Nonlinear Analysis: Modelling and Control, Vol. 25, No. 5, 860–883
https://doi.org/10.15388/namc.2020.25.19278
ISSN: 1392-5113
eISSN: 2335-8963

Joint universality of periodic zeta-functions with multiplicative coefficients

Antanas Laurinčikas[a,1], **Monika Tekorė**[b]

[a]Institute of Mathematics, Vilnius University,
Naugarduko str. 24, LT-03225 Vilnius, Lithuania
antanas.laurincikas@mif.vu.lt

[b]Institute of Regional Development, Šiauliai University,
P. Višinskio str. 25, LT-76351 Šiauliai, Lithuania
MatM18_Monika_Tekore@stud.su.lt

Received: October 23, 2019 / **Revised:** April 16, 2020 / **Published online:** September 1, 2020

Abstract. The periodic zeta-function is defined by the ordinary Dirichlet series with periodic coefficients. In the paper, joint universality theorems on the approximation of a collection of analytic functions by nonlinear shifts of periodic zeta-functions with multiplicative coefficients are obtained. These theorems do not use any independence hypotheses on the coefficients of zeta-functions.

Keywords: joint universality, periodic zeta-function, space of analytic functions, weak convergence.

1 Introduction

After a famous Voronin's work [27], it is known that the majority of classical zeta- and L-functions have the universality property, i.e., they approximate wide classes of analytic functions. Voronin obtained the universality property for the Riemann zeta-function

$$\zeta(s)=\sum_{m=1}^{\infty}\frac{1}{m^s},\quad s=\sigma+\mathrm{i}t,\ \sigma>1,$$

which has meromorphic continuation to the whole complex plane with unique simple pole at the point $s=1$ with residue 1. Let $D=\{s\in\mathbb{C}\colon 1/2<\sigma<1\}$. Voronin considered approximation of analytic functions defined on D by shifts $\zeta(s+\mathrm{i}\tau)$, $\tau\in\mathbb{R}$. For the last version of the Voronin universality theorem, it is convenient to use the following notation. Denote by $\mathcal{K}$ the class of compact subsets of the strip D with connected complements,

[1]The author is supported by the European Social Fund (project No. 09.3.3-LMT-K-712-01-0037) under grant agreement with the Research Council of Lithuania (LMT LT).

and by $H_0(K)$ with $K \in \mathcal{K}$ the class of continuous nonvanishing functions on K that are analytic in the interior of K. Moreover, let $\operatorname{meas} A$ stand for the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. Then the Voronin theorem asserts that if $K \in \mathcal{K}$ and $f(s) \in H_0(K)$, then, for every $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\Big\{\tau \in [0,T]\colon \sup_{s\in K} \big|\zeta(s+\mathrm{i}\tau) - f(s)\big| < \varepsilon\Big\} > 0.$$

A proof of the above statement by different methods is given in [1,6], see also [13,25].

A similar assertion is obtained for Dirichlet L-functions [1,6,11,27]

$$L(s,\chi) = \sum_{m=1}^{\infty} \frac{\chi(m)}{m^s}, \quad \sigma > 1,$$

where χ is a Dirichlet character.

More general there are zeta-functions attached to certain cusp forms F

$$\zeta(s,F) = \sum_{m=1}^{\infty} \frac{c(m)}{m^s}, \quad \sigma > \frac{\kappa+1}{2},$$

where $c(m)$ are Fourier coefficients of the form F, and κ denotes the weight of F. Also, the functions $\zeta(s,F)$ has analytic continuation to an entire function. The universality for $\zeta(s,F)$ with normalized Hecke eigen cusp forms was obtained in [19].

The above mentioned zeta-functions have a one common feature, they have the Euler product over prime numbers. For example,

$$\zeta(s,F) = \prod_p \left(1 - \frac{\alpha(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta(p)}{p^s}\right)^{-1},$$

where $\alpha(p)$ and $\beta(p)$ are conjugate complex numbers such that $c(p) = \alpha(p) + \beta(p)$, and p denotes a prime number.

A nonclassical generalization of the functions $\zeta(s)$ and $L(s,\chi)$ is the so-called periodic zeta-function with multiplicative coefficients. Let $\mathfrak{a} = \{a_m\colon m \in \mathbb{N}\}$ be a periodic sequence of complex numbers with minimal period $q \in \mathbb{N}$. Obviously, there exists a constant $c = c(\mathfrak{a}) > 0$ such that $|a_m| \leqslant c$ for all $m \in \mathbb{N}$. The periodic zeta-function $\zeta(s;\mathfrak{a})$ is defined by the Dirichlet series

$$\zeta(s;\mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m}{m^s},$$

which is absolutely convergent for $\sigma > 1$.

In virtue of the periodicity of $\mathfrak{a}$, the equality

$$\zeta(s;\mathfrak{a}) = \frac{1}{q^s} \sum_{l=1}^{q} a_l \zeta\left(s, \frac{l}{q}\right) \tag{1}$$

holds, where $\zeta(s,\alpha)$ is the classical Hurwitz zeta-function with parameter $0<\alpha\leqslant 1$ that has, as $\zeta(s)$, meromorphic continuation to the whole complex plane with unique simple pole at the point $s=1$ with residue 1. Thus, the function $\zeta(s;\mathfrak{a})$ can be analytically continued to the whole complex plane, except for a simple pole at the point $s=1$ with residue

$$r_{\mathfrak{a}} \stackrel{\text{def}}{=} \frac{1}{q}\sum_{l=1}^{q} a_l.$$

If $r_{\mathfrak{a}}=0$, then $\zeta(s;\mathfrak{a})$ is an entire function.

Bagchi obtained [1] the universality of the function

$$\zeta_1(s;\mathfrak{a}) = \sum_{\substack{m=1\\(m,q)=1}}^{\infty} \frac{a_m}{m^s}, \quad \sigma>1.$$

Steuding [24, 25] considered the function $\zeta(s;\mathfrak{a})$ with nonmultiplicative sequence $\mathfrak{a}$ and proved its universality. The paper [20] is devoted to the universality of $\zeta(s;\mathfrak{a})$ with multiplicative $\mathfrak{a}$ ($a_{mn}=a_m a_n$ for coprimes m and n, and $a_1=1$). If the sequence $\mathfrak{a}$ is multiplicative, then the function $\zeta(s;\mathfrak{a})$ has the Euler product, i.e., for $\sigma>1$,

$$\zeta(s;\mathfrak{a}) = \prod_p \left(1+\sum_{k=1}^{\infty}\frac{a_{p^k}}{p^{ks}}\right).$$

Kaczorowski [10] introduced new restricted type of universality for $\zeta(s;\mathfrak{a})$ involving the notion of height of the set K.

Zeta- and L-functions also have a joint universality property. In this case, a collection of analytic functions is approximated simultaneously by a collection of shifts of zeta- or L-functions. The first joint universality results were obtained for Dirichlet L-functions in [1, 2, 6, 26], see also [11, 15, 25]. It is clear that, in the case of joint universality, the approximating shifts must be in some sense independent. In the case of Dirichlet L-functions, the nonequivalence of Dirichlet characters is used (two Dirichlet characters are called equivalent if they are generated by the same primitive characters). The joint universality Voronin theorem [26] says that if $\chi_1,\dots,\chi_r$ are pairwise nonequivalent Dirichlet characters, for $j=1,\dots,r$, $K_j\in\mathcal{K}$ and $f_j(s)\in H_0(K_j)$, then, for every $\varepsilon>0$,

$$\liminf_{T\to\infty}\frac{1}{T}\,\mathrm{meas}\Bigl\{\tau\in[0,T]\colon \sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}\bigl|L(s+\mathrm{i}\tau;\chi_j)-f_j(s)\bigr|<\varepsilon\Bigr\}>0.$$

Pańkowski in [23] proposed a new way of joint universality for Dirichlet L-functions by using different shifts for L-functions with arbitrary characters $\chi_1,\dots,\chi_r$. Let $\alpha_1,\dots,\alpha_r\in\mathbb{R}$, $a_1,\dots,a_r\in\mathbb{R}^+$, and $b_1,\dots,b_r$ be such that

$$b_j \in \begin{cases}\mathbb{R} & \text{if } a_j\notin\mathbb{N},\\ (-\infty,0]\cup(1+\infty) & \text{if } a_j\in\mathbb{N},\end{cases}$$

and $a_j \neq a_k$ or $b_j \neq b_k$ if $k \neq j$. Moreover, let $K \in \mathcal{K}$, $f_1, \dots, f_r \in H_0(K)$. Then the Pańkowski theorem asserts that, for every $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\Big\{\tau \in [2, T]\colon \sup_{1\leqslant j\leqslant r} \sup_{s\in K} \big|L\big(s + \mathrm{i}\alpha_j \tau^{a_j} \log^{b_j} \tau; \chi_j\big) - f_j(s)\big| < \varepsilon\Big\} > 0.$$

Other joint universality results can be found in the excellent survey paper [21].

