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Solvability of fractional dynamic systems utilizing measure of noncompactness

Hemant Kumar Nashine[a], **Rabha W. Ibrahim**[b,c], **Reza Arab**[d], **Mohsen Rabbani**[d]

[a]Vellore Institute of Technology, Vellore-632014, TN, India, 50603

[b]Informetrics Research Group, Ton Duc Thang University, Ho Chi Minh City, Vietnam

[c]Faculty of Mathematics & Statistics, Ton Duc Thang University, Ho Chi Minh City, Vietnam
rabhaibrahim@tdtu.edu.vn

[d]Islamic Azad University, Sari, Iran

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Abstract. Fractional dynamics is a scope of study in science considering the action of systems. These systems are designated by utilizing derivatives of arbitrary orders. In this effort, we discuss the sufficient conditions for the existence of the mild solution (m-solution) of a class of fractional dynamic systems (FDS). We deal with a new family of fractional m-solution in $\mathbb{R}^n$ for fractional dynamic systems. To accomplish it, we introduce first the concept of (F, ψ)-contraction based on the measure of noncompactness in some Banach spaces. Consequently, we establish requisite fixed point theorems (FPTs), which extend existing results following the Krasnoselskii FPT and coupled fixed point results as a outcomes of derived one. Finally, we give a numerical example to verify the considered FDS, and we solve it by iterative algorithm constructed by semianalytic method with high accuracy. The solution can be considered as bacterial growth system when the time interval is large.

Keywords: fractional calculus, fractional differential operator, fixed point theorem, measure of noncompactness.

1 Introduction

Fractional calculus includes all fractional concepts, (operators) fractional formulas (equations, inequalities and inclusions) and fractional formal (logic concepts) (see [19, 20, 22, 23, 26]) can express the possessions of the history of materials. Practical problems take in classifications of the fractional operators (differential and integral) allowing the procedure of the entity and uniqueness of associations outcome based equity model. For example, fractional diffusion equations (derivatives with respect to time), where elements are more slowly than a traditional diffusion. This concept demonstrated its authority in all sciences.

The main problem in the classes of fractional differentiation arguments (equations, inclusions and systems) is the uniqueness of m-solution. This problem has been discussed by many authors. For recent work, one can see [3, 9–11, 13, 15–17, 28, 32]. Most of these efforts delivered by various types of FPTs in compact sets. Therefore, we shall develop a set of fixed point theorems of measure of noncompactness based on (F, ψ)-contraction functions (see [1–8, 12, 21]).

In our investigation, we establish some basic fixed point results, which generalize some well-known results. Our method is based on the new definition of (F, ψ)-contraction with respect to measures of noncompactness in Banach spaces. Consequently, a set of coupled FPTs is also derived from the main result. Applying our results, we deliver adequate conditions for a constructed mild solution (m-solution) of fractional dynamic systems.

2 Background

2.1 M-solution

A continuous function $\nu : [0, \infty) \to E$ is titled a *mild solution* (m-solution) of the Cauchy problem $\nu'(t) = \Delta\nu(t)$, $\nu(0) = \nu_0$ if

$$\int_0^t \nu(s)\,\mathrm{d}s \in \mathrm{Dom}(\Delta) \quad \text{and} \quad \Delta \int_0^t \nu(s)\,\mathrm{d}s = \nu(t) - \nu_0.$$

Any classical solution is m-solution. In [9], Araya and Lizama provided the idea of α-resolvent sets establishing the entity of m-solutions of equation [9]

$$D_t^\alpha \nu(t) = \Delta\nu(t) + t^n\phi(t) \quad \alpha \in [1, 2],\ n \in \mathbb{Z}_+,$$

in a Banach space E for automorphism functions $\phi : \mathbb{R} \to E$. Moreover, the researchers studied the m-solution of

$$D_t^\alpha \nu(t) = \Delta\nu(t) + \phi\bigl(t, \nu(t)\bigr), \quad \alpha \in [1, 2],$$

and

$$D_t^\alpha \nu(t) = \Delta\nu(t) + \phi\bigl(t, \nu(t), \nu'(t)\bigr), \quad \alpha \in [1, 2].$$

Many investigators imposed different criteria of m-solution for various classes of FDS (see [13, 28]). In [13], Cuevas and Lizama suggested the almost mild solutions for the following class of equation:

$$D_t^\alpha \nu(t) = \Delta\nu(t) + D_t^{\alpha-1}\phi(\cdot, \nu), \quad \alpha \in [1, 2],$$

where Δ is a linear operator, and $\phi(t, \nu)$ is Lipschitz in ν. In [3], Agarwal et al. imposed analytic operator establishing the integral formal

$$D_t^\alpha \nu(t) = \Delta\nu(t) + \int_0^t \Theta(t - \varsigma)\nu(\varsigma)\,\mathrm{d}\varsigma, \quad t > 0,$$
$$\nu(0) = \nu_0.$$

In [28], Ponce studied the solutions of the following equation:

$$D_t^\alpha \nu(t) = \Delta\nu(t) + \int_{-\infty}^{t} \rho(t-\varsigma)\Delta\nu(\varsigma)\,\mathrm{d}\varsigma + \phi\bigl(t, \nu(t)\bigr), \quad t \in \mathbb{R},$$

where the linear operator Δ is closed on a Banach space E, $\alpha > 0$, $\rho \in L^1(\mathbb{R}_+)$ is a kernel of the integral operator, and $\phi : \mathbb{R} \times E \to E$ achieves a special type of Lipschitz conditions. In [15], Dhanapalan et al. established m-solution of a class of nonlinear FDS of the form

$$\begin{aligned} &D_t^\alpha \nu(t) + \Delta\nu(t) \\ &\quad = \int_0^t \phi\bigl(t, \varsigma, \nu(\varsigma)\bigr)\,\mathrm{d}\varsigma + \int_0^t \rho(t-\varsigma)\psi\bigl(\varsigma, \nu(\varsigma)\bigr)\,\mathrm{d}\varsigma, \quad \varsigma, t \in [0,T],\ \varsigma < t, \\ &\nu(0) = \nu_0. \end{aligned}$$

2.2 Measure concept

The following abbreviations are utilized in this manuscript: $\mathbb{R}$ is the set of real numbers, $\mathbb{R}^+ = [0, +\infty)$; $(E, \|\cdot\|)$ is the Banach space (BC); $\mathcal{B}(x, r)$ – the closed ball, $\mathcal{B}_r = \mathcal{B}(0, r)$; $\mathfrak{M}_E$ and $\mathfrak{N}_E$ denote the family of nonempty bounded subsets of E and the subfamily connecting all relatively compact set, respectively; μ denotes the measure of noncompactness (MNC) (see [12]); BCC – the bounded closed convex set.

Definition 1. A mapping $\mu : \mathfrak{M}_E \to \mathbb{R}^+$ is called MNC in E if it fulfills the next conditions:

- (d1) The family $\ker\mu \subset \mathfrak{N}_E$ (the kernel of the MNC) is nonempty set satisfying $\ker\mu = \{\mathfrak{Y} \in \mathfrak{M}_E\colon \mu(\mathfrak{Y}) = 0\}$;
- (d2) $\mathfrak{Y} \subset \mathfrak{Z} \Rightarrow \mu(\mathfrak{Y}) \leqslant \mu(\mathfrak{Z})$;
- (d3) $\mu(\bar{\mathfrak{Y}}) = \mu(\mathfrak{Y})$;
- (d4) $\mu(\operatorname{Conv}\mathfrak{Y}) = \mu(\mathfrak{Y})$;
- (d5) $\mu(\eta\mathfrak{Y} + (1-\eta)\mathfrak{Z}) \leqslant \eta\mu(\mathfrak{Y}) + (1-\eta)(\mathfrak{Y})$ for $\eta \in [0, 1]$;
- (d6) Let $(\mathfrak{Y}_n)$ be a sequence of closed sets in $\mathfrak{M}_E$ achieving the inclusion $\mathfrak{Y}_{n+1} \subset \mathfrak{Y}_n$ $(n = 1, 2, \dots)$, and let $\lim_{n\to\infty}\mu(\mathfrak{Y}_n) = 0$, then the conclusion setting by $\mathfrak{Y}_\infty = \bigcap_{n=1}^\infty \mathfrak{Y}_n$ is nonempty, and $\mu(\mathfrak{Y}_\infty) \leqslant \mu(\mathfrak{Y}_n)$, $n \in \mathbb{N}$.

Denote $\Lambda = \{\mathfrak{C}\colon\ \mathfrak{C} \neq \emptyset, \text{BCC}, \ \mathfrak{C} \subset \mathfrak{E}\}$.

