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Persistence of nonautonomous logistic system with time-varying delays and impulsive perturbations*

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Abstract. In this paper, we develop the impulsive control theory to nonautonomous logistic system with time-varying delays. Some sufficient conditions ensuring the persistence of nonautonomous logistic system with time-varying delays and impulsive perturbations are derived. It is shown that the persistence of the considered system is heavily dependent on the impulsive perturbations. The proposed method of this paper is completely new. Two examples and the simulations are given to illustrate the proposed method and results.

Keywords: nonautonomous logistic system, persistence, impulsive perturbations, time-varying delays, impulsive control theory.

1 Introduction

In recent years, for modelling the dynamics of some biological populations, several delay differential systems of logistic type have been proposed and studied by many authors; see [2, 4, 5, 7, 9, 13, 19, 29, 31]. The classical logistic system with time delay can be described as follows:

$$\dot{N}(t) = N(t)r\left[1 - \frac{N(t-\tau)}{K}\right], \quad t \geqslant 0, \tag{1}$$

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where r, K, τ are some positive numbers. Hutchinson [13] has proposed that system (1) can be applied to simulate the dynamical behavior of a single species population that grows to a saturation level K with reproductive rate r. The term $[1-N(t-\tau)/K]$ denotes a density-dependent feedback term which chooses τ units of time to respond to changes in the population density represented in (1) by N. System (1) and its generalized forms have been studied in many applications. We refer to the monographs [9, 15] for detailed information.

On the other hand, it has been shown that most biological populations are usually affected by the outside environments such as weather variations, human activities (planting and harvesting), and some other factors [1, 15, 17, 26]. From the mathematical point of view, it is very essential and significant to study the dynamics of the population models under external influences. These influences are usually considered continuously by adding some items to the right hand of the models [3, 6, 11, 14], whereas one may note that there are many cases that cannot do as we like, such as in the real world it is unrealistic for fisherman to fish the whole day, and in fact, they only fish for some time. Besides, the seasons and weather variations will also affect the fishing. In this sense, it is significant to consider the discontinuous harvesting, i.e, impulsive harvesting [8, 38]. In addition, continuous changes in environmental parameters such as temperature or rainfall can also produce some discontinuous outbreaks in the biological populations. Such kind of problems can be described by impulsive differential systems, such as [10, 12, 18, 20–22, 28, 33, 35], which describe the evolution processes characterized in that they are transient and at certain moments to undergo mutation. Systems with impulses have been widely applied to many fields such as inspection process in operations research, drug administration, aircraft control, and secure communication [17, 21]. In recent years, some impulsive differential systems have been introduced into population dynamics related to disease chemotherapy [16], vaccination [37], population ecology [1, 23], and other places [27]. In [1], Bainov and Simeonov considered the nonautonomous impulsive logistic system

$$\begin{aligned} \dot{N}(t) &= N(t)r(t)\left[1-\frac{N(t)}{K(t)}\right], \quad t\in[t_{k-1},t_k), \\ N(t_k)-N\big(t_k^-\big) &= b_kN\big(t_k^-\big), \end{aligned} \tag{2}$$

where r, K, b_k are some periodic functions. When $b_k>0$, disturbance means planting, and $b_k<0$ means harvesting. Some sufficient conditions for the existence and asymptotic stability of periodic solution were obtained. Then the results were extended by Liu and Chen [24]. Considering the effects of time delay, Sun and Chen [30] further studied the dynamics of the following impulsive delay logistic model:

$$\begin{aligned} \dot{N}(t) &= N(t)r(t)\left[1-\frac{N(t-\tau)}{K(t)}\right], \quad t\in[t_{k-1},t_k), \\ N(t_k)-N\big(t_k^-\big) &= b_kN\big(t_k^-\big), \end{aligned} \tag{3}$$

where $\tau>0$ is a real constant. However, one may note that the results in [24, 25, 30, 32, 34, 39] are only focused on the investigation of dynamics of the periodic logistic systems with periodic impulsive perturbations, which cannot be applied to the general impulsive logistic systems. It is considered that, for fishery management and many other harvesting situations, it is unreasonable to assume that the harvesting rate (i.e., b_k) is periodic (it may be dependent on the population density), and sometimes it is also difficult to guarantee the harvesting time is periodic due to human factor and weather variations. Another example, considering the metapopulation models, a species may migrate from one place to another place according to the seasons. The impulsive perturbations are regarded as the death rate in the migration due to outside environments. Obviously, it is unreasonable to assume that the impulsive perturbations are periodic. Owing to the practical significance, it is necessary to study the dynamics of logistic system with nonperiodic impulsive perturbations or nonperiodic logistic system. Recently, Yang et al. [36] investigated the permanence of infinite delay impulsive logistic system with nonperiodic condition.

Inspired by the above discussions, this article will consider the delay logistic system governed by the following nonautonomous system:

$$\begin{aligned} \dot N(t) &= N(t)r(t)\left[1-\frac{N(t)}{K_1(t)}-\frac{N(t-\tau(t))}{K_2(t)}\right], \quad t\in[t_{k-1},t_k),\\ N(t_k)-N\left(t_k^-\right) &= I_k\left(N\left(t_k^-\right)\right), \end{aligned} \tag{4}$$

where N denotes population density at time t, K_1, K_2 the carrying capacity, r the intrinsic growth rate of population, and τ the time needed for immature individual to mature, I_k the magnitude of the impulse effects on the population. Obviously, system (4) includes systems (1)–(3) as the special cases. We will discuss the effects of impulsive perturbations such as harvesting and planting and establish conditions for persistence of system (4). Our result shows that the persistence of system (4) can be guaranteed if the impulsive functions I_k vary in a certain degree and the lower bound of the impulsive interval is greater than a certain constant. Two numerical simulations will prove the effectiveness and novelty of the approach we obtained. In addition, it should be pointed out that our developed result is different from the usual methods in other literatures and is very practical.

This paper is organized as follows. In the next section, we introduce some preliminary knowledge. A number of important lemmas and our main result are presented in Section 3. Several examples and the simulations are given to illustrate the effectiveness and novelty of the proposed results in Section 4. Conclusions are given in Section 5.

