusion equations. Finally, to check the theoretical conclusions, numerical approximations for system (3) with $\Omega = (0, \pi)$ and special parameters values of a and b are also included at the end of the paper by means of the MATLAB software package and the difference methods solving reaction–diffusion equations.

2 Stability and Hopf bifurcation analysis of the local ODE system

In the absence of the effect of spatial diffusion, the reaction–diffusion system (3) is reduced to the following local ODE system:

$$
\begin{aligned}
&\frac{\mathrm{d}u}{\mathrm{d}t} = 1 - (b+1)u + bu^2 v, \quad t > 0,\\
&\frac{\mathrm{d}v}{\mathrm{d}t} = a^2 u(1 - uv), \quad t > 0,\\
&u(0) = u_0 \geqslant 0, \qquad v(0) = v_0 \geqslant 0.
\end{aligned}
\tag{4}
$$

For system (4), we have the following result on the positivity of solutions.

Lemma 1. *Let* $(u(t), v(t))$ *be any solution of system* (4)*. Then* $u(t), v(t) > 0$ *for all* $t > 0$*, that is, the set* $\{(u,v) \mid u, v \geqslant 0\}$ *is a positive invariant set of system* (4).

Proof. If $u_0 = 0$, then from the first equation of system (4) we can observe that $\mathrm{d}u(0)/\mathrm{d}t = 1 > 0$. Therefore, it follows from the continuity of solutions for the initial value problems of the ODEs that there exists $t_1 > 0$ such that $u(t) > 0$ when $t \in (0, t_1)$. Assume that there exists $t_2 > t_1$ such that $u(t_2) = 0$. Then one can know that $\mathrm{d}u(t_2)/\mathrm{d}t \leqslant 0$. However, the first equation of system (4) shows that $\mathrm{d}u(t_2)/\mathrm{d}t = 1 > 0$, and thus, a contradiction is lead. Hence, we can derive that $u(t) > 0$ for any $t > 0$. If $u_0 > 0$, then one can get easily $u(t) > 0$ for any $t > 0$ by using the argument of contradiction.

In the sequel, we shall reveal the positivity of $v(t)$ when $t > 0$.

If $v_0 = 0$, then from the second equation of system (4) we can see that $\mathrm{d}v(0)/\mathrm{d}t = a^2 u_0$.

(i) Let $u_0 = 0$. Then $\mathrm{d}v(0)/\mathrm{d}t = 0$. Assume that there exists $t_* > 0$ such that $v(t) \leqslant 0$ when $0 < t < t_*$. Then from the positivity of $u(t)$ and the second equation of (4) we know that $\mathrm{d}v(t)/\mathrm{d}t > 0$ when $0 < t < t_*$. Accordingly, $v(t)$ is strictly monotonically increasing on $[0, t_*]$, and this implies that $v(t) > 0$ when $0 < t < t_*$. Thus, a contradiction is lead, and we know that there exists $t^* > 0$ such that $v(t) > 0$ when $0 < t < t^*$. Similarly to the proof of the positivity of $u(t)$, we can demonstrate that $v(t) > 0$ for any $t > 0$.

(ii) If $u_0 > 0$, then $\mathrm{d}v(0)/\mathrm{d}t = a^2 u_0 > 0$, and thus, we can know that there exists $t^{**} > 0$ such that $v(t) > 0$ for any $0 < t < t^{**}$. Similarly to the above argument of contradiction, it is easy to show that $v(t) > 0$ for any $t > 0$.

We can also further show that if $v_0 > 0$, then $v(t)$ when $t > 0$.

Summarizing the above arguments, the proof of the lemma is complete. □

Now we consider the local asymptotic stability and Hopf bifurcation of the unique positive equilibrium $E^* = (1, 1)$ of the ODE system (4). It is easy to derive that the Jacobian matrix of (4) at E^* has the form

$$J = \begin{pmatrix} b-1 & b \\ -a^2 & -a^2 \end{pmatrix}.$$

Notice that $D = \det J = a^2(1-b) + a^2 b = a^2 > 0$ and $T = \operatorname{trace} J = b - 1 - a^2$.

Let b_0 be defined by $b_0 = 1 + a^2$. Then one can know that the unique positive equilibrium E^* of system (4) is locally asymptotically stable when $0 < b < b_0$ and is unstable when $b > b_0$.

When $b = b_0$, make the change of variables $\bar{u} = u - 1$, $\bar{v} = v - 1$ and still use u, v to denote $\bar{u}$, $\bar{v}$. Then system (4) becomes

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}t} &= a^2 u + (a^2+1)v + (a^2+1)u^2 + 2(a^2+1)uv + (a^2+1)u^2 v, \\ \frac{\mathrm{d}v}{\mathrm{d}t} &= -a^2(u + v + u^2 + 2uv + u^2 v). \end{aligned} \tag{5}$$

Let
$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{a^2}{a^2+1} & -\frac{a}{a^2+1} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$
Then system (5) is transformed into
$$\frac{\mathrm{d}x}{\mathrm{d}t} = -ay + \left(1-a^2\right)x^2 - 2axy - a^2x^3 - ax^2y, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = ax.$$
From the reference [2] one can obtain that the third focal value for the multiple focus $(0,0)$ of system (5) is
$$\alpha_3 = -\frac{\pi}{4a}(a^2+2) < 0.$$
Thus, we know that when $b = b_0$, the unique positive equilibrium $E^* = (1,1)$ of system (4) is a locally asymptotically stable multiple focus of multiplicity 1.

In addition, notice that when $b = b_0$, the Jocabian matrix J has a pair of purely imaginary eigenvalues $\pm\mathrm{i}a$. Let $\lambda = \gamma(b) + \mathrm{i}\omega(b)$ be a pair of conjugate complex characteristic roots of J when b varies near b_0. Then we have
$$\gamma(b) = \frac{b-1-a^2}{2}, \qquad \omega(b) = \sqrt{D - \gamma^2(b)}.$$
Consequently,
$$\gamma'(b_0) = \gamma'(b)|_{b=b_0} = \frac{1}{2} > 0.$$
This demonstrates that a Hopf bifurcation occurs at the positive equilibrium E^* of system (4) when $b = b_0$.