Lemma 1. (*See* [12].) *Let* $\mathfrak{C} \in \Lambda$, *and* $\mathfrak{T} : \mathfrak{C} \to \mathfrak{C}$ *be a continuous and* μ*-set contraction operator such that there is a constant* $k \in [0, 1)$ *with*

$$\mu\bigl(\mathfrak{T}(\mathfrak{Y})\bigr) \leqslant k\mu(\mathfrak{Y})$$

for any nonempty subset $\mathfrak{Y}$ *of* $\mathfrak{C}$, *where* μ *be the Kuratowski MNC on* E. *Then* $\mathfrak{T}$ *admits a fixed point.*

Thereafter, various types of DFPT and their coupled version were considered by utilizing several types of contractive condition in the sense of MNC (for instant, see [1–26]). Here, we introduce a new μ-contraction operator know as (F, ψ)-contraction in the sense of MNC, and we prove some new fixed point, Krasnoselskii FPT and coupled FPTs that generalize the outcomes in [5, 8, 12, 27] mainly.

3 FPTs outcomes

Here, we deal with the set of functions $\psi, \varphi, \mathcal{F} : [0, +\infty) \to [0, +\infty)$ with the following properties:

(i) $\mathcal{F}$ is nondecreasing and continuous satisfying $\mathcal{F}(0) = 0 < \mathcal{F}(t)$. The set of all $\mathcal{F}$ is denoted by $\mathfrak{F}$;

(ii) ψ is right continuous, $\psi(0) = 0$ and bounded by t ($\psi(t) < t$). The set of all ψ is denoted by Ψ;

(iii) φ is a continuous mapping.

Theorem 1. *Define a continuous operator $\mathfrak{T} : \mathfrak{C} \to \mathfrak{C}$, $\mathfrak{C} \in \Lambda$ satisfying*

$$\mathcal{F}\bigl(\mu\bigl(\mathfrak{T}(\mathfrak{Y})\bigr) + \varphi\bigl(\mu\bigl(\mathfrak{T}(\mathfrak{Y})\bigr)\bigr)\bigr) \leqslant \psi\bigl(\mathcal{F}\bigl(\mu(\mathfrak{Y}) + \varphi\bigl(\mu(\mathfrak{Y})\bigr)\bigr)\bigr) \tag{1}$$

for all $\mathfrak{Y} \subseteq \mathfrak{C}$. Then $\mathfrak{T}$ admits at least one fixed point in $\mathfrak{C}$.

Proof. Starting from $\mathfrak{C}_0 = \mathfrak{C}$, we construct a sequence $\{\mathfrak{C}_n\}$ as $\mathfrak{C}_{n+1} = \mathrm{Conv}(\mathfrak{T}\mathfrak{C}_n)$ for $n \in \mathbb{N}^* = \mathbb{N} \cup \{0\}$. Let $n_0 \in \mathbb{N}^*$ and

$$\mu(\mathfrak{C}_{n_0}) + \varphi\bigl(\mu(\mathfrak{C}_{n_0})\bigr) = 0, \qquad \mu(\mathfrak{C}_{n_0}) = 0,$$

then $\mathfrak{C}_{n_0}$ is compact achieving the inclusion

$$\mathfrak{T}(\mathfrak{C}_{n_0}) \subseteq \mathrm{Conv}(\mathfrak{T}\mathfrak{C}_{n_0} = \mathfrak{C}_{n_0+1}) \subseteq \mathfrak{C}_{n_0}.$$

Thus, Schauder's fixed point theorem gives that $\mathfrak{T}$ admits a fixed point. Therefore, we let

$$\mu(\mathfrak{C}_n) + \varphi(\mu(\mathfrak{C}_n)) > 0 \quad \forall n \geqslant 1.$$

In view of (1), we conclude that

$$\begin{aligned}
&\mathcal{F}\bigl[\mu(\mathfrak{C}_{n+1}) + \varphi\bigl(\mu(\mathfrak{C}_{n+1})\bigr)\bigr] \\
&\quad = \mathcal{F}\bigl[\mu\bigl(\mathrm{Conv}(\mathfrak{T}\mathfrak{C}_n)\bigr) + \varphi\bigl(\mu\bigl(\mathrm{Conv}(\mathfrak{T}\mathfrak{C}_n)\bigr)\bigr)\bigr] = \mathcal{F}\bigl[\mu(\mathfrak{T}\mathfrak{C}_n) + \varphi\bigl(\mu(\mathfrak{T}\mathfrak{C}_n)\bigr)\bigr] \\
&\quad \leqslant \psi\bigl(\mathcal{F}\bigl[\mu(\mathfrak{C}_n) + \varphi\bigl(\mu(\mathfrak{C}_n)\bigr)\bigr]\bigr) < \mathcal{F}\bigl[\mu(\mathfrak{C}_n) + \varphi\bigl(\mu(\mathfrak{C}_n)\bigr)\bigr].
\end{aligned}$$

Now, in virtue of

$$\begin{aligned}
&\mathcal{F}\bigl(\mu(\mathfrak{C}_{n+1}) + \varphi\bigl(\mu(\mathfrak{C}_{n+1})\bigr)\bigr) \\
&\qquad \leqslant \psi\bigl(\mathcal{F}\bigl(\mu(\mathfrak{C}_n) + \varphi\bigl(\mu(\mathfrak{C}_n)\bigr)\bigr)\bigr) \leqslant \cdots \leqslant \psi^n\bigl(\mathcal{F}\bigl(\mu(\mathfrak{C}_0) + \varphi\bigl(\mu(\mathfrak{C}_0)\bigr)\bigr)\bigr),
\end{aligned}$$

we attain to

$$\lim_{n\to\infty} \mathcal{F}\big(\mu(\mathfrak{C}_{n+1}) + \varphi\big(\mu(\mathfrak{C}_{n+1})\big)\big) = 0,$$

which leads to

$$\lim_{n\to\infty} \mu(\mathfrak{C}_n) + \varphi\big(\mu(\mathfrak{C}_n)\big) = 0 \quad \text{and} \quad \lim_{n\to\infty} \mu(\mathfrak{C}_n) = 0.$$

Since $\mathfrak{C}_n \supseteq \mathfrak{C}_{n+1}$ and $\mathfrak{T}\mathfrak{C}_n \subseteq \mathfrak{C}_n$ for all $n = 1, 2, \dots$, then by (d6) of Definition 1, $\mathfrak{Y}_\infty = \bigcap_{n=1}^{\infty} \mathfrak{Y}_n$ is nonempty convex closed set, invariant under $\mathfrak{T}$ and belongs to $\ker \mu$. So, Schauder's FPT gives the requested result. □

Remark 1. Putting $\mathcal{F}(t) = t$ in Theorem 1, we achieve the result given in [8]. Special case when $\varphi(t) = 0$, then the inequality $\mathcal{F}(\mu(\mathfrak{T}(\mathfrak{Y}))) \leqslant \psi(\mathcal{F}(\mu(\mathfrak{Y})))$ implies that $\mathfrak{T}$ admits at least one fixed point in $\mathfrak{C}$. Moreover, if $\psi(t) = \lambda t$ and $\mathcal{F}(t) = t$, where $0 \leqslant \lambda < 1$, then we obtain the DFPT. If $\mathcal{F}(t) = t$ and $\varphi(t) = 0$ for each $t \geqslant 0$ in Theorem 1, we will have result given in [5].

Proposition 1. *If $\mathfrak{T} \in \mathfrak{C}$ satisfies the inequality*

$$\mathcal{F}\big(\mathrm{diam}\big(\mathfrak{T}(\mathfrak{Y})\big) + \varphi\big(\mathrm{diam}\big(\mathfrak{T}(\mathfrak{Y})\big)\big)\big) \leqslant \psi\big(\mathcal{F}\big(\mathrm{diam}\,\mathfrak{Y} + \varphi\big(\mathrm{diam}\,\mathfrak{Y}\big)\big)\big), \tag{2}$$

then $\mathfrak{T}$ admits a unique fixed point in $\mathfrak{C}$.

Proof. In view of (d6), it is well know that $\mathrm{diam}(\cdot)$ is a MNC, and thus, from Theorem 1 we get the existence of a $\mathfrak{T}$-invariant nonempty closed convex subset $\mathfrak{X}_\infty$ with $\mathrm{diam}\,\mathfrak{Y}_\infty = 0$. Consequently, $\mathfrak{X}_\infty$ is a singleton, and therefore, $\mathfrak{T}$ has a fixed point in $\mathfrak{C}$.

To attain the uniqueness, we assume that there exist two distinct fixed points $\zeta, \xi \in \mathfrak{C}$, then we may define the set $\mathcal{Y} := \{\zeta, \xi\}$. In this case, $\mathrm{diam}\,\mathcal{Y} = \mathrm{diam}(\mathfrak{T}(\mathcal{Y})) = \|\xi - \zeta\| > 0$. Using (2) and notion of $\mathcal{F}$ and ψ, we obtain

$$\begin{aligned} &\mathcal{F}\big(\mathrm{diam}\big(\mathfrak{T}(\mathcal{Y})\big) + \varphi\big(\mathrm{diam}\big(\mathfrak{T}(\mathcal{Y})\big)\big)\big) \\ &\quad \leqslant \psi\big(\mathcal{F}\big(\mathrm{diam}(\mathcal{Y}) + \varphi\big(\mathrm{diam}(\mathcal{Y})\big)\big)\big) < \mathcal{F}\big(\mathrm{diam}\,\mathcal{Y} + \varphi(\mathrm{diam}\,\mathcal{Y})\big), \end{aligned}$$

a contradiction and hence the result. □

Now we are in position to derive some classical fixed point result from Proposition 1 and Theorem 1.