The present paper is devoted to the joint universality for periodic zeta-functions. Suppose that, for $j = 1, \dots, r$, $\mathfrak{a}_j = \{a_{jm}\colon m \in \mathbb{N}\}$ is a periodic sequence of complex numbers with minimal period $q_j \in \mathbb{N}$. Denote by q the least common multiple of the periods $q_1, \dots, q_r$, by $l_1, \dots, l_{r_1}$ ($r_1 = \varphi(q)$ is the Euler totient function) the reduced system modulo q, and define the matrix

$$A = \begin{pmatrix} a_{1l_1} & a_{2l_1} & \dots & a_{rl_1} \\ a_{1l_2} & a_{2l_2} & \dots & a_{rl_2} \\ \dots & \dots & \dots & \dots \\ a_{1l_{r_1}} & a_{2l_{r_1}} & \dots & a_{rl_{r_1}} \end{pmatrix}.$$

Then, in [18], the following joint universality theorem has been proved.

Theorem 1. *Suppose that the sequences $\mathfrak{a}_1, \dots, \mathfrak{a}_r$ are multiplicative and* $\operatorname{rank} A = r$. *For $j = 1, \dots, r$, let $K_j \in \mathcal{K}$ and $f_j(s) \in H_0(K_j)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\Big\{\tau \in [0, T]\colon \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} \big|\zeta(s + \mathrm{i}\tau; \mathfrak{a}_j) - f_j(s)\big| < \varepsilon\Big\} > 0.$$

To be precise, in [18], a technical condition

$$\sum_{k=1}^{\infty} \frac{|a_{jp^k}|}{p^{k/2}} \leqslant c_j < 1, \quad j = 1, \dots, r,$$

was required, however, it can be easily removed.

Joint universality of more general collections of zeta-functions was studied in [12, 14, 16, 17] and [7–9]. We note that joint mixed universality theorems imply those for zeta-function with Euler product.

The aim of this paper is to replace the condition $\operatorname{rank} A = r$ in Theorem 1 by using more general, nonlinear shifts $\zeta(s + \mathrm{i}\gamma_j(\tau); \mathfrak{a}_j)$, with some functions $\gamma_j(\tau)$. In [18], the linear shifts $\zeta(s + \mathrm{i}\tau; \mathfrak{a}_j)$ were used. We propose two types of $\gamma_j(\tau)$.

Denote by $U_1(T_0)$, $T_0 > 0$, the class of real increasing to ∞ continuously differentiable functions $\gamma(\tau)$ with monotonic derivative $\gamma'(\tau)$ on $[T_0, \infty)$ such that $\gamma(2\tau) \times \max_{\tau\leqslant u\leqslant 2\tau} 1/\gamma'(u) \ll \tau$ as $\tau \to \infty$.

Theorem 2. *Suppose that the sequences $\mathfrak{a}_1, \dots, \mathfrak{a}_r$ are multiplicative, $a_1, \dots, a_r$ are real algebraic numbers linearly independent over the field of rational numbers $\mathbb{Q}$, and $\gamma(\tau) \in U_1(T_0)$. For $j = 1, \dots, r$, let $K_j \in \mathcal{K}$ and $f_j(s) \in H_0(K_j)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T - T_0} \operatorname{meas}\Big\{\tau \in [T_0, T]\colon \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} \big|\zeta\big(s + \mathrm{i}a_j\gamma(\tau); \mathfrak{a}_j\big) - f_j(s)\big| < \varepsilon\Big\} > 0.$$

Moreover, the limit

$$\lim_{T\to\infty}\frac{1}{T-T_0}\,\mathrm{meas}\Big\{\tau\in[T_0,T]\colon \sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}\big|\zeta\big(s+\mathrm{i}a_j\gamma(\tau);\mathfrak{a}_j\big)-f_j(s)\big|<\varepsilon\Big\}>0$$

exists for all but at most countably many $\varepsilon>0$.

Denote by $U_r(T_0)$ the class of real increasing to infinity continuously differentiable functions $\gamma_1(\tau),\dots,\gamma_r(\tau)$ on $[T_0,\infty)$ with derivatives $\gamma_j'(\tau)=\hat{\gamma}_j(\tau)(1+o(1))$, where $\hat{\gamma}_1(\tau),\dots,\hat{\gamma}_r(\tau)$ are monotonic and are compared in the sense that, for every subset $J\subset\{1,\dots,r\}$, $\#J\geqslant 2$, there exists $j_0=j_0(J)$ such that $\hat{\gamma}_j(\tau)=o(\hat{\gamma}_{j_0}(\tau))$ for $j\in J$, $j\neq j_0$, and $\gamma_j(2\tau)\max_{\tau\leqslant u\leqslant 2\tau}1/\hat{\gamma}_j(u)\ll\tau$, $j=1,\dots,r$, as $\tau\to\infty$.

Theorem 3. *Suppose that the sequences* $\mathfrak{a}_1,\dots,\mathfrak{a}_r$ *are multiplicative, and* $(\gamma_1(\tau),\dots,\gamma_r(\tau))\in U_r(T_0)$. *For* $j=1,\dots,r$, *let* $K_j\in\mathcal{K}$ *and* $f_j(s)\in H_0(K_j)$. *Then, for every* $\varepsilon>0$,

$$\liminf_{T\to\infty}\frac{1}{T-T_0}\,\mathrm{meas}\Big\{\tau\in[T_0,T]\colon \sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}\big|\zeta\big(s+\mathrm{i}\gamma_j(\tau);\mathfrak{a}_j\big)-f_j(s)\big|<\varepsilon\Big\}>0.$$

Moreover, the limit

$$\lim_{T\to\infty}\frac{1}{T-T_0}\,\mathrm{meas}\Big\{\tau\in[T_0,T]\colon \sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}\big|\zeta\big(s+\mathrm{i}\gamma_j(\tau);\mathfrak{a}_j\big)-f_j(s)\big|<\varepsilon\Big\}>0$$

exists for all but at most countably many $\varepsilon>0$.

For example, we may take $\underline{a}=(\sqrt{2},\sqrt{3},\sqrt{5},\dots,\sqrt{p_r})$, where p_r is the rth prime number, and $\gamma(\tau)=\tau\log\tau$, $\tau\geqslant 2$, in Theorem 2, and $\gamma_1(\tau)=\tau\log\tau$, $\gamma_2=\tau^2\log\tau$, $\dots,\gamma_r(\tau)=\tau^r\log\tau$ in Theorem 3.

Similar results can be obtained for more general zeta-functions with Euler product, for example, for the Matsumoto zeta-functions.

For the proof of Theorems 2 and 3, we will apply the probabilistic approach based on limit theorems for probability measures in the space of analytic functions. Denote by $\mathcal{B}(\mathbb{X})$ the Borel σ-field of the space $\mathbb{X}$, by $H(D)$ the space of analytic functions on $D=\{s\in\mathbb{C}\colon 1/2<\sigma<1\}$ endowed with the topology of uniform convergence on compacta, let, for brevity, $\underline{\mathfrak{a}}=(\mathfrak{a}_1,\dots,\mathfrak{a}_r)$, $\underline{a}=(a_1,\dots,a_r)$, $\underline{\gamma}(\tau)=(\gamma_1(\tau),\dots,\gamma_r(\tau))$, and

$$\underline{\zeta}(s;\underline{\mathfrak{a}})=\big(\zeta(s;\mathfrak{a}_1),\dots,\zeta(s;\mathfrak{a}_r)\big).$$

More precisely, we will consider the weak convergence for

$$P_T^1(A)\stackrel{\mathrm{def}}{=}\frac{1}{T-T_0}\,\mathrm{meas}\big\{\tau\in[T_0,T]\colon \underline{\zeta}\big(s+\mathrm{i}\underline{a}\gamma(\tau);\underline{\mathfrak{a}}\big)\in A\big\},\quad A\in\mathcal{B}\big(H^r(D)\big),$$

and

$$P_T^r(A)\stackrel{\mathrm{def}}{=}\frac{1}{T-T_0}\,\mathrm{meas}\big\{\tau\in[T_0,T]\colon \underline{\zeta}\big(s+\mathrm{i}\underline{\gamma}(\tau);\underline{\mathfrak{a}}\big)\in A\big\},\quad A\in\mathcal{B}\big(H^r(D)\big),$$

as $T\to\infty$.