In the following, we devote to considering the direction of the above Hopf bifurcation and the stability of bifurcating periodic solutions by means of the methods introduced in the literature [8]. Rewrite system (5) into the following form:
$$\frac{\mathrm{d}U}{\mathrm{d}t} = AU + \frac{1}{2}B(U,U) + \frac{1}{6}C(U,U,U),$$
where
$$A = \begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix}, \qquad U = \begin{pmatrix} u \\ v \end{pmatrix},$$
and for $X = (x_1,x_2)^{\mathrm{T}}$, $Y = (y_1,y_2)^{\mathrm{T}}$ and $Z = (z_1,z_2)^{\mathrm{T}} \in \mathbb{C}^2$, the multiple linear forms $B(X,Y)$ and $C(X,Y,Z)$ are defined as
$$B(X,Y) = \begin{pmatrix} 2(1-a^2)x_1y_1 - 2a(x_2y_1 + x_1y_2) \\ 0 \end{pmatrix} \tag{6}$$
and
$$C(X,Y,Z) = \begin{pmatrix} -6a^2x_1y_1z_1 - 2a[(x_2y_1 + x_1y_2)z_1 + x_1y_1z_2] \\ 0 \end{pmatrix}. \tag{7}$$

Obviously, the matrix A possesses two eigenvalues $\pm\mathrm{i}a$, and complex vectors $q = (i,1)^{\mathrm{T}}$ and $p = (1/2)(i,1)^{\mathrm{T}}$ are, respectively, the eigenvectors of A and A^{T} corresponding to the eigenvalues $\mathrm{i}\omega$ and $-\mathrm{i}\omega$, and $\langle q,p\rangle = 1$. Based on (6) and (7), one can derive

$$\begin{aligned} g_{20} &= \bigl\langle p, B(q,q)\bigr\rangle = -2a + \bigl(1-a^2\bigr)\mathrm{i},\\ g_{11} &= \bigl\langle p, B(q,\bar{q})\bigr\rangle = -\bigl(1-a^2\bigr)\mathrm{i},\\ g_{21} &= \bigl\langle p, C(q,q,\bar{q})\bigr\rangle = -3a^2 + a\mathrm{i}. \end{aligned} \tag{8}$$

Combining (8) and formulae (3.20) in [8], we can get that the first Lyapunov coefficient of system (5) at the origin is

$$l(0) = \frac{1}{2a^2}\mathrm{Re}(\mathrm{i}g_{20}g_{11} + ag_{21}) = -\frac{a^2+2}{2a} < 0.$$

According to the condition $\gamma'(b_0) > 0$, from [2] we have the following conclusion.

Theorem 1.

(i) *If* $0 < b \leqslant b_0$, *then the unique positive equilibrium* E^* *of system* (4) *is locally asymptotically stable*;

(ii) *If* $b > b_0$, *then the unique positive equilibrium* E^* *of system* (4) *is unstable*;

(iii) *System* (4) *can undergo a supercritical Hopf bifurcation at the positive equilibrium* E^*, *and the bifurcating periodic solutions are stable when* $b = b_0$.

3 Local asymptotic stability and Turing instability of the reaction–diffusion system

Define the real-valued Sobolev space X by

$$X = \left\{(u,v) \in H^2(\varOmega) \times H^2(\varOmega) \;\Big|\; \frac{\partial u}{\partial \gamma} = \frac{\partial v}{\partial \gamma} = 0,\ x \in \partial\varOmega\right\}.$$

The complex expansion space of X is described by

$$X_c = X \oplus \mathrm{i}X = \{x_1 + \mathrm{i}x_2 \mid x_1, x_2 \in X\}.$$

Let the linear operator L be defined by

$$L = \begin{pmatrix} \Delta + b - 1 & b \\ -a^2 & \sigma\Delta - a^2 \end{pmatrix}.$$

Notice that, under the homogeneous Neumann boundary condition, the eigenvalues of the operator $-\Delta$ satisfy

$$0 = \mu_0 < \mu_1 < \mu_2 < \cdots.$$

Define the sequence of matrices L_n for $n \in \mathbb{N}_0 = \{0,1,2,\dots\}$ by

$$L_n = \begin{pmatrix} b - 1 - \mu_n & b \\ -a^2 & -a^2 - \sigma\mu_n \end{pmatrix}.$$

We first may state the following result; see [14].

Lemma 2. *$\beta \in C$ is an eigenvalue of the operator L if and only if there exists some certain $n \in \mathbb{N}_0$ such that β is also the eigenvalue of the matrix L_n.*

From Lemma 2 and the analysis in Section 2, we can obtain immediately the following result.

Theorem 2. *If $b > b_0$, then the positive constant equilibrium E^* of the reaction–diffusion system* (3) *is unstable.*

In the next, we are concerned with the local asymptotic stability and Turing instability of the positive constant equilibrium E^* of the reaction–diffusion system (3) when $0 < b < b_0$. It is clear that the characteristic equation of L_n is

$$\beta^2 - T_n\beta + D_n = 0, \tag{9}$$

where

$$T_n = T - (1+\sigma)\mu_n, \qquad D_n = \sigma\mu_n^2 + \left[a^2 - \sigma(b-1)\right]\mu_n + a^2.$$

Under the condition $0 < b < b_0$, one can see that $T < 0$, and from the positivity of σ and the nonnegativity of μ_n we know that $T_n < 0$ for all $n \in \mathbb{N}_0$.

(i) If $0 < b \leqslant 1$, then we can see easily that $D_n > 0$ for all $n \in \mathbb{N}_0$. This implies that all the roots of the characteristic equation (9) have negative real parts, and we know that the positive constant equilibrium E^* of the reaction–diffusion system (3) is locally asymptotically stable.

(ii) If $1 < b < b_0$, then $D_n > 0$ for all $n \in \mathbb{N}_0$ when

$$0 < \sigma \leqslant \frac{a^2}{b-1}.$$

Therefore, one can know that the positive constant equilibrium E^* of the reaction–diffusion system (3) is locally asymptotically stable. Assume that

$$\sigma > \frac{a^2}{b-1}$$

and let

$$\mathscr{D} = \left(a^2 - \sigma b + \sigma\right)^2 - 4\sigma a^2 = \sigma^2(1-b)^2 - 2a^2\sigma(b+1) + a^4.$$

Consider the quadratic function

$$h(z) = (1-b)^2z^2 - 2a^2(b+1)z + a^4.$$

It is easy to see that the discriminant of $H(z)$ is

$$\begin{aligned}\mathcal{D} &= \left[-2a^2(b+1)\right]^2 - 4a^4(1-b)^2 = 4a^4(b+1)^2 - 4a^4(1-b)^2\\ &= 4a^4\left[(b+1)^2 - (1-b)^2\right] = 16a^4b > 0.\end{aligned}$$

Accordingly, the equation $h(z) = 0$ has two different real positive roots

$$z_1 = \frac{a^2(b+1) - 2a^2\sqrt{b}}{(1-b)^2}, \qquad z_2 = \frac{a^2(b+1) + 2a^2\sqrt{b}}{(1-b)^2}. \tag{10}$$

If $z_1 < \sigma < z_2$, then $h(\sigma) < 0$. So, $\mathscr{D} < 0$ when $z_1 < \sigma < z_2$, and thus, we have that $D_n > 0$ for all $n \in \mathbb{N}_0$. Note that

$$h\left(\frac{a^2}{b-1}\right) = -\frac{4a^4}{b-1} < 0.$$

Therefore, $z_1 < a^2/(b-1) < z_2$.

Summarizing the above discussions, one can obtain the following result.

Theorem 3. *Define* z_2 *by* (10) *when* $1 < b < b_0$. *Let one of the following conditions holds*:

(i) $0 < b \leqslant 1$ *and* $\sigma > 0$;
(ii) $1 < b < b_0$ *and* $0 < \sigma < z_2$.

Then the positive constant equilibrium E^* *of the reaction–diffusion system* (3) *is locally asymptotically stable.*

Now we consider the local asymptotic stability and Turing instability of the positive constant equilibrium E^* of the reaction–diffusion system (3) when $1 < b < b_0$ and $\sigma \geqslant z_2$.