Corollary 1. *Suppose that $\mathfrak{T} \in \mathfrak{C}$ achieves the inequality*

$$\mathcal{F}\big(\|\mathfrak{T}u - \mathfrak{T}v\| + \varphi\big(\|\mathfrak{T}u - \mathfrak{T}v\|\big)\big) \leqslant \psi\big(\mathcal{F}\big(\|u - v\| + \varphi\big(\|u - v\|\big)\big)\big) \tag{3}$$

for all $u, v \in \mathfrak{C}$. Then $\mathfrak{T}$ admits a unique fixed point.

Proof. Suppose that $\mu(\mathfrak{C}) = \mathrm{diam}\,\mathfrak{C}$, where

$$\mathrm{diam}\,\mathfrak{C} = \sup\big\{\|u - v\|:\ u, v \in \mathfrak{C}\big\}$$

is the diameter of $\mathfrak{C}$. Clearly, μ is a MNC in a space E in the sense of Definition 1. Therefore, from (3) we have

$$\begin{aligned}
&\sup_{\upsilon,\nu\in\mathfrak{C}} \mathcal{F}\big(\|\mathfrak{T}\upsilon-\mathfrak{T}\nu\|+\varphi\big(\|\mathfrak{T}\upsilon-\mathfrak{T}\nu\|\big)\big)\\
&\quad\leqslant \mathcal{F}\Big(\sup_{\upsilon,\nu\in\mathfrak{C}}\|\mathfrak{T}\upsilon-\mathfrak{T}\nu\|+\sup_{\upsilon,\nu\in\mathfrak{C}}\varphi\big(\|\mathfrak{T}\upsilon-\mathfrak{T}\nu\|\big)\Big)\\
&\quad\leqslant \sup_{\upsilon,\nu\in\mathfrak{C}}\psi\big(\mathcal{F}\big(\|\upsilon-\nu\|+\varphi\big(\|\upsilon-\nu\|\big)\big)\big)\\
&\quad\leqslant \psi\Big(\mathcal{F}\Big(\sup_{u,\upsilon\in\mathfrak{C}}\|\upsilon-\nu\|+\varphi\Big(\sup_{\upsilon,\nu\in\mathfrak{C}}\|\upsilon-\nu\|\Big)\Big)\Big),
\end{aligned}$$

which implies that

$$\mathcal{F}\big(\operatorname{diam}\big(\mathfrak{T}(\mathfrak{C})\big)+\varphi\big(\operatorname{diam}\big(\mathfrak{T}(\mathfrak{C})\big)\big)\big)\leqslant\psi\big(\mathcal{F}\big(\operatorname{diam}\mathfrak{C}+\varphi(\operatorname{diam}\mathfrak{C})\big)\big).$$

Thus, following Proposition 1, $\mathfrak{T}$ has an unique fixed point. □

Following is the Krasnoselskii FPT:

Corollary 2. *Let $\mathfrak{T}_1,\mathfrak{T}_2:\mathfrak{C}\to\mathfrak{C}$ be two operators satisfying:*

(i) $(\mathfrak{T}_1+\mathfrak{T}_2)(\mathfrak{C})\subseteq\mathfrak{C}$;

(ii) *There exist $\mathcal{F}\in\mathfrak{F}$, $\psi\in\Psi$, $\varphi:\mathbb{R}^+\to\mathbb{R}^+$, a continuous mapping such that*

$$\mathcal{F}\big(\|\mathfrak{T}_1\upsilon-\mathfrak{T}_1\nu\|+\varphi\big(\|\mathfrak{T}_1\upsilon-\mathfrak{T}_1\nu\|\big)\big)\leqslant\psi\big(\mathcal{F}\big(\|\upsilon-\nu\|+\varphi\big(\|\upsilon-\nu\|\big)\big)\big); \tag{4}$$

(iii) *$\mathfrak{T}_2$ is a continuous and compact operator.*

Then $\mathfrak{T}:=\mathfrak{T}_1+\mathfrak{T}_2:\mathfrak{C}\to\mathfrak{C}$ admits a fixed point $u\in\mathfrak{C}$.

Proof. Define new Kuratowski MNC by $\chi:\mathfrak{M}_E\to[0,\infty)$. Suppose that $\mathfrak{Y}$ is a subset of $\mathfrak{C}$ with $\chi(\mathfrak{Y})>0$. By the notion of Kuratowski MNC, for each $n\in\mathbb{N}$, there exist $\mathfrak{C}_1,\dots,\mathfrak{C}_{m(n)}$ bounded subsets such that $\mathfrak{Y}\subseteq\bigcup_{i=1}^{m(n)}\mathfrak{C}_i$ and $\operatorname{diam}(\mathfrak{C}_i)\leqslant\chi(\mathfrak{Y})+1/n$. Suppose that $\chi(\mathfrak{T}(\mathfrak{Y}))>0$. Since $\mathfrak{T}_1(\mathfrak{Y})\subseteq\bigcup_{i=1}^{m(n)}\mathfrak{T}_1(\mathfrak{C}_i)$, there exists $i_0\in\{1,2,\dots,m(n)\}$ such that $\chi(\mathfrak{T}(\mathfrak{Y}))\leqslant\operatorname{diam}(\mathfrak{T}_1(\mathfrak{C}_{i_0}))$. Using (4) condition of $\mathfrak{T}_1$ with discussed arguments, we have

$$\begin{aligned}
&\mathcal{F}\big(\chi\big(\mathfrak{T}_1(\mathfrak{Y})\big)+\varphi\big(\chi\big(\mathfrak{T}_1(\mathfrak{Y})\big)\big)\big)\\
&\quad\leqslant\mathcal{F}\big(\operatorname{diam}\big(\mathfrak{T}_1(\mathfrak{C}_{i_0})\big)+\varphi\big(\operatorname{diam}\big(\mathfrak{T}_1(\mathfrak{C}_{i_0})\big)\big)\big)\\
&\quad\leqslant\mathcal{F}\big(\operatorname{diam}\mathfrak{C}_{i_0}+\varphi\big(\operatorname{diam}\mathfrak{C}_{i_0}\big)\big)\\
&\quad\leqslant\psi\bigg(\mathcal{F}\bigg(\chi(\mathfrak{Y})+\frac{1}{n}+\varphi\bigg(\chi(\mathfrak{Y})+\frac{1}{n}\bigg)\bigg)\bigg).
\end{aligned}\tag{5}$$

Taking the limit in (5) as $n\to\infty$, we get

$$\mathcal{F}\big(\chi\big(\mathfrak{T}_1(\mathfrak{Y})\big)+\varphi\big(\chi\big(\mathfrak{T}_1(\mathfrak{Y})\big)\big)\big)\leqslant F\big(\chi(\mathfrak{Y})+\varphi\big(\chi(\mathfrak{Y})\big)\big). \tag{6}$$

By (iii) hypothesis and (6) we have by the notion of χ that

$$\begin{aligned}
&\mathcal{F}\big(\chi\big(\mathfrak{T}(\mathfrak{Y})\big) + \varphi\big(\chi\big(\mathfrak{T}(\mathfrak{Y})\big)\big)\big) \\
&\quad = \mathcal{F}\big(\chi\big(\mathfrak{T}_1(\mathfrak{Y}) + \mathfrak{T}_2(\mathfrak{Y})\big) + \varphi\big(\chi\big(\mathfrak{T}_1(\mathfrak{Y}) + \mathfrak{T}_2(\mathfrak{Y})\big)\big)\big) \\
&\quad \leqslant \mathcal{F}\big(\chi\big(\mathfrak{T}_1(\mathfrak{Y})\big) + \chi\big(\mathfrak{T}_2(\mathfrak{Y})\big) + \varphi\big(\chi\big(\mathfrak{T}_1(\mathfrak{Y})\big) + \chi\big(\mathfrak{T}_2(\mathfrak{Y})\big)\big)\big) \\
&\quad = \mathcal{F}\big(\chi\big(\mathfrak{T}_1(\mathfrak{Y})\big) + \varphi\big(\chi\big(\mathfrak{T}_1(\mathfrak{Y})\big)\big)\big) \\
&\quad \leqslant \psi\big(\mathcal{F}\big(\chi(\mathfrak{Y}) + \varphi\big(\chi(\mathfrak{Y})\big)\big)\big).
\end{aligned}$$

Therefore, from Theorem 1 $\mathfrak{T}$ has a fixed point $u \in \mathfrak{C}$. □

4 Coupled fixed point results

In this section, we introduce the result of Theorem 1 for $\varphi(t) = 0$.