2 Preliminaries

Notations. Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}_+$ positive real numbers, and $\mathbb{Z}_+$ positive integers. For any interval $S\subseteq\mathbb{R}$, set $C(S,\mathbb{R})=\{\psi: S\to\mathbb{R}$ is continuous$\}$, $PC(S,\mathbb{R})=\{\psi: S\to\mathbb{R}$ is continuous everywhere except at finite number of points t at which $\psi(t^+)$, $\psi(t^-)$ exist and $\psi(t^+)=\psi(t)\}$. In particular, for given $\tau>0$, let PC_τ be an open set in $PC([-\tau,0],\mathbb{R}_+)$. $[\cdot]^*$ denotes the integer function. The impulse times t_k, $k\in\mathbb{Z}_+$, satisfy $0\leqslant t_0<t_1<\cdots<t_k\to+\infty$ as $k\to+\infty$.

By a simple change, system (4) endowed with initial value may be rewritten as

$$\begin{aligned} &\dot{x}(t) = x(t)\big[r(t) - a(t)x(t) - b(t)x\big(t - \tau(t)\big)\big], \quad t \in [t_{k-1}, t_k),\\ &x(t_k) - x\big(t_k^-\big) = I_k\big(x\big(t_k^-\big)\big), \quad k \in \mathbb{Z}_+,\\ &x(t_0 + s) = \phi(s), \quad -\tau \leqslant s \leqslant 0, \end{aligned} \tag{5}$$

where $\phi \in PC_\tau$ and $0 \leqslant \tau(t) \leqslant \tau$, τ is a given positive constant. $x(t_k^-)$ and $x(t_k)$ (i.e., $N(t_k^{\mp})$) are numbers denoting the densities of population before and after impulsive effect at the moments t_k, respectively. $I_k : \mathbb{R}_+ \to \mathbb{R}$ are some continuous functions characterizing the gain of the impulse at the instant t_k and satisfy $I_k(s) + s > 0$ for any $s \in \mathbb{R}_+, k \in \mathbb{Z}_+$. In particular, when $I_k > 0$, the perturbation means planting of the species, while $I_k < 0$ means harvesting. r, a (i.e., r/K_1), and b (i.e., r/K_2) $: \mathbb{R}_+ \to \mathbb{R}_+$ are continuous functions, which have positive upper-lower bounds and are natural for biological meanings. By the basic theories of IFDEs in [21], system (5) has a unique solution on $[-\tau, \infty)$. Next, we set that the solution of system (5) is denoted by $x(t) = x(t, t_0, \phi)$ with the initial value (t_0, ϕ).

Given a function g, which is continuous, bounded, and defined on $J \in \mathbb{R}$, we set

$$g^I \doteq \inf_{s \in J} g(s), \qquad g^S \doteq \sup_{s \in J} g(s).$$

Definition 1. Assume that there exist positive constants m and M such that every solution $x(t) = x(t, t_0, \phi)$ of system (5) satisfies

$$0 < m \leqslant \liminf_{t \to +\infty} x(t) \leqslant \limsup_{t \to +\infty} x(t) \leqslant M.$$

Then system (5) is said to be persistent.

In the following, we will focus on the persistence of system (5).

3 Persistence results

We firstly present two lemmas. In particular, Lemma 2 plays an important part in the investigation of the permanence of system (5).

Lemma 1. *The set* $\mathbb{R}_+$ *is the positively invariant set of system* (5).

Proof. Note that for given $k \in \mathbb{Z}_+$, $I_k(s) + s > 0$ for all $s \in \mathbb{R}_+$. The proof of Lemma 1 is obvious and omitted here. □

Lemma 2. *Let there exist scalars* $\delta > 1$ *and* $\rho > 1$ *such that*

$$\begin{aligned} &\frac{1-\rho}{\rho} s \leqslant I_k(s) \leqslant (\delta - 1)s, \quad s > 0,\\ &\inf_{k \in \mathbb{Z}_+} \{t_k - t_{k-1}\} > \frac{\ln \rho}{r^I}. \end{aligned} \tag{6}$$

Then the set $\Omega = \{x \in \mathbb{R}_+\colon\ 0 < m < x < M\}$ *is the ultimately bounded set of system* (5) *in which* M *and* m *satisfy*

$$M > \delta^{\tau/\mu+4} \frac{\frac{\ln\delta}{\mu} + r^S}{a^I + b^I} \max\left\{1,\ \exp\left(\frac{r^S b^I - a^I \frac{\ln\delta}{\mu}}{a^I + b^I}\tau\right)\right\},$$
$$m < \frac{r^I - \frac{\ln\rho}{\mu}}{a^S + b^S} \frac{\exp(-((a^S + b^S)M - r^I)\tau)}{\rho^{\tau/\mu+3}} \min\left\{\frac{1}{\rho},\ \frac{1}{\delta^{\tau/\mu+3}} \exp\bigl(-r^S\tau\bigr)\right\},$$

where $\mu \doteq \inf_{k\in\mathbb{Z}_+}\{t_k - t_{k-1}\}$.

Proof. Consider the following two auxiliary definitions:

$$\underline{M} = \frac{\frac{\ln\delta}{\mu} + r^S}{a^I + b^I}, \qquad \overline{m} = \frac{r^I - \frac{\ln\rho}{\mu}}{a^S + b^S}.$$

From the definitions of M and $\underline{M}$, there exists a constant $\varepsilon > 0$ such that

$$M > \delta^{\tau/\mu+4}(\underline{M} + \varepsilon) \max\bigl\{1,\ \exp\bigl(\bigl(r^S - a^I(\underline{M} + \varepsilon)\bigr)\tau\bigr)\bigr\}. \tag{7}$$

Step 1. First, we prove that there is a constant $T_1 \geqslant t_0$ such that $x(t) < M$, $t \geqslant T_1$.

