Nonlinear Analysis: Modelling and Control, Vol. 25, No. 5, 806–826 ISSN: 1392-5113
https://doi.org/10.15388/namc.2020.25.18383 eISSN: 2335-8963

Multiple positive solutions for singular higher-order semipositone fractional differential equations with p-Laplacian*

Qiuyan Zhong[a], **Xingqiu Zhang**[b,1], **Lufeng Gu**[b], **Lei Lei**[c], **Zengqin Zhao**[d]

[a]Center for Information Technology,
Jining Medical University,
Jining 272067, Shandong, China
zhqy197308@163.com

[b]School of Medical Information Engineering,
Jining Medical University,
Rizhao 276826, Shandong, China
zhxq197508@163.com; gulufeng1214@163.com

[c]Mathematical Group, Jining No. 15 Middle School,
Jining 272000, Shandong, China
leilei2009ld@163.com

[d]School of Mathematical Sciences,
Qufu Normal University,
Qufu 273165, Shandong, China
zqzhaoy@163.com

Received: February 6, 2019 / **Revised:** August 22, 2020 / **Published online:** September 1, 2020

Abstract. In this article, together with Leggett–Williams and Guo–Krasnosel'skii fixed point theorems, height functions on special bounded sets are constructed to obtain the existence of at least three positive solutions for some higher-order fractional differential equations with p-Laplacian. The nonlinearity permits singularities both on the time and the space variables, and it also may change its sign.

Keywords: fractional differential equations, height functions, singularity on space variable, semipositone, triple positive solutions.

*This research was supported by Supporting Fund for Teachers' research of Jining Medical University (JYFC2018KJ015), a Project of Shandong Province Higher Educational Science and Technology Program (J18KA217), the National Natural Science Foundation of China (11571296, 11571197, 11871302), the Foundation for NSFC cultivation project of Jining Medical University (2016-05), the Natural Science Foundation of Jining Medical University (JY2015BS07, 2017JYQD22), the Natural Science Foundation of Shandong Province of China (ZR2015AL002) and the Key Research Project of Henan Higher Education Institutions (17A110001).

[1]Corresponding author.

1 Introduction

In this article, we are devoted to investigating the existence of multiple positive solutions for the following fractional differential equation with p-Laplacian (FPDE for short):

$$\begin{aligned} &-D^{\mu}_{0+}\big(\varphi_p\big(-D^{\alpha}_{0+}u(t)\big)\big) = f\big(t, u(t)\big), \quad 0 < t < 1, \\ &u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \qquad D^{\alpha}_{0+}u(0) = D^{\alpha}_{0+}u(1) = 0, \\ &D^{\beta}_{0+}u(1) = \lambda \int\limits_0^{\eta} D^{\gamma}_{0+}u(t)\,\mathrm{d}A(t), \end{aligned} \tag{1}$$

where D^{μ}_{0+}, D^{α}_{0+}, D^{β}_{0+} and D^{γ}_{0+} denote the standard Riemann–Liouville derivatives of orders μ, α, β and γ, respectively, $f \in C(I \times \mathbb{R}, \mathbb{R})$, $J = [0,1]$, $I = (0,1)$, $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}_+ = (0, +\infty)$, $1 < \mu \leqslant 2$, $n-1 < \alpha \leqslant n$, $n \geqslant 3$, $\beta \geqslant 1$, $\alpha-\beta-1 > 0$, $\alpha - \gamma - 1 > 0$, $0 < \eta \leqslant 1$, λ is a positive parameter with $0 \leqslant \lambda\Gamma(\alpha - \beta) \times \int_0^{\eta} t^{\alpha-\gamma-1}\,\mathrm{d}A(t) < \Gamma(\alpha - \gamma)$, $\varphi_p(s) = |s|^{p-2}s$, $p > 1$, A is a function of bounded variation, $\int_0^{\eta} D^{\gamma}_{0+}u(t)\,\mathrm{d}A(t)$ denotes the Riemann–Stieltjes integral with respect to A. It is clear, $\varphi_p(s)$ is invertible, and its inverse operator is $\varphi_q(s)$, where $q > 1$ with $1/p + 1/q = 1$. In this paper, the nonlinearity permits singularities both on the time, and the space variables and it also may change its sign.

Nowadays, there is a sharp increase in investigation of fractional nonlocal problems for their wide successful applications in tackling various physical phenomena in natural sciences and engineering; see [1–5, 7–13, 15–26, 29–55]. Very recently, by constructing height functions in different bounded sets, researchers obtained existence results on positive solutions and multiple positive solutions for some fractional nonlocal problems [31,33,34,46,51]. Recently, when f is semipositone, Luca [31] investigated the existence of positive solutions for the following fractional differential equations:

$$D^{\alpha}_{0+}u(t) + f\big(t, u(t)\big) = 0, \quad 0 < t < 1, \tag{2}$$

subject to Riemann–Stieltjes boundary conditions

$$u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \qquad D^{p}_{0+}u(1) = \int\limits_0^1 D^{q}_{0+}u(t)\,\mathrm{d}H(t),$$

where the BCs is the generalization of multi-point BCs

$$u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \qquad D^{p}_{0+}u(1) = \sum_{i=1}^{m} a_i D^{q}_{0+}u(\xi_i),$$

which is studied by Zhang et al. [46], Pu et al. [34], Henderson and Luca [25]. In a recent paper [50], by means of the property of u_0-positive linear operator and Banach contraction map principle, we discussed the uniqueness of solutio for BVP (2) under more general integral BCs

$$u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \qquad D_{0+}^{\beta} u(1) = \lambda \int_0^{\eta} h(t) D_{0+}^{\gamma} u(t)\,\mathrm{d}t.$$

Motivated by papers mentioned above and the multiple solutions results such as in [3–5, 28], the purpose of this paper is to investigate the existence of at least triple positive solutions for FPDE (1). The discussion based on height functions constructed on special bounded sets together with Leggett–Williams and Guo–Krasnosel'skii fixed point theorems. As far as we know, there are relatively few results on multiple solutions for fractional differential equation nonlocal problems when the nonlinearity permits singularities both on the time and the space variables. This paper admits the following features. Firstly, Riemann–Stieltjes integral is involved in BCs, which makes the problems discussed in this paper be a generalization of [25, 34, 46]. Secondly, two parameters λ, η and p-Laplacian operator are contained in FPDE (1), so the problems considered in this paper perform more general form compared with those in [7, 31, 47, 50, 51]. Thirdly, different from the traditional results obtained by Leggett–Williams, the nonlinearity permits singularity on space variable. It should be pointed out that the height function $\widehat{\Psi}(t, r_1, r_2)$ employed in this article is quite different from $\widehat{\psi}(t, r_1, r_2)$ used in [46]. This makes the verification of the condition easier and more efficient.

2 Auxiliary results

The basic Banach space used in this paper is $E = C[0, 1]$ equipped with the maximum norm $\|u\| = \max_{0 \leqslant t \leqslant 1} |u(t)|$. By a positive solution of BVP (1) we mean a function $u \in E$ satisfying BVP (1) with $u(t) > 0$ for all $t \in (0, 1]$.

Definition 1. (See [28].) A functional $\zeta : P \to [0, +\infty)$ is called a concave positive functional on a cone P if

$$\zeta\big(tx + (1-t)y\big) \geqslant t\zeta(x) + (1-t)\zeta(y) \quad \forall x, y \in P,\ 0 \leqslant t \leqslant 1.$$

Lemma 1. (*See* [6].) *Given $h \in L^1[0, 1]$ and $1 < \mu \leqslant 2$, the unique solution of the boundary value problems*

$$-D_{0+}^{\mu} u(t) = h(t), \quad 0 < t < 1, \qquad u(0) = u(1) = 0,$$

is

$$u(t) = \int_0^1 K_{\mu}(t, s) y(s)\,\mathrm{d}s, \quad t \in [0, 1],$$

where

$$K_{\mu}(t, s) = \begin{cases} \frac{t(1-s)^{\mu-1} - (t-s)^{\mu-1}}{\Gamma(\mu)}, & 0 \leqslant s \leqslant t \leqslant 1, \\ \frac{t(1-s)^{\mu-1}}{\Gamma(\mu)}, & 0 \leqslant t \leqslant s \leqslant 1. \end{cases} \tag{3}$$

Lemma 2. *Assume that* $\lambda\Gamma(\alpha-\beta)\int_0^\eta t^{\alpha-\gamma-1}\,\mathrm{d}A(t) \neq \Gamma(\alpha-\gamma)$. *Then for any* $y \in C[0,1]$, *the unique solution of the boundary value problems*

$$\begin{aligned} &-D_{0+}^{\alpha}u(t) = y(t), \quad 0<t<1,\\ &u(0)=u'(0)=\cdots=u^{(n-2)}(0)=0, \qquad D_{0+}^{\beta}u(1) = \lambda\int_0^\eta D_{0+}^{\gamma}u(t)\,\mathrm{d}A(t), \end{aligned} \tag{4}$$

solves

$$u(t) = \int_0^1 G(t,s)y(s)\,\mathrm{d}s, \quad t\in[0,1],$$

where

$$G(t,s) = G_1(t,s) + G_2(t,s), \tag{5}$$

$$G_1(t,s) = \begin{cases} \frac{t^{\alpha-1}(1-s)^{\alpha-\beta-1}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0\leqslant s\leqslant t\leqslant 1,\\ \frac{t^{\alpha-1}(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)}, & 0\leqslant t\leqslant s\leqslant 1, \end{cases} \tag{6}$$

$$G_2(t,s) = \frac{\lambda\Gamma(\alpha-\beta)t^{\alpha-1}}{\Gamma(\alpha-\gamma)-\lambda\Gamma(\alpha-\beta)\int_0^\eta t^{\alpha-\gamma-1}\,\mathrm{d}A(t)}\int_0^\eta H(t,s)\,\mathrm{d}A(t),$$

$$H(t,s) = \begin{cases} \frac{t^{\alpha-\gamma-1}(1-s)^{\alpha-\beta-1}-(t-s)^{\alpha-\gamma-1}}{\Gamma(\alpha)}, & 0\leqslant s\leqslant t\leqslant 1,\\ \frac{t^{\alpha-\gamma-1}(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)}, & 0\leqslant t\leqslant s\leqslant 1. \end{cases}$$

Here $G(t,s)$ *is called the Green function of BVP* (4). *Obviously,* $G(t,s)$ *is continuous on* $[0,1]\times[0,1]$.