Notice that $\mathscr{D} = 0$ when $\sigma = z_2 > a^2/(b-1)$, and so the equation

$$\sigma\mu^2 + \left[a^2 - \sigma(b-1)\right]\mu + a^2 = 0$$

has two equal positive real roots $\mu = \mu^* := (\sigma b - a^2 - \sigma)/(2\sigma)$. If $\mu^* \neq \mu_n$ for all $n \in \mathbb{N}_0$, then $D_n > 0$. Thus, we can obtain the following theorem.

Theorem 4. *Assume that* $1 < b < b_0$ *and* $\sigma = z_2$. *If* $\mu^* = (\sigma b - a^2 - \sigma)/(2\sigma) \neq \mu_n$ *for all* $n \in \mathbb{N}_0$, *then the positive constant equilibrium* E^* *of the reaction–diffusion system* (3) *is locally asymptotically stable.*

If $\sigma > z_2$, then $\mathscr{D} > 0$, and hence the equation

$$\sigma k^2 + \left(a^2 - \sigma b + \sigma\right)k + a^2 = 0$$

has two different real positive roots

$$k_-(\sigma) = \frac{-(a^2 - \sigma b + \sigma) - \sqrt{\mathscr{D}}}{2\sigma}, \qquad k_+(\sigma) = \frac{-(a^2 - \sigma b + \sigma) + \sqrt{\mathscr{D}}}{2\sigma}. \tag{11}$$

Lemma 3. *Assume that* $1 < b < b_0$, $\sigma > z_2$ *and* $k_\pm(\sigma)$ *is defined by* (11). *Then* $k_+(\sigma)$ *is strictly monotonically increasing, and* $k_-(\sigma)$ *is strictly monotonically decreasing.*

Proof. Notice that

$$k_+(\sigma) = \frac{-(a^2 - \sigma b + \sigma) + \sqrt{\mathscr{D}}}{2\sigma}$$
$$= -\frac{a^2}{2\sigma} + \frac{b-1}{2} + \frac{1}{2}\sqrt{\frac{a^4}{\sigma^2} - \frac{2a^2(b+1)}{\sigma} + (b-1)^2}.$$

It follows that

$$k'_+(\sigma) = \frac{a^2}{2\sigma^2}\left[1 + \frac{\sigma(b+1) - a^2}{\sqrt{\mathscr{D}}}\right].$$

Since $\sigma > z_2 > a^2/(b-1)$, one can know that $k'_+(\sigma) > 0$ when $\sigma > z_2$, which implies that $k_+(\sigma)$ is strictly monotonically increasing for $\sigma > z_2$.

On the other hand, notice that $k_+(\sigma)k_-(\sigma) = a^2/\sigma$. Thus,

$$k'_+(\sigma)k_-(\sigma) + k_+(\sigma)k'_-(\sigma) = -\frac{a^2}{\sigma^2}.$$

From the positivity of $k_+(\sigma)$, $k_-(\sigma)$ and $k'_+(\sigma)$ one can easily know that $k'_-(\sigma) < 0$, which demonstrates that $k_-(\sigma)$is strictly monotonically decreasing for $\sigma > z_2$. □

Let $\sigma \to \infty$. Then

$$\lim_{\sigma\to\infty} k_-(\sigma) = 0, \qquad \lim_{\sigma\to\infty} k_+(\sigma) = b - 1.$$

The following result can be obtained.

Theorem 5. *If* $1 < b < b_0$ *and* $\mu_1 > b - 1$*, then the positive constant equilibrium* E^* *of the reaction–diffusion system* (3) *is locally asymptotically stable for any* $\sigma > 0$.

If there exists $n \in \mathbb{N} = \{1, 2, \dots\}$ such that $k_-(\sigma) < \mu_n < k_+(\sigma)$, then $D_n < 0$, and thus, we have the following result.

Theorem 6. *Assume that* $1 < b < b_0$ *and* $\sigma > z_2$*. If there exists* $n \in \mathbb{N} = \{1, 2, \dots\}$ *such that* $k_-(\sigma) < \mu_n < k_+(\sigma)$*, then the positive constant equilibrium* E^* *of the reaction–diffusion system* (3) *is Turing unstable.*

Theorem 7. *Assume that* $1 < b < b_0$*. If there exits* $n \in \mathbb{N} = \{1, 2, \dots\}$ *such that* $0 < \mu_n < b - 1$*, then the positive constant equilibrium* E^* *of the reaction–diffusion system* (3) *is Turing unstable when* $\sigma \to \infty$.

4 Hopf bifurcation of reaction–diffusion system

This section devotes to considering the existence of Hopf bifurcation and the properties of spatially homogeneous Hopf bifurcation of system (3) at the positive constant equilibrium E^* by using the methods due to Yi, Wei and Shi [15]; see also Yan, Chen and Zhang [14] .

4.1 Existence of Hopf bifurcation

For $n \in \mathbb{N}_0$, let $T_n(b)$ and $D_n(b)$ be defined by

$$T_n(b) = b - 1 - a^2 - (1+\sigma)\mu_n$$

and

$$D_n(b) = \sigma\mu_n^2 + \left[a^2 - \sigma(b-1)\right]\mu_n + a^2.$$

We know from [15] that if there exist some certain $n \in \mathbb{N}_0$ and $b = b^H > 0$ such that

$$T_n\left(b^H\right) = 0, \qquad D_n\left(b^H\right) > 0$$

and for all $j \neq n$,

$$T_j\left(b^H\right) \neq 0, \qquad D_j\left(b^H\right) \neq 0$$

and

$$T_n'\left(b^H\right) \neq 0,$$

then the reaction–diffusion system (3) undergoes a Hopf bifurcation at the positive constant equilibrium E^* when $b = b^H$. In addition, the corresponding Hopf bifurcation is spatially homogeneous if $n = 0$ and is spatially inhomogeneous if $n \in \mathbb{N}$.

Let $b = b_0 = 1 + a^2$. Then it is easy to see from Section 2 that $T_0(b_0) = 0$, $D_0(b_0) > 0$ and $T_j(b_0) < 0$ for all $j \in \mathbb{N}$. Notice that

$$D_n(b_0) = \sigma\mu_n^2 + (1-\sigma)a^2\mu_n + a^2.$$

Therefore, one can see that if $0 < \sigma \leqslant 1$ or $\sigma > 1$ and $a^2(1-\sigma)^2 < 4\sigma$ holds, then $D_j(b_0) > 0$ for all $j \in \mathbb{N}$. Combining the fact $T_0'(b_0) = 1$, we can state the following result regarding the existence of spatially homogenous Hopf bifurcation.