To do this, we need first claim that there exists a constant $T_0 \geqslant t_0$ such that $x(T_0) < \underline{M} + \varepsilon$. We use the counter-evidence method, i.e., assume that $x(t) \geqslant \underline{M} + \varepsilon$ for all $t \geqslant t_0$. Then it follows from system (5), the definition of $\underline{M}$, and Lemma 1 that

$$\begin{aligned} \dot{x}(t) &\leqslant x(t)\bigl[r^S - (a^I + b^I)(\underline{M} + \varepsilon)\bigr] \\ &= -\mathcal{A}x(t), \quad t \in [t_{k-1}, t_k) \cap [t_0 + \tau, \infty),\ k \in \mathbb{Z}_+, \end{aligned} \tag{8}$$

where $\mathcal{A} = \ln\delta/\mu + (a^I + b^I)\varepsilon > 0$. Without loss of generality, we assume that $t_0 + \tau \in [t_m, t_{m+1})$ for some $m \in \mathbb{Z}_+$. Then by (6) and (8), we get

$$\begin{aligned} x(t_{m+1}^-) &\leqslant x(t_0 + \tau)\exp\bigl(-\mathcal{A}(t_{m+1} - t_0 - \tau)\bigr) \leqslant x(t_0 + \tau), \\ x(t_{m+2}^-) &\leqslant x(t_{m+1})\exp\bigl(-\mathcal{A}(t_{m+2} - t_{m+1})\bigr) \leqslant \delta\exp(-\mathcal{A}\mu)x(t_0 + \tau), \\ x(t_{m+3}^-) &\leqslant x(t_{m+2})\exp\bigl(-\mathcal{A}(t_{m+3} - t_{m+2})\bigr) \leqslant \bigl[\delta\exp(-\mathcal{A}\mu)\bigr]^2 x(t_0 + \tau), \\ &\cdots \\ x(t_{m+j}^-) &\leqslant \bigl[\delta\exp(-\mathcal{A}\mu)\bigr]^{j-1} x(t_0 + \tau), \quad j \in \mathbb{Z}_+. \end{aligned}$$

In view of the definition of $\mathcal{A}$, it can be deduced that

$$\begin{aligned} x(t_{m+j}) &\leqslant \delta x\bigl(t_{m+j}^-\bigr) \leqslant \delta\bigl[\delta\exp(-\mathcal{A}\mu)\bigr]^{j-1} x(t_0 + \tau) \\ &\leqslant \delta\exp\bigl(-(j-1)\bigl(a^I + b^I\bigr)\varepsilon\mu\bigr)x(t_0 + \tau) \to 0 \quad \text{as } j \to +\infty, \end{aligned}$$

which is a contradiction with our previous assumption that $x(t) \geqslant \underline{M} + \varepsilon$ for $t \geqslant t_0$. Hence, there exists a constant $T_0 \geqslant t_0$ such that $x(T_0) < \underline{M} + \varepsilon$. Next, we will show that

$x(t) < M$ for all $t \geqslant T_0$. Otherwise, then there exists $t' > T_0$ such that $x(t') \geqslant M$. Let $\bar{t} = \inf\{t \in [T_0, t')\colon x(t) \geqslant M\}$, then it follows that $x(\bar{t}^+) \geqslant M$ and $M/\delta \leqslant x(\bar{t}^-) \leqslant M$. Moreover, we know that $\bar{t} > T_0$ since $x(T_0) < \underline{M} + \varepsilon$. Note that $M > \delta(\underline{M} + \varepsilon)$, we further determine a definition that $\underline{t} = \sup\{t \in [T_0, \bar{t})\colon x(t) \leqslant \underline{M} + \varepsilon\}$. Then it holds that $x(\underline{t}^-) \leqslant \underline{M} + \varepsilon$ and $\underline{M} + \varepsilon \leqslant x(\underline{t}^+) \leqslant \delta(\underline{M} + \varepsilon)$. Moreover, it is obvious that $\underline{t} < \bar{t}$. By now, we get

$$\begin{aligned} &\frac{M}{\delta} \leqslant x\bigl(\bar{t}^-\bigr) \leqslant M, \qquad \underline{M} + \varepsilon \leqslant x\bigl(\underline{t}^+\bigr) \leqslant \delta(\underline{M} + \varepsilon), \\ &\underline{M} + \varepsilon \leqslant x(t) \leqslant M, \quad t \in [\underline{t}, \bar{t}). \end{aligned} \tag{9}$$

On account of (9), one may derive the following two assertions:

(i) $\bar{t} > \underline{t} + \tau$.

(ii) $x(\underline{t} + \tau) \leqslant \delta^{\tau/\mu+3}(\underline{M} + \varepsilon)\max\{1, \exp(\mathcal{B}\tau)\}$, where $\mathcal{B} = r^S - a^I(\underline{M} + \varepsilon)$.

First, we claim that (i) holds. Otherwise, we suppose on the contrary that $\bar{t} \leqslant \underline{t} + \tau$. From system (5) and (9), we note that

$$\dot{x}(t) \leqslant x(t)\bigl[r^S - a^I(\underline{M} + \varepsilon)\bigr] = \mathcal{B}x(t), \quad t \in [t_{k-1}, t_k) \cap [\underline{t}, \bar{t}),\ k \in \mathbb{Z}_+. \tag{10}$$

If it is continuous on the interval $[\underline{t}, \bar{t})$, then integrating (10) from $\underline{t}$ to $\bar{t}$, we get

$$\begin{aligned} x(\bar{t}^-) \leqslant x(\underline{t})\exp\bigl(\mathcal{B}(\bar{t} - \underline{t})\bigr) \leqslant &\begin{cases} x(\underline{t}), & \mathcal{B} \leqslant 0, \\ x(\underline{t})\exp(\mathcal{B}\tau), & \mathcal{B} > 0 \end{cases} \\ \leqslant &\, x(\underline{t})\max\bigl\{1, \exp(\mathcal{B}\tau)\bigr\}, \end{aligned}$$

which, together with (9), yields that

$$\frac{M}{\delta} \leqslant x(\bar{t}^-) \leqslant x(\underline{t})\max\bigl\{1, \exp(\mathcal{B}\tau)\bigr\} \leqslant \delta(\underline{M} + \varepsilon)\max\bigl\{1, \exp(\mathcal{B}\tau)\bigr\}.$$