Definition 2. (See [18].) An argument $(u^*, v^*) \in \mathcal{E}^2$ is said to be a coupled fixed point (CFP) of a mapping $\mathcal{G} : \mathcal{E}^2 \to \mathcal{E}$ if $\mathcal{G}(u^*, v^*) = u^*$ and $\mathcal{G}(v^*, u^*) = v^*$.

Theorem 2. (*See* [12].) *Assume that $\mu_1, \mu_2, \dots, \mu_n$ are MNCs in Banach spaces $\mathcal{E}_1, \mathcal{E}_2, \dots, \mathcal{E}_n$, respectively. Moreover, assume that the function $\mathcal{G} : [0,\infty)^n \to [0,\infty)$ is convex and $\mathcal{G}(\chi_1, \chi_2, \dots, \chi_n) = 0$ if and only if $\chi_i = 0$ for $i = 1, 2, 3, \dots, n$. Then*

$$\mu(\mathfrak{C}) = \mathcal{G}\big(\mu_1(\mathfrak{C}_1), \mu_2(\mathfrak{C}_2), \dots, \mu_n(\mathfrak{C}_n)\big)$$

defines a MNC in $\mathcal{E}_1 \times \mathcal{E}_2 \times \mathcal{E}_3 \times \cdots \times \mathcal{E}_n$, where $\mathfrak{C}_i$ denotes the natural projection of $\mathfrak{C}$ into $\mathcal{E}_i$ for $i = 1, 2, 3, \dots, n$.

Theorem 3. *Let $\mathfrak{C} \in \Lambda$ and $\mathcal{G} : \mathfrak{C}^2 \to \mathfrak{C}$ is continuous operator, and let there exist $\mathcal{F} \in \mathfrak{F}$, $\psi \in \Psi$, and $\mathcal{F}$ is subadditive such that*

$$\mathcal{F}\big(\mu\big(\mathcal{G}(\mathfrak{Y}_1 \times \mathfrak{Y}_2)\big)\big) \leqslant \frac{1}{2}\psi\big(\mathcal{F}\big(\mu(\mathfrak{Y}_1) + \mu(\mathfrak{Y}_2)\big)\big) \tag{7}$$

for all $\mathfrak{Y}_1, \mathfrak{Y}_2$ in $\mathfrak{C}$. Then $\mathcal{G}$ admits at least a CFP.

Proof. Consider the map $\mathcal{G} : \mathfrak{C}^2 \to \mathfrak{C}^2$ having the definition $\widehat{\mathcal{G}}(\upsilon, \nu) = (\mathcal{G}(\upsilon, \nu), \mathcal{G}(\nu, \upsilon))$. Also, we define a new MNC in the space $\mathfrak{C}^2$ (see [5]) as $\widehat{\mu}(\mathfrak{Y}) = \mu(\mathfrak{Y}_1) + \mu(\mathfrak{Y}_2)$, where $\mathfrak{Y}_i$, $i = 1, 2$, denote the natural projections of $\mathfrak{C}$. Now let $\emptyset \neq \mathfrak{Y}$, and thus, by (7) and condition (d2) of Definition 1 we conclude that

$$\begin{aligned}
\mathcal{F}\big(\widehat{\mu}\big(\widehat{\mathcal{G}}(\mathfrak{Y})\big)\big) &\leqslant \mathcal{F}\big(\widehat{\mu}\big(\mathcal{G}(\mathfrak{Y}_1 \times \mathfrak{Y}_2) \times \mathcal{G}(\mathfrak{Y}_2 \times \mathfrak{Y}_1)\big)\big) \\
&= \mathcal{F}\big(\mu\big(\mathcal{G}(\mathfrak{Y}_1 \times \mathfrak{Y}_2)\big)\big) + \mathcal{F}\big(\mu\big(\mathcal{G}(\mathfrak{Y}_2 \times \mathfrak{Y}_1)\big)\big) \\
&\leqslant \frac{1}{2}\psi\big(\mathcal{F}\big(\mu(\mathfrak{Y}_1) + \mu(\mathfrak{Y}_2)\big)\big) + \frac{1}{2}\psi\big(\mathcal{F}\big(\mu(\mathfrak{Y}_2) + \mu(\mathfrak{Y}_1)\big)\big) \\
&= \psi\big(\mathcal{F}\big(\mu(\mathfrak{Y}_1) + \mu(\mathfrak{Y}_2)\big)\big) = \psi\big(\mathcal{F}\big(\widehat{\mu}(\mathfrak{Y})\big)\big),
\end{aligned}$$

that is,

$$\psi\big(\widehat{\mu}\big(\widehat{\mathcal{G}}(\mathfrak{Y})\big)\big) \leqslant \psi\big(\mathcal{F}\big(\widehat{\mu}(\mathfrak{Y})\big)\big).$$

Hence, $\widehat{\mathcal{G}}$ admits a fixed point (CFP). □

Theorem 4. *Let $\mathfrak{C} \in \Lambda$ and $\mathcal{G} : \mathfrak{C}^2 \to \mathfrak{C}$ is continuous operator, and let there exist $\mathcal{F} \in \mathfrak{F}$, $\psi \in \Psi$, and $\mathcal{F}$ is subadditive such that*

$$\mathcal{F}\big(\mu\big(\mathcal{G}(\mathfrak{Y}_1 \times \mathfrak{Y}_2)\big)\big) \leqslant \psi\big(\mathcal{F}\big(\max\{\mu(\mathfrak{Y}_1), \mu(\mathfrak{Y}_2)\}\big)\big) \tag{8}$$

for all $\mathfrak{Y}_1, \mathfrak{Y}_2$ in $\mathfrak{C}$. Then $\mathcal{F}$ admits at least one CFP.

Proof. Consider the map $\mathcal{G} : \mathfrak{C}^2 \to \mathfrak{C}^2$ by the formal

$$\widehat{\mathcal{G}}(\upsilon, \nu) = \big(\mathcal{G}(\upsilon, \nu), \mathcal{G}(\nu, \upsilon)\big).$$

Define a MNC in the space $\mathfrak{C}^2$ (see [5]) by $\widehat{\mu}(\mathfrak{Y}) = \max\{\mu(\mathfrak{Y}_1), \mu(\mathfrak{Y}_2)\}$, where $\mathfrak{Y}_i$, $i = 1, 2$, denote the natural projections of $\mathfrak{C}$. Assume that $\mathfrak{Y} \subset \mathfrak{C}^2$ is a nonempty subset. Thus, by (8) and condition (d2) we obtain

$$\begin{aligned}
\mathcal{F}\big(\widehat{\mu}\big(\hat{\mathcal{G}}(\mathfrak{Y})\big)\big) &\leqslant \mathcal{F}\big(\widehat{\mu}\big(\mathcal{G}(\mathfrak{Y}_1 \times \mathfrak{Y}_2) \times \mathcal{G}(\mathfrak{Y}_2 \times \mathfrak{Y}_1)\big)\big) \\
&= \mathcal{F}\big(\max\big\{\mu\big(\mathcal{G}(\mathfrak{Y}_1 \times \mathfrak{Y}_2)\big), \mu\big(\mathcal{G}(\mathfrak{Y}_2 \times \mathfrak{Y}_1)\big)\big\}\big) \\
&= \max\big\{\mathcal{F}\big(\mu\big(\mathcal{G}(\mathfrak{Y}_1 \times \mathfrak{Y}_2)\big)\big), \mathcal{F}\big(\mu\big(\mathcal{G}(\mathfrak{Y}_2 \times \mathfrak{Y}_1)\big)\big)\big\} \\
&\leqslant \max\big\{\psi\big(\mathcal{F}\big(\max\big\{\mu(\mathfrak{Y}_1), \mu(\mathfrak{Y}_2)\big\}\big)\big), \psi\big(\mathcal{F}\big(\max\big\{\mu(\mathfrak{Y}_2), \mu(\mathfrak{Y}_1)\big\}\big)\big)\big\} \\
&= \psi\big(\mathcal{F}\big(\max\big\{\mu(\mathfrak{Y}_1), \mu(\mathfrak{Y}_2)\big\}\big)\big) = \psi\big(\mathcal{F}\big(\widehat{\mu}(\mathfrak{Y})\big)\big),
\end{aligned}$$

that is, $\mathcal{F}(\widehat{\mu}(\hat{\mathcal{G}}(\mathfrak{Y}))) \leqslant \psi(\mathcal{F}(\mu(\mathfrak{Y})))$. Consequently, $\mathcal{G}$ admits a CFP. □

5 Applications

In this section, we construct a m-solution for a class of FDS with delay.

