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Consensus tracking problem for linear fractional multi-agent systems with initial state error*

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Abstract. In this paper, we discuss the consensus tracking problem by introducing two iterative learning control (ILC) protocols (namely, D^α-type and PD^α-type) with initial state error for fractional-order homogenous and heterogenous multi-agent systems (MASs), respectively. The initial state of each agent is fixed at the same position away from the desired one for iterations. For both homogenous and heterogenous MASs, the D^α-type ILC rule is first designed and analyzed, and the asymptotical convergence property is carefully derived. Then, an additional P-type component is added to formulate a PD^α-type ILC rule, which also guarantees the asymptotical consensus performance. Moreover, it turns out that the PD^α-type ILC rule can further adjust the final performance. Two numerical examples are provided to verify the theoretical results.

Keywords: fractional-order, homogenous and heterogenous multi-agent systems, initial state error, convergence.

1 Introduction and problem formulation

With the developments in various fields such as unmanned aerial vehicle (UVA) formation, wireless sensor network, and micro-robot, multi-agent systems (MASs) have been became a hotly investigated topic in recent years.

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In accordance to the agent dynamics, MASs are classified into homogenous and heterogenous types [6, 31]. A majority of rich references concerns homogeneous MASs [2, 4, 19, 24, 28], where all agents are assumed to have the same time-domain dynamics. However, MASs are heterogeneous among numerous practical applications. That is, agents have different system dynamics. The conventional Kronecker product cannot be applied for the closed system dynamics because of different state evolution mechanisms. Thus, the design and analysis for heterogeneous MASs are more complex. Iterative learning control (ILC) has been proved an effective control strategy for various repetitive systems by trial and error to improve the system performance. As a consequence, ILC is also applied to achieve iteration-domain-based learning consensus for MASs whenever the MASs can repeat their coordination process. Indeed, the conventional MASs have been widely investigated in the past few years [11, 17, 18, 30].

To achieve the perfect tracking of system, the initial states for all agents are same as the desired initial state or are set via proper updating rule [1, 9]. Nevertheless, MASs can be regarded as "systems of systems", where the controllers are distributed in nature, and all agents are independent entities. Therefore, it is difficult to require all agents to realize the desired initial state because the communication among agents may not be integrated. Moreover, it is not feasible that all agents' initial states are adjusted arbitrarily even if all agents are conscious of the desired initial state. Thus, it is extremely important to relax the initial condition.

To describe memory and hereditary properties of the systems, in which some effects are ignored while using integer-order models, fractional-order differential equation filed a powerful tool [22]. Some systems such as electromagnetism, damped vibrations, and viscoelastic single mass systems should be described by fractional differential equations [5, 20, 34–36]. For example, aircrafts in a particular formulation fly under the influence of granular materials such as rain, snow, fog, and dust [3]. As a consequence, grouping the above objectives clearly forms fractional-order multi-agent systems, which have become important in practice. Recently, fractional-order MASs are widely used in cross-disciplinary nature and attract much attention on the coordination and control problems [7, 16, 21, 27, 31, 37]. Emerging results have been reported on the leader-follower solutions of fractional-order MASs [25, 26]. Available literatures, authors study the consensus problem of fractional-order systems with input delays by Laplace transform in [21]. The convergence of the iterative process for fractional-order system in time domain is discussed in [10]. Luo et al. [12] designed both P-type and PI-type ILC update laws for linear fractional-order MAS, where a direct channel from the input to the output was assumed for facilitation of the control design. In [33], a fractional-order iterative learning control framework with initial state learning for the tracking problem of integrate linear time-varying systems is present, where both open-loop and closed-loop D^{α}-type iterative learning updating laws are considered. Yang et al. in Chapter 4 of [30] address a leader-follower tracking problem by ILC approach in the integral multi-agent systems with initial state error, which shows the PD-type updating rule is able to tune the final control performance than D-type one. In addition to ILC approach, there are many interesting results for fractional-order MAS by using frequency domain theory, linear matrix inequality, observer-based protocols, and sampled-data control [15, 29, 32].

How to establish a framework to describe the communication topology is the key problem in ILC problem for fractional-order MASs. In fact, their communication networks are much more complicated than the first-order MAS due the memory and hereditary properties. First, the information is transmitted crossly for each MAS for digital control, and different MAS makes some necessary communication. Second, how to design ILC laws to display the reliable information channels for each MAS is important. It is a difficult to design the suitable ILC laws to deal with the singularity for the output digital message deriving from the fractional-order MASs. Thus, many techniques of fractional differential equations have been involved in establishing the convergence results. According to this investigation, general results of ILC for fractional-order MASs with initial state error have not been reported. This observation strongly motivates the investigation of this paper. In other words, we investigate the consensus tracking problem by designing D^α-type and PD^α-type ILC laws with initial state error for fractional-order homogenous MASs and fractional-order heterogenous MASs, respectively. This will provide another possible way to realizing the consensus control for fractional-order MASs.

In MASs, the graph theory is an assistant tool to depict the communication topology among agents. In this paper, we use the symbol $\mathcal{G}$ ($\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$) as a directed graph (or weighted graph), where $\mathcal{V} = \{v_1, v_2, \dots, v_N\}$ ($\{v_i,\ i = 1, 2, \dots, N\}$ is node of the ith agent, N is the number of agent)is the set of vertex, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges, and $\mathcal{A}$ is the adjacency matrix. A directed edge from v_i to v_j is indicated by an ordered pair $\langle v_i, v_j\rangle \in \mathcal{E}$, which denotes the ith agent transmits information to the jth agent. All neighboring nodes of the jth agent are denoted by $N_j = \{v_i \in \mathcal{V} \mid \langle v_i, v_j\rangle \in \mathcal{E}\}$. $\mathcal{A} = (a_{ij}) \in \mathbb{R}^{N\times N}$ ($a_{ij} \geqslant 0$) is the adjacency matrix of $\mathcal{G}$, where a_{ij} is the weight of the edge $\langle v_j, v_i\rangle$. If $\langle v_j, v_i\rangle \in \mathcal{E}$, we say a neighbour of the ith agent is the jth agent and $a_{ij} > 0$. Else, $a_{ij} = 0$.

There exists a kind of matrices which is called Laplacian matrix in the interaction of agents. The Laplacian matrix is defined that $L = \mathcal{D} - \mathcal{A}$, where the diagonal matrix $\mathcal{D} = \operatorname{diag}(d_1^{in}, \dots, d_N^{in})$ is the degree matrix of the graph $\mathcal{G}$, and $d_i^{in} = \sum_{j=1}^N a_{ij}$ is the in-degree of vertex v_i. Obviously, $L\mathbf{1}_N = 0$ ($\mathbf{1}_N$ is N dimension column vector whose all elements are "1").

We introduce a symbol "$\otimes$". The symbol $\otimes$ represents Kronecker product whose definition and property can be seen in [12]. There are N heterogenous time-invariant dynamic agents, which interaction topology can be described by $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, and the ith agent ($v_i \in \mathcal{V}$) at kth iteration is governed by the following linear fractional time-invariant model:

$$\begin{aligned} {}_0^C D_t^\alpha x_{k,i}(t) &= A_i x_{k,i}(t) + B_i u_{k,i}(t),\\ y_{k,i}(t) &= C_i x_{k,i}(t), \end{aligned} \tag{1}$$

where $x_{k,i}(t) \in \mathbb{R}^{n_i}$, $u_{k,i}(t) \in \mathbb{R}^m$, and $y_{k,i}(t) \in \mathbb{R}^p$ are the state vector, input vector, and output vector, respectively. $A_i \in \mathbb{R}^{n_i\times n_i}$, $B_i \in \mathbb{R}^{n_i\times m}$, and $C_i \in \mathbb{R}^{p\times n_i}$ are constant matrices.