This contradicts (7). If there are some impulses on $[\underline{t}, \bar{t})$. Let $t_{i_1}, \dots, t_{i_l}$ be the impulsive points satisfying $\underline{t} < t_{i_1} < \dots < t_{i_l} < \bar{t}$. Note that $t_k - t_{k-1} \geqslant \mu$, one may deduce that $\mu(l-1) \leqslant t_{i_l} - t_{i_1} \leqslant \bar{t} - \underline{t} \leqslant \tau$, which implies that $l \leqslant \tau/\mu + 1$. Then it can be deduced from (10) that

$$\begin{aligned} x(t_{i_1}^-) &\leqslant x(\underline{t})\exp\bigl(\mathcal{B}(t_{i_1} - \underline{t})\bigr), \\ x(t_{i_2}^-) &\leqslant x(t_{i_1})\exp\bigl(\mathcal{B}(t_{i_2} - t_{i_1})\bigr) \leqslant \delta x(\underline{t})\exp\bigl(\mathcal{B}(t_{i_2} - \underline{t})\bigr), \\ x(t_{i_3}^-) &\leqslant x(t_{i_2})\exp\bigl(\mathcal{B}(t_{i_3} - t_{i_2})\bigr) \leqslant \delta^2 x(\underline{t})\exp\bigl(\mathcal{B}(t_{i_3} - \underline{t})\bigr), \\ &\cdots \\ x(\bar{t}^-) &\leqslant x(t_{i_l})\exp\bigl(\mathcal{B}(\bar{t} - t_{i_l})\bigr) \leqslant \delta^l x(\underline{t})\exp\bigl(\mathcal{B}(\bar{t} - \underline{t})\bigr), \end{aligned}$$

which implies that

$$\begin{aligned} \frac{M}{\delta} \leqslant x(\bar{t}^-) &\leqslant \delta^{l+1}(\underline{M} + \varepsilon)\max\bigl\{1, \exp(\mathcal{B}\tau)\bigr\} \\ &\leqslant \delta^{\tau/\mu+2}(\underline{M} + \varepsilon)\max\bigl\{1, \exp(\mathcal{B}\tau)\bigr\}. \end{aligned}$$

This also contradicts (7). Hence, we have proven that assertion (i) holds.

Next, we claim that (ii) holds. In fact, the proof of assertion (ii) is similar to the proof of assertion (i). If it is continuous on $[\underline{t}, \underline{t}+\tau)$, then it is obvious from (10) that

$$\begin{aligned} x(\underline{t}+\tau) &\leqslant \delta x\bigl(\underline{t}+\tau^-\bigr) \leqslant \delta x(\underline{t})\exp(\mathcal{B}\tau) \leqslant \delta^2(\underline{M}+\varepsilon)\max\bigl\{1,\exp(\mathcal{B}\tau)\bigr\} \\ &< \delta^{\tau/\mu+3}(\underline{M}+\varepsilon)\max\bigl\{1,\exp(\mathcal{B}\tau)\bigr\}. \end{aligned}$$

If some impulsive points exist, let $t_{i_1},\dots,t_{i_l}$ be the impulse instants on $[\underline{t}, \underline{t}+\tau)$ satisfying $\underline{t} \leqslant t_{i_1} < \dots < t_{i_l} < \underline{t}+\tau$. Then similar to the proof of assertion (i), we have

$$x(\underline{t}+\tau) \leqslant \delta x\bigl(\underline{t}+\tau^-\bigr) \leqslant \delta^{\tau/\mu+3}(\underline{M}+\varepsilon)\max\bigl\{1,\exp(\mathcal{B}\tau)\bigr\}.$$

This then completes the proof of assertion (ii).

Since $\bar{t} > \underline{t}+\tau$, it is meaningful to take the interval $[\underline{t}+\tau, \bar{t})$ into account. From system (5) and the third inequality in (9), it is not difficult to derive that

$$\begin{aligned} \dot{x}(t) &\leqslant x(t)\bigl[r^S - \bigl(a^I+b^I\bigr)(\underline{M}+\varepsilon)\bigr] \\ &= -\mathcal{A}x(t) < 0, \quad t \in [t_{k-1}, t_k) \cap [\underline{t}+\tau, \bar{t}),\ k \in \mathbb{Z}_+. \end{aligned} \tag{11}$$

If no impulsive point exists in the interval $[\underline{t}+\tau, \bar{t})$, then it obviously holds that $x(\underline{t}+\tau^+) > x(\bar{t}^-)$. On the other hand, if there is some impulse instants existing in the interval, assume that $t_{i_1},\dots,t_{i_m}$ be the impulsive points at the interval $[\underline{t}+\tau, \bar{t})$ satisfying $\underline{t}+\tau < t_{i_1} < \dots < t_{i_m} < \bar{t}$. It follows from (11) and the fact that $\mu \leqslant t_k - t_{k-1}$ that

$$\begin{aligned} x(t_{i_1}^-) &\leqslant x(\underline{t}+\tau)\exp\bigl(-\mathcal{A}(t_{i_1}-\underline{t}-\tau)\bigr) \leqslant x(\underline{t}+\tau), \\ x(t_{i_2}^-) &\leqslant x(t_{i_1})\exp\bigl(-\mathcal{A}(t_{i_2}-t_{i_1})\bigr) \leqslant \delta x\bigl(t_{i_1}^-\bigr)\exp(-\mathcal{A}\mu) < x(\underline{t}+\tau), \\ x(t_{i_3}^-) &\leqslant x(t_{i_2})\exp\bigl(-\mathcal{A}(t_{i_3}-t_{i_2})\bigr) \leqslant \delta x\bigl(t_{i_2}^-\bigr)\exp(-\mathcal{A}\mu) < x(\underline{t}+\tau), \\ &\cdots \\ x(\bar{t}^-) &\leqslant x(t_{i_m})\exp\bigl(-\mathcal{A}(\bar{t}-t_{i_m})\bigr) \leqslant \delta x\bigl(t_{i_m}^-\bigr)\exp(-\mathcal{A}\mu) < x(\underline{t}+\tau). \end{aligned}$$

Thus, regardless of whether there are impulsive points, it follows that $x(\underline{t}+\tau^+) > x(\bar{t}^-)$. By assertion (ii) and (9), we get

$$\delta^{\tau/\mu+3}(\underline{M}+\varepsilon)\max\bigl\{1,\exp(\mathcal{B}\tau)\bigr\} \geqslant x\bigl(\underline{t}+\tau^+\bigr) > x\bigl(\bar{t}^-\bigr) \geqslant \frac{M}{\delta},$$

which contradicts (7). Hence, we obtain that $x(t) < M$ for all $t \geqslant T_0$. Let $T_1 = T_0 + \tau$, then it certainly holds that $x(t) < M$ for all $t \geqslant T_1$.

Step 2. Next we show that there is a constant T_2 satisfying $T_1 \leqslant T_2$ such that $x(t) > m$, $t > T_2$.