5.1 Construction

- Let $\nu(t) = (\nu_1(t), \dots, \nu_n(t))^{\mathfrak{T}} \in \mathbb{R}^n$ be a vector of variables with continuously differential components in the partition intervals $[\eta_1,\, t-\tau_1], \dots, [\eta_k,\, t-\tau_k]$, $t > \tau_i$, $i = 1, \dots, k$, and $C_i = C_i^{\mathfrak{T}}$ (transpose matrix of C_i) be a constant $n \times n$ matrix satisfying the following operational equation:

  $$(\Delta_i \nu)(t) = \nu(t) - C_i \nu(t - \tau_i) \in \mathbb{R}^n,$$

  where $\Delta : \mathbb{R}^n \to \mathbb{R}^n$. Adding the above relation, we take out the operator $\mathbf{D} : \mathbb{R}^n \to \mathbb{R}^n$ possessing the $n \times n$ summation

  $$(\mathbf{D}\nu)(t) := \sum_{i=1}^{k} (\Delta_i \nu)(t) \in \mathbb{R}^n. \tag{9}$$

- Let $\eta_1 < t - \tau_1, \dots, \eta_k < t - \tau_k$, $\eta_i \geqslant 0$, and M_i be a constant $n \times n$ matrix such that $M_i = M_i^{\mathfrak{T}} > 0$, $i = 1, \dots, k$, fulfilling the integral formula

  $$\big(\mathbf{I}_i^{\alpha} \nu\big)(t) := \int_{t-\eta_i}^{t} (t - \varrho)^{\alpha - 1} \big[M_i\, \nu(\varrho)\big]\, \mathrm{d}\varrho \in \mathbb{R}^n.$$

Repeating the above construction integral k times, we attain the general summation formal

$$\left(\mathcal{I}^{\alpha}\nu\right)(t) = \sum_{i=1}^{k} \left(\mathbf{I}_i^{\alpha}\nu\right)(t) \in \mathbb{R}^n. \tag{10}$$

- Let $\eta_1 < t - \tau_1, \dots, \eta_k < t - \tau_k$, $\eta_i \geqslant 0$, and R_i be a constant $n \times n$ matrix satisfying $R_i = R_i^{\mathfrak{T}} > 0$, $i = 1, \dots, k$, achieving the formal integral equation

$$\left(\mathbf{J}_i^{\alpha}\nu\right)(t) := \int\limits_{\eta_i}^{t-\tau_i} (t-\varrho)^{\alpha} \left[R_i\,\nu(\varrho)\right]' \,\mathrm{d}\varrho \in \mathbb{R}^n.$$

Adding k times, we bring out

$$\left(\mathcal{J}^{\alpha}\nu\right)(t) = \sum_{i=1}^{k} \left(\mathbf{J}_i^{\alpha}\nu\right)(t) \in \mathbb{R}^n. \tag{11}$$

Combine (9)–(11) to obtain a new mild solution $\nu : [0, \infty) \to \mathbb{R}^n$:

$$\begin{aligned} \nu(t) &:= (\mathbf{D}\nu)(t) + \left(\mathcal{I}^{\alpha}\nu\right)(t) + \left(\mathcal{J}^{\alpha}\nu\right)(t) + \wp_{\alpha} \int\limits_0^t (t-\varrho)^{\alpha-1} F\left(\varrho, \nu(\varrho)\right) \mathrm{d}\varrho, \\ t_0 &= 0, \end{aligned} \tag{12}$$

$\alpha \in (0, 1]$, $\nu \in \mathbb{R}^n$, $F \in C(\mathbb{R}^n)$, $\wp_{\alpha} \in [0, \infty)$, Obviously, $\nu(t)$ is a continuous and differential function. Our aim is to show that the following dynamic system has a mild solution in the frame of (12):

$$D_0^{\alpha}\nu(t) - C_i D_0^{\alpha}\nu(t - \tau_i) = A_i\nu(t) + F(t, \nu), \tag{13}$$

where $F : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous,

$$\left\|F(t, \nu) - F(t, \chi)\right\| \leqslant \ell\|\nu - \chi\|, \quad \ell > 0,$$

and $A_{n\times n}$ and $C_{n\times n}$ are constant matrices achieving $cij > aij > 0$ with the property

$$\hat{C} := \sum_{i=1}^{k} \|C_i\|_1 = \sum_{i=1}^{k} \left(\max_{1 \leqslant j \leqslant n} \sum_{\text{ß}=1}^{n} |c_{\text{ß}j}| \right) < 1. \tag{14}$$

5.2 M-Solutions

In this subsection, we establish the m-solution of FDS (13).

Theorem 5. *Let inequality* (14) *hold. Then the fractional dynamic system* (13) *admits at least one m-solution $\nu \in BC(\mathbb{R}^n)$ taking the form* (12).

Proof. Define an operator $Q : \mathbb{R}^n \to \mathbb{R}^n$ as follows:

$$(Q\nu)(t) := (\mathbf{D}\nu)(t) + \bigl(\mathcal{I}^\alpha \nu\bigr)(t) + \bigl(\mathcal{J}^\alpha \nu\bigr)(t) + \wp_\alpha \int_0^t (t-\varrho)^{\alpha-1} F\bigl(\varrho, \nu(\varrho)\bigr)\,\mathrm{d}\varrho.$$

Our aim is to show that Q admits at least one fixed point in a BCC set in $\mathbb{R}^n$.

Boundedness. By (12) we get

$$\begin{aligned}
\bigl|(Q\nu)(t)\bigr| &= \left|(\mathbf{D}\nu)(t) + \bigl(\mathcal{I}^\alpha \nu\bigr)(t) + \bigl(\mathcal{J}^\alpha \nu\bigr)(t) + \wp_\alpha \int_0^t (t-\varrho)^{\alpha-1} F\bigl(\varrho, \nu(\varrho)\bigr)\,\mathrm{d}\varrho\right| \\
&= \Biggl|\sum_{i=1}^k (\Delta_i \nu)(t) + \sum_{i=1}^k \Biggl(\int_{t-\eta_i}^t (t-\varrho)^{\alpha-1}\bigl[M_i\,\nu(\varrho)\bigr]\,\mathrm{d}\varrho \\
&\quad + \int_{\eta_i}^{t-\tau_i} (t-\varrho)^\alpha \bigl[R_i\,\nu(\varrho)\bigr]'\,\mathrm{d}\varrho\Biggr) + \wp_\alpha \int_0^t (t-\varrho)^{\alpha-1} F\bigl(\varrho, \nu(\varrho)\bigr)\,\mathrm{d}\varrho\Biggr| \\
&\leqslant \sum_{i=1}^k \bigl(\nu(t) - C_i \nu(t-\tau_i)\bigr)^2 + \sum_{i=1}^k \Biggl(\int_{t-\eta_i}^t (t-\varrho)^{\alpha-1}\bigl|M_i\,\nu(\varrho)\bigr|\,\mathrm{d}\varrho \\
&\quad + \alpha \int_{\eta_i}^{t-\tau_i} (t-\varrho)^{\alpha-1}\bigl|R_i\,\nu(\varrho)\bigr|\,\mathrm{d}\varrho \\
&\quad + \tau_i^\alpha \bigl|R_i \nu(t-\tau_i) - (t-\eta_i)^\alpha R_i \nu(\eta_i)\bigr|\Biggr) + \wp_\alpha |F| \frac{t^\alpha}{\alpha} \\
&\leqslant \|\nu\|(1-\hat{C}) + \|\nu\|\hat{M} \sum_{i=1}^k \frac{(\eta_i)^\alpha}{\alpha} + \|\nu\|\hat{R} \sum_{i=1}^k (t-\eta_i)^\alpha - (\tau_i)^\alpha \\
&\quad + \bigl(\alpha_1(t) + \alpha_2(t)\bigr) k \|\nu\|\hat{R} + \wp_\alpha \|F\| \frac{t^\alpha}{\alpha},
\end{aligned}$$

where $\alpha_1(t) = \max_i \tau_i^\alpha$ and $\alpha_2(t) = \min_i (t-\eta_i)^\alpha$. Since $\tau_i < t - \eta_i$, then if let

$$\alpha_3(t) := \max_i \bigl[(t-\eta_i)^\alpha - (\tau_i)^\alpha\bigr],$$

we obtain

$$\begin{aligned}
\bigl|(Q\nu)(t)\bigr| &\leqslant \|\nu\|(1-\hat{C}) + \|\nu\|\hat{M} \frac{k\hat{\eta}^\alpha}{\alpha} + k\bigl(\alpha_1(t) + \alpha_2(t)\bigr)\|\nu\|\hat{R} \\
&\quad + k\alpha_3(t)\|\nu\|\hat{R} + \wp_\alpha \|F\| \frac{t^\alpha}{\alpha} \\
&:= \beta(t)\|\nu\| + \wp_\alpha \|F\| \frac{t^\alpha}{\alpha} \leqslant 2\biggl[\bar{\beta}\|\nu\| + \wp_\alpha \|F\| \frac{T^\alpha}{\alpha}\biggr],
\end{aligned}$$

where

$$\beta(t) := (1-\hat{C}) + k\left(\frac{\hat{\eta}^\alpha}{\alpha}\hat{M} + \big(\alpha_1(t)+\alpha_2(t)+\alpha_3(t)\big)\hat{R}\right), \qquad \bar{\beta} = \max_t \beta(t).$$

Taking the sup norm over $t \in [0,T]$, $T<\infty$, we have

$$\|Q\| \leqslant \frac{2\frac{\wp_\alpha T^\alpha}{\alpha}\|F\|}{1-2\bar{\beta}} := r, \quad \bar{\beta} < \frac{1}{2}.$$

Hence, $Q : B_r \to B_r$ is bounded.