Proof. The proof is similar to that in Lemma 2.3 in [50]. To complete the proof, we need only to replace $\int_0^\eta h(t)t^{\alpha-\gamma-1}\,\mathrm{d}t$ and $\int_0^\eta h(t)H(t,s)\,\mathrm{d}t$ with $\int_0^\eta t^{\alpha-\gamma-1}\,\mathrm{d}A(t)$ and $\int_0^\eta H(t,s)\,\mathrm{d}A(t)$, respectively. We omit the details. □

Lemma 3. *Let* $r\in L^1[0,1]$. *Then the unique solution of fractional boundary value problem*

$$\begin{aligned} &-D_{0+}^{\mu}\bigl(\varphi_p\bigl(-D_{0+}^{\alpha}u(t)\bigr)\bigr) = r(t), \quad 0<t<1,\\ &u(0)=u'(0)=\cdots=u^{(n-2)}(0)=0, \qquad D_{0+}^{\alpha}u(0)=D_{0+}^{\alpha}u(1)=0,\\ &D_{0+}^{\beta}u(1) = \lambda\int_0^\eta D_{0+}^{\gamma}u(t)\,\mathrm{d}A(t), \end{aligned} \tag{7}$$

is

$$u(t) = \int_0^1 G(t,s)\varphi_q\Biggl(\int_0^1 K_\mu(s,\tau)r(\tau)\,\mathrm{d}\tau\Biggr)\,\mathrm{d}s.$$

Proof. Let $v = \varphi_p(-D^{\alpha}_{0+}u)$. Then we have by (7)

$$-D^{\mu}_{0+}v(t) = r(t), \quad 0 < t < 1, \qquad v(0) = v(1) = 0. \tag{8}$$

It follows from Lemma 1 that the unique solution of (8) is

$$v(t) = \int_0^1 K_\mu(t,s)r(s)\,\mathrm{d}s, \quad t \in [0,1]. \tag{9}$$

We know from (9) that the solution of (7) meets

$$\begin{gathered}
-D^{\alpha}_{0+}u(t) = \varphi_q\left(\int_0^1 K_\mu(t,s)r(s)\,\mathrm{d}s\right), \quad 0 < t < 1, \\
u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \qquad D^{\beta}_{0+}u(1) = \lambda\int_0^\eta D^{\gamma}_{0+}u(t)\,\mathrm{d}A(t).
\end{gathered} \tag{10}$$

By means of Lemma 2 the solution of (10) admits the following form:

$$u(t) = \int_0^1 G(t,s)\varphi_q\left(\int_0^1 K_\mu(s,\tau)r(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s. \qquad \square$$

Lemma 4. (*See* [27].) *The Green function $K_\mu(t,s)$ defined by* (3) *has the following properties*: $K_\mu(t,s) = K_\mu(1-s,1-t)$ *and for* $t,s \in (0,1)$,

$$\frac{\mu-1}{\Gamma(\mu)}t^{\mu-1}(1-t)(1-s)^{\mu-1}s \leqslant K_\mu(t,s) \leqslant \frac{1}{\Gamma(\mu)}t^{\mu-1}(1-t)(1-s)^{\mu-2}.$$

Lemma 5. *The functions $G_1(t,s)$ and $G(t,s)$ given by* (5) *and* (6)*, respectively, admit the following properties*:

(a1) $G_1(t,s) \geqslant \frac{1}{\Gamma(\alpha)}t^{\alpha-1}s(1-s)^{\alpha-\beta-1} \quad \forall t,s \in [0,1]$;

(a2) $G_1(t,s) \leqslant \frac{1}{\Gamma(\alpha)}(\alpha-1)s(1-s)^{\alpha-\beta-1} \quad \forall t,s \in [0,1]$;

(a3) $G(t,s) \leqslant J(s)$,

$$J(s) = \frac{1}{\Gamma(\alpha)}(\alpha-1)s(1-s)^{\alpha-\beta-1} + \frac{\lambda\Gamma(\alpha-\beta)\int_0^\eta H(t,s)\,\mathrm{d}A(t)}{\Gamma(\alpha-\gamma) - \lambda\Gamma(\alpha-\beta)\int_0^\eta t^{\alpha-\gamma-1}\,\mathrm{d}A(t)} \quad \forall t,s \in [0,1];$$

(a4) $\frac{1}{(\alpha-1)}t^{\alpha-1}J(s) \leqslant G(t,s) \leqslant \frac{1}{\Gamma(\alpha)}\Lambda t^{\alpha-1}(1-s)^{\alpha-\beta-1}$, *here*

$$\Lambda = (\alpha - 1) + \frac{\lambda\Gamma(\alpha-\beta)\int_0^\eta t^{\alpha-\gamma-1}\,\mathrm{d}A(t)}{\Gamma(\alpha-\gamma) - \lambda\Gamma(\alpha-\beta)\int_0^\eta t^{\alpha-\gamma-1}\,\mathrm{d}A(t)} \quad \forall\, t, s \in [0,1].$$

Proof. The proof is similar to the proof of Lemma 2.4 in [50], we need only to replace $\int_0^\eta h(t)t^{\alpha-\gamma-1}\,\mathrm{d}t$, $\int_0^\eta h(t)H(t,s)\,\mathrm{d}t$ with $\int_0^\eta t^{\alpha-\gamma-1}\,\mathrm{d}A(t)$ and $\int_0^\eta H(t,s)\,\mathrm{d}A(t)$, respectively. We omit it here. □

Lemma 6. *Suppose that $w \in C[0,1]$ be the solution of*

$$\begin{aligned}
&-D_{0+}^{\mu}\big(\varphi_p\big(-D_{0+}^{\alpha}u(t)\big)\big) = k(t), \quad 0 < t < 1,\\
&u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \qquad D_{0+}^{\alpha}u(0) = D_{0+}^{\alpha}u(1) = 0,\\
&D_{0+}^{\beta}u(1) = \lambda\int_0^\eta D_{0+}^{\gamma}u(t)\,\mathrm{d}A(t),
\end{aligned}$$

where $k \in L^1(0,1)$, $k(t) > 0$. Then $w(t) \leqslant \vartheta t^{\alpha-1}$, $0 \leqslant t \leqslant 1$, where

$$\vartheta = \frac{\Lambda}{\Gamma(\alpha)}\cdot\left(\frac{1}{\Gamma(\mu)}\right)^{q-1}\int_0^1 (1-s)^{\alpha-\beta+q-2}s^{(\mu-1)(q-1)}\,\mathrm{d}s\cdot\left(\int_0^1(1-\tau)^{\mu-2}k(\tau)\,\mathrm{d}\tau\right)^{q-1}.$$

Proof. For any $0 \leqslant t \leqslant 1$, we get by Lemmas 3, 4 and 5,

$$\begin{aligned}
w(t) &= \int_0^1 G(t,s)\varphi_q\left(\int_0^1 K_\mu(s,\tau)k(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s\\
&\leqslant \frac{\Lambda}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-\beta-1}\varphi_q\left(\frac{1}{\Gamma(\mu)}\int_0^1 s^{\mu-1}(1-s)(1-\tau)^{\mu-2}k(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s t^{\alpha-1}\\
&= \frac{\Lambda(\frac{1}{\Gamma(\mu)})^{q-1}}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-\beta-1}\varphi_q\left(\int_0^1 s^{\mu-1}(1-s)(1-\tau)^{\mu-2}k(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s t^{\alpha-1}\\
&= \vartheta t^{\alpha-1}.
\end{aligned}$$

□

Denote

$$K = \left\{u \in E\colon\ u(t) \geqslant \frac{1}{\alpha-1}t^{\alpha-1}\|u\|,\ t \in [0,1]\right\}.$$

Then K is a cone in E. For simplicity, write $K_r = \{u \in K\colon\ \|u\| < r\}$ and

$$\begin{aligned}
K(\zeta, a, b) &= \big\{u \in K\colon\ a \leqslant \zeta(u),\ \|u\| \leqslant b\big\},\\
\mathring{K}(\zeta, a, b) &= \big\{u \in K\colon\ a < \zeta(u),\ \|u\| \leqslant b\big\}.
\end{aligned} \tag{11}$$

We formulate assumptions used throughout this paper.

(H1) $A: J \to \mathbb{R}$ is a nondecreasing function, and $0 < \lambda\Gamma(\alpha-\beta)\int_0^\eta t^{\alpha-\gamma-1}\,\mathrm{d}A(t) < \Gamma(\alpha-\gamma)$.

(H2) The function $f \in C(I \cdot \mathbb{R}_+, \mathbb{R})$, and there exists a function $k \in C(I, \mathbb{R}_+) \cap L^1(0,1)$ such that $f(t,u) \geqslant -k(t)$ for all $t \in I$, $u \in \mathbb{R}_+$ with $\int_0^1 (1-\tau)^{\mu-2} \times k(\tau)\,\mathrm{d}\tau < +\infty$.

(H3) For any positive numbers $r_1 < r_2$, there exists a nonnegative continuous function $\gamma_{r_1/(\alpha-1),r_2} \in L^1(0,1)$ such that $f(t,u) \leqslant \gamma_{r_1/(\alpha-1),r_2}(t)$, $0 < t < 1$, $r_1 t^{\alpha-1}/(\alpha-1) \leqslant u \leqslant r_2$ with $\int_0^1 (1-s)^{\mu-2}\gamma_{r_1/(\alpha-1),r_2}(s)\,\mathrm{d}s < +\infty$.