From the definitions of m and $\bar{m}$, a constant $\varepsilon_0 \in (0, \bar{m})$ can be selected such that

$$m < \frac{\bar{m}-\varepsilon_0}{\rho^{\tau/\mu+3}}\exp\bigl(-\bigl(\bigl(a^S+b^S\bigr)M - r^I\bigr)\tau\bigr)\min\biggl\{\frac{1}{\rho}, \frac{1}{\delta^{\tau/\mu+3}}\exp\bigl(-r^S\tau\bigr)\biggr\}. \tag{12}$$

In the first place, one may claim that there is a constant $T' \geqslant T_1$ such that $x(T') > \bar{m} - \varepsilon_0$. Otherwise, it holds that $x(t) \leqslant \bar{m} - \varepsilon_0$, $t \geqslant T_1$. By system (5), it can be deduced that

$$\begin{aligned}\dot{x}(t) &\geqslant x(t)\bigl[r^I - \bigl(a^S + b^S\bigr)(\bar{m} - \varepsilon_0)\bigr] \\ &= \mathcal{C}x(t), \quad t \in [t_{k-1}, t_k) \cap [T_1 + \tau, \infty),\ k \in \mathbb{Z}_+,\end{aligned}$$

where $\mathcal{C} = \ln\rho/\mu + \varepsilon_0(a^S + b^S) > 0$. Without loss of generality, assume that $T_1 + \tau \in [t_k, t_{k+1})$ for some $k \in \mathbb{Z}_+$. Then we get

$$\begin{aligned}
x(t_{k+1}^-) &\geqslant x(T_1 + \tau)\exp\bigl(\mathcal{C}(t_{k+1} - T_1 - \tau)\bigr) > x(T_1 + \tau), \\
x(t_{k+2}^-) &\geqslant x(t_{k+1})\exp\bigl(\mathcal{C}(t_{k+2} - t_{k+1})\bigr) > \frac{\exp(\mathcal{C}\mu)}{\rho} x(T_1 + \tau), \\
x(t_{k+3}^-) &\geqslant x(t_{k+2})\exp\bigl(\mathcal{C}(t_{k+3} - t_{k+2})\bigr) > \left[\frac{\exp(\mathcal{C}\mu)}{\rho}\right]^2 x(T_1 + \tau), \\
&\dots \\
x(t_{k+j}^-) &> \left[\frac{\exp(\mathcal{C}\mu)}{\rho}\right]^{j-1} x(T_1 + \tau) = \exp\bigl((j-1)\varepsilon_0\bigl(a^S + b^S\bigr)\mu\bigr)x(T_1 + \tau) \\
&\to +\infty \quad \text{as } j \to +\infty.
\end{aligned}$$

This is contradictory, and so we have proved that there is a constant $T' \geqslant T_1$ such that $x(T') > \bar{m} - \varepsilon_0$. Next, we will show that $x(t) > m$, for $t \geqslant T'$. Otherwise, we suppose on the contrary that there exists a $t^0 \geqslant T'$ such that $x(t^0) \leqslant m$. Define $\hat{t} = \inf\{t \in [T', t^0]\colon x(t) \leqslant m\}$, then it holds that $x(\hat{t}^+) \leqslant m$ and $m \leqslant x(\hat{t}^-) \leqslant \rho m$. Moreover, we obtain $\hat{t} > T'$ since $x(T') > \bar{m} - \varepsilon_0 > m$. Then define $\tilde{t} = \sup\{t \in [T', \hat{t})\colon x(t) > \bar{m} - \varepsilon_0\}$. Then it holds that $x(\tilde{t}^-) \geqslant \bar{m} - \varepsilon_0$ and $(\bar{m} - \varepsilon_0)/\rho \leqslant x(\tilde{t}^+) \leqslant \bar{m} - \varepsilon_0$. In view of the fact that $\bar{m} - \varepsilon_0 > \rho^2 m$, we know that $\tilde{t} < \hat{t}$. By now, we get

$$\begin{aligned}
&m \leqslant x\bigl(\hat{t}^-\bigr) \leqslant \rho m, \qquad \frac{\bar{m} - \varepsilon_0}{\rho} \leqslant x\bigl(\tilde{t}^+\bigr) \leqslant \bar{m} - \varepsilon_0, \\
&m \leqslant x(t) \leqslant \bar{m} - \varepsilon_0, \quad t \in [\tilde{t}, \hat{t}).
\end{aligned} \tag{13}$$

Based on (13), we claim that

$$x(\tilde{t} + \tau) \geqslant \frac{\bar{m} - \varepsilon_0}{\rho^{\tau/\mu + 3}} \exp(-\mathcal{D}\tau), \tag{14}$$

where $\mathcal{D} = (a^S + b^S)M - r^I > 0$. In fact, since $x(t) < M$ for all $t \geqslant T' \geqslant T_1$, it can be obtained that

$$\dot{x}(t) \geqslant x(t)\bigl\{r^I - \bigl(a^S + b^S\bigr)M\bigr\} = -\mathcal{D}x(t), \quad t \in [t_{k-1}, t_k) \cap [\tilde{t}, \infty),\ k \in \mathbb{Z}_+.$$

If it is continuous on $[\tilde{t}, \tilde{t} + \tau)$, then it is obvious that

$$x(\tilde{t} + \tau) \geqslant \frac{1}{\rho} x(\tilde{t} + \tau^-) \geqslant \frac{1}{\rho} x(\tilde{t}) \exp(-\mathcal{D}\tau) \geqslant \frac{\bar{m} - \varepsilon_0}{\rho^2} \exp(-\mathcal{D}\tau).$$

Otherwise, it can be easily deduced that

$$x(\tilde{t}+\tau) \geqslant \frac{1}{\rho}x(\tilde{t}+\tau^-) \geqslant \frac{1}{\rho^{l+1}}x(\tilde{t})\exp(-\mathcal{D}\tau) \geqslant \frac{\bar{m}-\varepsilon_0}{\rho^{l+2}}\exp(-\mathcal{D}\tau),$$

where l represents the number of the impulses on $[\tilde{t}, \tilde{t}+\tau)$. Thus, assertion (14) can be derived since $l < \tau/\mu + 1$. In order to obtain the ideal contradictions, we consider the following two cases: (I) $\tilde{t}+\tau < \hat{t}$; (II) $\tilde{t}+\tau \geqslant \hat{t}$.