Continuity. Let $\delta > 0$ and $\nu, \upsilon \in B_r$ such that $\|\nu - \upsilon\| \leqslant \delta$. Then a computation implies

$$\begin{aligned}
&\big|(Q\nu)(t) - (Q\upsilon)(t)\big| \\
&\quad\leqslant \|\nu-\upsilon\|(1-\hat{C}) + \|\nu-\upsilon\|\hat{M}\sum_{i=1}^k \frac{(\eta_i)^\alpha}{\alpha} - \|\nu-\upsilon\|\hat{R}\sum_{i=1}^k (\tau_i)^\alpha - (t-\eta_i)^\alpha \\
&\quad+ \big(\alpha_1(t)\|\nu-\upsilon\| + \alpha_2(t)\big)k\|\nu-\upsilon\|\hat{R} + \big\|F(\nu)-F(\upsilon)\big\|\frac{\wp_\alpha T^\alpha}{\alpha} \\
&\quad\leqslant \|\nu-\upsilon\|(1-\hat{C}) + \|\nu-\upsilon\|\hat{M}\sum_{i=1}^k \frac{(\eta_i)^\alpha}{\alpha} - \|\nu-\upsilon\|\hat{R}\sum_{i=1}^k (\tau_i)^\alpha - (t-\eta_i)^\alpha \\
&\qquad+ \big(\bar{\alpha}_1 + \bar{\alpha}_2\big)k\|\nu-\upsilon\|\hat{R} + \|\nu-\upsilon\|\frac{\wp_\alpha T^\alpha \ell}{\alpha} \\
&\quad\leqslant \delta\left((1-\hat{C}) + \hat{M}\frac{k\hat{\eta}^\alpha}{\alpha} + k(\bar{\alpha}_1+\bar{\alpha}_2)\hat{R} + k\bar{\alpha}_3|\hat{R} + \frac{\wp_\alpha T^\alpha \ell}{\alpha}\right) := \varepsilon,
\end{aligned}$$

where $\bar{\alpha}_j := \max_t \alpha_j(t)$, $j = 1,2,3$. Hence, Q is continuous in B_r.

Measurement. Here, we aim to prove

$$\mu(Q)(B_r) \leqslant \mu(B_r).$$

For ν and $\upsilon \in B_r$, we have

$$\begin{aligned}
&\big|(Q\nu)(t) - (Q\upsilon)(t)\big| \\
&\quad\leqslant \|\nu-\upsilon\|(1-\hat{C}) + \|\nu-\upsilon\|\hat{M}\sum_{i=1}^k \frac{(\eta_i)^\alpha}{\alpha} - \|\nu-\upsilon\|\hat{R}\sum_{i=1}^k (\tau_i)^\alpha - (t-\eta_i)^\alpha \\
&\qquad+ \big(\bar{\alpha}_1 + \bar{\alpha}_2\big)k\|\nu-\upsilon\|\hat{R} + \|\nu-\upsilon\|\frac{\wp_\alpha T^\alpha \ell}{\alpha} \\
&\quad= \|\nu-\upsilon\|\Bigg((1-\hat{C}) + \hat{M}\sum_{i=1}^k \frac{(\eta_i)^\alpha}{\alpha} + \hat{R}\sum_{i=1}^k (t-\eta_i)^\alpha - (\tau_i)^\alpha \\
&\qquad+ \big(\bar{\alpha}_1+\bar{\alpha}_2\big)k\hat{R} + \frac{\wp_\alpha T^\alpha \ell}{\alpha}\Bigg),
\end{aligned}$$

$(\tau_i)^\alpha < (t-\eta_i)^\alpha$, then we conclude that

$$\operatorname{diam}(Q(B_r)) \leqslant K_\alpha \operatorname{diam} B_r,$$

where

$$\begin{aligned} K_\alpha :&= (1-\hat{C}) + \hat{M}\sum_{i=1}^{k}\frac{(\eta_i)^\alpha}{\alpha} + \hat{R}\sum_{i=1}^{k}(T-\eta_i)^\alpha - (\tau_i)^\alpha \\ &\quad + (\bar{\alpha}_1+\bar{\alpha}_2)k\hat{R} + \frac{\wp_\alpha T^\alpha \ell}{\alpha}. \end{aligned}$$

To satisfy the condition of Theorem 1, we follow the same technique in [27]. Consequently, we obtain the desired result.

Equicontinuous. Let t_1 and $t_2 \in [0,T]$ with $t_1 > t_2$, then we attain

$$\begin{aligned} &\bigl|(Q\nu)(t_1) - (Q\nu)(t_2)\bigr| \\ &= \Biggl|\Biggl(\sum_{i=1}^{k}(\Delta_i\nu)(t_1) - (\Delta_i\nu)(t_2)\Biggr) \\ &\quad + \sum_{i=1}^{k}\Biggl(\int\limits_{t_1-\eta_i}^{t_1}(t_1-\varrho)^{\alpha-1}\bigl[M_i\,\nu(\varrho)\bigr]\,\mathrm{d}\varrho - \int\limits_{t_2-\eta_i}^{t_2}(t_2-\varrho)^{\alpha-1}\bigl[M_i\,\nu(\varrho)\bigr]\,\mathrm{d}\varrho \\ &\quad + \int\limits_{\eta_i}^{t_1-\tau_i}(t_1-\varrho)^{\alpha}\bigl[R_i\,\nu(\varrho)\bigr]'\,\mathrm{d}s - \int\limits_{\eta_i}^{t_2-\tau_i}(t_2-\varrho)^{\alpha}\bigl[R_i\,\nu(\varrho)\bigr]'\,\mathrm{d}\varrho\Biggr) \\ &\quad + \wp_\alpha\Biggl(\int\limits_0^{t_1}(t_1-\varrho)^{\alpha-1}F\bigl(\varrho,\nu(\varrho)\bigr)\,\mathrm{d}\varrho - \int\limits_0^{t_2}(t_2-\varrho)^{\alpha-1}F\bigl(\varrho,\nu(\varrho)\bigr)\,\mathrm{d}\varrho\Biggr)\Biggr| \\ &\leqslant 2\biggl(\|\nu\|(1-\hat{C}) + \|\nu\|\hat{M}\frac{k\hat{\eta}^\alpha}{\alpha} + k\bigl(\alpha_1(t)+\alpha_2(t)\bigr)\|\nu\|\hat{R} + k\alpha_3(t)\|\nu\|\hat{R}\biggr) \\ &\quad + \wp_\alpha\|F\|\frac{t_1^\alpha + t_2^\alpha}{\alpha} \\ &\leqslant 2\biggl[\bar{\beta}\|\nu\| + \wp_\alpha\frac{T^\alpha}{\alpha}\|F\|\biggr]. \end{aligned}$$

This conclude that the operator Q is equicontinuous in B_r. As a consequence, Theorem 1 yields that Q admits at least one fixed point. □

Next, we provide the sufficient condition on Q to has a unique fixed point.

Theorem 6. *Let inequality* (14) *hold. If*

$$L := (1-\hat{C}) + (k\hat{M} + \wp_\alpha\ell)\frac{T^\alpha}{\alpha} + \bigl(T^\alpha + \bar{\alpha}_1 + \bar{\alpha}_2\bigr)k\hat{R} < 1,$$

then the FDS (13) *has a unique m-solution* $\nu \in BC(\mathbb{R}^n)$ *taking the form* (12).