Lemma 7. (*See* [28].) *Suppose that $T: \overline{K}_c \to K$ is completely continuous, and suppose there exist a concave positive functional ζ with $\zeta(u) \leqslant \|u\|$, $u \in K$, and numbers $b > a > 0$, $b \leqslant c$ satisfying the following conditions*:

(i) $\{u \in K(\zeta, a, b)\colon \zeta(u) > a\} \neq \emptyset$, *and* $\zeta(Tu) > a$ *if* $u \in K(\zeta, a, b)$;
(ii) $Tu \in \overline{K}_c$ *if* $u \in K(\zeta, a, c)$;
(iii) $\zeta(Tu) > a$ *for all* $u \in K(\zeta, a, c)$ *with* $\|Tu\| > b$.

Then $i(T, \mathring{K}(\zeta, a, c), \overline{K}_c) = 1$.

Lemma 8. (*See* [14].) *Let K be a cone in Banach space X, $T: K \to K$ be a completely continuous operator. Let a, b, c be three positive numbers with $a < b < c$.*

(i) *If* $\|Tu\| > \|u\|$ *for* $u \in \partial(K_a)$ *and* $\|Tu\| \leqslant \|u\|$ *for* $u \in \partial(K_b)$, *then*
$$i(T, \overline{K}_b \setminus \overline{K}_a, \overline{K}_b) = 1,$$
(ii) *If* $\|Tu\| > \|u\|$ *for* $u \in \partial(K_a)$ *and* $\|Tu\| < \|u\|$ *for* $u \in \partial(K_b)$, *then*
$$i(T, K_b \setminus \overline{K}_a, \overline{K}_c) = 1.$$

3 Main result

Suppose that $0 < a^* < b^* \leqslant 1$. In applications, we can choose a^* and b^* in terms of the properties of $f(t,u)$. Denote
$$\sigma^* = \min_{t \in [a^*, b^*]} \frac{1}{\alpha-1} t^{\alpha-1}.$$

For any $r, r_1, r_2 > 0$ with $r_1 < r_2$, define the height functions as follows:
$$\widehat{\varphi}(t,r) = \max\left\{f(t,u)\colon \left(\frac{1}{\alpha-1}r - \vartheta\right)t^{\alpha-1} \leqslant u \leqslant r\right\} + k(t),$$
$$\widehat{\psi}(t,r) = \min\left\{f(t,u)\colon \left(\frac{1}{\alpha-1}r - \vartheta\right)t^{\alpha-1} \leqslant u \leqslant r\right\} + k(t),$$
$$\widehat{\varphi}(t,r_1,r_2) = \max\left\{f(t,u)\colon \left(\frac{1}{\alpha-1}r_1 - \vartheta\right)t^{\alpha-1} \leqslant u \leqslant r_2\right\} + k(t),$$
$$\widehat{\Psi}(t,r_1,r_2) = \min\bigl\{f(t,u)\colon (r_1 - \vartheta) \leqslant u \leqslant r_2\bigr\} + k(t).$$

Theorem 1. *Suppose that* (H1)–(H3) *hold. In addition, there exist five positive numbers* $(\alpha-1)\vartheta < e_1 < e_2 < e_3 < e_4 \leqslant e_5$ *with* $e_4 \geqslant e_3{\sigma^*}^{-1}$ *satisfying*

(A1) $$\int_0^1 (1-\tau)^{\mu-2}\widehat{\varphi}(\tau, e_2)\,\mathrm{d}\tau < \varphi_p\left[\left(\frac{1}{\Gamma(\mu)}\right)^{1-q}\varrho^{-1}e_2\right];$$

(A2) $$\int_0^1 (1-\tau)^{\mu-1}\tau\widehat{\psi}(\tau, e_1)\,\mathrm{d}\tau \geqslant \varphi_p\left[(\alpha-1)\left(\frac{\mu-1}{\Gamma(\mu)}\right)^{1-q}\varrho^{-1}e_1\right];$$

(A3) $$\int_0^1 (1-\tau)^{\mu-2}\widehat{\varphi}(\tau, e_3, e_5)\,\mathrm{d}\tau \leqslant \varphi_p\left[\left(\frac{1}{\Gamma(\mu)}\right)^{1-q}\varrho^{-1}e_5\right];$$

(A4) $$\int_{a^*}^{b^*} (1-\tau)^{\mu-1}\tau\widehat{\Psi}(\tau, e_3, e_4)\,\mathrm{d}\tau > \varphi_p\left[(\alpha-1){\sigma^*}^{-1}\left(\frac{\mu-1}{\Gamma(\mu)}\right)^{1-q}\varrho^{-1}e_3\right].$$

Here $\varrho = \int_0^1 J(s)\varphi_q(s^{\mu-1}(1-s))\,\mathrm{d}s$. *Then BVP* (1) *has at least three positive solutions* $\widehat{u}_1$, $\widehat{u}_2$, $\widehat{u}_3$ *with* $e_1-\vartheta \leqslant \|\widehat{u}_1\| \leqslant e_2$, $e_3-\vartheta \leqslant \|\widehat{u}_2\| \leqslant e_5$, $e_2-\vartheta \leqslant \|\widehat{u}_3\| \leqslant e_5$ *and*

$$\min_{t\in[a^*,b^*]} \widehat{u}_2(t) \geqslant e_3-\vartheta, \qquad \min_{t\in[a^*,b^*]} \widehat{u}_3(t) \leqslant e_3.$$

Proof. First, consider the following modified approximating BVP (MABVP for short):

$$\begin{aligned}
&D_{0+}^{\mu}\bigl(\varphi_p\bigl(-D_{0+}^{\alpha}u(t)\bigr)\bigr) + f\bigl(t, \chi_n(u-w)(t)\bigr) + k(t) = 0, \quad 0<t<1,\\
&u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \qquad D_{0+}^{\alpha}u(0) = D_{0+}^{\alpha}u(1) = 0,\\
&D_{0+}^{\beta}u(1) = \lambda\int_0^{\eta} D_{0+}^{\gamma}u(t)\,\mathrm{d}A(t),
\end{aligned}$$

where w is the same as that in Lemma 6, and

$$\chi_n(u) = \begin{cases} u, & u \geqslant \frac{1}{n},\\ \frac{1}{n}, & u < \frac{1}{n},\ n\in\mathbb{N}^+. \end{cases}$$

For $t\in[0,1]$, $n\in\mathbb{N}^+$, define operators T_n as follows:

$$(T_nu)(t) = \int_0^1 G(t,s)\varphi_q\left(\int_0^1 K_\mu(s,\tau)\bigl[f\bigl(\tau, \chi_n(u-w)(\tau)\bigr) + k(\tau)\bigr]\,\mathrm{d}\tau\right)\mathrm{d}s.$$

In the sequel, we will give the proof by the following three steps.

(I) We show that for any $(\alpha-1)\vartheta < r_1 < r_2$ and sufficiently large n, $T_n : (\overline{K}_{r_2} \setminus K_{r_1}) \to K$ is completely continuous.

First, we show that $T_n : (\overline{K}_{r_2} \setminus K_{r_1}) \to K$ is well defined. Let $u \in \overline{K}_{r_2} \setminus K_{r_1}$, $n \geqslant [1/r_1] + 1$. Then for $t \in [0,1]$, one has

$$u(t) - w(t) \geqslant \frac{1}{\alpha - 1} t^{\alpha-1} \|u\| - \vartheta t^{\alpha-1} \geqslant \left(\frac{r_1}{\alpha-1} - \vartheta\right) t^{\alpha-1} \geqslant 0.$$

Therefore,

$$\left(\frac{r_1}{\alpha-1} - \vartheta\right) t^{\alpha-1} \leqslant \max\left\{u(t) - w(t), \frac{1}{n}\right\} \leqslant r_2, \quad n \geqslant \left[\frac{1}{r_1}\right] + 1,$$

i.e.,

$$\left(\frac{r_1}{\alpha-1} - \vartheta\right) t^{\alpha-1} \leqslant \chi_n(u-w)(t) \leqslant r_2, \quad n \geqslant \left[\frac{1}{r_1}\right] + 1,\ t \in (0,1).$$

By Lemmas 5 and 2 we have

$$\begin{aligned} J(s) &\leqslant \frac{1}{\Gamma(\alpha)} + \frac{\lambda\Gamma(\alpha-\beta)}{\Gamma(\alpha-\gamma) - \lambda\Gamma(\alpha-\beta)\int_0^\eta t^{\alpha-\gamma-1}\,\mathrm{d}A(t)} \cdot \int\limits_0^\eta \frac{1}{\Gamma(\alpha)}\,\mathrm{d}A(t) \\ &\leqslant \frac{1}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)} \cdot \frac{\lambda\Gamma(\alpha-\beta)}{\Gamma(\alpha-\gamma) - \lambda\Gamma(\alpha-\beta)\int_0^\eta t^{\alpha-\gamma-1}\,\mathrm{d}A(t)} \cdot \bigl(A(\eta) - A(0)\bigr) \\ &\triangleq M_0 \quad \forall s \in [0,1]. \end{aligned}$$