Then, the leader's state vector $x_d(t)$ and the leader's trajectory (the desired consensus trajectory) $y_d(t)$ defined on a finite-time interval $[0, T]$ are resolved from following equation:

$$
\begin{aligned}
{}_0^C D_t^\alpha x_d(t) &= A_d x_d(t) + B_d u_d(t), \\
y_d(t) &= C_d x_d(t),
\end{aligned}
\tag{2}
$$

where $u_d(t) \in \mathbb{R}^m$ is the continuous and unique desired control input, the $x_d(0)$ is provided, and $A_d \in \mathbb{R}^{n_d \times n_d}$, $B_d \in \mathbb{R}^{n_d \times m}$, $C_d \in \mathbb{R}^{p \times n_d}$ are the constant matrices.

Here, we regard the desired trajectory $y_d(t)$ as a virtual leader and index it by vertex v_0 in the graph representation. Thus, the whole graph depicting the information flow among leader and agents can be expressed $\tilde{\mathcal{G}} = (\mathcal{V} \cup \{v_0\}, \tilde{\mathcal{E}}, \tilde{\mathcal{A}})$, where $\tilde{\mathcal{E}}$ and $\tilde{\mathcal{A}}$ are the set of edge and the weighted adjacency matrix of $\tilde{\mathcal{G}}$, respectively.

In this work, our contributions are listed as follows: (i) We consider the consensus tracking problem by proposing fractional-type ILC rules with initial state error for fractional-order homogenous and heterogenous MASs. This is a basic work for this topic and provide a general approach to deal with the continued topic in this fields. (ii) We establish two fractional versions of convergence results under mild assumptions by using fractional calculus techniques, which display the difference between fractional-order MASs and first-order MASs. One has to deal with the singular kernel involving the formula of the state function. (iii) We show that the PD^α-type updating rule can adjust the final consensus performance, which display fractional ILC law is good strategy for fractional-order MASs. (iv) We reveal that the relationship between virtual input (output) and desired input (output) can be displayed via an appropriate equation for homogenous MASs but not for heterogeneous MASs.

2 Preliminary

In this paper, $\rho(M) = \max_{1 \leqslant j \leqslant n}\{|\Lambda_j|\}$, and Λ_j ($j = 1, 2, \dots, n$) is the eigenvalue of the n-dimensional square matrix M. The λ-norm of vector function h: $[0, T] \to \mathbb{R}^n$, $\|h\|_\lambda = \sup_{t \in [0,T]}\{\mathrm{e}^{\lambda t}\|h(t)\|\}$ ($\lambda > 0$), where $\|\cdot\|$ is a norm in $\mathbb{R}^n$.

To simplify the controller design and convergence analysis, we need the following assumptions, lemmas, and definitions.

Assumption 1. The fixed and directed communication graph $\tilde{\mathcal{G}}$ involves a spanning tree, which takes the virtual leader as the root node.

Assumption 2. The initial state of every agent is set to the same position, which is no equal to the desired state at every iteration, that is, $x_{k,i}(0) = x_{1,i}(0) \neq x_d(0)$ for all $k \geqslant 2, k \in N$.

Assumption 3. CB is of full column rank.

Lemma 1. (*See* [8, *Lemma* 2.21]) *Let* $0 < \alpha < 1$ *and* $g(\cdot) \in L_\infty(a, b)$ *or* $g(\cdot) \in C^1[a, b]$, *then*

$$\left({}^C D_{a^+}^\alpha I_{a^+}^\alpha g\right)(t) = g(t),$$

where $(I_{a^+}^\alpha g)(t) := \int_a^t g(\tau)/(t-\tau)^{1-\alpha}\,\mathrm{d}\tau/\Gamma(\alpha)$, *and* $({}^C D_{a^+}^\alpha g)(t) = \int_a^t g'(\tau)/(t-\tau)^\alpha\,\mathrm{d}\tau/\Gamma(1-\alpha) =: (I_{a^+}^{1-\alpha} Dg)(t)$.

Lemma 2. *Consider a positive sequence* $\{\varphi_i\}$ *satisfying*

$$\varphi_{i+1} = c_1\rho_0^i\varphi_1 + \frac{c_2}{\sqrt[q]{q(\lambda-\rho_1)}}\left(\rho_0^{i-1}\varphi_1 + \rho_0^{i-2}\varphi_2 + \cdots + \varphi_i\right)$$

for $i \geqslant 1$*, where* c_1*,* c_2*, and* ρ_1 *are positive constants,* $0 < \rho_0 < 1$ *and the* q *satisfied that* $1/p + 1/q = 1$ *(*$p \in (1, 1/(1-\alpha)$*),* $0 < \alpha < 1$*). If* $\lambda > \rho_1 + (1/q)(c_2/(1-\rho_0)^q$*, then* $\varphi_i \to 0$.

Proof. The proof is similar to that of Lemma 3.2 in [30]. □

Let $E_{\alpha,\alpha}(\cdot)$ be the generalized Mittag–Leffler function defined by

$$E_{\alpha,\alpha}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\alpha+\alpha)}, \quad z \in \mathbb{C},\ \mathfrak{Re}(\alpha) > 0.$$

Lemma 3. (*See* [23, *Lemma* 2], [13].) *There exist constants* $N_1, N_2 > 1$ *such that for any* $0 < \alpha < 1$*,*

$$\left\|E_{\alpha,1}\left(At^\alpha\right)\right\| \leqslant N_1\left\|\mathrm{e}^{At}\right\|, \qquad \left\|E_{\alpha,\alpha}\left(At^\alpha\right)\right\| \leqslant N_2\left\|\mathrm{e}^{At}\right\|,$$

where A *denotes matrix, and* $\|\cdot\|$ *denotes any vector or induced matrix norm.*

Lemma 4. (*See* [30, *Lemma* 3.1].) *For a given square matrix* M*, if its spectral radius* $\rho(M) < 1$*, then there exist positive constants* c_0 *and* $0 < \rho_0 < 1$ *such that* $\|M^k\| \leqslant c_0\rho_0^k$.

3 Main results

In this section, the learning law is first designed for MASs with identical agents in Section 3.1. Then, the results will be extended to heterogeneous MASs in Section 3.2.

3.1 Controller design for homogeneous MASs

Assume that in (1), each agent has identical dynamics. That is, $A_i = A_d = A$, $B_i = B_d = B$, and $C_i = C_d = C$. Then the dynamics model of N agents is rewritten that

$$\begin{aligned} {}_0^C D_t^\alpha x_k(t) &= (I_N \otimes A)x_k(t) + (I_N \otimes B)u_k(t),\\ y_k(t) &= (I_N \otimes C)x_k(t), \end{aligned} \tag{3}$$

In this section, we design suitable protocols and analyze convergence for the fractional-order linear MASs (3). In Section 3.1.1, the distributed D^α-type protocol is considered, and its convergence properties are fully analyzed. To improve the final performance, we add the P-type structure to the D^α-type protocol to analyze the convergence properties in Section 3.1.2.

3.1.1 Distributed D^α-type updating rule for homogenous agents

Considering the distributed construction of fractional-order MAS, we let $\eta_{k,i}(t)$ be the distribute error by the ith agent at the $(k+1)$th iteration, which is defined as

$$\eta_{k,i}(t) = \sum_{j\in N_i} \tilde{a}_{i,j}\big(y_{k,j}(t) - y_{k,i}(t)\big) + s_i\big(y_d(t) - y_{k,i}(t)\big), \tag{4}$$

where $s_i = 1$ if $\langle v_0, v_i\rangle \in \bar{\mathcal{E}}$ (it means the ith agent directly links to the virtual leader), and $s_i = 0$ otherwise.