2 Limit theorems on the torus

Let $\gamma = \{s \in \mathbb{C}\colon |s| = 1\}$ be the unit circle, $\mathbb{P}$ denote the set of all prime numbers, and

$$\Omega = \prod_{p\in\mathbb{P}} \gamma_p,$$

where $\gamma_p = \gamma$ for all $p \in \mathbb{P}$. With the product topology and pointwise multiplication, the torus Ω is a compact topological group, therefore on $(\Omega, \mathcal{B}(\Omega))$, the probability Haar measure exists. For the proof of Theorem 1 in [18], a limit theorem for probability measures on $(\Omega, \mathcal{B}(\Omega))$ was applied. In our case, the above theorem is not sufficient. Define,

$$\underline{\Omega}^r = \Omega_1 \times \cdots \times \Omega_r,$$

where $\Omega_j = \Omega$ for $j = 1, \dots, r$. Then, again, $\underline{\Omega}^r$ is a compact topological group, therefore, on $(\underline{\Omega}^r, \mathcal{B}(\underline{\Omega}^r))$, the probability Haar measure m_H^r can be defined. This gives the probability space $(\underline{\Omega}^r, \mathcal{B}(\underline{\Omega}^r), m_H^r)$. For $A \in \mathcal{B}(\underline{\Omega}^r)$, define

$$Q_T^1(A) = \frac{1}{T - T_0} \operatorname{meas}\bigl\{\tau \in [T_0, T]\colon \bigl(p^{-\mathrm{i}a_1\gamma(\tau)}\colon p \in \mathbb{P}\bigr), \dots, \bigl(p^{-\mathrm{i}a_r\gamma(\tau)}\colon p \in \mathbb{P}\bigr) \in A\bigr\}.$$

Lemma 1. *Suppose that $\underline{a}$ and $\gamma(\tau)$ satisfy the hypotheses of Theorem 2. Then Q_T^1 converges weakly to the Haar measure m_H^r as $T \to \infty$.*

Proof. The dual group of $\underline{\Omega}^r$ is isomorphic to

$$\bigoplus_{j=1}^r \bigoplus_{p\in\mathbb{P}} \mathbb{Z}_{jp},$$

where $\mathbb{Z}_{jp} = \mathbb{Z}$ for all $j = 1, \dots, r$ and $p \in \mathbb{P}$. Therefore, the Fourier transform $g_T^1(\underline{k})$ of Q_T^1, $\underline{k} = (\underline{k}_1, \dots, \underline{k}_r)$, $\underline{k}_j = \{k_{jp} \in \mathbb{Z}\colon p \in \mathbb{P}\}$, is of the form

$$g_T^1(\underline{k}) = \int\limits_{\underline{\Omega}^r} \prod_{j=1}^r {\prod_{p\in\mathbb{P}}}^* \omega_j^{k_{jp}}(p)\, \mathrm{d}Q_T^1,$$

where $\omega_j(p)$ is the pth component of an element $\omega_j \in \Omega_j$, $p \in \mathbb{P}$, and the star "$*$" shows that only a finite number of integers k_{jp} are distinct from zero. Hence, by the definition of Q_T^1,

$$\begin{aligned} g_T^1(\underline{k}) &= \frac{1}{T - T_0} \int\limits_{T_0}^T \prod_{j=1}^r {\prod_{p\in\mathbb{P}}}^* p^{-\mathrm{i}a_j\gamma(\tau)k_{jp}}\, \mathrm{d}\tau \\ &= \frac{1}{T - T_0} \int\limits_{T_0}^T \exp\Biggl\{-\mathrm{i}\gamma(\tau) \sum_{j=1}^r a_j {\sum_{p\in\mathbb{P}}}^* k_{jp} \log p\Biggr\}\, \mathrm{d}\tau. \end{aligned} \tag{2}$$

Clearly,

$$g_T^1\big((\underline{0},\dots,\underline{0})\big)=1. \tag{3}$$

Now, suppose that $\underline{k}\neq(\underline{0},\dots,\underline{0})$. We have

$$A_{\underline{k}}\stackrel{\mathrm{def}}{=}\sum_{j=1}^{r}a_j\sum_{p\in\mathbb{P}}{}^{*}k_{jp}\log p=\sum_{p\in\mathbb{P}}{}^{*}\log p\sum_{j=1}^{r}a_jk_{jp}.$$

Let

$$p_{\min}=\min_{1\leqslant j\leqslant r}\min_{p}\big\{p\colon\ k_{jp}\in\underline{k}_j,\ k_{jp}\neq 0\big\}$$

and

$$p_{\max}=\max_{1\leqslant j\leqslant r}\max_{p}\big\{p\colon\ k_{jp}\in\underline{k}_j,\ k_{jp}\neq 0\big\}.$$

Then there exists at least one $p\in[p_{\min},p_{\max}]$ such that $k_{jp}\neq0$ for some j, thus, by the linear independence of the numbers $a_1,\dots,a_r$,

$$\beta_p\stackrel{\mathrm{def}}{=}\sum_{j=1}^{r}a_jk_{jp}\neq 0.$$

The numbers β_p are algebraic, moreover, it is well known that the set $\{\log p\colon\ p\in\mathbb{P}\}$ is linearly independent over $\mathbb{Q}$. Therefore, by the Baker theorem, see, for example, [3], the form

$$A_{\underline{k}}=\sum_{p\in\mathbb{P}}{}^{*}\beta_p\log p\neq 0.$$

Using the monotonicity of $\gamma'(\tau)$ and the mean value theorem, we find by (2)

$$g_T^1(\underline{k})\ll\frac{1}{|A(\underline{k})|T}\max\biggl(\frac{1}{\gamma'(T)},\frac{1}{\gamma'(T_0)}\biggr). \tag{4}$$

Since $\gamma(\tau)\in U_1(T_0)$, we have $1/\gamma'(T)=o(T)$. This, together with (3) and (4), shows that

$$\lim_{T\to\infty}g_T^1(\underline{k})=\begin{cases}1 & \text{if } \underline{k}=(\underline{0},\dots,\underline{0}),\\ 0 & \text{if } \underline{k}\neq(\underline{0},\dots,\underline{0}).\end{cases}$$

Since the right-hand side of the above equality is the Fourier transform of the Haar measure m_H^r, the lemma is proved. □

For $A\in\mathcal{B}(\underline{\Omega}^r)$, define

$$Q_T^r(A)=\frac{1}{T-T_0}\operatorname{meas}\big\{\tau\in[T_0,T]\colon\ \underline{\zeta}\big(s+\mathrm{i}\underline{\gamma}(\tau);\underline{\mathfrak{a}}\big)\in A\big\}.$$

Lemma 2. *Suppose that* $(\gamma_1(\tau),\dots,\gamma_r(\tau))\in U_r(T_0)$. *Then* Q_T^r *converges weakly to the Haar measure* m_H^r *as* $T\to\infty$.

Proof. As in the proof of Lemma 1, we consider the Fourier transform of Q_T^r

$$g_T^r(\underline{k}) = \frac{1}{T - T_0} \int_{T_0}^{T} \exp\left\{-\mathrm{i} \sum_{j=1}^{r} \gamma_j(\tau) \sum_{p\in\mathbb{P}}^{*} k_{jp} \log p\right\} \mathrm{d}\tau. \tag{5}$$