First, if $\tilde{t}+\tau < \hat{t}$, consider the interval $[\tilde{t}+\tau, \hat{t})$, and it follows from the third inequality of (13) that

$$\begin{aligned}\dot{x}(t) &\geqslant x(t)\big[r^I - \big(a^S + b^S\big)(\bar{m}-\varepsilon_0)\big]\\ &= \mathcal{C}x(t), \quad t \in [t_{k-1}, t_k) \cap [\tilde{t}+\tau, \hat{t}),\ k \in \mathbb{Z}_+.\end{aligned}$$

Note that $\exp(\mathcal{C}\mu) > \rho$, no matter which case of impulsive points it is, one may easily derive that

$$x(\hat{t}^-) > x(\tilde{t}+\tau),$$

which, together with (13) and (14), yields that

$$\rho m > \frac{\bar{m}-\varepsilon_0}{\rho^{\tau/\mu+3}}\exp(-\mathcal{D}\tau).$$

Obviously, this is a contradiction with (12). Thus, case (I) is impossible.

Next, we will consider the case where $\tilde{t}+\tau \geqslant \hat{t}$. Note that

$$x(\hat{t}^+) \leqslant m < \frac{\bar{m}-\varepsilon_0}{\rho^{\tau/\mu+4}}\exp(-\mathcal{D}\tau) \leqslant \frac{1}{\rho}x(\tilde{t}+\tau).$$

Thus $\tilde{t}+\tau \neq \hat{t}$. In the following, we only need consider the case that $\tilde{t}+\tau > \hat{t}$. For convenience, we define an auxiliary function:

$$\mathcal{F} = \frac{\bar{m}-\varepsilon_0}{\rho^{\tau/\mu+3}}\exp(-\mathcal{D}\tau).$$

Since $x(\hat{t}^+) \leqslant m < \mathcal{F}/\rho$ and $x(\tilde{t}+\tau) \geqslant \mathcal{F}$, there must be two constants t^* and $t^\star$ such that $\mathcal{F}/\delta \leqslant x(t^{\star -}) \leqslant \mathcal{F}$, $x(t^{\star +}) \geqslant \mathcal{F}$, $m \leqslant x(t^{*+}) \leqslant \delta m$, $x(t^{*-}) \leqslant m$, and $m \leqslant x(t) \leqslant \mathcal{F}$, $t \in [t^*, t^\star)$. By system (5), we know that

$$\dot{x}(t) \leqslant x(t)r^S, \quad t \in [t_{k-1}, t_k) \cap [t^*, t^\star),\ k \in \mathbb{Z}_+.$$

Considering the fact that $t^\star - t^* \leqslant \tilde{t}+\tau-\hat{t} < \tau$, no matter which case of impulsive points it is on $[t^*, t^\star)$, we can deduce that

$$x\big(t^{\star -}\big) \leqslant \delta^l x\big(t^*\big)\exp\big(r^S\tau\big),$$

where l represents the number of the impulses in the interval $[t^*, t^\star)$ satisfying $l < \tau/\mu+1$. Therefore, we obtain that

$$\frac{\mathcal{F}}{\delta} \leqslant x\big(t^{\star -}\big) \leqslant \delta^{\tau/\mu+2} m\exp\big(r^S\tau\big),$$

which contradicts (12). Thus, case (II) also is impossible. This completes the proof. □

In particular, consider the following delay logistic systems:

$$\begin{aligned} &\dot{x}(t) = x(t)\bigl[r(t) - b(t)x\bigl(t-\tau(t)\bigr)\bigr], \quad t \in [t_{k-1}, t_k),\\ &x(t_k) - x\bigl(t_k^-\bigr) = I_k\bigl(x\bigl(t_k^-\bigr)\bigr), \quad k \in \mathbb{Z}_+,\\ &x(t_0+s) = \phi(s), \quad -\tau \leqslant s \leqslant 0. \end{aligned} \tag{15}$$

For system (15), we can draw the following corollary

Corollary 1. *If there are two scalars* $\delta > 1$ *and* $\rho > 1$ *satisfying* (6)*, then system* (15) *admits a ultimately bounded set* Ω*, where* $\Omega = \{x \in \mathbb{R}_+\colon 0 < m < x < M\}$*, and* $M > 0$ *and* $m > 0$ *satisfy*

$$\begin{aligned} M &> \delta^{\tau/\mu+4}\frac{\frac{\ln\delta}{\mu} + r^S}{b^I}\max\bigl\{1, \exp\bigl(r^S\tau\bigr)\bigr\},\\ m &< \frac{r^I - \frac{\ln\rho}{\mu}}{b^S}\frac{\exp(-(b^S M - r^I)\tau)}{\rho^{\tau/\mu+3}}\min\biggl\{\frac{1}{\rho}, \frac{1}{\delta^{\tau/\mu+3}}\exp\bigl(-r^S\tau\bigr)\biggr\}, \end{aligned}$$

where $\mu \doteq \inf_{k\in\mathbb{Z}_+}\{t_k - t_{k-1}\}$.

Furthermore, if $\tau = 0$, then system (15) becomes

$$\begin{aligned} &\dot{x}(t) = x(t)\bigl[r(t) - b(t)x(t)\bigr], \quad t \in [t_{k-1}, t_k),\\ &x(t_k) - x\bigl(t_k^-\bigr) = I_k\bigl(x\bigl(t_k^-\bigr)\bigr), \quad k \in \mathbb{Z}_+,\\ &x(t_0+s) = \phi(s), \quad -\tau \leqslant s \leqslant 0. \end{aligned} \tag{16}$$

For system (16), we have

Corollary 2. *If there exist scalars* $\rho > 1$ *and* $\delta > 1$ *satisfying* (6)*, then system* (15) *admits a ultimately bounded set* Ω*, where* $\Omega = \{x \in \mathbb{R}_+\colon 0 < m < x < M\}$*, and* $M > 0$ *and* $m > 0$ *satisfy*

$$M > \delta^4\frac{\frac{\ln\delta}{\mu} + r^S}{b^I}, \qquad m < \frac{r^I - \frac{\ln\rho}{\mu}}{b^S}\frac{1}{\rho^3}\min\biggl\{\frac{1}{\rho}, \frac{1}{\delta^3}\biggr\},$$

where $\mu \doteq \inf_{k\in\mathbb{Z}_+}\{t_k - t_{k-1}\}$.