This, together with (H1)–(H3) and Lemma 4 yields that

$$\begin{aligned} (T_n u)(t) &= \int\limits_0^1 G(t,s)\varphi_q\left(\int\limits_0^1 K_\mu(s,\tau)\bigl[f\bigl(\tau, \chi_n(u-w)(\tau)\bigr) + k(\tau)\bigr]\,\mathrm{d}\tau\right)\mathrm{d}s \\ &\leqslant \int\limits_0^1 J(s)\varphi_q\left(\int\limits_0^1 \frac{1}{\Gamma(\mu)} s^{\mu-1}(1-s)(1-\tau)^{\mu-2}\right. \\ &\qquad\qquad \left.\times \bigl[f\bigl(\tau, \chi_n(u-w)(\tau)\bigr) + k(\tau)\bigr]\,\mathrm{d}\tau\right)\mathrm{d}s \\ &\leqslant M_0\left(\frac{1}{\Gamma(\mu)}\right)^{q-1} \int\limits_0^1 \bigl[s^{\mu-1}(1-s)\bigr]^{q-1}\,\mathrm{d}s \\ &\quad \times \varphi_q\left(\int\limits_0^1 (1-\tau)^{\mu-2}\Bigl[\gamma_{r_1/(\alpha-1)-\vartheta,\, r_2}(\tau) + k(\tau)\Bigr]\,\mathrm{d}\tau\right) \\ &< +\infty. \end{aligned} \tag{12}$$

Besides, for any $u \in \overline{K}_{r_2} \setminus K_{r_1}$, $t \in [0,1]$, by (12) and Lemma 5 we have

$$(T_n u)(t) \leqslant \int_0^1 J(s)\varphi_q\left(\int_0^1 K_\mu(s,\tau)\left[f\left(\tau, \chi_n(u-w)(\tau)\right) + k(\tau)\right] \mathrm{d}\tau\right) \mathrm{d}s,$$

i.e.,

$$\|T_n u\| \leqslant \int_0^1 J(s)\varphi_q\left(\int_0^1 K_\mu(s,\tau)\left[f\left(\tau, \chi_n(u-w)(\tau)\right) + k(\tau)\right] \mathrm{d}\tau\right) \mathrm{d}s,$$

$$\begin{aligned}(T_n u)(t) &\geqslant \frac{1}{\alpha-1}t^{\alpha-1}\int_0^1 J(s)\varphi_q\left(\int_0^1 K_\mu(s,\tau)\left[f\left(\tau, \chi_n(u-w)(\tau)\right) + k(\tau)\right] \mathrm{d}\tau\right) \mathrm{d}s \\ &\geqslant \frac{1}{\alpha-1}\|T_n u\|. \end{aligned} \tag{13}$$

It follows from (12) and (13) that $T_n : (\overline{K}_{r_2} \setminus K_{r_1}) \to K$ is well defined.

Next, we prove that T_n is completely continuous. Given $D \subset \overline{K}_{r_2} \setminus K_{r_1}$, we deduce from (12) that $T_n(D)$ is uniformly bounded. It follows from Arzelà–Ascoli theorem, we need only to show the equicontinuity of $T_n(D)$. We have from (5), (6), (H3) for $t \in [0,1]$:

$$\begin{aligned}
&\left|(T_n u)'(t)\right| \\
&\quad = \left|\int_0^1 \frac{\partial}{\partial t}G(t,s)\varphi_q\left(\int_0^1 K_\mu(s,\tau)\left[f\left(\tau, \chi_n(u-w)(\tau)\right) + k(\tau)\right] \mathrm{d}\tau\right) \mathrm{d}s\right| \\
&\quad \leqslant \left|\left(\int_0^1 \frac{\partial}{\partial t}G_1(t,s) + \frac{(\alpha-1)\lambda\Gamma(\alpha-\beta)t^{\alpha-2}\int_0^\eta H(t,s)\,\mathrm{d}A(t)}{\Gamma(\alpha-\gamma) - \lambda\Gamma(\alpha-\beta)\int_0^\eta t^{\alpha-\gamma-1}\,\mathrm{d}A(t)}\right)\right. \\
&\qquad \left. \times \varphi_q\left(\int_0^1 K_\mu(s,\tau)\left[f\left(\tau, \chi_n(u-w)(\tau)\right) + k(\tau)\right] \mathrm{d}\tau\right) \mathrm{d}s\right| \\
&\quad \leqslant \left(\int_0^1 \frac{(\alpha-1)t^{\alpha-2}(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)} + \frac{(\alpha-1)\lambda\Gamma(\alpha-\beta)t^{\alpha-2}\int_0^\eta \frac{1}{\Gamma(\alpha)}\,\mathrm{d}A(t)}{\Gamma(\alpha-\gamma) - \lambda\Gamma(\alpha-\beta)\int_0^\eta t^{\alpha-\gamma-1}\,\mathrm{d}A(t)}\right) \\
&\qquad \times \varphi_q\left(\int_0^1 \frac{1}{\Gamma(\mu)}(1-\tau)^{\mu-2}\left(\gamma_{r_1/(\alpha-1)-\vartheta, r_2}(\tau) + k(\tau)\right) \mathrm{d}\tau\right) \\
&\quad \leqslant (\alpha-1)\left[\frac{1}{\Gamma(\alpha)} + \frac{\lambda\Gamma(\alpha-\beta)}{\Gamma(\alpha-\gamma) - \lambda\Gamma(\alpha-\beta)\int_0^\eta t^{\alpha-\gamma-1}\,\mathrm{d}A(t)} \cdot \frac{A(\eta)-A(0)}{\Gamma(\alpha)}\right] \\
&\qquad \times \left(\frac{1}{\Gamma(\mu)}\right)^{q-1}\varphi_q\left(\int_0^1 (1-\tau)^{\mu-2}\left(\gamma_{r_1/(\alpha-1)-\vartheta, r_2}(\tau) + k(\tau)\right) \mathrm{d}\tau\right) \triangleq M_1.
\end{aligned}$$

Then for any $t_1, t_2 \in [0,]$, we have

$$\left|T_n u(t_1) - T_n u(t_2)\right| = \left|\int\limits_{t_1}^{t_2} (T_n u)'(s)\,\mathrm{d}s\right| \leqslant M_1|t_1 - t_2| \quad \forall t_1, t_2 \in [0, 1].$$

Thus, $T_n(D)$ is equicontinuous. Therefore, we have proved that $T_n : (\overline{K}_{r_2} \setminus K_{r_1}) \to K$ is completely continuous.

For sufficiently large n, the same conclusion is valid for $T_n : \overline{K}_{e_5} \setminus K_{e_1} \to K$. Define $\zeta(u) = \min_{t\in[a^*,b^*]} u(t)$ for any $u \in K$. In the following, $K(\zeta, e_3, e_4)$, $\mathring{K}(\zeta, e_3, e_4)$, $K(\zeta, e_3, e_5)$ have the same meaning as those in (11).

(II) We demonstrate that for sufficiently large n, T_n has three fixed points.

First, we are in position to show that for sufficiently large n,

$$i\bigl(T_n, \mathring{K}(\zeta, e_3, e_5), \overline{K}_{e_5}\bigr) = 1. \tag{14}$$

Set $u_0(t) \equiv (e_3 + e_4)/2$. Then $u_0 \in \mathring{K}(\zeta, e_3, e_4)$, which means that $\mathring{K}(\zeta, e_3, e_4) \neq \emptyset$. If $u \in K(\zeta, e_3, e_4)$, we have $e_3 \leqslant \min_{t\in[a^*,b^*]} u(t) \leqslant \max_{t\in[0,1]} u(t) = \|u\| \leqslant e_4$. Thus, for $t \in [a^*, b^*]$, by Lemma 6 we know that $0 < (e_3 - \vartheta) \leqslant u(t) - w(t) \leqslant e_4$, and $(e_3 - \vartheta) \leqslant \max\{u(t) - w(t), 1/n\} \leqslant e_4 (n > N_1 = [1/e_3] + 1)$, which means

$$(e_3 - \vartheta) \leqslant \chi_n(u - w)(t) \leqslant e_4, \quad a^* < t < b^*,\ n > N_1.$$

It follows from Lemmas 4, 5 and (A4) that

$$\begin{aligned}
\zeta(T_n u) &= \min_{t\in[a^*,b^*]} (T_n u)(t) \geqslant \min_{t\in[a^*,b^*]} \frac{1}{\alpha - 1} t^{\alpha-1} \|T_n u\| = \sigma^* \|T_n u\| \\
&= \sigma^* \max_{t\in[0,1]} \int\limits_0^1 G(t, s)\varphi_q\Biggl(\int\limits_0^1 K_\mu(s, \tau)\bigl[f\bigl(\tau, \chi_n(u - w)(\tau)\bigr) + k(\tau)\bigr]\,\mathrm{d}\tau\Biggr)\,\mathrm{d}s \\
&\geqslant \sigma^* \max_{t\in[0,1]} \frac{1}{\alpha - 1} t^{\alpha-1} \int\limits_0^1 J(s)\varphi_q\Biggl(\int\limits_0^1 \frac{\mu - 1}{\Gamma(\mu)} s^{\mu-1}(1 - s)(1 - \tau)^{\mu-1}\tau \\
&\qquad \times \bigl[f\bigl(\tau, \chi_n(u - w)(\tau)\bigr) + k(\tau)\bigr]\,\mathrm{d}\tau\Biggr)\,\mathrm{d}s \\
&\geqslant \sigma^* \frac{1}{\alpha - 1} \left(\frac{\mu - 1}{\Gamma(\mu)}\right)^{q-1} \varrho \cdot \varphi_q\Biggl(\int\limits_{a^*}^{b^*} (1 - \tau)^{\mu-1}\tau\widehat{\Psi}(\tau, e_3, e_4)\,\mathrm{d}\tau\Biggr) > e_3.
\end{aligned}$$

If $u \in K(\zeta, e_3, e_5)$, then by the construction of cone K and Lemma 6 we know that $0 < (e_3/(\alpha - 1) - \vartheta)t^{\alpha-1} \leqslant u(t) - w(t) \leqslant e_5$, $t \in [0, 1]$, and $(e_3/(\alpha - 1) - \vartheta)t^{\alpha-1} \leqslant \max\{u(t) - w(t), 1/n\} \leqslant e_5$, $n > N_1$, which means

$$\left(\frac{e_3}{\alpha - 1} - \vartheta\right)t^{\alpha-1} \leqslant \chi_n(u - w)(t) \leqslant e_5, \quad 0 < t < 1,\ n > N_1. \tag{15}$$