Obviously,

$$g_T^r\big((\underline{0}, \dots, \underline{0})\big) = 1. \tag{6}$$

Therefore, it remains to consider the case $\underline{k} \neq (\underline{0}, \dots, \underline{0})$. For brevity, let

$$b_j = \sum_{p\in\mathbb{P}}^{*} k_{jp} \log p.$$

Since, the set $\{\log p\colon\ p \in \mathbb{P}\}$ is linearly independent over $\mathbb{Q}$, we have $b_j \neq 0$ for $\underline{k}_j \neq \underline{0}$, $j = 1, \dots, r$. Put

$$A(\tau) = \sum_{j=1}^{r} b_j \gamma_j(\tau).$$

Suppose that $\underline{k}_j \neq \underline{0}$ for $j \in J \subset \{1, \dots, r\}$, $\#J \geqslant 2$. Then there exists $j_0 \in J$ such that $\hat{\gamma}_j(\tau) = o(\hat{\gamma}_{j_0}(\tau))$, $\tau \to \infty$, for $j \in J \setminus \{j_0\}$. Therefore,

$$A'(\tau) = \sum_{j\in J} b_j \gamma_j'(\tau) = \sum_{j\in J} b_j \hat{\gamma}_j(\tau)\big(1 + o(1)\big) = b_{j_0} \hat{\gamma}_{j_0}(\tau)\big(1 + o(1)\big),$$

$$\big(A'(\tau)\big)^{-1} = \frac{1}{b_{j_0} \hat{\gamma}_{j_0}(\tau)(1 + o(1))} = \frac{1}{b_{j_0} \hat{\gamma}_{j_0}(\tau)}\big(1 + o(1)\big)$$

and

$$\frac{1}{b_{j_0} \hat{\gamma}_{j_0}(\tau)} = \frac{(A(\tau))^{-1}}{(1 + o(1))} = \big(A(\tau)\big)^{-1}\big(1 + o(1)\big)$$

as $\tau \to \infty$. Hence, using the monotonicity of $\hat{\gamma}_{j_0}(\tau)$ and the second mean value theorem, we find

$$\begin{aligned}
\int_{T_0}^{T} \cos A(\tau)\, \mathrm{d}\tau &= \int_{\log T}^{T} \cos A(\tau)\, \mathrm{d}\tau + O(\log T) \\
&= \int_{\log T}^{T} \frac{1}{A'(\tau)} \cos A(\tau)\, \mathrm{d}A(\tau) + O(\log T) \\
&= \int_{\log T}^{T} \frac{1}{b_{j_0} \hat{\gamma}_{j_0}(\tau)} \cos A(\tau)\, \mathrm{d}A(\tau) \\
&\quad + \int_{\log T}^{T} \frac{o(1)}{b_{j_0} \hat{\gamma}_{j_0}(\tau)} \cos A(\tau)\, \mathrm{d}A(\tau) + O(\log T)
\end{aligned}$$

$$
\begin{aligned}
&= \int\limits_{\log T}^{T} \frac{1}{b_{j_0}\hat{\gamma}_{j_0}(\tau)}\, \mathrm{d}\bigl(\sin A(\tau)\bigr) \\
&\quad + \int\limits_{\log T}^{T} \frac{o(1)(1+o(1))}{A'(\tau)} \cos A(\tau)\, \mathrm{d}A(\tau) + O(\log T) \\
&= o(T) + \int\limits_{\log T}^{T} o(1)\cos A(\tau)\, \mathrm{d}\tau + O(\log T) \\
&= o(T), \quad T \to \infty,
\end{aligned}
$$

because $1/(\hat{\gamma_0}(\tau)) = o(\tau)$ as $\tau \to \infty$. By the same lines, we obtain

$$
\int\limits_{T_0}^{T} \sin A(\tau)\, \mathrm{d}\tau = o(T).
$$

This, (6) and (5) show that, for $\underline{k} \neq (\underline{0}, \ldots, \underline{0})$,

$$
\lim_{T\to\infty} g_T^r(\underline{k}) = 0,
$$

and the lemma follows from (6) in the same way as Lemma 1, because, in the case $\#J = 1$, $A(\tau) = b_j\gamma_j(\tau)$ for some j. $\square$

3 Case of absolutely convergent series

Lemmas 1 and 2 allow to prove limit theorems in the space $H^r(D)$ for measures defined by means of absolutely convergent Dirichlet series.

For fixed $\theta > 1/2$, and $m, n \in \mathbb{N}$, let $v_n(m) = \exp\{-(m/n)^\theta\}$. Define the series

$$
\zeta_n(s; \mathfrak{a}_j) = \sum_{m=1}^{\infty} \frac{a_{jm} v_n(m)}{m^s}, \quad j = 1, \ldots, r.
$$

Then, in view of the definition of $v_n(m)$, the latter series are absolutely convergent for $\sigma > 1/2$ [20]. For brevity, let

$$
\underline{\zeta}_n(s; \underline{\mathfrak{a}}) = \bigl(\zeta_n(s; \mathfrak{a}_1), \ldots, \zeta_n(s; \mathfrak{a}_r)\bigr)
$$

and, for $\mathcal{B}(H^r(D))$,

$$
P_{T,n}^1(A) = \frac{1}{T - T_0} \operatorname{meas}\bigl\{\tau \in [T_0, T]\colon \underline{\zeta}_n\bigl(s + \mathrm{i}\underline{a}\gamma(\tau); \underline{\mathfrak{a}}\bigr) \in A\bigr\}
$$

and

$$
P_{T,n}^r(A) = \frac{1}{T - T_0} \operatorname{meas}\bigl\{\tau \in [T_0, T]\colon \underline{\zeta}_n\bigl(s + \mathrm{i}\underline{\gamma}(\tau); \underline{\mathfrak{a}}\bigr) \in A\bigr\}.
$$

Denote by $\underline{\omega} = (\omega_1, \dots, \omega_r)$, $\omega_j \in \Omega_j$, $j = 1, \dots, r$, the elements of $\underline{\Omega}^r$. Together with series $\zeta_n(s; \mathfrak{a}_j)$, we consider the series

$$\zeta_n(s, \omega_j; \mathfrak{a}_j) = \sum_{m=1}^{\infty} \frac{a_{jm}\omega_j(m)v_n(m)}{m^s}, \quad j = 1, \dots, r,$$

that are absolutely convergent for $\sigma > 1/2$ as well. Here, for $m \in \mathbb{N}$,

$$\omega_j(m) = \prod_{\substack{p^l \mid m \\ p^{l+1} \nmid m}} \omega_j^l(p), \quad j = 1, \dots, r.$$

Analogically, let, for $\underline{\omega} \in \underline{\Omega}^r$,

$$\underline{\zeta}_n(s, \underline{\omega}; \underline{\mathfrak{a}}) = \big(\zeta_n(s, \omega_1; \mathfrak{a}_1), \dots, \zeta_n(s, \omega_r; \mathfrak{a}_r)\big)$$

and, for $\mathcal{B}(H^r(D))$,

$$P^1_{T,n,\underline{\omega}}(A) = \frac{1}{T - T_0} \operatorname{meas}\big\{\tau \in [T_0, T]\colon \underline{\zeta}_n\big(s + \mathrm{i}\underline{a}\gamma(\tau), \underline{\omega}; \underline{\mathfrak{a}}\big) \in A\big\}$$

and

$$P^r_{T,n,\underline{\omega}}(A) = \frac{1}{T - T_0} \operatorname{meas}\big\{\tau \in [T_0, T]\colon \underline{\zeta}_n\big(s + \mathrm{i}\underline{\gamma}(\tau), \underline{\omega}; \underline{\mathfrak{a}}\big) \in A\big\}.$$

Let the mapping $u_n : \underline{\Omega}^r \to H^r(D)$ be given by the formula

$$u_n(\underline{\omega}) = \underline{\zeta}_n(s, \underline{\omega}; \underline{\mathfrak{a}}).$$

Then the mapping u_n is continuous because of the absolute convergence of the series $\zeta_n(s, \omega_j; \mathfrak{a}_j)$. Therefore, the definitions of $P^1_{T,n}$, $P^1_{T,n,\underline{\omega}}$ and Q^1_T, and $P^r_{T,n}$, $P^r_{T,n,\underline{\omega}}$ and Q^r_T, Lemmas 1 and 2, and properties of weak convergence of probability measures [4, Thm. 5.1] lead to the following limit theorems on $(H^r(D), \mathcal{B}(H^r(D)))$.

Lemma 3. *Suppose that $\underline{a}$ and $\gamma(\tau)$ satisfy the hypotheses of Theorem 2. Then $P^1_{T,n}$ and $P^1_{T,n,\underline{\omega}}$ converge weakly to the measure $m^r_H u_n^{-1}$ as $T \to \infty$.*

Lemma 4. *Suppose that $(\gamma_1(\tau), \dots, \gamma_r(\tau)) \in U_r(T_0)$. Then $P^r_{T,n}$ and $P^r_{T,n,\underline{\omega}}$ converge weakly to the measure $m^r_H u_n^{-1}$ as $T \to \infty$.*

4 Mean square estimates

To pass from weak convergence for $P^1_{T,n}$ and $P^r_{T,n}$ to for P^1_T and P^r_T, respectively, as $T \to \infty$, a certain approximation of $\underline{\zeta}(s; \underline{\mathfrak{a}})$ by $\underline{\zeta}_n(s; \underline{\mathfrak{a}})$ is needed. This approximation is based on the mean square estimates for $\zeta(s, \mathfrak{a}_j)$.

Thus, let $\mathfrak{a}$ be an arbitrary periodic sequence of complex numbers, and $a \in \mathbb{R} \setminus \{0\}$.