Theorem 8. *If $0 < \sigma \leqslant 1$ or $\sigma > 1$ and $a^2(1-\sigma)^2 < 4\sigma$ is satisfied, then system (3) undergoes a spatially homogenous Hopf bifurcation at the positive constant equilibrium E^* when $b = b_0$.*

In the sequel, we consider the existence of spatially heterogenous Hopf bifurcation of system (3) at the positive constant equilibrium E^*. For $n \in \mathbb{N}$, define b_n by

$$b_n = 1 + a^2 + (1+\sigma)\mu_n. \tag{12}$$

Then $T_n(b_n) = 0$ and $T_j(b_n) \neq 0$ for any $j \neq n$. Furthermore, let a, σ and μ_n satisfy one of the following conditions:

$$\sigma < 1 \quad \text{and} \quad a^2(1-\sigma) \geqslant \sigma(1+\sigma)\mu_n \tag{13}$$

or

$$\sigma \geqslant 1 \quad \text{and} \quad \sigma(1+\sigma)\mu_n - a^2(1-\sigma) < 2a. \tag{14}$$

Then for any $j \in \mathbb{N}_0$,

$$D_j(b_n) = \sigma\mu_j^2 + \left[a^2(1-\sigma) - \sigma(1+\sigma)\mu_n\right]\mu_j + a^2 > 0.$$

Thus, we can derive the following result regarding the existence of spatially heterogenous Hopf bifurcation of system (3) at the positive constant equilibrium E^*.

Theorem 9. *For $n \in \mathbb{N}$, let b_n be defined by* (12)*. If a, σ and μ_n satisfy one of the conditions* (13) *or* (14)*, then system* (3) *can undergo a spatially heterogenous Hopf bifurcation at the positive constant equilibrium E^* when $b = b_n$.*

4.2 Properties of spatially homogeneous Hopf bifurcation

In this subsection, we shall analyze mainly the properties of spatially homogeneous Hopf bifurcation of system (3) at the positive constant equilibrium E^* when $b = b_0$. In order to guarantee the existence of spatially homogeneous Hopf bifurcation of system (3) at the positive constant equilibrium E^*, we always assume that $0 < \sigma \leqslant 1$ or $\sigma > 1$ and $a^2(1-\sigma)^2 < 4\sigma$ holds throughout this subsection. Moreover, for the sake of convenience, we assume that the spatial domain Ω can be chosen such that all the eigenvalues μ_i ($i \in \mathbb{N}_0$) of $-\Delta$ on Ω with homogenous Neumann boundary condition are simple.

Let the linear operator $L(b)$ be defined by

$$L(b) = \begin{pmatrix} \Delta + b - 1 & b \\ -a^2 & \sigma\Delta - a^2 \end{pmatrix},$$

and the inner product $\langle\cdot,\cdot\rangle$ on $X_{\mathbb{C}}^2$ be given by $\langle U, V\rangle = \int_0^\pi (\bar{u_1} u_2 + \bar{v_1} v_2)\,\mathrm{d}x$ for any $U = (u_1, u_2)$, $V = (v_1, v_2) \in X_{\mathbb{C}}$. Then the conjugate operator $L^*(b)$ of $L(b)$ under the meaning of the inner product $\langle\cdot,\cdot\rangle$ is given by

$$L^*(b) = \begin{pmatrix} \Delta + b - 1 & -a^2 \\ b & \sigma\Delta - a^2 \end{pmatrix}. \tag{15}$$

From the discussions in Section 2 we see easily that $\pm \mathrm{i}a$ are a pair of purely imaginary eigenvalues of $L(b_0)$ and $L^*(b_0)$. Take q and q^* respectively as

$$q = \begin{pmatrix} 1 \\ \frac{-a^2 + \mathrm{i}a}{1+a^2} \end{pmatrix}, \qquad q^* = \frac{1}{2a|\Omega|} \begin{pmatrix} a + \mathrm{i}a^2 \\ \mathrm{i}(1+a^2) \end{pmatrix}.$$

Then one can easily verify that q and q^* satisfy the following equalities

$$\langle q^*, q\rangle = 1, \quad \langle q^*, \bar{q}\rangle = 0, \quad L(b_0)q = \mathrm{i}aq \quad \text{and} \quad L^*(b_0)q^* = -\mathrm{i}aq^*. \tag{16}$$

Let $U = (u, v)^{\mathrm{T}}$,

$$\begin{aligned} f_1(U) &= \left(1 + a^2\right)\left(u^2 + 2uv + u^2 v\right), \\ f_2(U) &= -a^2\left(u^2 + 2uv + u^2 v\right) \end{aligned} \tag{17}$$

and $f(U) = (f_1(U), f_2(U))^{\mathrm{T}}$. So, when $b = b_0$, system (3) can be rewritten into the following abstract form:

$$\frac{\mathrm{d}U}{\mathrm{d}t} = L(b_0)U + f(U). \tag{18}$$

Define X^c and X^s by

$$X^c = \{zq + \bar{z}\bar{q} \mid z \in \mathbb{C}\} \quad \text{and} \quad X^s = \{u \in X \mid \langle q^*, u\rangle = 0\}.$$

Then the Sobolev space X can be represented as the direct sum of X^c and X^s, that is, $X = X^c \bigoplus X^s$. Thus, we can know that for any $U = (u,v)^{\mathrm{T}} \in X$, there exists a complex number $z \in \mathbb{C}$ and $w = (w_1, w_2)^{\mathrm{T}} \in X^s$ such that

$$U = zq + \bar{z}\bar{q} + w. \tag{19}$$

Combining (16) and (19), the abstract system (18) can be changed into the system of the form

$$\begin{aligned} \frac{\mathrm{d}z}{\mathrm{d}t} &= \mathrm{i}az + \langle q^*,\, f(zq + \bar{z}\bar{q} + w)\rangle, \\ \frac{dw}{\mathrm{d}t} &= L(b_0)w + H(z, \bar{z}, w), \end{aligned} \tag{20}$$

where

$$\begin{aligned} H(z, \bar{z}, w) = f(zq + \bar{z}\bar{q} + w) &- \langle q^*,\, f(zq + \bar{z}\bar{q} + w)\rangle q \\ &- \langle \bar{q}^*,\, f(zq + \bar{z}\bar{q} + w)\rangle \bar{q}. \end{aligned} \tag{21}$$

By the center manifold theorem in [5], we know that system (20) has a local center manifold near the origin O and it is tangent to the center subspace of (20) at the origin O. Consequently, w can be expressed into the following form:

$$w = \frac{w_{20}}{2} z^2 + w_{11} z\bar{z} + \frac{w_{02}}{2} \bar{z}^2 + O(|z|^3). \tag{22}$$

In view of (17) and (22), $H(z, \bar{z}, w)$ in (21) can be expanded as

$$H(z, \bar{z}, w) = \frac{H_{20}}{2} z^2 + H_{11} z\bar{z} + \frac{H_{02}}{2} \bar{z}^2 + O\bigl(|z|^3\bigr). \tag{23}$$

Substituting (22) and (23) into the second equation of (20), we can obtain

$$\bigl(2\mathrm{i}\omega_0 - L(b_0)\bigr) w_{20} = H_{20}, \quad \bigl(-L(b_0)\bigr) w_{11} = H_{11} \quad \text{and} \quad w_{02} = \overline{w}_{20}. \tag{24}$$

Let $q_2 = (-a^2 + \mathrm{i}a)/(1 + a^2)$ and define c_0, d_0, e_0, f_0, g_0 and h_0 by

$$\begin{aligned} c_0 &= \bigl(1 + a^2\bigr)(1 + 2q_2), & d_0 &= -a^2(1 + 2q_2), \\ e_0 &= 2\bigl(1 + a^2\bigr)(1 + q_2 + \bar{q}_2), & f_0 &= -2a^2(1 + q_2 + \bar{q}_2), \\ g_0 &= \bigl(1 + a^2\bigr)(2q_2 + \bar{q}_2), & h_0 &= -a^2(2q_2 + \bar{q}_2). \end{aligned}$$