This, together with Lemmas 4, 5 and (A3), means that

$$\begin{aligned}\|T_n u\| &= \max_{t\in[0,1]} \int_0^1 G(t,s)\varphi_q\left(\int_0^1 K_\mu(s,\tau)\bigl[f\bigl(\tau,\chi_n(u-w)(\tau)\bigr)+k(\tau)\bigr]\,\mathrm{d}\tau\right)\mathrm{d}s \\ &\leqslant \int_0^1 J(s)\varphi_q\left(\int_0^1 \frac{1}{\Gamma(\mu)}s^{\mu-1}(1-s)(1-\tau)^{\mu-2}\widehat{\varphi}(\tau,e_3,e_5)\,\mathrm{d}\tau\right)\mathrm{d}s \\ &= \left(\frac{1}{\Gamma(\mu)}\right)^{q-1}\int_0^1 J(s)\varphi_q\bigl(s^{\mu-1}(1-s)\bigr)\,\mathrm{d}s\cdot\varphi_q\left(\int_0^1(1-\tau)^{\mu-2}\widehat{\varphi}(\tau,e_3,e_5)\,\mathrm{d}\tau\right) \\ &\leqslant e_5, \quad n>N_1. \end{aligned} \tag{16}$$

Consequently, $T_n u \in \overline{K}_{e_5}$.

For $u \in K(\zeta, e_3, e_5)$ with $\|T_n u\| > e_4$, noticing that $e_4 \geqslant e_3\sigma^{*-1}$, we have $\|T_n u\| > e_3\sigma^{*-1}$. Therefore,

$$\zeta(T_n u) = \min_{t\in[a^*,b^*]}(T_n u)(t) \geqslant \sigma^*\|T_n u\| > \sigma^* e_3 \sigma^{*-1} = e_3.$$

Thus, for $n > N_1$, we know from Lemma 7 that (14) holds.

If $u \in \partial(K_{e_5})$, then $\|u\| = e_5$ and $(e_3/(\alpha-1))t^{\alpha-1} \leqslant (e_5/(\alpha-1))t^{\alpha-1} \leqslant u(t) \leqslant e_5$, $t \in [0,1]$. Thus, (15) holds. By (15), (A3), Lemmas 4 and 5, similar to the proof of (16), for any $n > N_1$, one gets

$$\begin{aligned}\|T_n u\| \leqslant \left(\frac{1}{\Gamma(\mu)}\right)^{q-1}&\int_0^1 J(s)\varphi_q\bigl(s^{\mu-1}(1-s)\bigr)\,\mathrm{d}s \\ &\times\varphi_q\left(\int_0^1(1-\tau)^{\mu-2}\widehat{\varphi}(\tau,e_3,e_5)\,\mathrm{d}\tau\right) \leqslant e_5 \quad \forall u\in\partial(K_{e_5}). \end{aligned}\tag{17}$$

If $u \in \partial(K_{e_2})$, then $\|u\| = e_2$ and $(e_2/(\alpha-1))t^{\alpha-1} \leqslant u(t) \leqslant e_2$, $t \in [0,1]$. Thus, we have $0 < (e_2/(\alpha-1) - \vartheta)t^{\alpha-1} \leqslant u(t) - w(t) \leqslant e_2$ for $t \in [0,1]$ and $(e_2/(\alpha-1) - \vartheta)t^{\alpha-1} \leqslant \max\{u(t) - w(t), 1/n\} \leqslant e_2$, $n > N_2 = [1/e_2] + 1$, i.e.,

$$\left(\frac{e_2}{\alpha-1}-\vartheta\right)t^{\alpha-1} \leqslant \chi_n(u-w)(t) \leqslant e_2, \quad 0<t<1,\ n>N_2.$$

By (A1), Lemmas 4 and 5, for any $n > N_2$, one has

$$\begin{aligned}\|T_n u\| &= \max_{t\in[0,1]} \int_0^1 G(t,s)\varphi_q\left(\int_0^1 K_\mu(s,\tau)\bigl[f\bigl(\tau,\chi_n(u-w)(\tau)\bigr)+k(\tau)\bigr]\,\mathrm{d}\tau\right)\mathrm{d}s \\ &\leqslant \int_0^1 J(s)\varphi_q\left(\int_0^1 \frac{1}{\Gamma(\mu)}s^{\mu-1}(1-s)(1-\tau)^{\mu-2}\widehat{\varphi}(\tau,e_2)\,\mathrm{d}\tau\right)\mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{1}{\Gamma(\mu)}\right)^{q-1} \varrho \cdot \varphi_q\left(\int_0^1 (1-\tau)^{\mu-2}\widehat{\varphi}(\tau, e_2)\,\mathrm{d}\tau\right)\\
&< e_2 \quad \forall u \in \partial(K_{e_2}).
\end{aligned} \tag{18}
$$

If $u \in \partial(K_{e_1})$, then $(e_1/(\alpha-1))t^{\alpha-1} \leqslant u(t) \leqslant e_1$, $t \in [0,1]$. Thus, we have $0 < (e_1/(\alpha-1)-\vartheta)t^{\alpha-1} \leqslant u(t)-w(t) \leqslant e_1$ for $t \in [0,1]$ and $(e_1/(\alpha-1)-\vartheta)t^{\alpha-1} \leqslant \max\{u(t)-w(t), 1/n\} \leqslant e_1$, $n > N_3 = [1/e_1]+1$, i.e.,

$$
\left(\frac{e_1}{\alpha-1} - \vartheta\right)t^{\alpha-1} \leqslant \chi_n(u-w)(t) \leqslant e_1, \quad 0<t<1,\ n>N_3.
$$

By (A2), Lemmas 4 Lemma 5, for any $n > N_3$, one gets

$$
\begin{aligned}
\|T_n u\| &= \max_{t\in[0,1]} \int_0^1 G(t,s)\varphi_q\left(\int_0^1 K_\mu(s,\tau)\big[f\big(\tau, \chi_n(u-w)(\tau)\big) + k(\tau)\big]\,\mathrm{d}\tau\right)\mathrm{d}s\\
&\geqslant \max_{t\in[0,1]} \frac{1}{\alpha-1}t^{\alpha-1}\\
&\quad \times \int_0^1 J(s)\varphi_q\left(\int_0^1 \frac{\mu-1}{\Gamma(\mu)} s^{\mu-1}(1-s)(1-\tau)^{\mu-1}\tau\widehat{\psi}(\tau, e_1)\,\mathrm{d}\tau\right)\mathrm{d}s\\
&= \frac{1}{\alpha-1}\left(\frac{\mu-1}{\Gamma(\mu)}\right)^{q-1} \varrho \cdot \varphi_q\left(\int_0^1 (1-\tau)^{\mu-1}\tau\widehat{\psi}(\tau, e_1)\,\mathrm{d}\tau\right)\\
&\geqslant e_1 \quad \forall u \in \partial(K_{e_1}).
\end{aligned} \tag{19}
$$

By (17), (18), (19) and Lemma 8, for any $n > N = \max\{N_1, N_2, N_3\}$, we know (14) and the following two equalities hold simultaneously:

$$
i(T_n, \overline{K}_{e_5} \setminus \overline{K}_{e_1}, \overline{K}_{e_5}) = 1, \qquad i(T_n, K_{e_2} \setminus \overline{K}_{e_1}, \overline{K}_{e_5}) = 1. \tag{20}
$$

It is clear, (A1) implies that T_n has no fixed point on $\partial(K_{e_2})$. In addition, for $u \in K(\zeta, e_3, e_4)$, we have that $\zeta(T_n u) > e_3$ and for $u \in K(\zeta, e_3, e_5)$ with $\|T_n u\| > e_4$, we also have that $\zeta(T_n u) > e_3$. This is to say, T_n has no fixed point on $K(\zeta, e_3, e_5) \setminus \mathring{K}(\zeta, e_3, e_5)$. Thus, for $n > N$, it follows from (14), (20) and the addition property of the topological degree that

$$
\begin{aligned}
&i\big(T_n, \overline{K}_{e_5} \setminus \big(K(\zeta, e_3, e_5) \cup \overline{K}_{e_2}\big), \overline{K}_{e_5}\big)\\
&\quad = i(T_n, \overline{K}_{e_5} \setminus \overline{K}_{e_1}, \overline{K}_{e_5}) - i(T_n, K_{e_2} \setminus \overline{K}_{e_1}, \overline{K}_{e_5})\\
&\qquad - i\big(T_n, \mathring{K}(\zeta, e_3, e_5), \overline{K}_{e_5}\big)\\
&\quad = -1.
\end{aligned}
$$

As a consequence, for $n > N$, T_n has at least three fixed points $u_{1n}^* \in K_{e_2} \setminus \overline{K}_{e_1}$, $u_{2n}^* \in \mathring{K}(\zeta, e_3, e_5)$, $u_{3n}^* \in \overline{K}_{e_5} \setminus (K(\zeta, e_3, e_5) \cup \overline{K}_{e_2})$ satisfying $e_1 \leqslant \|u_{1n}^*\| < e_2$, $e_3 \leqslant \|u_{2n}^*\| \leqslant e_5$, $e_2 \leqslant \|u_{3n}^*\| < e_5$ with $\min_{t\in[a^*,b^*]} u_{2n}^*(t) > e_3$, $\min_{t\in[a^*,b^*]} u_{3n}^*(t) < e_3$.

(III) We prove that BVP (1) has at least triple positive solutions.