The actual tracking error is defined as $e_{k,i}(t) = y_d(t) - y_{k,i}(t)$, which cannot be utilized in the controller design due to two reasons. The first one is that such a tracking error is not available for all agents because only a small number of followers can access the leader's trajectory. In other words, the distributed error $\eta_{k,i}(t)$ rather than $e_{k,i}(t)$ is used in the following algorithms. The other reason is that we require a suitable causality between the input and tracking error according to (1) so that the input signals can be updated effectively [13, 14]. In other words, it is the suitable derivation of the distributed error rather than itself will be employed.

Hence, we first adopt the D^α-type ILC protocol:

$$u_{k+1,i}(t) = u_{k,i}(t) + W_{D^\alpha}\, {}_0^C D_t^\alpha \eta_{k,i}(t), \quad u_{0,i}(t) = 0 \quad \forall v_i \in \mathcal{V}, \tag{5}$$

where $W_{D^\alpha} \in \mathbb{R}^{m\times p}$ is the learning gain matrix.

Let the $\tilde{x}_{v,i}(t)$, $\tilde{u}_{v,i}(t)$, and $\tilde{y}_{v,i}(t) = C\tilde{x}_{v,i}(t)$ satisfy the following virtual dynamics:

$$\tilde{x}_{v,i}(t) = x_{1,i}(0) + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{A\tilde{x}_{v,i}(\tau) + B\tilde{u}_{v,i}(\tau)}{(t-\tau)^{1-\alpha}}\, \mathrm{d}\tau, \tag{6}$$

and

$$W_{D^\alpha}\big({}_0^C D_t^\alpha y_d(t) - {}_0^C D_t^\alpha \tilde{y}_{v,i}(t)\big) = 0. \tag{7}$$

Theorem 1. *The fractional-order homogeneous MAS* (3)*, under Assumptions* 1–3*, the communication graph $\tilde{\mathcal{G}}$, and distributed D^α-type updating rule* (5) *are considered. If the learning rule W_{D^α} is chosen such that*

$$\rho\big(I_{pN} - (L+S)\otimes W_{D^\alpha} CB\big) < 1, \tag{8}$$

then the control input $u_{k,i}(t)$ and the output $y_{k,i}(t)$ converge to $\tilde{u}_{v,i}(t)$ and $\tilde{y}_{v,i}(t)$, respectively, as the iteration number tends to infinity.

Proof. According to the definition of the distributed measurement $\eta_{k,i}(t)$ in (4), condition (7), and the property of linearity in the fractional calculus, we can obtain that

$$\begin{aligned} W_{D^\alpha}\, {}_0^C D_t^\alpha \eta_{k,i}(t) = \sum_{j\in N_i} \tilde{a}_{i,j}\big(&W_{D^\alpha}\, {}_0^C D_t^\alpha \tilde{y}_{v,i}(t) - W_{D^\alpha}\, {}_0^C D_t^\alpha y_{k,i}(t) \\ &- W_{D^\alpha}\, {}_0^C D_t^\alpha \tilde{y}_{v,j}(t) + W_{D^\alpha}\, {}_0^C D_t^\alpha y_{k,j}(t)\big) \\ &+ s_i\big(W_{D^\alpha}\, {}_0^C D_t^\alpha \tilde{y}_{v,i}(t) - W_{D^\alpha}\, {}_0^C D_t^\alpha y_{k,i}(t)\big). \end{aligned} \tag{9}$$

The virtual tracking error is defined as $\tilde{e}_{k,i}(t) = \tilde{y}_{v,i}(t) - y_{k,i}(t)$, then (9) can be simplified as

$$W_{D^\alpha}\,{}_0^C D_t^\alpha \eta_{k,i}(t) = W_{D^\alpha}\Biggl(\sum_{j\in N_i}\tilde{a}_{i,j}\bigl({}_0^C D_t^\alpha \tilde{e}_{k,i}(t) - {}_0^C D_t^\alpha \tilde{e}_{k,j}(t)\bigr) + s_i\bigl({}_0^C D_t^\alpha \tilde{e}_{k,i}(t)\bigr)\Biggr). \tag{10}$$

Define the following notations: $\delta u_{k,i}(t) = \tilde{u}_{v,i}(t) - u_{k,i}(t)$, and $\delta x_{k,i}(t) = \tilde{x}_{v,i}(t) - x_{k,i}(t)$.

From (5) and (10) we have

$$\delta u_{k+1,i}(t) = \delta u_{k,i}(t) - W_{D^\alpha}\Biggl(\sum_{j\in N_i}\tilde{a}_{i,j}\bigl({}_0^C D_t^\alpha \tilde{e}_{k,i}(t) - {}_0^C D_t^\alpha \tilde{e}_{k,j}(t)\bigr) + s_i\bigl({}_0^C D_t^\alpha \tilde{e}_{k,i}(t)\bigr)\Biggr). \tag{11}$$

Thus, Eq. (11) can be rewritten in the following columnar form:

$$\delta u_{k+1}(t) = \delta u_k(t) - \bigl((L+S)\otimes W_{D^\alpha}\bigr){}_0^C D_t^\alpha \tilde{e}_k(t). \tag{12}$$

The result, which we take the Caputo fractional calculus from both sides of (6) combining with Lemma 1, does substraction with Eq. (1),

$${}_0^C D_t^\alpha \delta x_{k,i}(t) = A\delta x_{k,i}(t) + B\delta u_{k,i}(t). \tag{13}$$

Thus, (13) is rewritten by the column stack vectors united from (3), and we have

$$\begin{aligned}{}_0^C D_t^\alpha \delta x_k(t) &= {}_0^C D_t^\alpha x_v(t) - {}_0^C D_t^\alpha x_k(t)\\ &= (I_N\otimes A)\delta x_k(t) + (I_N\otimes B)\delta u_k(t).\end{aligned} \tag{14}$$

Note that $\tilde{e}_k(t) = \tilde{y}_v(t) - y_k(t) = (I_N\otimes C)\delta x_k(t)$. For (12), we have

$$\begin{aligned}\delta u_{k+1}(t) = M_1^k\delta u_1(t) - \bigl(&M_1^{k-1}M_2\delta x_1(t)\\ &+ M_1^{k-2}M_2\delta x_2(t) + \cdots + M_2\delta x_k\bigr),\end{aligned} \tag{15}$$

where $M_1 = I_{pN} - (L+S)\otimes W_{D^\alpha}CB$, $M_2 = (L+S)\otimes W_{D^\alpha}CA$. In terms of Assumption 1, $L+S$ must be a matrix of full rank.