Remark 1. Let $a = 0$ or $\tau = 0$ in Lemma 2, then one may derive Corollaries 1 and 2 easily. Since systems (15) and (16) include systems (2) and (3) as the special cases, Corollaries 1 and 2 are also suitable for systems (2) and (3), respectively. In the following, we will give our main result for the persistence of system (5).

Theorem 1. *If there exists a scalar* $\rho > 1$ *such that*

$$\frac{1-\rho}{\rho} \leqslant \frac{I_k(s)}{s} < \infty, \quad s > 0,\ k \in \mathbb{Z}_+,$$

$$\inf_{k\in\mathbb{Z}_+}\{t_k - t_{k-1}\} > \frac{\ln\rho}{r^I},$$

then system (5) *is persistent.*

Proof. Note that $I_k(s)/s < \infty$ for $s > 0$, $k \in \mathbb{Z}_+$. One may choose a $\delta > 1$ such that

$$\frac{I_k(s)}{s} \leqslant \delta - 1, \quad k \in \mathbb{Z}_+.$$

By Lemma 2, Theorem 1 can be obtained. □

Remark 2. From the impulsive harvesting point of view, i.e., $I_k < 0$, Theorem 1 tells us that the persistence of system (5) can be guaranteed, provided that the impulsive harvesting rate keeps in a certain proportion which depends on the lower bound of the harvesting intervals. Moreover, since Theorem 1 is independent of constant δ, it implies that system (5) is persistent if the impulsive perturbations only include impulsive planting. On the other hand, from Remark 1 it is obvious that Theorem 1 can be directly applied to systems (2) and (3).

Remark 3. In this paper, the persistence of system (5) is investigated via the analysis techniques on impulsive delay differential equations. A simple but practical condition is derived. The ideas used in this paper is completely new and can be extended to investigate the impulsive effects on dynamics of other biological models such as Lasota–Wazewska models, predator–prey models, LV cooperative models, and so on.

Remark 4. The article [24] studied the global behaviors of the periodic logistic system with periodic impulsive perturbations and time delay, which extended the results in [1]. [32] studied the existence of almost periodic solutions of a delay logistic model with fixed moments of impulsive perturbations. [39] investigated a stochastic nonautonomous Holling–Tanner predator–prey system with impulsive effects. However, those results only focused on the investigation of dynamics of the periodic logistic systems with periodic impulsive perturbations, which cannot be applied to the impulsive logistic systems with general impulses. Our result shows that the persistence of system (4) with time-varying delays and impulsive perturbations can be guaranteed if the impulsive functions I_k vary in a certain degree and the lower bound of the impulsive interval is greater than a certain constant. The process of exploration and analysis is rather complicated, but the results are actually very simple and practical.

4 Examples

In this section, two examples and simulations are presented to demonstrate the advantages and validity of the obtained result.

Example 1. Consider the following logistic system:

$$\begin{aligned} \dot{x}(t) &= x(t)\big[1 - \big(1 + 0.8[\sin 3t]^*\big)x(t) \\ &\quad - (1 + 0.8\cos 20t)x\big(t - \tau(t)\big)\big], \quad t > 0, \\ x(t_k) - x(t_k^-) &= \left(\frac{1}{k} - \frac{1}{2}\right)x\big(t_k^-\big), \quad k \in \mathbb{Z}_+, \\ x(s) &= \phi(s), \quad -\tau \leqslant s \leqslant 0, \end{aligned} \tag{17}$$

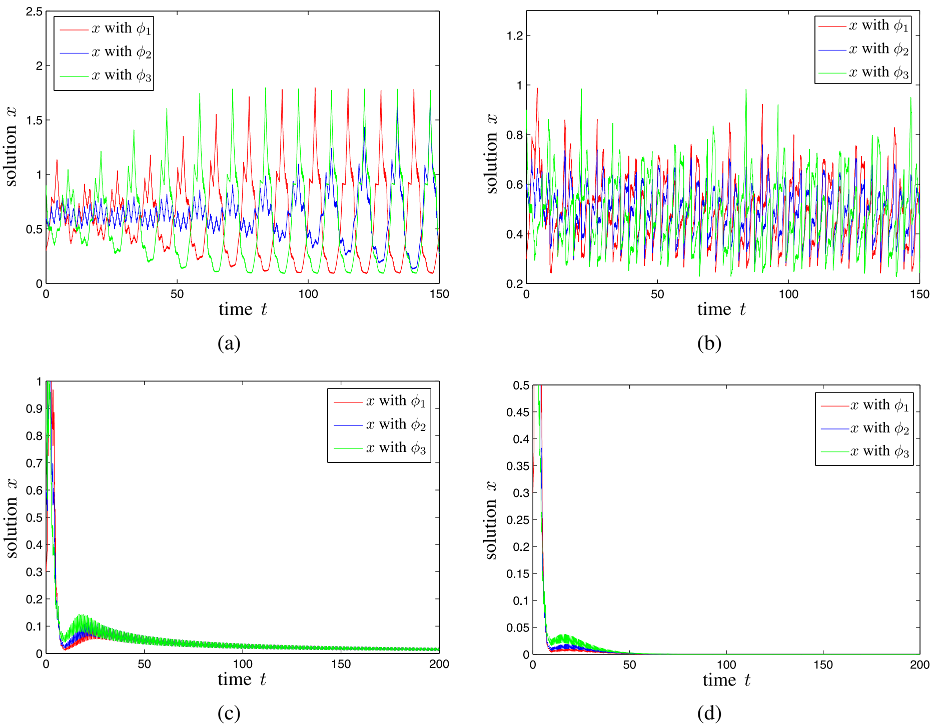

Figure 1. (a) State trajectories of system (17) without impulsive effects with initial values ϕ_i, $i = 1, 2, 3$. (b) State trajectories of system (17) with $t_k = 3k$ and initial values ϕ_i, $i = 1, 2, 3$. (c) State trajectories of system (17) with $t_k = 0.7k$ and initial values ϕ_i, $i = 1, 2, 3$. (d) State trajectories of system (17) with $t_k = 0.6k$ and initial values ϕ_i, $i = 1, 2, 3$.

where $\tau(t) = 5 + 2[\sin t]^*$. By Theorem 1, the following result can be derived easily, and its proof is omitted here.