Respectively, denote Q_{qq}, $Q_{q\bar{q}}$ and $C_{qq\bar{q}}$ by

$$Q_{qq} = \begin{pmatrix} c_0 \\ d_0 \end{pmatrix}, \quad Q_{q\bar{q}} = \begin{pmatrix} e_0 \\ f_0 \end{pmatrix} \quad \text{and} \quad C_{qq\bar{q}} = \begin{pmatrix} g_0 \\ h_0 \end{pmatrix}.$$

Then

$$f(zq + \bar{z}\bar{q} + w) = Q_{qq} z^2 + Q_{q\bar{q}} z\bar{z} + \bar{Q}_{qq} \bar{z}^2 + O\bigl(|z|^3\bigr).$$

Define q_1^* and q_2^* by

$$q_1^* = \frac{1+ai}{2} \quad \text{and} \quad q_2^* = \frac{(1+a^2)i}{2a}.$$

Thus, from the definitions of q_2, q_1^* and q_2^* we can get

$$\begin{aligned}
&c_0 - \langle q^*, Q_{qq}\rangle - \langle \bar{q}^*, Q_{qq}\rangle \\
&\quad = c_0 - (q_1^* + \bar{q}_1^*)c_0 - (q_2^* + \bar{q}_2^*)d_0 = 0, \\
&d_0 - \langle q^*, Q_{qq}\rangle q_2 - \langle \bar{q}^*, Q_{qq}\rangle \bar{q}_2 \\
&\quad = d_0 - (q_1^*\bar{q}_2 + \bar{q}_1^* q_2)c_0 - (q_2^*\bar{q}_2 + \bar{q}_2^* q_2)d_0 = 0, \\
&e_0 - \langle q^*, Q_{q\bar{q}}\rangle - \langle \bar{q}^*, Q_{q\bar{q}}\rangle \\
&\quad = e_0 - (q_1^* + \bar{q}_1^*)e_0 - (q_2^* + \bar{q}_2^*)f_0 = 0, \\
&f_0 - \langle q^*, Q_{q\bar{q}}\rangle q_2 - \langle \bar{q}^*, Q_{q\bar{q}}\rangle \bar{q}_2 \\
&\quad = f_0 - (q_1^*\bar{q}_2 + \bar{q}_1^* q_2)e_0 - (q_2^*\bar{q}_2 + \bar{q}_2^* q_2)f_0 = 0.
\end{aligned} \tag{25}$$

(25) together with (21) yield that $H_{20} = H_{11} = 0$, and further, we can also obtain from (24) that $w_{20} = w_{11} = 0$ since $L(b_0)$ has only a pair of purely imaginary eigenvalues $\pm ia$ and has no the other eigenvalues. Now, we know that the reaction–diffusion system (15) restricted to the center manifold in z, $\bar{z}$ coordinates is

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \mathrm{i}az + \frac{1}{2}g_{20}z^2 + g_{11}z\bar{z} + \frac{1}{2}g_{02}\bar{z}^2 + \frac{1}{2}g_{21}z^2\bar{z} + \mathcal{O}\big(|z|^4\big),$$

where

$$\begin{aligned}
g_{20} &= \langle q^*, Q_{qq}\rangle = \bar{q}_1^* c_0 + \bar{q}_2^* d_0 = \frac{1-a^2}{2} + \mathrm{i}a, \\
g_{11} &= \langle q^*, Q_{q\bar{q}}\rangle = \bar{q}_1^* e_0 + \bar{q}_2^* f_0 = 1 - a^2, \\
g_{21} &= \langle q^*, C_{qq\bar{q}}\rangle = \bar{q}_1^* g_0 + \bar{q}_2^* h_0 = \frac{-3a^2 + \mathrm{i}a}{2}.
\end{aligned} \tag{26}$$

According to (26), we can compute

$$\begin{aligned}
\mathrm{Re}c_1(0) &= \mathrm{Re}\left[\frac{\mathrm{i}}{2a}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2\right) + \frac{g_{21}}{2}\right] \\
&= \mathrm{Re}\left(\frac{\mathrm{i}}{2a}g_{20}g_{11} + \frac{g_{21}}{2}\right) = -\frac{a^2+2}{4} < 0.
\end{aligned} \tag{27}$$

Combining the references [5, 15], (27) and $T_0'(b_0) = 1$, we can state the following result.

Theorem 10. *Assume that the conditions* $0 < \sigma \leqslant 1$ *or* $\sigma > 1$ *and* $a^2(1-\sigma)^2 < 4\sigma$ *are satisfied. Then the spatially homogeneous Hopf bifurcation of system* (3) *at the positive constant equilibrium* E^* *is supercritical, and the corresponding spatially homogeneous bifurcating periodic solutions are orbitally asymptotically stable.*

5 Examples and numerical simulations

In this section, we provide some numerical simulations for particular cases of system (3) to support the theoretical conclusions obtained in Sections 2–4 by means of the MATLAB software package and the numerical methods solving differential equations. In order to complete our numerical verifications, we restrict the spatial domain Ω as $(0, \pi)$ and consider the following model:

$$\begin{aligned}
&\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 1 - (b+1)u + bu^2 v, \quad x \in (0,\pi),\ t > 0,\\
&\frac{\partial v}{\partial t} = \sigma \frac{\partial^2 v}{\partial x^2} + a^2\bigl(u - u^2 v\bigr), \quad x \in (0,\pi),\ t > 0,\\
&\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0, \quad x = 0,\ \pi, t > 0,\\
&u(x,0) = u_0(x) \geqslant 0, \quad v(x,0) = v_0(x) \geqslant 0, \quad x \in (0,\pi).
\end{aligned} \tag{28}$$

Example 1. Let $a = 1$ in system (4). Then $b_0 = 1 + a^2 = 2$. Therefore, by Theorem 1 we know that the equilibrium E^* of system (4) is locally asymptotically stable when $0 < b < b_0 = 2$, while is unstable as $b > b_0 = 2$, and a stable limit cycle can bifurcate from E^* if $0 < b - b_0 \ll 1$; see Fig. 1.

Example 2.

(i) Take $a = 1$ and $b = 0.5$ in system (28). Then $0 < b < 1$, and from Theorem 3 we know that the positive constant equilibrium E^* of system (28) is locally asymptotically stable for any $\sigma > 0$; see Figs. 2–3.

(ii) Fix $a = 2$ and $b = 4$ in system (28). Then $1 < b < b_0 = 1 + a^2 = 5$ and z_2 defined by (10) is equal to 4. Thus, by Theorem 3 one can know that the positive constant equilibrium E^* of system (28) is locally asymptotically stable when $0 < \sigma < z_2 = 4$; see Fig. 4.