Taking the λ-norm on (15), we have

$$\begin{aligned}\|\delta u_{k+1}\|_\lambda \leqslant \bigl\|\|M_1\|^k\|\delta u_1\bigr\|_\lambda + \|M_2\|\bigl(&\|M_1\|^{k-1}\|\delta x_1\|_\lambda\\ &+ \|M_1\|^{k-2}\|\delta x_2\|_\lambda + \cdots + \|\delta x_k\|_\lambda\bigr).\end{aligned} \tag{16}$$

Due to $\tilde{x}_{v,i}(0) = x_{1,i}(0)$, from Assumption 2, $\delta x_{k,i}(0) = 0$. Solving $\delta x_k(t)$ from (14), we have (see [8])

$$\delta x_k(t) = \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}\bigl((I_N\otimes A)(t-\tau)^\alpha\bigr)(I_N\otimes B)\delta u_k(\tau)\,\mathrm{d}\tau. \tag{17}$$

Taking any generic norm on the both side of (17) via Lemma 3, there exists a constant N_2 such that

$$\big\|\delta x_k(t)\big\| \leqslant N_2\|I_N \otimes B\| \int_0^t (t-\tau)^{\alpha-1} \mathrm{e}^{\|I_N\otimes A\|(t-\tau)+\lambda\tau}\,\mathrm{d}\tau \|\delta u_k\|_\lambda. \tag{18}$$

Multiplying $\mathrm{e}^{-\lambda t}$ by the both of (18) and via the Hölder inequality, we have

$$\begin{aligned}\big\|\delta x_k(t)\big\|\mathrm{e}^{-\lambda t} &= \frac{N_2\|I_N\otimes B\|t^{1/p-(1-\alpha)}}{\sqrt[p]{1-p(1-\alpha)}}\frac{\sqrt[q]{1-\mathrm{e}^{-q(\lambda-\|I_N\otimes A\|)t}}}{\sqrt[q]{q(\lambda-\|I_N\otimes A\|)}}\|\delta u_k\|_\lambda \\ &\leqslant \frac{N_2\|I_N\otimes B\|t^{1/p-(1-\alpha)}}{\sqrt[p]{1-p(1-\alpha)}\,\sqrt[q]{q(\lambda-\|I_N\otimes A\|)}}\|\delta u_k\|_\lambda. \end{aligned} \tag{19}$$

Taking supremum for the both side of (19) on the interval $[0,T]$, we have

$$\|\delta x_k\|_\lambda \leqslant \frac{N_2\|I_N\otimes B\|T^{1/p-(1-\alpha)}}{\sqrt[p]{1-p(1-\alpha)}\,\sqrt[q]{q(\lambda-\|I_N\otimes A\|)}}\|\delta u_k\|_\lambda. \tag{20}$$

Now one can substitute (20) into (16) , which yields

$$\begin{aligned}\|\delta u_{k+1}\|_\lambda &\leqslant \|M_1\|^k\|\delta u_1\|_\lambda + \frac{N_2\|M_2\|\|I_N\otimes B\|T^{1/p-(1-\alpha)}}{\sqrt[p]{1-p(1-\alpha)}\,\sqrt[q]{q(\lambda-\|I_N\otimes A\|)}} \\ &\quad\times \big(\|M_1\|^{k-1}\|\delta u_1\|_\lambda + \|M_1\|^{k-2}\|\delta u_2\|_\lambda + \cdots + \|\delta u_k\|_\lambda\big). \end{aligned} \tag{21}$$

Due to Assumptions 1, 3 and condition (8), the spectral radius of M_1 is less than 1. Uniting Lemma 4 leads to $\|M_1\|^k \leqslant c_1\rho_0^k$. Therefore, (21) can be rewritten as

$$\begin{aligned}\|\delta u_{k+1}\|_\lambda &\leqslant c_1\rho_0^k\|\delta u_1\|_\lambda + \frac{c_2}{\sqrt[q]{q(\lambda-\|I_N\otimes A\|)}} \\ &\quad\times \big(\rho_0^{k-1}\|\delta u_1\|_\lambda + \rho_0^{k-2}\|\delta u_2\|_\lambda + \cdots + \|\delta u_k\|_\lambda\big), \end{aligned} \tag{22}$$

where $c_2 = c_1N_2\|M_2\|\|B\|T^{1/p-(1-\alpha)}/\sqrt[p]{1-p(1-\alpha)}$.

Choosing the $\lambda > \|I_N\otimes A\| + (1/q)(c_2/(1-\rho_0))^q$ and applying Lemma 2, we can conclude that $\|\delta u_{k+1}\|_\lambda \to 0$ as $k\to\infty$; that is, $u_{k,i}(t)\to\tilde{u}_{v,i}(t)$ and $y_{k,i}(t)\to\tilde{y}_{v,i}(t)$. □

Remark 1. Under Assumption 3, we can take

$$W_{D^\alpha} = \frac{((CB)^{\mathrm{T}}CB)^{-1}(CB)^{\mathrm{T}}}{\max(|\sigma(L+S)|)+\min(|\sigma(L+S)|)},$$

where $\sigma(L+S)$ denotes the eigenvalue of matrix $(L+S)$.

Theorem 2. *If the D^α-type rule converges and $W_{D^\alpha}CB$ is nonsingular, there exist unique $\tilde{u}_{v,i}(t)$ and $\tilde{y}_{v,i}(t)$ satisfying the virtual dynamics* (6) *and* (7)*. Specifically,*

$$\tilde{u}_{v,i}(t) = u_d(t) + (W_{D^\alpha}CB)^{-1}W_{D^\alpha}CAE_{\alpha,1}(F_{D^\alpha}t^\alpha)\big(x_d(0) - x_{k,i}(0)\big),$$

and

$$\tilde{y}_{v,i}(t) = y_d(t) - CE_{\alpha,1}(F_{D^\alpha}t^\alpha)\big(x_d(0) - x_{k,i}(0)\big),$$

where $F_{D^\alpha} = (I_n - B(W_{D^\alpha}CB)^{-1}W_{D^\alpha}C)A$.

Proof. From (7) and the dynamics (6) and (2) we have

$$W_{D^\alpha}CB\big(u_d(t) - \tilde{u}_{v,i}(t)\big) = -W_{D^\alpha}CA\big(x_d(t) - \tilde{x}_{v,i}(t)\big).$$

Define $\delta\tilde{u}_{v,i}(t) = u_d(t) - \tilde{u}_{v,i}(t)$ and $\delta\tilde{x}_{v,i}(t) = x_d(t) - \tilde{x}_{v,i}(t)$. Note that $W_{D^\alpha}CB$ is nonsingular; hence, we have

$$\delta\tilde{u}_{v,i}(t) = -(W_{D^\alpha}CB)^{-1}W_{D^\alpha}CA\delta\tilde{x}_{v,i}(t). \tag{23}$$

On account of ${}_0^C D_t^\alpha \delta\tilde{x}_{v,i}(t) = A\delta\tilde{x}_{v,i}(t) + B\delta\tilde{u}_{v,i}(t)$ with initial condition $\delta\tilde{x}_{v,i}(0) = x_d(0) - \tilde{x}_{v,i}(0)$, we have

$${}_0^C D_t^\alpha \delta\tilde{x}_{v,i}(t) = \big(I_p - B(W_{D^\alpha}CB)^{-1}W_{D^\alpha}C\big)A\delta\tilde{x}_{v,i}(t). \tag{24}$$

The solution of (24) is

$$\delta\tilde{x}_{v,i}(t) = E_{\alpha,1}(F_{D^\alpha}t^\alpha)\big(x_d(0) - x_{1,i}(0)\big),$$

where $F_{D^\alpha} = (I_p - B(W_{D^\alpha}CB)^{-1}W_{D^\alpha}C)A$.

Therefore, from (23) we have

$$\tilde{u}_{v,i}(t) = u_d(t) + (W_{D^\alpha}CB)^{-1}W_{D^\alpha}CAE_{\alpha,1}(F_{D^\alpha}t^\alpha)\big(x_d(0) - x_{1,i}(0)\big),$$

and

$$\tilde{y}_{v,i}(t) = y_d(t) - CE_{\alpha,1}(F_{D^\alpha}t^\alpha)\big(x_d(0) - x_{1,i}(0)\big). \qquad \square$$

Remark 2. If W_{D^α} is of full column rank, then

$$\tilde{y}_{v,i}(t) = y_d(t) - C\big(x_d(0) - x_{1,i}(0)\big).$$

3.1.2 Distributed PD^α-type updating rule for homogenous agents

In this section, the PD^α-type updating rule is considered to improve the final performance. The distributed measurement is consistent with the D^α-type case. The proposed PD^α-type updating rule for ith agent at kth iteration is

$$u_{k+1,i}(t) = u_{k,i}(t) + W_{D^\alpha}\big({}_0^C D_t^\alpha \eta_{k,i}(t) + W_{PD^\alpha}\eta_{k,i}(t)\big), \tag{25}$$

where W_{PD^α} is a learning gain matrix with suitable dimension.