Taking into account the construction of the cone K, for $t \in [0,1]$, $n > N$ $(i = 1,2,3)$, one has that

$$u_{in}^*(t) \geqslant \|u_{in}^*\| \frac{1}{\alpha-1} t^{\alpha-1} \geqslant \frac{e_1}{\alpha-1} t^{\alpha-1} \geqslant \vartheta t^{\alpha-1} \geqslant w(t) \tag{21}$$

and

$$u_{in}^*(t) = \int_0^1 G(t,s)\varphi_q\left(\int_0^1 K_\mu(s,\tau)\big[f\big(\tau, \chi_n(u_{in}^* - w)(\tau)\big) + k(\tau)\big]\,\mathrm{d}\tau\right)\mathrm{d}s. \tag{22}$$

It is easy to know from (H2) that $\{u_{in}^*\colon n > N\}$ $(i = 1,2,3)$ are bounded and equicontinuous on $[0,1]$. Thus, Arzelà–Ascoli theorem implies that there exists a subsequence N_0 of N and corresponding continuous functions u_i^* $(i = 1,2,3)$ such that u_{in}^* converges to u_i^* $(i = 1,2,3)$ uniformly on $[0,1]$ as $n \to \infty$ through N_0. Let $n \to \infty$ on both sides of (22), for $t \in [0,1]$, $i = 1,2,3$, one has

$$u_i^*(t) = \int_0^1 G(t,s)\varphi_q\left(\int_0^1 K_\mu(s,\tau)\big[f\big(\tau, (u_i^* - w)(\tau)\big) + k(\tau)\big]\,\mathrm{d}\tau\right)\mathrm{d}s \tag{23}$$

and

$$e_1 \leqslant \|u_1^*\| \leqslant e_2, \qquad e_3 \leqslant \|u_2^*\| \leqslant e_5, \qquad e_2 \leqslant \|u_3^*\| \leqslant e_5 \tag{24}$$

with

$$\min_{t\in[a^*,b^*]} u_2^*(t) \geqslant e_3, \qquad \min_{t\in[a^*,b^*]} u_3^*(t) \leqslant e_3. \tag{25}$$

It can be easily seen from (21) that $u_i^*(t) \geqslant (1/(\alpha-1))t^{\alpha-1}\|u_i^*\| \geqslant (e_1/(\alpha-1))t^{\alpha-1} \geqslant \vartheta t^{\alpha-1} \geqslant w(t)$ $(i = 1,2,3)$. Let $\widehat{u}_i(t) = u_i^*(t) - w(t)$, then we know from (23) that $\widehat{u}_i(t)$ $(i = 1,2,3)$ are positive solutions for BVP (1). Combined with (24), (25) and Lemma 6, we know that $e_1 - \vartheta \leqslant \|\widehat{u}_1\| \leqslant e_2$, $e_3 - \vartheta \leqslant \|\widehat{u}_2\| \leqslant e_5$, $e_2 - \vartheta \leqslant \|\widehat{u}_3\| \leqslant e_5$ and $\min_{t\in[a^*,b^*]} \widehat{u}_2(t) \geqslant e_3 - \vartheta$, $\min_{t\in[a^*,b^*]} \widehat{u}_3(t) \leqslant e_3$. □

4 An example

Consider the following singular fractional differential equation:

$$\begin{gathered} -D_{0+}^{7/4}\big(\varphi_3\big(-D_{0+}^{11/3}u(t)\big)\big) = g\big(t,u(t)\big) - \frac{1}{10^5\sqrt[4]{t}}, \quad 0 < t < 1, \\ u(0) = u'(0) = u''(0) = 0, \qquad D_{0+}^{2/3}u(1) = \frac{1}{2}\int_0^{3/4} D_{0+}^{3/2}u(t)\,\mathrm{d}A(t). \end{gathered} \tag{26}$$

Here $\alpha = 11/3$, $n = 4$, $\beta = 2/3$, $\gamma = 1/3$, $\mu = 7/4$, $\eta = 3/4$, $p = 3$, $q = 3/2$, $\lambda = 1/2$, $f(t, u(t)) = g(t, u(t)) - 1/(10^5 \sqrt[4]{t})$, $g(t, u(t)) = \theta(u(t))/(25 \sqrt[4]{t(1-t)})$, where

$$\theta(u) = \begin{cases} u^{1/2} + u^{-1/6}, & 0 < u \leqslant 1, \\ u^{13} + 1, & 1 < u \leqslant 6, \\ u^{1/2} + 6^{13} + 1 - \sqrt{6}, & u > 6, \end{cases} \quad \text{and} \quad A(t) = \begin{cases} 1, & t \in [0, \frac{1}{4}), \\ 2, & t \in [\frac{1}{4}, \frac{1}{2}), \\ 4t, & t \in [\frac{1}{2}, \frac{3}{4}), \\ 4, & t \in [\frac{3}{4}, 1]. \end{cases}$$

Next, we are in position to check all the conditions of Theorem 1. Direct calculation shows that $\lambda\Gamma(\alpha - \beta) \int_0^\eta t^{\alpha-\gamma-1} \,\mathrm{d}A(t) = 0.3804 < 2.7782 = \Gamma(\alpha - \gamma) = \Gamma(10/3)$, which means (H1) holds.

Since $\int_0^1 (1-\tau)^{\mu-2} k(\tau) \,\mathrm{d}\tau = 10^{-5} \int_0^1 (1-\tau)^{-1/4} \tau^{-1/4} \,\mathrm{d}\tau = 1.6944 \cdot 10^{-5} < +\infty$, we know that (H2) meets for $k(t) = 1/(10^5 \sqrt[4]{t})$. It is easy to check that (H3) is valid for $\gamma_{r_1/(\alpha-1), r_2}(t) = (1/(25 \sqrt[4]{t(1-t)}))[r_2^{1/2} + ((3/8) r_1 t^{8/3})^{-1/6} + r_2^{13} + r_2^{1/2} + 6^{13} + 1 - \sqrt{6}]$. Considering that $\int_0^\eta H(t,s) \,\mathrm{d}A(t) = H(1/4, s) + 4 \int_{1/2}^{3/4} H(t,s) \,\mathrm{d}t$, we have

$$\int_0^\eta H(t,s) \,\mathrm{d}A(t) = \begin{cases} 0.0946(1-s)^2 - 0.2492(\frac{1}{4} - s)^{7/3} \\ \quad -0.2991(\frac{3}{4} - s)^{10/3} + 0.2991(\frac{1}{2} - s)^{10/3}, & 0 \leqslant s \leqslant \frac{1}{4}, \\ 0.0946(1-s)^2 - 0.2991(\frac{3}{4} - s)^{10/3} \\ \quad +0.2991(\frac{1}{2} - s)^{10/3}, & \frac{1}{4} \leqslant s \leqslant \frac{1}{2}, \\ 0.0946(1-s)^2 - 0.2991(\frac{3}{4} - s)^{10/3}, & \frac{1}{2} \leqslant s \leqslant \frac{3}{4}, \\ 0.0946(1-s)^2, & \frac{3}{4} \leqslant s \leqslant 1. \end{cases}$$

This implies that

$$J(s) = \begin{cases} 0.6646s(1-s)^2 + 0.0394(1-s)^2 - 0.1039(\frac{1}{4} - s)^{7/3} \\ \quad -0.1247(\frac{3}{4} - s)^{10/3} + 0.1247(\frac{1}{2} - s)^{10/3}, & 0 \leqslant s \leqslant \frac{1}{4}, \\ 0.6646s(1-s)^2 + 0.0394(1-s)^2 - 0.1247(\frac{3}{4} - s)^{10/3} \\ \quad +0.1247(\frac{1}{2} - s)^{10/3}, & \frac{1}{4} \leqslant s \leqslant \frac{1}{2}, \\ 0.6646s(1-s)^2 + 0.0394(1-s)^2 - 0.1247(\frac{3}{4} - s)^{10/3}, & \frac{1}{2} \leqslant s \leqslant \frac{3}{4}, \\ 0.6646s(1-s)^2 + 0.0394(1-s)^2, & \frac{3}{4} \leqslant s \leqslant 1. \end{cases}$$

Direct calculation means that

$$\int_0^1 J(s) \varphi_q\bigl(s^{\mu-1}(1-s)\bigr) \,\mathrm{d}s = \int_0^1 J(s) s^{3/8} (1-s)^{1/2} \,\mathrm{d}s$$

$$\begin{aligned}
&= \int_0^1 0.6646 s^{11/8}(1-s)^{5/2} + 0.0394 s^{3/8}(1-s)^{5/2}\,\mathrm{d}s \\
&\quad - \int_0^{1/4} 0.1039\left(\frac{1}{4}-s\right)^{7/3} s^{3/8}(1-s)^{1/2}\,\mathrm{d}s \\
&\quad - \int_0^{3/4} 0.1247\left(\frac{3}{4}-s\right)^{10/3} s^{3/8}(1-s)^{1/2}\,\mathrm{d}s \\
&\quad + \int_0^{1/2} 0.1247\left(\frac{1}{2}-s\right)^{10/3} s^{3/8}(1-s)^{1/2}\,\mathrm{d}s \\
&\geqslant 0.0278 + 0.0058 - 0.1039\int_0^{1/4}\left(\frac{1}{4}-s\right)^{7/3}\mathrm{d}s \\
&\quad - 0.1247\int_0^{3/4}\left(\frac{3}{4}-s\right)^{10/3}\mathrm{d}s \\
&\approx 0.0250,
\end{aligned} \tag{27}$$

and

$$\begin{aligned}
&\int_0^1 J(s)\varphi_q\big(s^{\mu-1}(1-s)\big)\,\mathrm{d}s \\
&\quad = 0.0278 + 0.0058 - 0.1039\int_0^{1/4}\left(\frac{1}{4}-s\right)^{7/3} s^{3/8}(1-s)^{1/2}\,\mathrm{d}s \\
&\qquad - 0.1247\int_0^{3/4}\left(\frac{3}{4}-s\right)^{10/3} s^{3/8}(1-s)^{1/2}\,\mathrm{d}s \\
&\qquad + 0.1247\int_0^{1/2}\left(\frac{1}{2}-s\right)^{10/3} s^{3/8}(1-s)^{1/2}\,\mathrm{d}s \\
&\quad \leqslant 0.0278 + 0.0058 + 0.1247\int_0^{1/2}\left(\frac{1}{2}-s\right)^{10/3}\mathrm{d}s \approx 0.0350.
\end{aligned} \tag{28}$$