Proof. Our aim is to satisfy inequality (3). For ν and $\upsilon \in B_r$, we have

$$\begin{aligned}
&\big|(Q\nu)(t) - (Q\upsilon)(t)\big| \\
&\quad \leqslant \|\nu - \upsilon\|(1 - \hat{C}) + \|\nu - \upsilon\|\hat{M}\sum_{i=1}^{k}\frac{(\eta_i)^\alpha}{\alpha} - \|\nu - \upsilon\|\hat{R}\sum_{i=1}^{k}(\tau_i)^\alpha - (t - \eta_i)^\alpha \\
&\qquad + (\bar{\alpha}_1 + \bar{\alpha}_2)k\|\nu - \upsilon\|\hat{R} + \|\nu - \upsilon\|\frac{\wp_\alpha T^\alpha \ell}{\alpha} \\
&\quad = \|\nu - \upsilon\|\Bigg((1 - \hat{C}) + \hat{M}\sum_{i=1}^{k}\frac{(\eta_i)^\alpha}{\alpha} + \hat{R}\sum_{i=1}^{k}(t - \eta_i)^\alpha - (\tau_i)^\alpha \\
&\qquad + (\bar{\alpha}_1 + \bar{\alpha}_2)k\hat{R} + \frac{\wp_\alpha T^\alpha \ell}{\alpha}\Bigg), \\
&\qquad (\tau_i)^\alpha < (t - \eta_i)^\alpha \\
&\qquad\quad \leqslant \|\nu - \upsilon\|\bigg((1 - \hat{C}) + (k\hat{M} + \wp_\alpha \ell)\frac{T^\alpha}{\alpha} + \big(T^\alpha + \bar{\alpha}_1 + \bar{\alpha}_2\big)k\hat{R}\bigg) \\
&\qquad\quad := L\|\nu - \upsilon\|.
\end{aligned}$$

Thus system (13) admits a unique m-solution $\nu \in BC(\mathbb{R}^n)$. $\square$

5.3 Numerical example

Consider the following system:

$$D_0^\alpha \nu(t) - C_i D_0^\alpha \nu(t - \tau_i) = A_i \nu(t) + F(t, \nu) \tag{15}$$

with the following data: $n = 2$, $t \in [0, 1]$, $\|F\| = 1/4$, $\tau_i < 1/2$,

$$A_i = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \qquad C_i = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}.$$

We have

$$\|Q\| \leqslant \frac{2\frac{\wp_\alpha T^\alpha}{\alpha}\|F\|}{1 - 2\bar{\beta}} = 1.3 = r,$$

where $\wp = 1$, $\alpha = 0.5$, $\hat{M} = 0.5$, $\hat{R} = 1/30$ with a simple calculation $\bar{\beta} = 0.4 < 0.5$. Moreover, condition (14) is satisfied, hence, in view Theorem 5, system (15) admits at least one m-solution taking the form (12).

6 An iterative algorithm to find solution of equation (15)

In fact, the above problem is a fractional delay singular integral equations system. We decide to find solution of it by an iterative algorithm. At first, we introduce $F(t, \nu)$

function as follows:

$$F(t,\nu) = 0.25 \begin{bmatrix} \nu_1(t) + \nu_2(t) \\ \nu_1(t) + \nu_2(t) \end{bmatrix} + \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}, \tag{16}$$

where

$$\begin{aligned} f_1(t) &= \frac{0.0857568}{\sqrt{t}} + 1.17351\sqrt{t} - 1.025t - 1.2036t^{3/2} - 0.00355339t^2, \\ f_2(t) &= \frac{0.064738}{\sqrt{t}} + 2.06073\sqrt{t} - 1.075t + 1.70215t^{3/2} - 0.244975t^2. \end{aligned} \tag{17}$$

By substituting (16) and (17) into (15) and concept of $D_0^\alpha \nu(t) = DI^{1-\alpha}\nu(t)$, $\alpha = 0.5$, we obtain a system of fractional singular delay integral equations of the form

$$\begin{aligned} &\int \nu_1(t)\,\mathrm{d}t \\ &\quad - h_1(t) + \int_0^t \frac{-35\nu_1(s) + 7\nu_1(s-\tau_1) + 25\nu_2(s) - 5\nu_2(s-\tau_1)}{6\sqrt{\pi}(t-s)^{1/2}}\,\mathrm{d}s = 0, \\ &\int \nu_2(t)\,\mathrm{d}t \\ &\quad - h_2(t) + \int_0^t \frac{25\nu_1(s) - 5\nu_1(s-\tau_1) - 35\nu_2(s) + 7\nu_2(s-\tau_1)}{6\sqrt{\pi}(t-s)^{1/2}}\,\mathrm{d}s = 0, \end{aligned} \tag{18}$$

where $\tau_1 = 0.4$, $\eta_1 = 0.2$ and $t \in (0.6, 1]$, also

$$\begin{aligned} h_1(t) &= -0.4610126329751105\sqrt{t} + 1.1605823875518624t^{3/2} + 0.75t^2 \\ &\quad + 5.6453332542923516t^{5/2} - 0.3333333333333333t^3, \\ h_2(t) &= -0.04063684742891729\sqrt{t} - 4.754186899317495t^{3/2} + t^2 \\ &\quad - 5.9776994441662845t^{5/2} + 0.4714045207910317t^3. \end{aligned}$$

Now, to solve (18), we use a modified technique constructed an important concept of topology and perturbations theory that named modified homotopy perturbation method. To introduce some applications of the similar method, [29–31] hcan be seen. To make the above iterative algorithm, we consider the nonlinear problem in the general form

$$A(\nu) - H(t) = 0, \quad t \in (0.6, 1], \tag{19}$$

where A is a general differential operator, H is a known function,

$$H(t) = \big(h_1(t), h_2(t)\big)^{\mathrm{T}} \in C\big(\mathbb{R}^2\big), \qquad \nu(t) = \big(\nu_1(t), \nu_2(t)\big)^{\mathrm{T}} \in BC\big(\mathbb{R}^2\big).$$

We distribute the common operator A to N_1 and N_2 nonlinear operators, and correspondingly H function adapts to some functions such as H_1 and H_2 in order to (19) can be represented by $N_1(\nu) - H_1(t) + N_2(\nu) - H_2(t) = 0$. Consequently, we define an adapted homotopy perturbation as tails:

$$H(u,p) = N_1(u) - H_1(t) + p\big(N_2(u) - H_2(t)\big) = 0, \quad p \in [0,1], \tag{20}$$

and

$$\nu(t) \simeq u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} \sum_{j=0}^{\infty} p^j u_{1j}(t) \\ \sum_{j=0}^{\infty} p^j u_{2j}(t) \end{bmatrix}, \tag{21}$$

where p is an inserting parameter. According to variations of $p = 0$ to $p = 1$, we deliver $N_1(u) = H_1(t)$ to $A(u) = H(t)$. Therefore, we can develop a solution of (19) (numerical-solution) for $p = 1$ and $\nu(t) \simeq \lim_{p\to 1} u(t)$. Currently, considering the system of fractional singular delay integral equations (18), we introduce operators N_1 and N_2 and also functions H_1 and H_2 in these forms:

$$\begin{aligned}
&N_1(u) = \begin{bmatrix} \int u_1(t)c \\ \int u_2(t)\,\mathrm{d}t \end{bmatrix}, \\
&N_2(u) = \begin{bmatrix} \int_0^t \frac{-35u_1(s)+7u_1(s-\tau_1)+25u_2(s)-5u_2(s-\tau_1)}{6\sqrt{\pi}(t-s)^{1/2}}\mathrm{d}s \\ \int_0^t \frac{25u_1(s)-5u_1(s-\tau_1)-35u_2(s)+7u_2(s-\tau_1)}{6\sqrt{\pi}(t-s)^{1/2}}\mathrm{d}s \end{bmatrix}, \\
&H_1(t) = \begin{bmatrix} h_{11}(t) \\ h_{21}(t) \end{bmatrix}, \qquad H_2(t) = \begin{bmatrix} h_{12}(t) \\ h_{22}(t) \end{bmatrix}, \\
&h_1(t) = h_{11}(t) + h_{12}(t), \qquad h_2(t) = h_{21}(t) + h_{22}(t).
\end{aligned} \tag{22}$$

Here $h_{11}(t)$ and $h_{21}(t)$ are simple functions, which are chosen as a prate of functions $h_1(t)$ and $h_2(t)$, respectively. Substituting (22) and (21) in (20) concludes that

$$\begin{aligned}
&\sum_{j=0}^{\infty} p^j \int u_{1j}(t)\,\mathrm{d}t - h_{11}(t) \\
&\quad + p\Bigg(\frac{1}{6\sqrt{\pi}} \sum_{j=0}^{\infty} p^j \int\limits_0^t \frac{-35u_{1j}(s) + 7u_{1j}(s-\tau_1) + 25u_{2j}(s) - 5u_{2j}(s-\tau_1)}{(t-s)^{1/2}}\,\mathrm{d}s \\
&\quad - h_{12}(t)\Bigg) = 0, \\
&\sum_{j=0}^{\infty} p^j \int u_{2j}(t)\,\mathrm{d}t - h_{21}(t) \\
&\quad + p\Bigg(\frac{1}{6\sqrt{\pi}} \sum_{j=0}^{\infty} p^j \int\limits_0^t \frac{25u_{1j}(s) - 5u_{1j}(s-\tau_1) - 35u_{2j}(s) + 7u_{2j}(s-\tau_1)}{(t-s)^{1/2}}\,\mathrm{d}s \\
&\quad - h_{22}(t)\Bigg) = 0.
\end{aligned} \tag{23}$$