The desired trajectory $y_d(t)$ and virtual output trajectory $y_{v,i}(t)$ satisfy the following formula:

$$W_{D^\alpha}\big({}_0^C D_t^\alpha y_d(t) - {}_0^C D_t^\alpha \tilde{y}_{v,i}(t) + W_{PD^\alpha}\big(y_d(t) - \tilde{y}_{v,i}(t)\big)\big) = 0. \tag{26}$$

Remark 3. Condition (26) is inspired from [30, Chap. 4] for case of $\alpha = 1$, i.e.,

$$W_D\big(\dot{y}_d(t) - \dot{y}_{v,i}(t) + W_{PD}\big(y_d(t) - y_{v,i}(t)\big)\big) = 0.$$

Theorem 3. *Consider the MAS* (3)*, if the learning gain matrix W_{D^α} is chose such that*

$$\rho\big(I_{pN} - (L+S) \otimes W_{D^\alpha}CB\big) < 1,$$

then the ILC rule is stable, and the output trajectory of any follower converges to the virtual output trajectory as the iteration number tends to infinity. In particular,

$$\begin{aligned}\tilde{u}_{v,i}(t) &= u_d(t)\\ &\quad + (W_{D^\alpha}CB)^{-1}W_{D^\alpha}(CA + W_{PD^\alpha}C)E_{\alpha,1}(F_{PD^\alpha}t^\alpha)\big(x_d(0) - x_{k,i}(0)\big),\end{aligned}$$

and

$$\tilde{y}_{v,i}(t) = y_d(t) - CE_{\alpha,1}(F_{PD^\alpha}t^\alpha)\big(x_d(0) - x_{k,i}(0)\big),$$

where $F_{PD^\alpha} = A - (W_{D^\alpha}CB)^{-1}W_{D^\alpha}(CA + W_{PD^\alpha}C)$.

Proof. The proof is similar to that of Theorems 1 and 2. $\tilde{e}_{k,i}(t)$, $\delta u_{k,i}(t)$, and $\delta x_{k,i}(t)$ are defined as in Theorem 1, as well as $\delta\tilde{u}_{k,i}(t)$ and $\delta\tilde{x}_{k,i}(t)$ are defined as in Theorem 2.

According to condition (26), we have

$$W_{D^\alpha}\big({}_0^CD_t^\alpha y_d(t) + W_{PD^\alpha}y_d(t)\big) = W_{D^\alpha}\big({}_0^CD_t^\alpha \tilde{y}_{v,i}(t) + W_{PD^\alpha}\tilde{y}_{v,i}(t)\big).$$

Then,

$$\begin{aligned}&W_{D^\alpha}\big({}_0^CD_t^\alpha \eta_{k,i}(t) + W_{PD^\alpha}\eta_{k,i}(t)\big)\\ &\quad = W_{D^\alpha}\bigg(\sum_{j\in N_i}\tilde{a}_{i,j}\big({}_0^CD_t^\alpha\tilde{e}_{k,i}(t) - {}_0^CD_t^\alpha\tilde{e}_{k,j}(t)\big) + s_i{}_0^CD_t^\alpha\tilde{e}_{k,i}(t)\\ &\qquad + W_{PD^\alpha}\bigg(\sum_{j\in N_i}\tilde{a}_{i,j}\big(\tilde{e}_{k,i}(t) - \tilde{e}_{k,j}(t)\big) + s_i\tilde{e}_{k,i}(t)\bigg)\bigg).\end{aligned} \tag{27}$$

From (25) and (27) the error of input $\delta u_{k+1}(t)$ is rewritten in the following compact form by (13) and (14):

$$\begin{aligned}\delta u_{k+1}(t) &= \big(I_{pN} - (L+S)\otimes W_{D^\alpha}CB\big)\delta u_k(t)\\ &\quad - \big((L+S)\otimes W_{D^\alpha}(CA + W_{PD^\alpha}C)\big)\delta x_k(t).\end{aligned} \tag{28}$$

Taking λ-norm for (28), we have

$$\begin{aligned}\|\delta u_{k+1}\|_\lambda &\leqslant \|M_1'\|^k\|\delta u_1\|_\lambda + \|M_2'\|\big(\|M_1'\|^{k-1}\|\delta x_1\|_\lambda\\ &\quad + \|M_1'\|^{k-2}\|\delta x_2\|_\lambda + \cdots + \|\delta x_k\|_\lambda\big),\end{aligned}$$

where $M_1' = I_{pN} - (L+S)\otimes W_{D^\alpha}CB$ and $M_2' = (L+S)\otimes W_{D^\alpha}(CA + W_{PD^\alpha}C)$.

From Theorem 1, Eq. (20) still is established. Obviously, we have,

$$\begin{aligned}
\|\delta u_{k+1}\|_\lambda &\leqslant \|M_1'\|^k\|\delta u_1\|_\lambda + \frac{N_2\|M_2'\|\|B\|T^{1/p-(1-\alpha)}}{\sqrt[p]{1-p(1-\alpha)}\sqrt[q]{q(\lambda-\|A\|)}} \\
&\quad \times \bigl(\|M_1'\|^{k-1}\|\delta u_1\|_\lambda + \|M_1'\|^{k-2}\|\delta u_2\|_\lambda + \cdots + \|\delta u_k\|_\lambda\bigr) \\
&\leqslant c_1'\rho_0^k\|\delta u_1\|_\lambda \\
&\quad + \frac{c_2'}{\sqrt[q]{q(\lambda-\|A\|)}}\bigl(\rho_0^{k-1}\|\delta u_1\|_\lambda + \rho_0^{k-2}\|\delta u_2\|_\lambda + \cdots + \|\delta u_k\|_\lambda\bigr),
\end{aligned}$$

where $c_2' = c_1' N_2\|M_2'\|\|B\|T^{1/p-(1-\alpha)}/\sqrt[p]{1-p(1-\alpha)}$.

Now one can choose $\lambda > \|A\| + (c_2'/(1-\rho_0))^q/q$ and apply Lemma 2 to conclude that $\|\delta u_{k+1}\|_\lambda \to 0$ as $k \to \infty$; that is, $u_{k,i}(t) \to \tilde{u}_{v,i}(t)$ and $y_{k,i}(t) \to \tilde{y}_{v,i}(t)$.

Further, according to (26), we have

$$\delta\tilde{u}_{v,i}(t) = -(W_{D^\alpha}CB)^{-1}W_{D^\alpha}(CA + W_{PD^\alpha}C)\delta\tilde{x}_{v,i}(t).$$

This yields that

$$ {}_0^C D_t^\alpha \delta\tilde{x}_{k,i}(t) = \bigl(A - B(W_{D^\alpha}CB)^{-1}W_{D^\alpha}(CA + W_{PD^\alpha}C)\bigr)\delta\tilde{x}_{k,i}(t). \tag{29}$$

Solving (29), we have

$$\delta\tilde{x}_{k,i}(t) = E_{\alpha,1}\bigl(F_{PD^\alpha}t^\alpha\bigr)\bigl(x_d(0) - x_{k,i}(0)\bigr),$$

where $F_{PD^\alpha} = A - (W_{D^\alpha}CB)^{-1}W_{D^\alpha}(CA + W_{PD^\alpha}C)$.