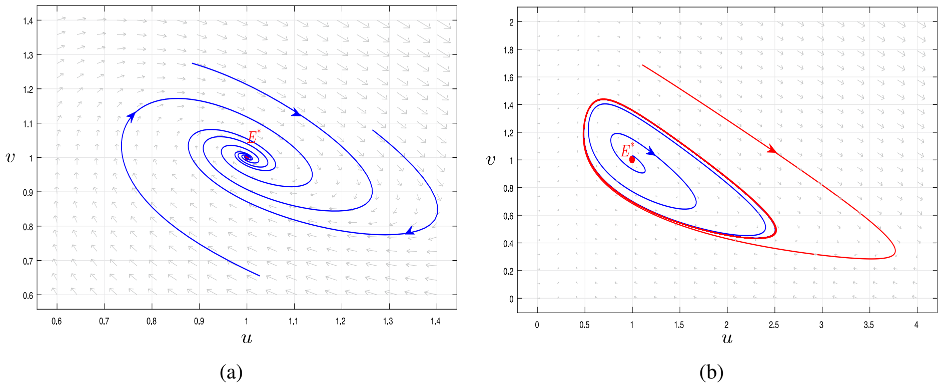

Figure 1. Phase portraits of system (4) when $a = 1$: (a) $b = 1.5$, the equilibrium E^* is locally asymptotically stable; (b) $b = 2.5$, E^* is unstable and (4) has a stable limit cycle bifurcating from E^*.

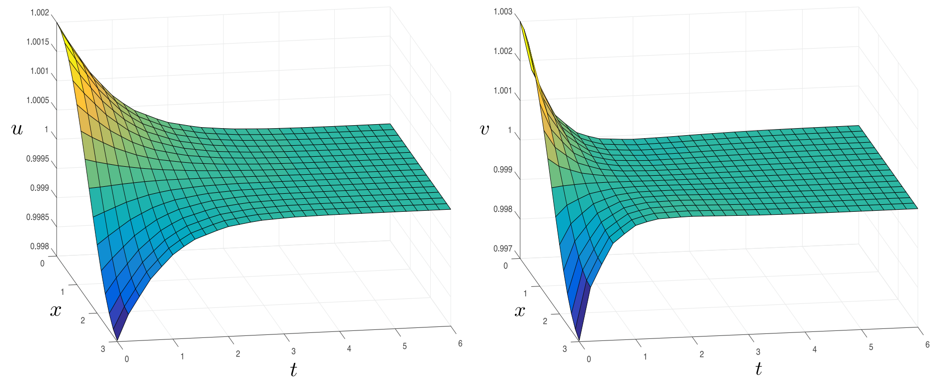

Figure 2. Stable behavior of the constant positive equilibrium E^* of (28) when $a = 1$, $b = 0.5$ and $\sigma = 1$.

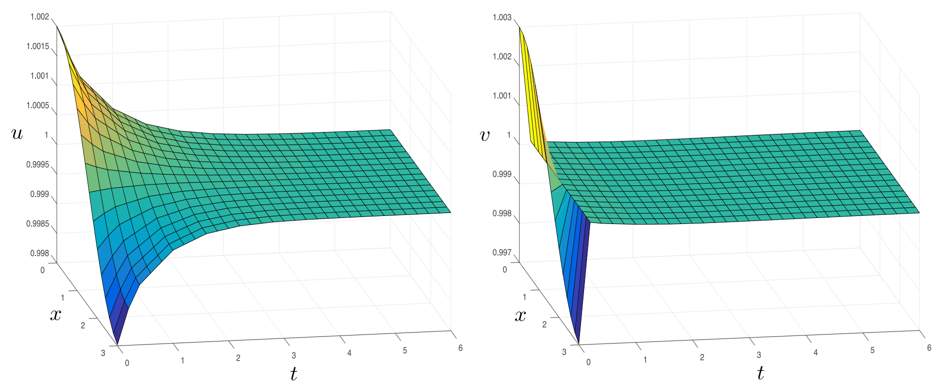

Figure 3. Stable behavior of the constant positive equilibrium E^* of (28) when $a = 1$, $b = 0.5$ and $\sigma = 100$.

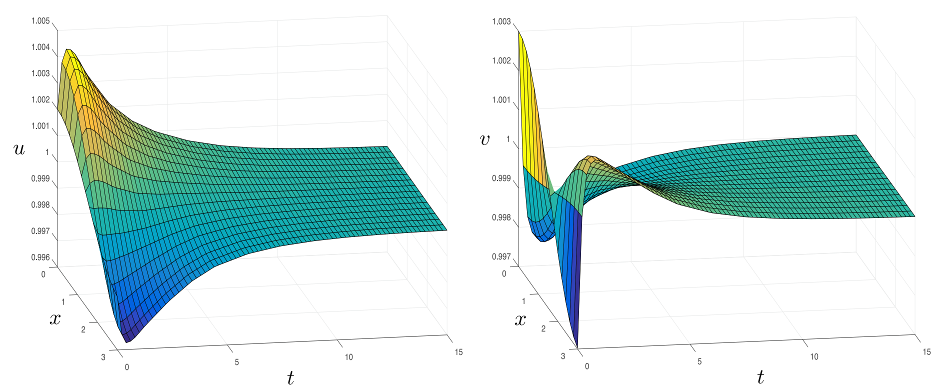

Figure 4. Local stability of the constant positive equilibrium E^* of (28) when $a = 2$, $b = 4$ and $\sigma = 3$.

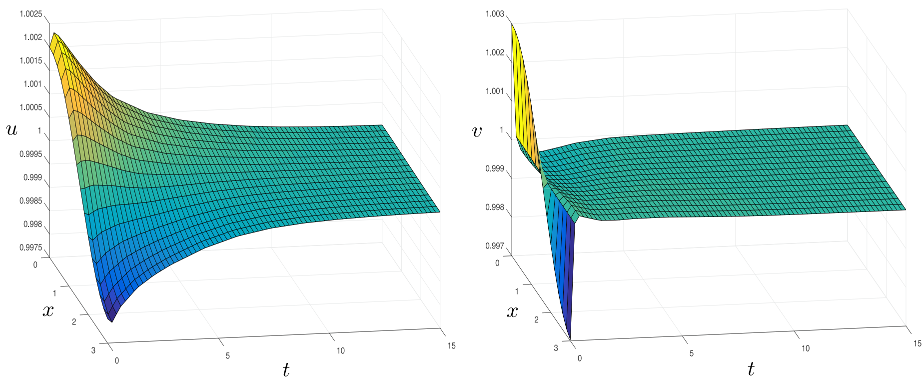

Figure 5. Stable behavior of system (28) when $a = \sqrt{2}$, $b = 2$ and $\sigma = z_2 = 6 + 4\sqrt{2}$.

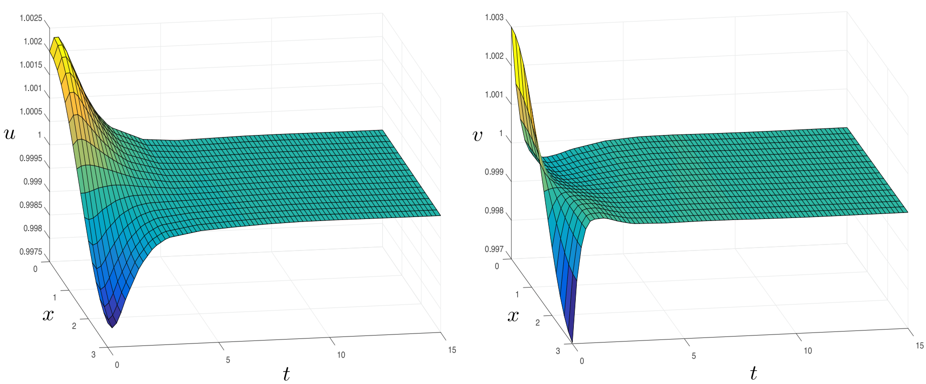

Figure 6. Stability of the positive constant equilibrium E^* of system (28) when $a = 1$, $b = 1.5$ and $\sigma = 2$.