Lemma 5. *Suppose that $\gamma(\tau) \in U_1(T_0)$. Then, for every fixed σ, $1/2 < \sigma < 1$, and $t \in \mathbb{R}$,*

$$\int_{T_0}^{T} \left|\zeta\bigl(\sigma + \mathrm{i}a\gamma(\tau) + \mathrm{i}t; \mathfrak{a}\bigr)\right|^2 \mathrm{d}\tau \ll_{\sigma,\mathfrak{a}} T\bigl(1 + |t|\bigr).$$

Proof. It is well known that, for fixed σ, $1/2 < \sigma < 1$, the Hurwitz zeta-function $\zeta(s, \alpha)$ satisfies

$$\int_{T_0}^{T} \left|\zeta(\sigma + \mathrm{i}t, \alpha)\right|^2 \mathrm{d}t \ll_{\sigma,\alpha} T.$$

This, together with (1), implies the bound

$$\int_{T_0}^{T} \left|\zeta(\sigma + \mathrm{i}t; \mathfrak{a})\right|^2 \mathrm{d}t \ll_{\sigma,\mathfrak{a}} T.$$

From this it follows

$$\int_{T_0}^{|t|+|a|\gamma(\tau)} \left|\zeta(\sigma + \mathrm{i}u; \mathfrak{a})\right|^2 \mathrm{d}u \ll_{\sigma,\mathfrak{a}} \bigl(|t| + |a|\gamma(\tau)\bigr).$$

Therefore, for $X \geqslant T_0$, we have that

$$\begin{aligned}
&\int_{X}^{2X} \left|\zeta\bigl(\sigma + \mathrm{i}a\gamma(\tau) + \mathrm{i}t; \mathfrak{a}\bigr)\right|^2 \mathrm{d}\tau \\
&\quad = \frac{1}{a} \int_{X}^{2X} \frac{1}{\gamma'(\tau)} \left|\zeta\bigl(\sigma + \mathrm{i}a\gamma(\tau) + \mathrm{i}t; \mathfrak{a}\bigr)\right|^2 \mathrm{d}\gamma(\tau) \\
&\quad \ll_a \max_{X \leqslant \tau \leqslant 2X} \frac{1}{\gamma'(\tau)} \left| \int_{X}^{2X} \mathrm{d}\left(\int_{T_0}^{t+a\gamma(\tau)} \left|\zeta(\sigma + \mathrm{i}u; \mathfrak{a})\right|^2 \mathrm{d}u \right) \right| \\
&\quad \ll_{a,\sigma,\mathfrak{a}} \bigl(|t| + |a|\gamma(2X)\bigr) \max_{X \leqslant \tau \leqslant 2X} \frac{1}{\gamma'(\tau)} \ll_{a,\sigma,\mathfrak{a}} X\bigl(1 + |t|\bigr)
\end{aligned}$$

because $\gamma(\tau) \in U_1(T_0)$. Taking $T2^{-k-1}$ and summing over $k = 0, 1, \dots,$ give the estimate of the lemma. □

Lemma 6. *Let $(\gamma_1(\tau), \dots, \gamma_r(\tau)) \in U_r(T_0)$. Then, for every fixed σ, $1/2 < \sigma < 1$, and $t \in \mathbb{R}$,*

$$\int_{T_0}^{T} \left|\zeta\bigl(\sigma + \mathrm{i}\gamma_j(\tau) + \mathrm{i}t; \mathfrak{a}\bigr)\right|^2 \mathrm{d}\tau \ll_{\sigma} T\bigl(1 + |t|\bigr)$$

for $j = 1, \dots, r$.

Proof. Using the notation of Lemma 5, we have

$$\begin{aligned}
&\int_X^{2X} \left|\zeta\bigl(\sigma + \mathrm{i}\gamma_j(\tau) + \mathrm{i}t; \mathfrak{a}\bigr)\right|^2 \mathrm{d}\tau \\
&\quad = \int_X^{2X} \frac{1}{\gamma_j'(\tau)} \left|\zeta\bigl(\sigma + \mathrm{i}\gamma_j(\tau) + \mathrm{i}t; \mathfrak{a}\bigr)\right|^2 \mathrm{d}\gamma_j(\tau) \\
&\quad = \int_X^{2X} \frac{(1 + o(1))}{\hat{\gamma}_j(\tau)} \,\mathrm{d}\Biggl(\int_{T_0}^{t+\gamma_j(\tau)} \left|\zeta(\sigma + \mathrm{i}u; \mathfrak{a})\right|^2 \mathrm{d}u\Biggr) \\
&\quad = \int_X^{2X} \frac{1}{\hat{\gamma}_j(\tau)} \,\mathrm{d}\Biggl(\int_{T_0}^{t+\gamma_j(\tau)} \left|\zeta(\sigma + \mathrm{i}u; \mathfrak{a})\right|^2 \mathrm{d}u\Biggr) \\
&\qquad + \int_X^{2X} \frac{o(1)(1 + o(1))}{\gamma_j'(\tau)} \,\mathrm{d}\Biggl(\int_{T_0}^{t+\gamma_j(\tau)} \left|\zeta(\sigma + \mathrm{i}u; \mathfrak{a})\right|^2 \mathrm{d}u\Biggr) \\
&\quad \ll_{\sigma,\mathfrak{a}} |t| + \gamma_j(2X) \max_{X \leqslant \tau \leqslant 2X} \frac{1}{\hat{\gamma}_j(\tau)} + \int_X^{2X} o(1) \left|\zeta\bigl(\sigma + \mathrm{i}\gamma_j(\tau) + \mathrm{i}t; \mathfrak{a}\bigr)\right|^2 \mathrm{d}\tau.
\end{aligned}$$

Hence,

$$\int_X^{2X} \left|\zeta\bigl(\sigma + \mathrm{i}\gamma_j(\tau) + \mathrm{i}t; \mathfrak{a}\bigr)\right|^2 \mathrm{d}\tau \ll_{\sigma,\mathfrak{a}} X\bigl(1 + |t|\bigr)\bigl(1 + r(X)\bigr) \ll_{\sigma,\mathfrak{a}} X\bigl(1 + |t|\bigr),$$

where $r(X) \to 0$ as $X \to \infty$. This proves the lemma. □

Lemmas 5 and 6 have their modifications for

$$\zeta(s, \omega; \mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m \omega(m)}{m^s}, \quad \sigma > 1,$$

with $\omega \in \Omega$. We note that the latter series is uniformly convergent on compact subsets of the strip D for almost all ω with respect to the Haar measure on $(\Omega, \mathcal{B}(\Omega))$.

Lemma 7. *Suppose that $\gamma(\tau) \in U_1(T_0)$. Then, for every fixed σ, $1/2 < \sigma < 1$, and $t \in \mathbb{R}$,*

$$\int_{T_0}^{T} \left|\zeta\bigl(\sigma + \mathrm{i}a\gamma(\tau) + \mathrm{i}t, \omega; \mathfrak{a}\bigr)\right|^2 \mathrm{d}\tau \ll_{\sigma,a,\mathfrak{a}} T\bigl(1 + |t|\bigr)$$

for almost all $\omega \in \Omega$.

Proof. Since, for almost all $\omega \in \Omega$, see [20],

$$\int_{T_0}^{T} \left|\zeta(\sigma + \mathrm{i}t, \omega; \mathfrak{a})\right|^2 \mathrm{d}t \ll_{\sigma,\mathfrak{a}} T, \tag{7}$$

the proof coincides with that of Lemma 5. □

Lemma 8. *Let $(\gamma_1(\tau), \dots, \gamma_r(\tau)) \in U_r(T_0)$. Then, for every fixed σ, $1/2 < \sigma < 1$, and $t \in \mathbb{R}$,*

$$\int_{T_0}^{T} \left|\zeta\bigl(\sigma + \mathrm{i}a\gamma_j(\tau) + \mathrm{i}t, \omega; \mathfrak{a}\bigr)\right|^2 \mathrm{d}\tau \ll_{\sigma,\mathfrak{a}} T\bigl(1 + |t|\bigr)$$

for almost all $\omega \in \Omega$, $j = 1, \dots, r$.