Then

$$\begin{aligned}
\tilde{u}_{v,i}(t) &= u_d(t) + (W_{D^\alpha}CB)^{-1}W_{D^\alpha}(CA + W_{PD^\alpha}C)E_{\alpha,1}\bigl(F_{PD^\alpha}t^\alpha\bigr) \\
&\quad \times \bigl(x_d(0) - x_{k,i}(0)\bigr),
\end{aligned}$$

and

$$\tilde{y}_{v,i}(t) = y_d(t) - CE_{\alpha,1}\bigl(F_{PD^\alpha}t^\alpha\bigr)\bigl(x_d(0) - x_{k,i}(0)\bigr). \qquad \square$$

3.2 Controller design for heterogenous agents

This section is extension of Section 3.1. The model of MASs (1) consisting of N heterogeneous agents is rewritten as

$$\begin{aligned}
{}_0^C D_t^\alpha x_k(t) &= \breve{A}x_k(t) + \breve{B}u_k(t), \\
y_k(t) &= \breve{C}x_k(t),
\end{aligned} \tag{30}$$

where $\breve{A} = \operatorname{diag}\{A_1, A_2, \ldots, A_N\}$, $\breve{B} = \operatorname{diag}\{B_1, B_2, \ldots, B_N\}$, and $\breve{C} = \operatorname{diag}\{C_1, C_2, \ldots, C_N\}$ are constant matrices with suitable dimensions. The definition of three column stack vectors $x_k(t)$, $y_k(t)$, and $u_k(t)$ is consistent with the previous section.

3.2.1 Distributed D^α-type updating rule for heterogeneous agents

The distribute measurement $\eta_{k,i}(t)$ is the same with (4). We still apply D^α-type updating rule for every agent but the gain matrix of each agent is different:

$$u_{k+1,i}(t) = u_{k,i}(t) + W_{D^\alpha,i}{}_0^C D_t^\alpha \eta_{k,i}(t). \tag{31}$$

The P-type updating law is then added to the D^α-type updating law to improve the final performance:

$$u_{k+1,i}(t) = u_{k,i}(t) + W_{D^\alpha,i}\ \big({}_0^C D_t^\alpha \eta_{k,i}(t) + W_{PD^\alpha,i}\ \eta_{k,i}(t)\big). \tag{32}$$

Similarly, let the $\tilde{x}_{v,i}(t)$, $\tilde{u}_{v,i}(t)$, and $\tilde{y}_{v,i}(t) = C_i\tilde{x}_{v,i}(t)$ satisfy the following virtual dynamics:

$$\tilde{x}_{v,i}(t) = x_{1,i}(0) + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{A_i x_{v,i}(\tau) + B_i u_{v,i}(\tau)}{(t-\tau)^{1-\alpha}}\,\mathrm{d}\tau,$$

and

$$W_{D^\alpha,i}\big({}_0^C D_t^\alpha y_d(t) - {}_0^C D_t^\alpha y_{v,i}(t)\big) = 0. \tag{33}$$

Theorem 4. *Assume that Assumption* 1–3 *hold for the time-invariant linear MASs* (30) *with the D^α-type updating law. If the learning gain matrix $\breve{W}_{D^\alpha}$ ($\breve{W}_{D^\alpha} = \operatorname{diag}\{W_{D^\alpha,1}, W_{D^\alpha,2}, \dots, W_{D^\alpha,N}\}$) satisfies the following condition*:

$$\rho\big(I_{pN} - \breve{W}_{D^\alpha}\big((L+S)\otimes I_p\big)\breve{C}\breve{B}\big) \leqslant \breve{\rho}_0 < 1$$

for some $\breve{\rho}_0 \in (0,1)$, then there exists a positive constant λ such that

$$\|\delta u_k\|_\lambda \to 0,$$

which indicates that $\lim_{k\to\infty} u_{k,i} = \tilde{u}_{v,i}(t)$, and then $\lim_{k\to\infty} y_{k,i}(t) = \tilde{y}_{v,i}(t)$ for all $t \in [0,T]$, $i = 1,2,\dots,N$.

Proof. The procedure that prove this theorem is similar to that of Theorem 1. □

3.2.2 Distributed $\boldsymbol{PD^\alpha}$-type updating rule for heterogeneous agents

For the PD^α-type updating rule, we have the following results.

Theorem 5. *Assume that Assumptions* 1–3 *hold for the time-invariant linear fractional-order MASs* (30) *with N heterogeneous agents and PD^α-type updating law* (32)*. If the learning gain W_{D^α} still choose that*

$$\rho\big(I_{pN} - \breve{W}_{D^\alpha}\big((L+S)\otimes I_p\big)\breve{C}\breve{B}\big) < 1,$$

then the ILC rule is stable and the output trajectory of any follower convergence.

Proof. The steps for this theorem is omitted here because it can be derived similar to that of Theorems 3 and 4. □

Remark 4. In Theorem 5, we need the following equation, which is also motivated by Remark 3:

$$W_{D^\alpha,i}\big({}_0^C D_t^\alpha y_d(t) - {}_0^C D_t^\alpha \tilde{y}_{v,i}(t) + W_{PD^\alpha,i}\big(y_d(t) - \tilde{y}_{v,i}(t)\big)\big) = 0,$$

$i = 1, 2, \dots, N$.

The relation between $y_d(t)$ and $\tilde{y}_{v,i}(t)$ (or between $u_d(t)$ and $\tilde{u}_{v,i}(t)$) cannot be showed via an equality due to different agent submits to different fractional-order dynamical system in Theorems 4 and 5.

Remark 5. The explicit solution of model (1) (see (17)) can also be obtained in the form of Mittag–Leffler functions.

Remark 6. Compared with the Chapter 4 in [30], this paper extends the integral-order model to the fractional-order model and corrects the inappropriateness, such as expressions for $u_{i,j}$ and $y_{i,j}$ in Theorem 4.3 of the original paper.

4 Simulation examples

In this section, two example are provided under uniform order and communication topology graph.

We select an arbitrary order in $(0, 1)$, say $\alpha = 0.7$, and initial input $u_0(t) = 0$. Firstly, we consider the D^α-type updating rule. Next, we add P-type updating rule to adjust the finial performance base on the example of the D^α-type updating rule.

For Fig. 1, the Laplacian for followers is

$$L = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 4 & -1 \\ 0 & 0 & -1 & -1 & 2 \end{bmatrix},$$

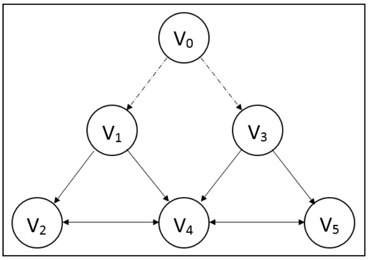


Figure 1. Directed fixed communication topology among agents in the network.

and $S = \mathrm{diag}(1, 0, 0, 1, 0)$ indicates the information flow from leader to followers and satisfies Assumption 1.