By simple computation, we have $\int_0^\eta t^{\alpha-\gamma-1}\,\mathrm{d}A(t) = \int_0^{3/4} t^{7/3}\,\mathrm{d}A(t) = (1/4)^{7/3} + 4\int_{1/2}^{3/4} t^{7/3}\,\mathrm{d}t = 0.3804$. Thus, $\Lambda = 8/3 + (1/(2.7782 - 0.3804))\cdot 0.3804 \approx 2.6667 + 0.1586 = 2.8253$, $\vartheta = (2.8253/4.0122)\cdot(1/0.9191)^{1/2}\cdot 0.1483\cdot 1.6944\cdot 10^{-5} \approx 1.8460\cdot 10^{-6}$. Take $e_1 = 10^{-3}$, $e_2 = 1/2$, $e_3 = 6$, $e_4 = 55$, $e_5 = 10^4$. Next, we check all the conditions of Theorem 1. We have

$$\begin{aligned}
&\int_0^1 (1-\tau)^{\mu-2}\widehat{\varphi}(\tau,e_2)\,\mathrm{d}\tau \\
&\quad= \int_0^1 (1-\tau)^{-1/4}\left[\max\left\{f(\tau,u)\colon\ 0.1875\tau^{8/3}\leqslant u\leqslant \frac{1}{2}\right\}+k(\tau)\right]\mathrm{d}\tau \\
&\quad= \frac{1}{25}\int_0^1 (1-\tau)^{-1/4}\frac{1}{\sqrt[4]{\tau(1-\tau)}}\left[\left(\frac{1}{2}\right)^{1/2}+\left(0.1875\tau^{8/3}\right)^{-1/6}\right]\mathrm{d}\tau \\
&\quad\approx 0.3052,
\end{aligned}$$

and $\varphi_q(\int_0^1(1-\tau)^{\mu-2}\widehat{\varphi}(\tau,e_2)\,\mathrm{d}\tau)=0.5524$. This, together with (28) and $(1/\Gamma(\mu))^{1-q}\times \varrho^{-1}e_2\geqslant (1/\Gamma(7/4))^{-1/2}0.0350^{-1}/2=13.6957$, implies that (A1) holds. It can be easily known from (27) that

$$\begin{aligned}
&\int_0^1 (1-\tau)^{\mu-1}\tau\widehat{\psi}(\tau,e_1)\,\mathrm{d}\tau \\
&\quad= \int_0^1 (1-\tau)^{\mu-1}\tau\left[\frac{\min\{u^{1/2}+u^{-1/6}\colon\ 3.7315\cdot 10^{-4}\tau^{8/3}\leqslant u\leqslant 10^{-3}\}}{25\sqrt[4]{\tau(1-\tau)}}\right]\mathrm{d}\tau \\
&\quad= \frac{1}{25}\int_0^1 (1-\tau)^{3/4}\tau\cdot\tau^{-1/4}\cdot(1-\tau)^{-1/4}\left[\left(3.7315\cdot 10^{-4}\tau^{8/3}\right)^{1/2}+\left(10^{-3}\right)^{-1/6}\right]\mathrm{d}\tau \\
&\quad\approx 0.0407,
\end{aligned}$$

and $(\alpha-1)((\mu-1)/\Gamma(\mu))^{1-q}\varrho^{-1}e_1\leqslant (8/3)\cdot((3/4)/\Gamma(7/4))^{-1/2}\cdot 0.0250^{-1}\cdot 10^{-3}\approx 0.1181$. Thus, $\varphi_q(\int_0^1(1-\tau)^{\mu-1}\tau\widehat{\psi}(\tau,e_1)\,\mathrm{d}\tau)=0.2017>0.1181$, which means that (A2) holds. We also have

$$\begin{aligned}
&\int_0^1 (1-\tau)^{\mu-2}\widehat{\varphi}(\tau,e_3,e_5)\,\mathrm{d}\tau \\
&\quad= \frac{1}{25}\int_0^1 (1-\tau)^{\mu-2}\tau\tau^{-1/4}(1-\tau)^{-1/4} \\
&\qquad\quad\times\max\left\{u^{1/2}+6^{13}+1-\sqrt{6}\colon\ 2.250\tau^{8/3}\leqslant u\leqslant 10^4\right\}\mathrm{d}\tau \\
&\quad= \frac{1}{25}\int_0^1 (1-\tau)^{-1/4}\tau^{-1/4}(1-\tau)^{-1/4}\,\mathrm{d}\tau\cdot\left(10^2+6^{13}+1-\sqrt{6}\right) \\
&\quad< \frac{1}{25}\cdot 2.3963\cdot\left(6^{13}+101\right)\approx 1.2519\cdot 10^9,
\end{aligned}$$

and $\varphi_q(\int_0^1 (1-\tau)^{\mu-2}\widehat{\varphi}(\tau,e_3,e_5)\,\mathrm{d}\tau) = (1.2519\cdot 10^9)^{1/2} \approx 3.5382\cdot 10^4$. Considering (28) and $(1/\Gamma(\mu))^{1-q}\varrho^{-1}e_5 \geqslant (1/\Gamma(7/4))^{-1/2}0.0350^{-1}\cdot 10^4 \approx 2.2391\cdot 10^5$, we know that (A3) is valid. Take $a^* = 2/3$, $b^* = 1$, then $\sigma^* = \min_{t\in[a^*,b^*]} t^{\alpha-1}/(\alpha-1) = (3/8)\cdot(2/3)^{8/3} \approx 0.1272$, $\sigma^{*-1} \approx 7.8616$. Thus, $\sigma^{*-1}e_3 = 47.1696 < 55 = e_4$. Finally, we get from (27) that

$$\begin{aligned}
&\int_{a^*}^{b^*} (1-\tau)^{\mu-1}\tau\widehat{\Psi}(\tau,e_3,e_4)\,\mathrm{d}\tau \\
&\quad = \int_{2/3}^{1} (1-\tau)^{3/4}\tau\left[\frac{\min\{u^{13}+1\colon\ 6-1.8460\cdot 10^{-6}\leqslant u\leqslant 55\}}{25\sqrt[4]{\tau(1-\tau)}}\right]\mathrm{d}\tau \\
&\quad \geqslant \frac{1}{25}\int_{2/3}^{1} (1-\tau)^{1/2}\tau^{3/4}\,\mathrm{d}\tau\cdot 6^{13} \\
&\quad \geqslant \frac{1}{25}\cdot\left(\frac{2}{3}\right)^{3/4}\int_{2/3}^{1} (1-\tau)^{1/2}\,\mathrm{d}\tau\cdot 6^{13} \approx 4.9453\cdot 10^7,
\end{aligned}$$

$\varphi_q(\int_{a^*}^{b^*}(1-\tau)^{\mu-1}\tau\widehat{\Psi}(\tau,e_3,e_4)\,\mathrm{d}\tau) \approx 7.0323\cdot 10^3$, and $(\alpha-1)\sigma^{*-1}((\mu-1)/\Gamma(\mu))^{1-q}\times \varrho^{-1}e_3 \leqslant (8/3)\cdot((3/4)/\Gamma(7/4))^{-1/2}\cdot 7.8616\cdot 0.0250^{-1}\cdot 6 \approx 5.5697\cdot 10^3$. Hence, (A4) is checked. It follows from Theorem 1 that FPDE (26) has at least three positive solutions $\widehat{u}_1$, $\widehat{u}_2$, $\widehat{u}_3$ with $9.9815\cdot 10^{-4} \leqslant \|\widehat{u}_1\| \leqslant 1/2$, $6 \leqslant \|\widehat{u}_2\| \leqslant 10^4$, $0.5000 \leqslant \|\widehat{u}_3\| \leqslant 10^4$ with $\min_{t\in[2/3,1]}\widehat{u}_2(t) \geqslant 6$, $\min_{t\in[2/3,1]}\widehat{u}_3(t) \leqslant 6$.

Acknowledgment. The authors would like to thank the referee for his/her valuable comments and suggestions.

References

1. B. Ahmad, M. Alghanmi, S.K. Ntouyas, A. Alsaedi, Fractional differential equations involving generalized derivative with Stieltjes and fractional integral boundary conditions, *Appl. Math. Lett.*, **84**:111–117, 2018.
2. B. Ahmad, S. K. Ntouyas, J. Tariboon, Nonlocal fractional-order boundary value problems with generalized Riemann–Liouville integral boundary conditions, *J. Comput. Anal. Appl.*, **23**(7):1281–1296, 2017.
3. R.I. Avery, A generalization of the Leggett–Williams fixed point theorem, *Math. Sci. Res. Hot-Line*, **3**(7):9–14, 1999.
4. R.I. Avery, J. Henderson, D. O'Regan, Four functionals fixed point theorem, *Math. Comput. Model.*, **48**(7–8):1081–1089, 2008.
5. R.I. Avery, J. Henderson, D. O'Regan, Six functionals fixed point theorem, *Commun. Appl. Anal.*, **12**(1):69–82, 2008.