Rearranging (23) in terms of p powers concludes that

$$
\begin{aligned}
p^0\colon\ & \biggl(\int u_{10}(t)\,\mathrm{d}t - h_{11}(t)\biggr),\\
p^1\colon\ & \Biggl(\int u_{11}(t)\,\mathrm{d}t\\
&\quad + \frac{1}{6\sqrt{\pi}}\int\limits_0^t \frac{-35u_{10}(s) + 7u_{10}(s-\tau_1) + 25u_{20}(s) - 5u_{20}(s-\tau_1)}{(t-s)^{1/2}}\,\mathrm{d}s\\
&\quad - h_1(t) + h_{11}(t)\Biggr),\\
p^j\colon\ & \Biggl(\int u_{1j+1}(t)\,\mathrm{d}t\\
&\quad + \frac{1}{6\sqrt{\pi}}\int\limits_0^t \frac{-35u_{1j}(s) + 7u_{1j}(s-\tau_1) + 25u_{2j}(s) - 5u_{2j}(s-\tau_1)}{(t-s)^{1/2}}\,\mathrm{d}s\Biggr),
\end{aligned}
$$

$j = 1, 2, 3, \dots$. Also,

$$
\begin{aligned}
p^0\colon\ & \biggl(\int u_{20}(t)\,\mathrm{d}t - h_{21}(t)\biggr),\\
p^1\colon\ & \Biggl(\int u_{21}(t)\,\mathrm{d}t\\
&\quad + \frac{1}{6\sqrt{\pi}}\int\limits_0^t \frac{25u_{10}(s) - 5u_{10}(s-\tau_1) - 35u_{20}(s) + 7u_{20}(s-\tau_1)}{(t-s)^{1/2}}\,\mathrm{d}s\\
&\quad - h_2(t) + h_{21}(t)\Biggr),\\
p^j\colon\ & \Biggl(\int u_{2j+1}(t)\,\mathrm{d}t\\
&\quad + \frac{1}{6\sqrt{\pi}}\int\limits_0^t \frac{25u_{1j}(s) - 5u_{1j}(s-\tau_1) - 35u_{2j}(s) + 7u_{2j}(s-\tau_1)}{(t-s)^{1/2}}\,\mathrm{d}s\Biggr),
\end{aligned}
$$

$j = 1, 2, 3, \dots$.

By the construction of generalized homotopy perturbation (20) the coefficients of p powers are amounting to zero. Thus, we obtain an iterative process for numerical solution of (18).

6.1 Algorithm

$$
\begin{aligned}
\int u_{i0}(t)\,\mathrm{d}t &= h_{i1}(t), \quad i=1,2,\\
\int u_{11}(t)\,\mathrm{d}t &= \frac{1}{6\sqrt{\pi}}\int_0^t \frac{35u_{10}(s)-7u_{10}(s-\tau_1)-25u_{20}(s)+5u_{20}(s-\tau_1)}{(t-s)^{1/2}}\,\mathrm{d}s\\
&\quad + h_1(t)-h_{11}(t),\\
\int u_{21}(t)\,\mathrm{d}t &= \frac{1}{6\sqrt{\pi}}\int_0^t \frac{-25u_{10}(s)+5u_{10}(s-\tau_1)+35u_{20}(s)-7u_{20}(s-\tau_1)}{(t-s)^{1/2}}\,\mathrm{d}s\\
&\quad + h_2(t)-h_{21}(t),\\
\int u_{1j+1}(t)\,\mathrm{d}t &= \frac{1}{6\sqrt{\pi}}\int_0^t \frac{35u_{1j}(s)-7u_{1j}(s-\tau_1)-25u_{2j}(s)+5u_{2j}(s-\tau_1)}{(t-s)^{1/2}}\,\mathrm{d}s,\\
\int u_{2j+1}(t)\,\mathrm{d}t &= \frac{1}{6\sqrt{\pi}}\int_0^t \frac{-25u_{1j}(s)+5u_{1j}(s-\tau_1)+35u_{2j}(s)-7u_{2j}(s-\tau_1)}{(t-s)^{1/2}}\,\mathrm{d}s,
\end{aligned}
\tag{24}
$$

$j=1,2,3,\dots$.

In algorithm (24), to solve example (18), we choose functions $h_{11}(t)$ and $h_{21}(t)$ as a part of functions $h_1(t)$ and $h_2(t)$ in this form:

$$
\begin{aligned}
h_{11}(t) &= 0.75t^2 - 0.3333333333333333t^3,\\
h_{21}(t) &= t^2 + 0.4714045207910317t^3.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
u_{10}(t) &= 1.5t - t^2, \qquad u_{20}(t) = 2t + 1.41421t^2,\\
u_{11}(t) &= -\frac{2.77556\cdot 10^{-17}}{\sqrt{t}} - \frac{3.33067\cdot 10^{-16}}{\sqrt{t}},\\
u_{21}(t) &= -\frac{3.46945\cdot 10^{-17}}{\sqrt{t}}.
\end{aligned}
\tag{25}
$$

In (25), $u_{11}(t)$ and $u_{21}(t)$ are approximately zero because $t\in(0.6,1]$. Consequently, according to algorithm (24), $u_{1j}(t)$ and $u_{2j}(t)$ are zero for all $j\geqslant 2$. We can give an approximation of solution as follows:

$$
\begin{aligned}
\nu(t) &\simeq \lim_{p\to 1} u(t) = \begin{bmatrix}\lim_{p\to 1} u_1(t)\\ \lim_{p\to 1} u_2(t)\end{bmatrix} = \begin{bmatrix}\sum_{j=0}^{\infty} u_{1j}(t)\\ \sum_{j=0}^{\infty} u_{2j}(t)\end{bmatrix},\\
&= \begin{bmatrix}u_{10}(t)+u_{11}(t)+u_{12}(t)+\cdots\\ u_{20}(t)+u_{21}(t)+u_{22}(t)+\cdots\end{bmatrix} = \begin{bmatrix}1.5t-t^2\\ 2t+1.41421t^2\end{bmatrix}.
\end{aligned}
\tag{26}
$$

Table 1. Computing the absolute errors in the some points.

t	Absolute errors	
	$\nu_1(t)$	$\nu_2(t)$
0.6	$2.2 \cdot 10^{-16}$	0.0
0.7	0.0	$8.8 \cdot 10^{-16}$
0.8	$8.8 \cdot 10^{-16}$	$2.6 \cdot 10^{-16}$
0.9	$8.8 \cdot 10^{-16}$	0.0
1.0	$1.7 \cdot 10^{-15}$	0.0

For validity of solution (26), we replace it in system of fractional delay singular integral equations (18) and compute the absolute errors in the some points of interval $(0.6, 1]$ (see Table 1).

6.2 Bacterial growth system

From the solution $\nu(t)$ in (26) we consider a realistic model of Bacterial growth population as follows:

$$\nu(t) = \begin{bmatrix} \kappa t - \ell t^2 \\ 2t + 1.41421t^2 \end{bmatrix}, \tag{27}$$

where $\nu(t)$ is the next-state function of the growth for two experiences, κ is a positive constant, while ℓ is a negative constant. The quadratic term is called a corrected term for the linear term. If the constant ℓ is negative, then the growth occurs; otherwise, there is no growth (death). Figure 1 shows the bacterial growth of a population. Moreover, the solution (27) converges to a fixed point of system (15). In fact, that there is an equilibrium state corresponding to a fixed point. The accuracy of the growth is given by the ratio

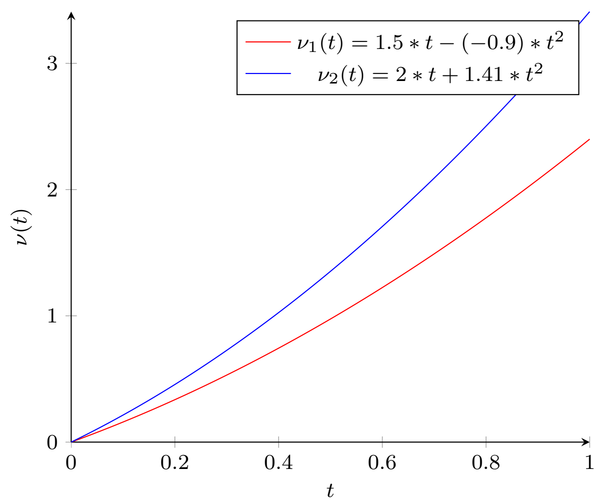

Figure 1. Solution of (15) when $\alpha = 0.5$, $t \in [0, 1]$.

$Q := \kappa/|\ell|$. Our example has accuracy = $1.5/0.9 = 1.6$, which approximated to the value of the golden ration. The degree of noncompactness of a set is measured by incomes of functions entitled measures of noncompactness. This type of measure can describe the behavior of the growth at infinity.

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