Example 1. Consider a group of five followers at kth iteration with their dynamics governed by the following model:

$$
\begin{aligned}
{}_0^C D_t^\alpha x_{k,i}(t) &= \begin{bmatrix} -0.5 & 0.8 \\ 0.4 & -0.2 \end{bmatrix} x_{k,i}(t) + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u_{k,i}(t), \\
y_{k,i}(t) &= \begin{bmatrix} 1 & 0.5 \\ 1.5 & 1 \end{bmatrix} x_{k,i}(t),
\end{aligned}
$$

and the leader's input is chosen as $u_d(t) = 0.2t + \cos t$ for all $t \in [0, 5]$, the initial state $x_d(0) = 0$. The initial condition for followers are $x_{k,1}(0) = [1.4, 1.2]^{\mathrm{T}}$, $x_{k,2}(0) = [1, 0.7]^{\mathrm{T}}$, $x_{k,3}(0) = [0.4, 0.3]^{\mathrm{T}}$, $x_{k,4}(0) = [-0.5, 0]^{\mathrm{T}}$, and $x_{k,5}(0) = [-1, -0.4]^{\mathrm{T}}$.

Obviously, the initial condition for followers satisfies Assumption 2. $\mathrm{rank}(CB) = 1$ satisfies Assumption 3.

We apply the updating rules (5) and (25) with learning gains matrix $W_{D^\alpha} = [0.0215, 0.0376]$, which is provide by Remark 1, and $W_{PD^\alpha} = \mathrm{diag}(3, 3)$ is chose correctly by many tests. Then the convergence condition is calculated:

$$\rho(I_{10} - (L + S) \otimes W_{D^\alpha} CB) = 0.8255 < 1,$$

which satisfies the convergence requirement in Theorems 1 and 2.

Figure 2 shows the output profiles of all agents at the 50th iteration under D^α-type updating rule. The output of followers is capable to track the general trend of the leader. According to the property of linearity in fractional calculus, (7) is rewritten that $W_{D^\alpha} {}_0^C D_t^\alpha (y_d(t) - y_{v,i}(t)) = 0$. Equation (7) is easily demonstrated by it. But the deviations between the followers and the leader is large. Simple calculation shows that the characteristic value of F_{D^α} is $\sigma(F_{D^\alpha}) = \{0, -1.1908\}$.

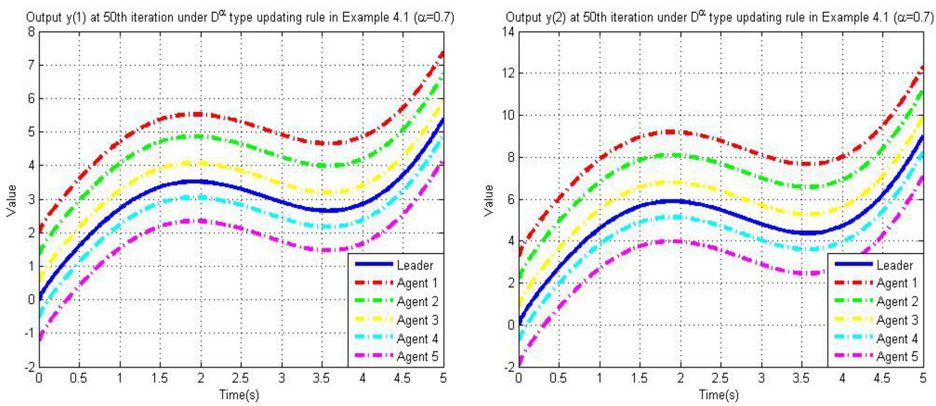


Figure 2. Output trajectories at 50th iteration under D^α-type ILC learning rule Example 1.

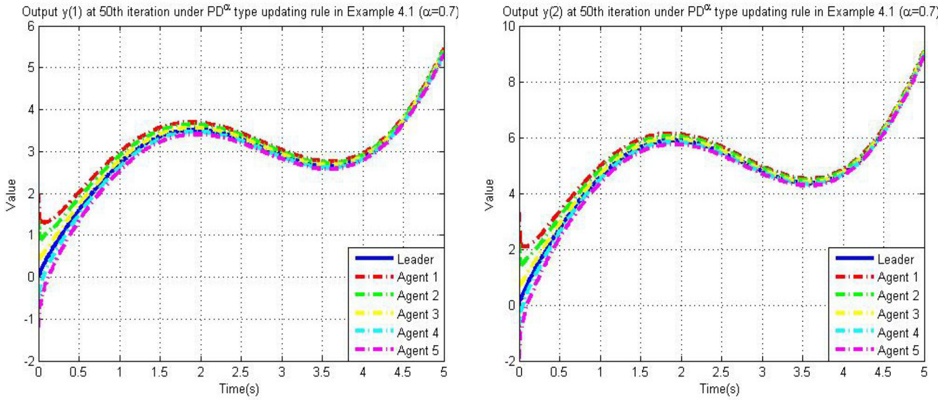


Figure 3. Output trajectories at 100th iteration under PD^{α}-type ILC learning rule in Example 1.

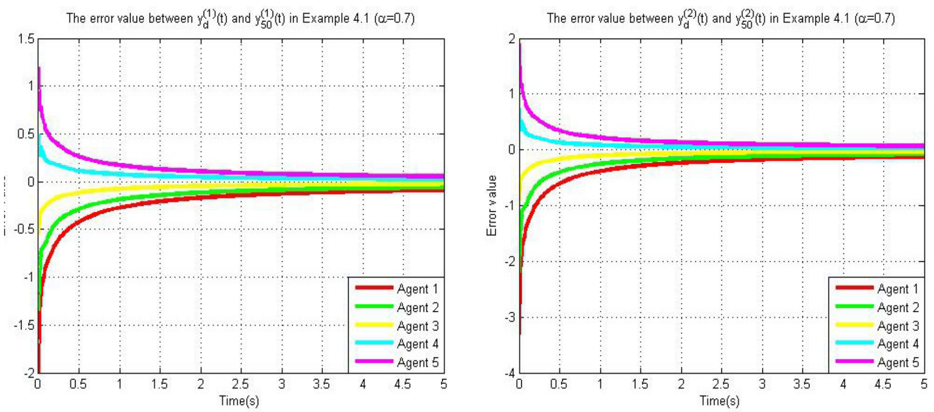


Figure 4. The error value between $y_d(t)$ and $y_{50}(t)$ under PD^{α}-type ILC learning rule in Example 1.

Figure 3 shows the output profiles of all agents at the 50th iteration under a PD^{α}-type updating rule. Different with the Fig. 2, we add the P-type updating rule to adjust the output profiles. It can be seen that the tracking error becomes smaller and smaller. The characteristic value of $F_{PD^{\alpha}}$ is $\sigma(F_{PD^{\alpha}}) = \{-1.1908, -4\}$. Due to the memorability of fractional calculus, the tracking trajectory of followers can roughly approach the leader's tracking trajectory with the growth of the iteration, but the deviation still exists.

Figure 4 shows the error value between $y_d(t)$ and $y_{50}(t)$ under PD^{α}-type updating rule. Obviously, the error value becomes smaller and smaller with the change of time. These pictures illustrate the PD^{α}-type updating rule can improve the finial tracking trajectory from Figs. 3 and 4.

Example 2. In this example, we consider a network consisting of one leader and five heterogeneous follower agents, which communication topology is showed by Fig. 1.