Property 1. System (17) *is persistent if* $\inf_{k\in\mathbb{Z}_+}\{t_k - t_{k-1}\} > \ln 2$.

Remark 5. Since system (17) is nonperiodic and delay $\tau(t)$ is nondifferentiable time-varying, all of the results in [24, 25, 30, 34] cannot be applied to ascertain the dynamics of system (17). Whereas Property 1 tells us that system (17) is persistent if the impulsive interval is greater than $\ln 2$, which can be illustrated in Fig. 1. Among them, Fig. 1(a) shows that system (17) is persistent without impulsive perturbation. Figures 1(b), 1(c) show that the persistence of the system (17) can be guaranteed when there exist some impulsive harvesting, where the harvesting time is $t_k = 3k$ and $0.7k$, respectively. However, when $t_k = 0.6k$, Property 1 becomes invalid since $t_k - t_{k-1} = 0.6 < \ln 2 \approx 0.6931$. In this case, it happened that all of the solutions of system (17) will go extinct; see Fig. 1(d). This situation matches our theoretical result perfectly.

In the simulations of Example 1, we take the time step size $h = 0.01$ and the initial values $\phi_j = 0.3j$, $j = 1, 2, 3$.

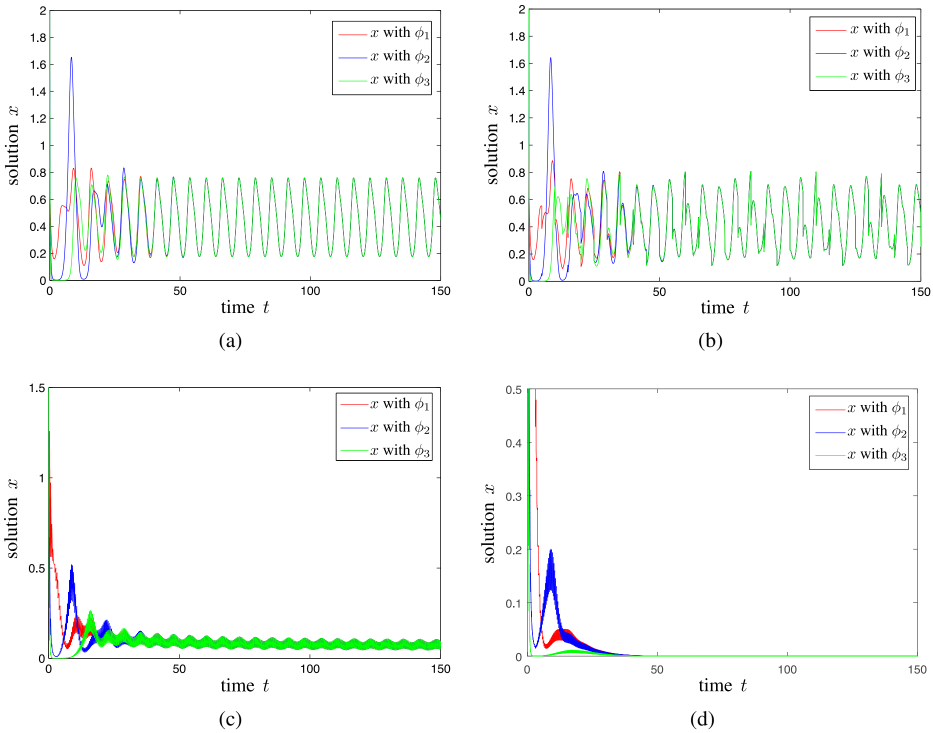

Figure 2. (a) State trajectories of system (18) without impulsive effects with initial values φ_j, $j = 1, 2, 3$. (b) State trajectories of system (18) with $\rho = 2$, $t_k = 5k$ and initial values φ_j, $j = 1, 2, 3$. (c) State trajectories of system (18) with $\rho = 2$, $t_k = 0.4k$ and initial values φ_j, $j = 1, 2, 3$. (d) State trajectories of system (18) with $\rho = 2$, $t_k = 0.3k$ and initial values φ_j, $j = 1, 2, 3$.

Example 2. Consider the following logistic system:

$$\begin{aligned} &\dot{x}(t) = x(t)\big[2 - (2 - \sin t)x(t) - (2 + \cos t)x(t-3)\big], \quad t > 0, \\ &x(t_k) - x(t_k^-) = \left(\frac{1}{\rho} + \frac{k}{e^k} - 1\right)x(t_k^-), \quad k \in \mathbb{Z}_+, \\ &x(s) = \varphi(s), \quad -3 \leqslant s \leqslant 0, \end{aligned} \tag{18}$$

where $\rho > 1$ is a constant. By Theorem 1, the following result can be derived easily, and its proof is also omitted here.

Property 2. System (18) *is persistent if* $\inf_{k \in \mathbb{Z}_+}\{t_k - t_{k-1}\} > 0.5 \ln \rho$.

Remark 6. It is obvious that system (18) without impulsive perturbation is 2π-periodic; see Fig. 2(a). When there are some impulsive perturbations such as $\rho = 2$, Property 2 tells us that the persistence of system (18) can be guaranteed if $t_k - t_{k-1} > 0.347$. In particular, Fig. 2(b) shows that system (18) is persistent but nonperiodic and has a attractor when $t_k = 5k$, and Fig. 2(c) shows the persistence when $t_k = 0.4k$. However, if we take

$t_k = 0.3k$ such that $t_k - t_{k-1} = 0.3 < 0.347$, then Property 2 becomes invalid, and in this case, the numerical simulation in Fig. 2(d) shows that system (18) is nonpersistent and all of the solutions will go extinct. It reflects not only the effectiveness but also the advantages of our development method.

In the simulations of Example 2, we take the time step size $h = 0.01$ and the initial values $\varphi_j = j$, $j = 1, 2, 3$.

5 Conclusion

In this paper, we considered a class of nonautonomous logistic systems with time-varying delays and impulsive perturbations. A new sufficient condition ensuring the persistence was derived by using the analysis techniques on impulsive delay differential equations. Our developed method is different from the usual methods in other literatures. The proof and analysis are rather complicated but the result is very simple and practical. Finally, we presented two examples to illustrate the applications. An interesting topic is to extend the approach in this paper to some complex logistic systems involving large delay or unknown delay.

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