Proof. We repeat the proof of Lemma 6 and apply the estimate (7). □

Now, we will apply Lemmas 5–8 for the approximation of $\underline{\zeta}(s; \underline{\mathfrak{a}})$ by $\underline{\zeta}_n(s; \underline{\mathfrak{a}})$. For $g_1, g_2 \in H(D)$, let

$$\rho(g_1, g_2) = \sum_{l=1}^{\infty} 2^{-l} \frac{\sup_{s \in K_l} |g_1(s) - g_2(s)|}{1 + \sup_{s \in K_l} |g_1(s) - g_2(s)|},$$

where $\{K_l\colon\ l \in \mathbb{N}\} \subset D$ is a sequence of compact sets such that

$$D = \bigcup_{l=1}^{\infty} K_l,$$

$K_l \subset K_{l+1}$, for all $l \in \mathbb{N}$, and if $K \subset D$ is a compact set, then $K \subset K_l$ for some l. Then ρ is a metric in $H(D)$ inducing its topology of uniform convergence on compacta. Let $\underline{g}_1 = (g_{11}, \dots, g_{1r})$, $\underline{g}_2 = (g_{21}, \dots, g_{2r}) \in H^r(D)$. Then taking

$$\underline{\rho}(\underline{g}_1, \underline{g}_2) = \max_{1 \leqslant j \leqslant r} \rho_j(g_{1j}, g_{2j})$$

gives the metric in the space $H^r(D)$ inducing its product topology.

Lemma 9. *Suppose that $a_1, \dots, a_r \in \mathbb{R} \setminus \{0\}$ and $\gamma(\tau) \in U_1(T_0)$. Then*

$$\lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{T - T_0} \int_{T_0}^{T} \underline{\rho}\bigl(\underline{\zeta}\bigl(s + \mathrm{i}\underline{a}\gamma(\tau); \underline{\mathfrak{a}}\bigr), \underline{\zeta}_n\bigl(s + \mathrm{i}\underline{a}\gamma(\tau); \underline{\mathfrak{a}}\bigr)\bigr) \,\mathrm{d}\tau = 0.$$

Moreover, for almost all $\underline{\omega} \in \underline{\Omega}$,

$$\lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{T - T_0} \int_{T_0}^{T} \underline{\rho}\bigl(\underline{\zeta}\bigl(s + \mathrm{i}\underline{a}\gamma(\tau), \underline{\omega}; \underline{\mathfrak{a}}\bigr), \underline{\zeta}_n\bigl(s + \mathrm{i}\underline{a}\gamma(\tau), \underline{\omega}; \underline{\mathfrak{a}}\bigr)\bigr) \,\mathrm{d}\tau = 0.$$

Proof. From the definitions of the metrics $\underline{\rho}$ and ρ it follows that it is sufficient to prove that, for every compact set $K \subset D$,

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T-T_0}\int_{T_0}^{T}\sup_{s\in K}\left|\zeta\left(s+\mathrm{i}a_j\gamma(\tau);\mathfrak{a}_j\right)-\zeta_n\left(s+\mathrm{i}a_j\gamma(\tau);\mathfrak{a}_j\right)\right|\mathrm{d}\tau=0$$

for all $j=1,\dots,r$.

Let $\mathfrak{a}$ and $a\neq 0$ be arbitrary. The definition of $\zeta_n(s;\mathfrak{a})$ and the classical Mellin formula

$$\frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\Gamma(s)b^{-s}\,\mathrm{d}s=\mathrm{e}^{-b},\quad b,c>0,$$

where $\Gamma(s)$ denotes the Euler gamma-function, yield the integral representation [20]

$$\zeta_n(s;\mathfrak{a})=\frac{1}{2\pi\mathrm{i}}\int_{\theta-\mathrm{i}\infty}^{\theta+\mathrm{i}\infty}\zeta(s+z;\mathfrak{a})l_n(z)\frac{\mathrm{d}z}{z},\quad l_n(s)=\frac{s}{\theta}\Gamma\left(\frac{s}{\theta}\right)n^s.$$

Therefore, taking $\theta_1>0$, we have

$$\zeta_n(s;\mathfrak{a})-\zeta(s;\mathfrak{a})=\frac{1}{2\pi\mathrm{i}}\int_{-\theta_1-\mathrm{i}\infty}^{-\theta_1+\mathrm{i}\infty}\zeta(s+z;\mathfrak{a})l_n(z)\frac{\mathrm{d}z}{z}+R_n(s;\mathfrak{a}),\tag{8}$$

where

$$R_n(s;\mathfrak{a})=r_{\mathfrak{a}}\frac{l_n(1-s)}{1-s}.$$

Let $K\subset D$ be an arbitrary compact set. Denote by $\sigma+\mathrm{i}v$ the points of K, and fix $\varepsilon>0$ such that $1/2+2\varepsilon\leqslant\sigma\leqslant 1-\varepsilon$. Then, by (8),

$$\begin{aligned}&\left|\zeta\left(s+\mathrm{i}a\gamma(\tau);\mathfrak{a}\right)-\zeta_n\left(s+\mathrm{i}a\gamma(\tau);\mathfrak{a}\right)\right|\\&\quad\ll\int_{-\infty}^{\infty}\left|\zeta\left(s+\mathrm{i}a\gamma(\tau)-\theta_1+\mathrm{i}t;\mathfrak{a}\right)\right|\frac{|l_n(-\theta_1+\mathrm{i}t)|}{|-\theta_1+\mathrm{i}t|}\,\mathrm{d}t+\left|R_n\left(s+\mathrm{i}a\gamma(\tau);\mathfrak{a}\right)\right|.\end{aligned}$$

Thus,

$$\frac{1}{T-T_0}\int_{T_0}^{T}\sup_{s\in K}\left|\zeta\left(s+\mathrm{i}a\gamma(\tau);\mathfrak{a}\right)-\zeta_n v\left(s+\mathrm{i}a\gamma(\tau);\mathfrak{a}\right)\right|\mathrm{d}\tau\ll I_1+I_2,\tag{9}$$

where

$$I_1=\int_{-\infty}^{\infty}\frac{1}{T-T_0}\int_{T_0}^{T}\left(\left|\zeta\left(\frac{1}{2}+\varepsilon+\mathrm{i}\left(t+a\gamma(\tau)\right);\mathfrak{a}\right)\right|\mathrm{d}\tau\right)\sup_{s\in K}\frac{|l_n(\frac{1}{2}+\varepsilon-s+\mathrm{i}t)|}{|\frac{1}{2}+\varepsilon-s+\mathrm{i}t|}\,\mathrm{d}t$$

and

$$I_2 = \frac{1}{T-T_0}\int_{T_0}^{T} \sup_{s\in K}\bigl|R_n\bigl(s+\mathrm{i}a\gamma(\tau);\mathfrak{a}\bigr)\bigr|\,\mathrm{d}\tau.$$

Since in the definition of $l_n(s)$ the gamma-function occurs, we can use the estimate

$$\Gamma(\sigma+\mathrm{i}t) \ll \exp\bigl\{-c|t|\bigr\}, \quad c>0,$$

which is uniform in σ, $\sigma_1 \leqslant \sigma \leqslant \sigma_2$, for arbitrary $\sigma_1 < \sigma_2$. Therefore, for $s \in K$,

$$\begin{aligned}\frac{|l_n(\frac{1}{2}+\varepsilon-s+\mathrm{i}t)|}{|\frac{1}{2}+\varepsilon-s+\mathrm{i}t|} &= \frac{n^{1/2+\varepsilon-\sigma}}{\theta}\left|\Gamma\left(\frac{\frac{1}{2}+\varepsilon-\sigma}{\theta}+\frac{\mathrm{i}(t-v)}{\theta}\right)\right| \\ &\ll_{\theta,K} n^{-\varepsilon}\exp\left\{-\frac{c_1}{\theta}|t|\right\}, \quad c_1>0. \qquad (10)\end{aligned}$$

Similarly, we find

$$R_n\bigl(s+\mathrm{i}a\gamma(\tau);\mathfrak{a}\bigr) \ll_{\theta,\mathfrak{a},K} n^{1-\sigma}\exp\left\{-\frac{c_2}{\theta}|a|\gamma(\tau)\right\}, \quad c_2>0. \qquad (11)$$

Now, putting $\theta = 1/2+\varepsilon$, and estimate (10) together with Lemma 5 yield

$$I_1 \ll_{\varepsilon,K,\mathfrak{a}} n^{-\varepsilon}\int_{-\infty}^{\infty}\bigl(1+|t|\bigr)\exp\bigl\{-c_3|t|\bigr\}\,\mathrm{d}t \ll_{\varepsilon,K,\mathfrak{a}} n^{-\varepsilon}, \quad c_3>0. \qquad (12)$$

Moreover, properties of the functions $\gamma(\tau)$ and (11) show that with $c_4>0$

$$\begin{aligned}I_2 &\ll_{\varepsilon,\mathfrak{a},K} n^{1/2-2\varepsilon}\frac{1}{T-T_0}\int_{T_0}^{T}\exp\bigl\{-c_4|a|\gamma(\tau)\bigr\}\,\mathrm{d}\tau \\ &\ll_{\varepsilon,\mathfrak{a},K} n^{1/2-2\varepsilon}\left(\frac{\log T}{T}+\frac{1}{T}\int_{\log T}^{T}\exp\bigl\{-c_4|a|\gamma(\tau)\bigr\}\,\mathrm{d}\tau\right) \\ &\ll n^{1/2-2\varepsilon}\left(\frac{\log T}{T}+\frac{1}{T}\exp\left\{-\frac{c_4}{2}|a|\gamma(\log T)\right\}\int_{\log T}^{T}\exp\left\{-\frac{c_4}{2}|a|\gamma(\tau)\right\}\mathrm{d}\tau\right) \\ &= o(T)\end{aligned}$$

as $T\to\infty$. This, (12) and (9) prove the first assertion of the lemma.