Example 3. Let $a = \sqrt{2}$ and $b = 2$ in system (28). Then $1 < b < b_0 = 1 + a^2 = 3$ and $z_2 = 6+4\sqrt{2}$. If $\sigma = z_2 = 6+4\sqrt{2}$, then $\mu^* = (\sigma b - a^2 - \sigma)/(2\sigma) = \sqrt{2} - 1 \neq \mu_n = n^2$ for all $n \in \mathbb{N}_0$. Theorem 4 tells us that the positive constant equilibrium E^* of system (28) is locally asymptotically stable; see Fig. 5.

Example 4. Let $a = 1$ and $b = 1.5$ in (28). Then one can observe that $1 < b < b_0 = 1 + a^2 = 2$ and $\mu_1 = 1 > b - 1 = 0.5$. Therefore, from Theorem 5 we know that the positive constant equilibrium E^* of the reaction–diffusion system (28) is locally asymptotically stable for any $\sigma > 0$; see Figs. 6–7.

Example 5. Let $a = 3$, $b = 6$ and $\sigma = 8$ in system (28). Then $1 < b = 6 < b_0 = 1 + a^2 = 10$ and $\sigma > z_2 = (63 + 18\sqrt{6})/5 \approx 4.2836$. In addition, $k_-(\sigma)$ and $k_+(\sigma)$ given as in (11) are respectively 0.3161 and 3.5589. Thus, we can see $k_-(\sigma) < \mu_1 = 1 < k_+(\sigma)$, and from Theorem 6 we know that the positive constant equilibrium E^* of the reaction–diffusion system (28) is Turing unstable; see Fig. 8.

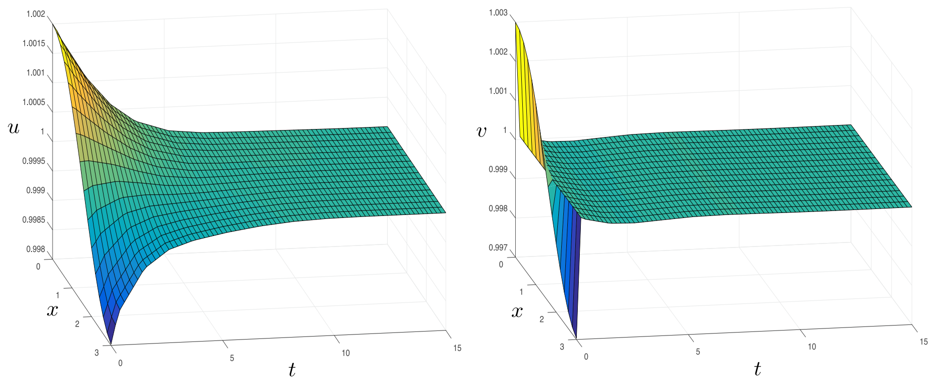

Figure 7. Stability of the positive constant equilibrium E^* of system (28) when $a = 1$, $b = 1.5$ and $\sigma = 100$.

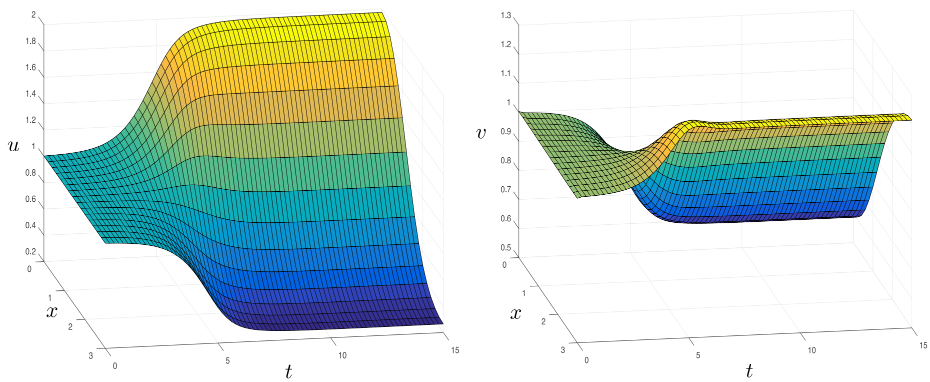

Figure 8. Turing instability of the positive constant equilibrium E^* of (28) when $a = 3$, $b = 6$ and $\sigma = 8$.

Example 6. Let $a = 2$ and $b = 3.5$. Then $0 < \mu_1 = 1 < b - 1 = 2.5$, and from Theorem 7 we know that the positive constant equilibrium E^* of the reaction–diffusion system (28) is Turing unstable when $\sigma \to \infty$; see Figs. 9–10.

Example 7. Let $a = 1$ and $\sigma = 0.5$ in system (28). Then $0 < \sigma < 1$, $b_0 = 1 + a^2 = 2$, and by Theorems 2 and 10 one can see that the positive constant equilibrium E^* of system (28) is unstable when $b > b_0 = 2$. Meanwhile, when $b = b_0 = 2$, the system can undergo a supercritical Hopf bifurcation at the positive equilibrium E^*, and the bifurcating periodic solutions are stable; see Figs. 11–12.

Example 8. Take $a = 2$ and $\sigma = 2$ in system (28). Then $\sigma > 1$ and $a^2(1 - \sigma)^2 = 4 < 4\sigma = 8$ hold. Therefore, by Theorems 2 and 10 we know that the positive constant equilibrium E^* of system (28) is unstable when $b > b_0 = 5$. Meanwhile, when $0 < b - 5 \ll 1$, the system can bifurcate a spatially homogeneous period solution form the positive equilibrium E^*, and the bifurcating periodic solutions are stable; see Figs. 13–14.

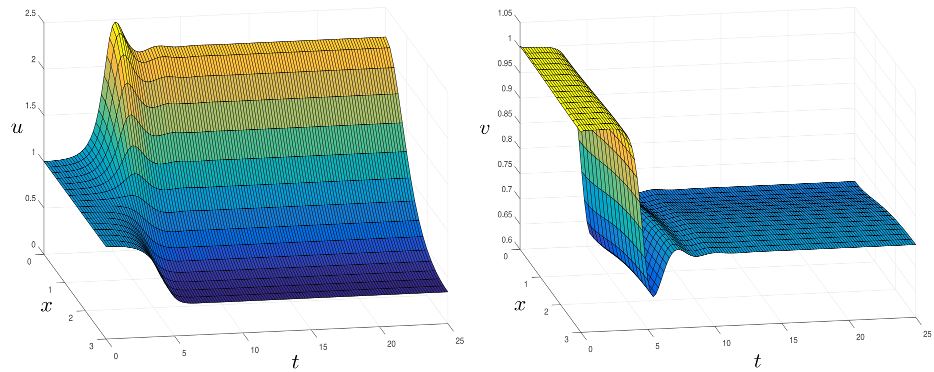

Figure 9. Turing instability of the positive constant equilibrium E^* of (28) when $a = 2$, $b = 3.5$ and $\sigma = 100$.