6. Z. Bai, H. Lv, Positive solutions for boundary value problem of nonlinear fractional differential equation, *J. Math. Anal. Appl.*, **311**:495–505, 2005.
7. A. Cabada, G. Wang, Positive solutions of nonlinear fractional differential equations with integral boundary value conditions, *J. Math. Anal. Appl.*, **389**(1):403–411, 2012.
8. Y. Cui, W. Ma, Q. Sun, X. Su, New uniqueness results for boundary value problem of fractional differential equation, *Nonlinear Anal. Model. Control*, **23**(1):31–39, 2018.
9. P. Eloe, J. Neugebauer, Smallest eigenvalues for a right focal boundary value problem, *Fract. Calc. Appl. Anal.*, **19**(1):11–18, 2016.
10. C.S. Goodrich, Coupled systems of boundary value problems with nonlocal boundary conditions, *Appl. Math. Lett.*, **41**:17–22, 2015.
11. J. Graef, L. Kong, B. Yang, Positive solutions for a semipositone fractional boundary value problem with a forcing term, *Fract. Calc. Appl. Anal.*, **15**(1):8–24, 2012.
12. J. Graef, L. Kong, B. Yang, Positive solutions for a fractional boundary value problem, *Appl. Math. Lett.*, **56**:49–55, 2016.
13. M. Günendi, Ï. Yaslan, Positive solutions of higher-order nonlinear multi-point fractional equations with integral boundary conditions, *Fract. Calc. Appl. Anal.*, **19**(4):989–1009, 2016.
14. D. Guo, V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, San Diego, CA, 1988.
15. L. Guo, L. Liu, Unique iterative positive solutions for a singular p-Laplacian fractional differential equation system with infinite-point boundary conditions, *Bound. Value Probl.*, **2019**(1): 113, 2019.
16. L. Guo, L. Liu, Y. Wu, Existence of positive solutions for singular fractional differential equations with infinite-point boundary conditions, *Nonlinear Anal. Model. Control*, **21**(5):635–650, 2016.
17. L. Guo, L. Liu, Y. Wu, Iterative unique positive solutions for singular p-Laplacian fractional differential equation system with several parameters, *Nonlinear Anal. Model. Control*, **23**(2): 182–203, 2018.
18. X. Hao, H. Sun, L. Liu, Existence results for fractional integral boundary value problem involving fractional derivatives on an infinite interval, *Math. Meth. Appl. Sci.*, **41**(16):6984–6996, 2018.
19. X. Hao, H. Sun, L. Liu, D. Wang, Positive solutions for semipositone fractional integral boundary value problem on the half-line, *Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat., RACSAM*, **113**(4):3055–3067, 2019.
20. X. Hao, H. Wang, Positive solutions of semipositone singular fractional differential systems with a parameter and integral boundary conditions, *Open Math.*, **16**(1):581–596, 2018.
21. X. Hao, L. Zhang, Positive solutions of a fractional thermostat model with a parameter, *Symmetry*, **11**(1):122, 2019.
22. X. Hao, L. Zhang, L. Liu, Positive solutions of higher order fractional integral boundary value problem with a parameter, *Nonlinear Anal. Model. Control*, **24**(2):210–223, 2019.
23. X. Hao, M. Zuo, L. Liu, Multiple positive solutions for a system of impulsive integral boundary value problems with sign-changing nonlinearities, *Appl. Math. Lett.*, **82**:24–31, 2018.

24. J. Henderson, R. Luca, Positive solutions for a system of nonlocal fractional boundary value problems, *Fract. Calc. Appl. Anal.*, **16**(4):985–1008, 2013.

25. J. Henderson, R. Luca, Existence of positive solutions for a singular fractional boundary value problem, *Nonlinear Anal. Model. Control*, **22**(1):99–114, 2017.

26. J. Henderson, R. Luca, Systems of Riemann–Liouville fractional equations with multi-point boundary conditions, *Appl. Math. Comput.*, **309**:303–323, 2017.

27. D. Jiang, C. Yuan, The positive properties of the green function for Dirichlet-type boundary value problems of nonlinear fractional differential equations and its application, *Nonlinear Anal., Theory Methods Appl.*, **72**(2):710–719, 2010.

28. R. Leggett, L. Williams, Multiple positive fixed points of nonlinear operators on ordered Banach spaces, *Indiana Univ. Math. J.*, **28**:673–688, 1979.

29. L. Liu, X. Hao, Y. Wu, Positive solutions for singular second order differential equations with integral boundary conditions, *Math. Comput. Model.*, **57**(3-4):836–847, 2013.

30. L. Liu, F. Sun, X. Zhang, Y. Wu, Bifurcation analysis for a singular differential system with two parameters via to topological degree theory, *Nonlinear Anal. Model. Control*, **22**(1):31–50, 2017.

31. R. Luca, On a class of nonlinear singular Riemann–Liouville fractional differential equations, *Results Math.*, **73**(3):125, 2018.

32. R. Luca, Positive solutions for a system of fractional differential equations with p-Laplacian operator and multi-point boundary conditions, *Nonlinear Anal. Model. Control*, **23**(5):771–801, 2018.

33. S. Padhi, J. Graef, S. Pati, Multiple positive solutions for a boundary value problem with nonlinear nonlocal Riemann–Stieltjes integral boundary conditions, *Fract. Calc. Appl. Anal.*, **21**(3):716–745, 2018.

34. R. Pu, X. Zhang, P. Cui, Y.and Li, W. Wang, Positive solutions for singular semipositone fractional differential equation subject to multipoint boundary conditions, *J. Funct. Spaces*, **2017**:15, 2017.

35. F. Wang, L. Liu, D. Kong, Y. Wu, Existence and uniqueness of positive solutions for a class of nonlinear fractional differential equations with mixed-type boundary value conditions, *Nonlinear Anal. Model. Control*, **24**(1):73–94, 2019.

36. F. Wang, L. Liu, Y. Wu, Iterative unique positive solutions for a new class of nonlinear singular higher order fractional differential equations with mixed-type boundary value conditions, *J. Inequal. Appl.*, **2019**(1):210, 2019.

37. J.R.L. Webb, G. Infante, Positive solutions of nonlocal boundary value problems: A unified approach, *J. Lond. Math. Soc.*, **74**(3):673–693, 2006.

38. J.R.L. Webb, G. Infante, Existence of positive solutions for a thermostat model, *J. Lond. Math. Soc.*, **13**(2):923–938, 2012.

39. J. Wu, X. Zhang, L. Liu, Y. Wu, Y. Cui, Convergence analysis of iterative scheme and error estimation of positive solution for a fractional differential equation, *Math. Model. Anal.*, **23**(4): 611–626, 2018.

40. X. Zhang, Positive solutions for a class of singular fractional differential equation with infinite-point boundary value conditions, *Appl. Math. Lett.*, **39**:22–27, 2015.

41. X. Zhang, J. Jiang, Y. Wu, Y. Cui, Existence and asymptotic properties of solutions for a nonlinear Schrödinger elliptic equation from geophysical fluid flows, *Appl. Math. Lett.*, **90**: 229–237, 2019.
42. X. Zhang, L. Liu, Y. Wu, Existence results for multiple positive solutions of nonlinear higher order perturbed fractional differential equations with derivatives, *Appl. Math. Comput.*, **219**(4):1420–1433, 2012.
43. X. Zhang, L. Liu, Y. Wu, The uniqueness of positive solution for a singular fractional differential system involving derivatives, *Commun. Nonlinear Sci. Numer. Simul.*, **18**(6):1400–1409, 2013.
44. X. Zhang, L. Liu, Y. Wu, Y. Cui, The existence and nonexistence of entire large solutions for a quasilinear Schrödinger elliptic system by dual approach, *J. Math. Anal. Appl.*, **464**(2):1089–1106, 2018.
45. X. Zhang, Z. Shao, Q. Zhong, Positive solutions for semipositone $(k, n-k)$ conjugate boundary value problems with singularities on space variables, *Appl. Math. Lett.*, **72**:50–57, 2017.
46. X. Zhang, Z. Shao, Q. Zhong, Z. Zhao, Triple positive solutions for semipositone fractional differential equations m-point boundary value problems with singularities and $p{-}q$-order derivatives, *Nonlinear Anal. Model. Control*, **23**(6):889–903, 2018.
47. X. Zhang, L. Wang, Q. Sun, Existence of positive solutions for a class of nonlinear fractional differential equations with integral boundary conditions and a parameter, *Appl. Math. Comput.*, **226**:708–718, 2014.
48. X. Zhang, Y. Wu, L. Caccetta, Nonlocal fractional order differential equations with changing-sign singular perturbation, *Appl. Math. Modelling*, **39**(21):6543–6552, 2015.
49. X. Zhang, Y. Wu, Y. Cui, Existence and nonexistence of blow-up solutions for a Schrödinger equation involving a nonlinear operator, *Appl. Math. Lett.*, **82**:85–91, 2018.
50. X. Zhang, Q. Zhong, Uniqueness of solution for higher-order fractional differential equations with conjugate type integral conditions, *Fract. Calc. Appl. Anal.*, **20**(6):1471–1484, 2017.
51. X. Zhang, Q. Zhong, Triple positive solutions for nonlocal fractional differential equations with singularities both on time and space variables, *Appl. Math. Lett.*, **80**:12–19, 2018.
52. B. Zhu, L. Liu, Periodic boundary value problems for fractional semilinear integro-differential equations with non-instantaneous impulses, *Bound. Value Probl.*, **2018**(1):128, 2018.
53. B. Zhu, L. Liu, Y. Wu, Local and global existence of mild solutions for a class of nonlinear fractional reaction-diffusion equations with delay, *Appl. Math. Lett.*, **61**:73–79, 2016.
54. B. Zhu, L. Liu, Y. Wu, Local and global existence of mild solutions for a class of semilinear fractional integro-differential equations, *Fract. Calc. Appl. Anal.*, **20**(6):1338–1355, 2017.
55. Y. Zou, G. He, On the uniqueness of solutions for a class of fractional differential equations, *Appl. Math. Lett.*, **74**:68–73, 2017.