For almost all $\omega \in \Omega$, the function $\zeta(s,\omega;\mathfrak{a})$ is analytic in the half-plane $\sigma > 1/2$. Therefore, the second assertion of the lemma is obtained similarly to that of the first with using Lemma 7. In this case, we have not the integral I_2. □

Lemma 10. *Suppose that* $(\gamma_1(\tau), \ldots, \gamma_r(\tau)) \in U_r(T_0)$. *Then*

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T-T_0} \int_{T_0}^{T} \underline{\rho}\big(\underline{\zeta}\big(s+\mathrm{i}\underline{\gamma}(\tau); \underline{\mathfrak{a}}\big), \underline{\zeta}_n\big(s+\mathrm{i}\underline{\gamma}(\tau); \underline{\mathfrak{a}}\big)\big)\,\mathrm{d}\tau = 0.$$

Moreover, for almost all $\underline{\omega} \in \underline{\Omega}$,

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T-T_0} \int_{T_0}^{T} \underline{\rho}\big(\underline{\zeta}\big(s+\mathrm{i}\underline{\gamma}(\tau), \underline{\omega}; \underline{\mathfrak{a}}\big), \underline{\zeta}_n\big(s+\mathrm{i}\underline{\gamma}(\tau), \underline{\omega}; \underline{\mathfrak{a}}\big)\big)\,\mathrm{d}\tau = 0.$$

Proof. We use Lemmas 6 and 8 and follow the proof of Lemma 9. □

5 Limit theorems for $\underline{\zeta}(s; \mathfrak{a})$

The results of Sections 3 and 4 are sufficient to prove limit theorems for $\underline{\zeta}(s; \mathfrak{a})$ without explicit forms of limit measures. Together with P_T^1 and P_T^r, we will prove the weak convergence, as $T \to \infty$, for

$$P_{T,\underline{\omega}}^1(A) = \frac{1}{T-T_0} \operatorname{meas}\big\{\tau \in [T_0, T]\colon\ \underline{\zeta}\big(s+\mathrm{i}\underline{a}\gamma(\tau), \underline{\omega}; \underline{\mathfrak{a}}\big) \in A\big\},$$

and

$$P_{T,\underline{\omega}}^r(A) = \frac{1}{T-T_0} \operatorname{meas}\big\{\tau \in [T_0, T]\colon\ \underline{\zeta}\big(s+\mathrm{i}\underline{\gamma}(\tau), \underline{\omega}; \underline{\mathfrak{a}}\big) \in A\big\},$$

where $A \in \mathcal{B}(H^r(D))$ and $\underline{\omega} \in \underline{\Omega}$.

Theorem 4. *Suppose that* $\underline{a}$ *and* $\gamma(\tau)$ *satisfy hypotheses of Theorem* 2. *Then, on* $(H^r(D), \mathcal{B}(H^r(D)))$, *there exists a probability measure* P^1 *such that* P_T^1 *and* $P_{T,\underline{\omega}}^1$ *both converge weakly to* P^1 *as* $T \to \infty$.

Proof. Let, for brevity, $V_n = m_H^r u_n^{-1}$, where u_n is the mapping from Lemma 3. Using the absolute convergence for the series $\zeta_n(s; \mathfrak{a}_j)$, we obtain by a standard way, see, for example, [14], that the sequence of probability measures $\{V_n\colon\ n \in \mathbb{N}\}$ is tight, i.e., for every $\varepsilon > 0$, there exists a compact set $K = K(\varepsilon) \subset H^r(D)$ such that $V_n(K) > 1-\varepsilon$ for all $n \in \mathbb{N}$. Then, by the Prokhorov theorem [4], the sequence $\{V_n\}$ is relatively compact. In what follows, we will use the language of random elements. Let θ_T be a random variable on a certain probability space with measure μ, and uniformly distributed on $[T_0, T]$. Define the $H^r(D)$-valued random element

$$\underline{X}_{T,n}^1 = \underline{X}_{T,n}^1(s) = \underline{\zeta}_n\big(s+\mathrm{i}\underline{a}\gamma(\theta_T); \underline{\mathfrak{a}}\big),$$

and denote by $\underline{X}_n^1 = \underline{X}_n^1(s)$ the $H^r(D)$-valued random element with the distribution V_n. Then the assertion of Lemma 3 can be written in the form

$$\underline{X}_{T,n}^1 \xrightarrow[T\to\infty]{\mathcal{D}} \underline{X}_n^1. \tag{13}$$

The relative compactness of $\{V_n\}$ implies the existence of subsequences $\{V_{n_k}\}$ such that V_{n_k} converges weakly to a certain probability measure P^1 on $(H^r(D), \mathcal{B}(H^r(D)))$ as $k \to \infty$. Thus,

$$\underline{X}^1_{n_k} \xrightarrow[k\to\infty]{\mathcal{D}} P^1. \tag{14}$$

Define one more $H^r(D)$-valued random element

$$\underline{X}^1_T = \underline{X}^1_T(s) = \underline{\zeta}\big(s + \mathrm{i}\underline{a}\gamma(\theta_T); \underline{\mathfrak{a}}\big).$$

Then, by the first assertion of Lemma 9, we find that, for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \limsup_{T\to\infty} \mu\big\{\underline{\rho}\big(\underline{X}^1_T, \underline{X}^1_{T,n}\big) \geqslant \varepsilon\big\}$$
$$\leqslant \lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T-T_0} \int\limits_{T_0}^{T} \underline{\rho}\big(\underline{\zeta}\big(s + \mathrm{i}\underline{a}\gamma(\tau), \underline{\mathfrak{a}}\big), \underline{\zeta}_n\big(s + \mathrm{i}\underline{a}\gamma(\tau), \underline{\mathfrak{a}}\big)\big)\, \mathrm{d}\tau = 0.$$

This, (13) and (14) show that all hypotheses of Theorem 4.2 from [4] are satisfied. Therefore, we have the relation

$$\underline{X}^1_T \xrightarrow[T\to\infty]{\mathcal{D}} P^1, \tag{15}$$

or that P^1_T converges weakly to P^1 as $T \to \infty$. Also, in view of (15), the measure P^1 is independent of the subsequence $\{V_{n_k}\}$. Thus,

$$\underline{X}^1_n \xrightarrow[T\to\infty]{\mathcal{D}} P^1. \tag{16}$$

To obtain the weak convergence for $P^1_{T,\underline{\omega}}$, introduce the $H^r(D)$-valued random elements

$$\underline{X}^1_{T,n,\underline{\omega}} = \underline{X}^1_{T,n,\underline{\omega}}(s) = \underline{\zeta}_n\big(s + \mathrm{i}\underline{a}\gamma(\theta_T), \underline{\omega}; \underline{\mathfrak{a}}\big)$$

and

$$\underline{X}^1_{T,\underline{\omega}} = \underline{X}^1_{T,\underline{\omega}}(s) = \underline{\zeta}\big(s + \mathrm{i}\underline{a}\gamma(\theta_T), \underline{\omega}; \underline{\mathfrak{a}}\big).$$

Then, repeating the above arguments for $\underline{X}^1_{T,n,\underline{\omega}}$ and $\underline{X}^1_{T,\underline{\omega}}$ (all relations are true for almost all $\underline{\omega} \in \underline{\Omega}^r$) and using (16), we obtain the weak convergence of $P^1_{T,\underline{\omega}}$ to P^1 as $T \to \infty$. The theorem is proved. □

Theorem 5. *Suppose that $(\gamma_1(\tau), \dots, \gamma_r(\tau)) \in U_r(T_0)$. Then, on $(H^r(D), \mathcal{B}(H^r(D)))$, there exists a probability measure P^r such that P^r_T and $P^r_{T,\underline{\omega}}$ both converge weakly to P^r as $T \to \infty$.*

Proof. We use arguments similar to those of the proof of Theorem 4 with application of Lemmas 4 and 10. □

6 Identification of the limit measures

In this section, we identify the measures P^1 and P^r in Theorems 4 and 5. For this, we will use some results of ergodic theory.