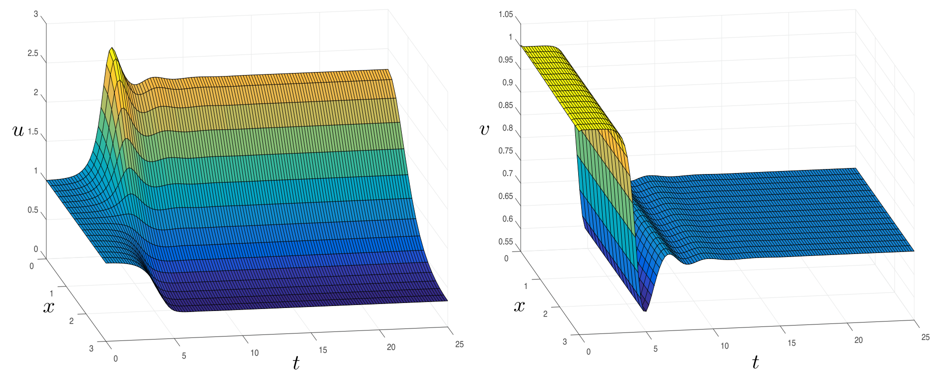

Figure 10. Turing instability of the positive constant equilibrium E^* of (28) when $a = 2$, $b = 3.5$ and $\sigma = 8000$.

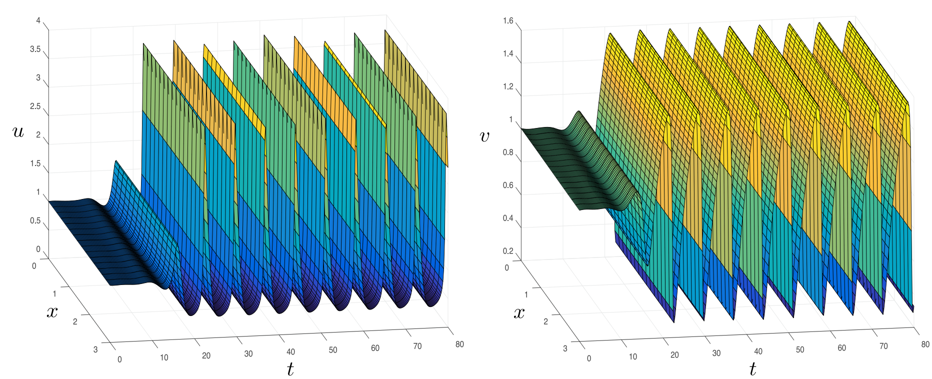

Figure 11. Instability of the constant positive equilibrium E^* and the stable time-period solution of system (28) when $a = 1$, $b = 3$ and $\sigma = 0.5$.

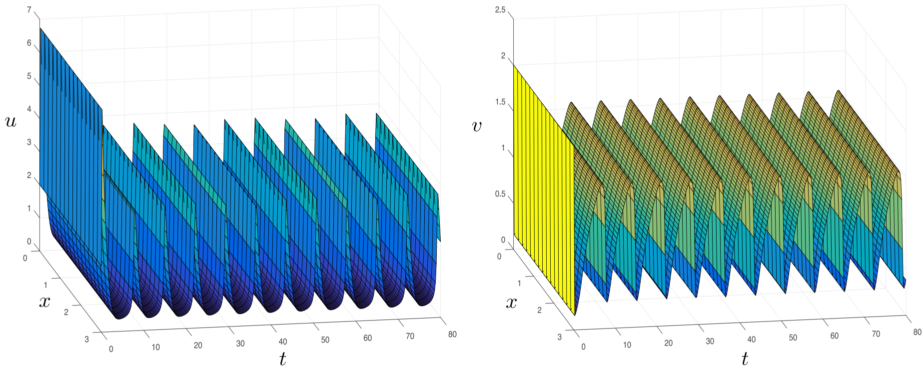

Figure 12. Instability of the constant positive equilibrium E^* and the stable time-period solution of system (28) when $a = 1$, $b = 3$ and $\sigma = 0.5$.

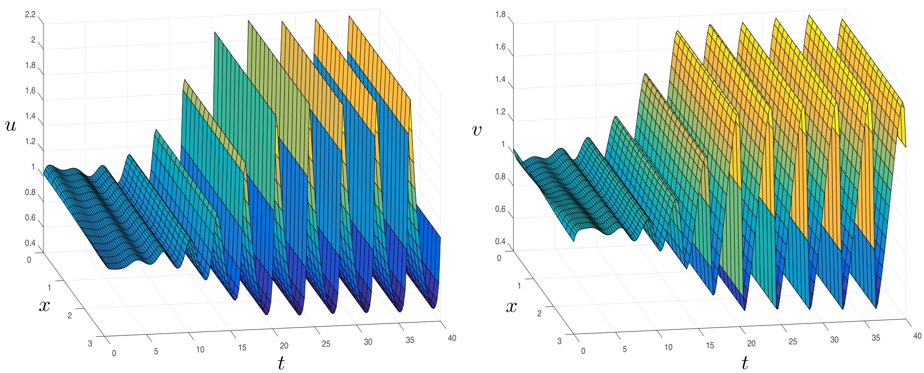

Figure 13. Instability of the constant positive equilibrium E^* and the stable time-period solution of system (28) when $a = 2$, $b = 5.5$ and $\sigma = 2$.

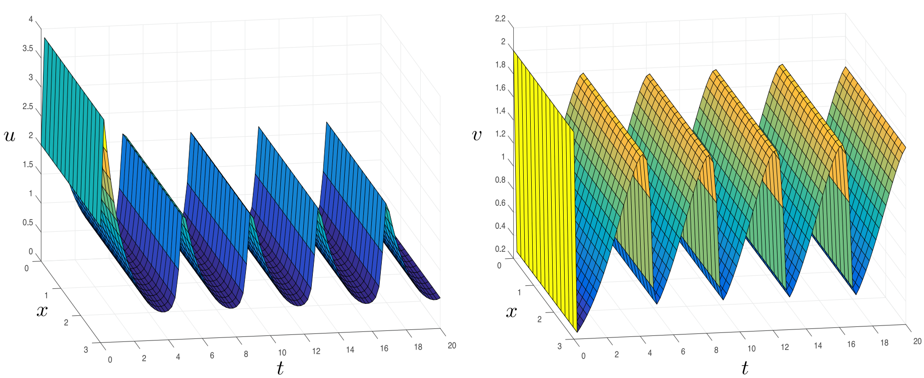

Figure 14. Instability of the constant positive equilibrium E^* and the stable time-period solution of system (28) when $a = 2$, $b = 5.5$ and $\sigma = 2$.

Acknowledgment. The authors would like to express great appreciation to the associated editors and anonymous referees for their valuable comments and suggestions, which resulted in much improvement of the manuscript.

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