The model of the first agent at kth iteration is governed by

$$
\begin{aligned}
{}_0^C D_t^\alpha x_{k,1}(t) &= \begin{bmatrix} 0.4 & 0 \\ 0 & 0.6 \end{bmatrix} x_{k,1}(t) + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u_{k,1}(t), \\
y_{k,1}(t) &= \begin{bmatrix} 0.2 & 0.6 \end{bmatrix} x_{k,1}(t);
\end{aligned}
$$

the model of the second agent at kth iteration is governed by

$$
\begin{aligned}
{}_0^C D_t^\alpha x_{k,2}(t) &= \begin{bmatrix} 0.5 \end{bmatrix} x_{k,2}(t) + \begin{bmatrix} 0.8 \end{bmatrix} u_{k,2}(t), \\
y_{k,2}(t) &= \begin{bmatrix} 0.7 \end{bmatrix} x_{k,2}(t);
\end{aligned}
$$

the model of the third agent at kth iteration is governed by

$$
\begin{aligned}
{}_0^C D_t^\alpha x_{k,3}(t) &= \begin{bmatrix} 0.1 & 0.2 \\ 0 & 0.9 \end{bmatrix} x_{k,3}(t) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u_{k,3}(t), \\
y_{k,3}(t) &= \begin{bmatrix} 0.4 & 0.8 \end{bmatrix} x_{k,3}(t);
\end{aligned}
$$

the model of the fourth agent at kth iteration is governed by

$$
\begin{aligned}
{}_0^C D_t^\alpha x_{k,4}(t) &= \begin{bmatrix} 0.1 & 0 \\ 0 & 0.2 \end{bmatrix} x_{k,4}(t) + \begin{bmatrix} 1.5 \\ -1.2 \end{bmatrix} u_{k,4}(t), \\
y_{k,4}(t) &= \begin{bmatrix} 0.5 & 0.7 \end{bmatrix} x_{k,4}(t);
\end{aligned}
$$

the model of the fifth agent at kth iteration is governed by

$$
\begin{aligned}
{}_0^C D_t^\alpha x_{k,5}(t) &= \begin{bmatrix} 0.5 & 0.7 & 0.8 \\ 0 & 0.4 & 1 \\ 0.3 & 0.8 & 0.2 \end{bmatrix} x_{k,5}(t) + \begin{bmatrix} 1.1 \\ -0.8 \\ 0.6 \end{bmatrix} u_{k,5}(t), \\
y_{k,5}(t) &= \begin{bmatrix} 0.6 & 0.5 & 0.9 \end{bmatrix} x_{k,5}(t).
\end{aligned}
$$

The model of desired reference trajectory is governed that

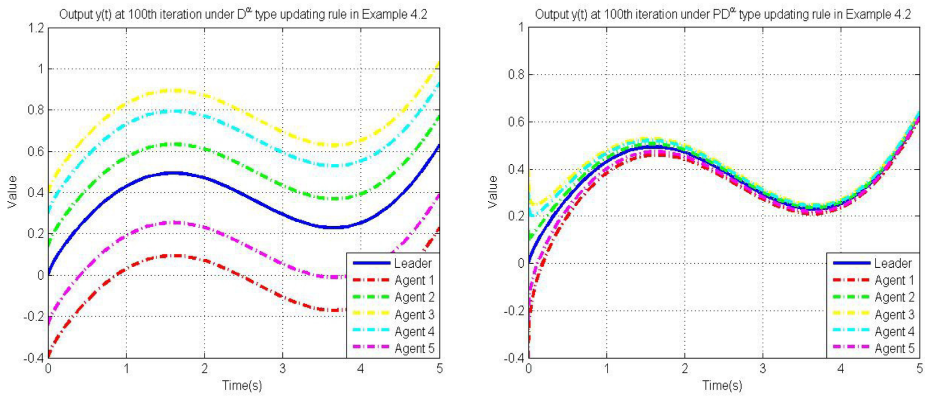
$$
\begin{aligned}
{}_0^C D_t^\alpha x_d(t) &= \begin{bmatrix} 0.2 & 0 \\ 0 & 0.1 \end{bmatrix} x_d(t) + \begin{bmatrix} 0.5 \\ 0.7 \end{bmatrix} u_d(t), \\
y_d(t) &= \begin{bmatrix} 0.1 & 0.5 \end{bmatrix} x_d(t).
\end{aligned}
$$

The leader's input $u_d(t)$ is still chosen as in Example 1, and the initial state $x_d(0) = 0$. The initial conditions for followers are $x_{k,1}(0) = [-2, 0]^{\mathrm{T}}$, $x_{k,2}(0) = [0.2]^{\mathrm{T}}$, $x_{k,3}(0) = [3, -1]^{\mathrm{T}}$, $x_{k,4}(0) = [2, -1]^{\mathrm{T}}$, and $x_{k,5}(0) = [0.6, -3, 1]^{\mathrm{T}}$.

Evidently, the initial ... satisfies Assumption 2. The $\operatorname{rank}(\breve{C}\breve{B}) = 5$ satisfies Assumption 3. The updating rules (31) and (32) with learning gains matrix $\breve{W}_{D^\alpha} = \operatorname{diag}(0.1246, 0.3115, -0.4361, -1.9384, 0.2181)$ are applied, and $\breve{W}_{PD^\alpha} = \operatorname{diag}(3, 3, 3, 3, 3)$ chose correctly by many tests. Then the convergence condition is calculated as

$$
\rho\big(I_5 - \breve{W}_{D^\alpha}(L + S)\breve{C}\breve{B}\big) = 0.8255 < 1,
$$

which satisfies the convergence requirement in Theorems 4 and 5.



(a) (b)

Figure 5. Output trajectories at 100th iteration in Example 2 under: (a) D^{α}-type ILC learning rule; (b) PD^{α}-type ILC learning rule.

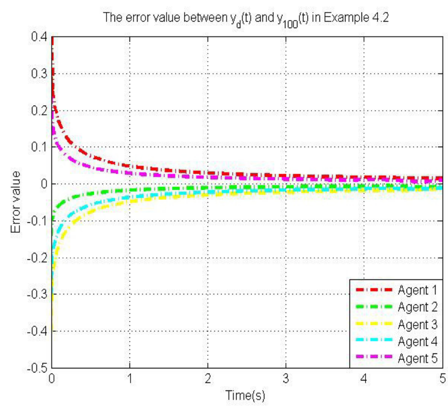


Figure 6. The error value between $y_d(t)$ and $y_{100}(t)$ under PD^{α}-type ILC learning rule in Example 2.

Figure 5(a) shows the output trajectory of heterogeneous agents at the 100th iteration under the D^{α}-type updating rule (31). Even though every agent is governed by different fractional-order model with larger the error, the output of followers is able to track the general trend of the leader. It easily shows that the formula (33) is equalled.

Figure 5(b) shows the output profiles of all agent at the 100th iteration under PD^{α}-type updating rule (32). Obviously, the tracking error which is adjusted by adding P-type updating rule becomes small and small. Similarly, in consider of the memorability of the fractional-order model, the error between the leader's tracking trajectory $y_d(t)$, and the followers' tracking trajectory $y_k(t)$ still is occurred.

Figure 6 shows the error of tracking trajectory between the leader's trajectory $y_d(t)$, and the followers' trajectory $y_{100}(t)$ becomes smaller and smaller.

Evidently, the PD^{α}-type updating rule which is introduced can adjust the final tracking performance than the D^{α} updating rule from Figs. 5(a), 5(b).

5 Conclusion

This paper studies the learning consensus problem for both fractional-order homogeneous and heterogeneous multi-agent systems, where the initialization position of each agent is fixed away from the desired one for all iterations. This condition implies an imperfect state resetting. It is proved that the D^{α}-type updating rule is convergent but the final trajectories are not the desired reference. To further improve tracking performance, the PD^{α}-type updating rule is then proposed, which provides the designer more freedom to tune the final performance. The theoretical results are verified by two numerical simulations. For further research, it is of interest to consider the fractional-order nonlinear MASs. In addition to fractional-order differential equations to describe engineering systems under complex environments, fuzzy differential equations are also suitable to simulate the process of change under uncertain conditions which can more accurately reflect the reality. Multi-agent systems using the fuzzy differential equation will be explored in the future.

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