For brevity, let, for $\tau \geqslant T_0$,

$$\underline{a}_\tau^1 = \{(p^{-\mathrm{i}a_1\gamma(\tau)}\colon p \in \mathbb{P}), \dots, (p^{-\mathrm{i}a_r\gamma(\tau)}\colon p \in \mathbb{P})\}$$

and

$$\underline{a}_\tau^r = \{(p^{-\mathrm{i}\gamma_1(\tau)}\colon p \in \mathbb{P}), \dots, (p^{-\mathrm{i}\gamma_r(\tau)}\colon p \in \mathbb{P})\}.$$

Clearly, $\underline{a}_\tau^1, \underline{a}_\tau^r \in \underline{\Omega}^r$. On $\underline{\Omega}^r$, define the families of transformations $\{\Phi_\tau^1\colon \tau \geqslant T_0\}$ and $\{\Phi_\tau^r\colon \tau \geqslant T_0\}$, where

$$\Phi_\tau^1(\underline{\omega}) = \underline{a}_\tau^1 \underline{\omega} \quad \text{and} \quad \Phi_\tau^r(\underline{\omega}) = \underline{a}_\tau^r \underline{\omega}, \quad \underline{\omega} \in \underline{\Omega}^r.$$

Then $\{\Phi_\tau^1\}$ and $\{\Phi_\tau^r\}$ are families of measurable measure preserving (because of invariance of the Haar measure m_H^r) transformations on $\underline{\Omega}^r$. Recall that a set $A \in \mathcal{B}(\underline{\Omega}^r)$ is called invariant with respect to $\{\Phi_\tau^k\colon \tau \geqslant T_0\}$ if, for every $\tau \geqslant T_0$, the sets A and $A_\tau = \Phi_\tau^k(A)$ can differ one from other at most by a set of m_H^r-measure zero, $k = 1$ or $k = r$. All invariant sets forms a σ-field. The family $\{\Phi_\tau^k\}$ is called ergodic if its σ-field of invariant sets consists only from sets of m_H^r-measure zero or one.

Lemma 11. *The families $\{\Phi_\tau^1\}$ and $\{\Phi_\tau^r\}$ are ergodic.*

Proof. We consider only $\{\Phi_\tau^1\}$ because the case $\{\Phi_\tau^r\}$ is similar, and apply the Fourier transform method. In the proof of Lemma 1, we already have used that the characters χ of the group $\underline{\Omega}^r$ are of the form

$$\chi(\underline{\omega}) = \prod_{j=1}^r {\prod_{p\in\mathbb{P}}}^* \omega_j^{k_{jp}}(p).$$

Thus, if the character χ is nontrivial ($\chi(\underline{\omega}) \not\equiv 1$), we have

$$\chi(\underline{a}_\tau^1) = \prod_{j=1}^r {\prod_{p\in\mathbb{P}}}^* p^{-\mathrm{i}a_j k_{jp}\gamma(\tau)} = \exp\left\{-\mathrm{i}\gamma(\tau)\sum_{j=1}^r a_j {\sum_{p\in\mathbb{P}}}^* k_{jp}\log p\right\}.$$

Since the character χ is nontrivial, $\underline{k} \neq (\underline{0}, \dots, \underline{0})$. Thus, in the proof of Lemma 1, we have seen that

$$\sum_{j=1}^r a_j {\sum_{p\in\mathbb{P}}}^* k_{jp}\log p \neq 0.$$

Therefore, there exists a value $\tau_0 \geqslant T_0$ such that

$$\chi(\underline{a}_{\tau_0}^1) \neq 1. \tag{17}$$

Now, let A be a invariant set with respect to $\{\Phi_\tau^1\}$, and let I_A is its indicator function. Then, for almost all $\underline{\omega} \in \underline{\Omega}^r$,

$$I_A\big(\underline{a}_\tau^1 \underline{\omega}\big) = I_A(\underline{\omega}).$$

Thus, in view of the invariance of m_H^r, the Fourier transform $\hat{I}_A(\chi)$ is

$$\begin{aligned}\hat{I}_A(\chi) &= \int\limits_{\underline{\Omega}^r} \chi(\underline{\omega}) I_A(\underline{\omega})\,\mathrm{d}m_H^r = \int\limits_{\underline{\Omega}^r} \chi\big(\underline{a}_{\tau_0}^1 \underline{\omega}\big) I_A\big(\underline{a}_{\tau_0}^1 \underline{\omega}\big)\,\mathrm{d}m_H^r \\ &= \chi\big(\underline{a}_{\tau_0}^1\big) \int\limits_{\underline{\Omega}^r} \chi(\underline{\omega}) I_A(\underline{\omega})\,\mathrm{d}m_H^r = \chi\big(\underline{a}_{\tau_0}^1\big) \hat{I}_A(\chi).\end{aligned}$$

Therefore, taking into account (17), we have

$$\hat{I}_A(\chi) = 0 \tag{18}$$

for all nontrivial characters of $\underline{\Omega}^r$.

Denote by χ_0 the trivial character of $\underline{\Omega}^r$, and suppose that $\hat{I}(\chi_0) = c$. Then using the orthogonality of characters and (18) give the equality

$$\hat{I}_A(\chi) = c \int\limits_{\underline{\Omega}^r} \chi(\underline{\omega})\,\mathrm{d}m_H^r = c\hat{1}(\chi) = \hat{c}(\chi)$$

for every character χ of $\underline{\Omega}^r$. This shows that $I_A(\underline{\omega}) = c$ for almost all $\underline{\omega} \in \underline{\Omega}^r$. Since $c = 0$ or $c = 1$, we obtain that $m_H^r(A) = 0$ or $m_H^r(A) = 1$. The lemma is proved. □

Lemma 11 allows to identify the limit measures in Theorems 4 and 5. On the probability space $(\underline{\Omega}^r, \mathcal{B}(\underline{\Omega}^r), m_H^r)$, define the $H^r(D)$-valued random element

$$\underline{\zeta}(s, \underline{\omega}; \underline{\mathfrak{a}}) = \big(\zeta(s, \omega_1; \mathfrak{a}_1), \ldots, \zeta(s, \omega_r; \mathfrak{a}_r)\big),$$

where

$$\zeta(s, \omega_j; \mathfrak{a}_j) = \sum_{m=1}^{\infty} \frac{a_{jm}\omega_j(m)}{m^s}, \quad j = 1, \ldots, r.$$

We note that the latter series, for almost all ω_j, are uniformly convergent on compact subsets of D. Moreover, in view of multiplicativity of a_{jm}, for almost all ω_j, the equality

$$\zeta(s, \omega_j; \mathfrak{a}_j) = \prod_{p \in \mathbb{P}} \left(1 + \sum_{k=1}^{\infty} \frac{a_{jp^k}\omega_j^k(p)}{p^{ks}}\right)$$

holds. Let $P_{\underline{\zeta}}$ be the distribution of the random element $\underline{\zeta}(s, \underline{\omega}; \underline{\mathfrak{a}})$, i.e.,

$$P_{\underline{\zeta}}(A) = m_H^r\big\{\underline{\omega} \in \underline{\Omega}^r\colon \underline{\zeta}(s, \underline{\omega}; \underline{\mathfrak{a}}) \in A\big\}, \quad A \in \mathcal{B}\big(H^r(D)\big).$$

Theorem 6. *Under hypotheses of Theorems* 2 *and* 3*,* P_T^1 *and* P_T^r *converge weakly to the measure* $P_{\underline{\zeta}}$ *as* $T \to \infty$*.*

Proof. In view of Theorems 4 and 5, it suffices to prove that P^1 and P^r coincides with $P_{\underline{\zeta}}$. We consider only the case of P^1.

Let A be a fixed continuity set of the measure P^1, i.e., $P^1(\partial A) = 0$, where ∂A is the boundary of A. Then the equivalent of weak convergence of probability measures in terms of continuity sets [4] and Theorem 4 imply

$$\lim_{T\to\infty} \frac{1}{T-T_0} \operatorname{meas}\bigl\{\tau \in [T_0, T]\colon \underline{\zeta}\bigl(s + \mathrm{i}\underline{a}\gamma(\tau), \underline{\omega}; \underline{\mathfrak{a}}\bigr) \in A\bigr\} = P^1(A). \tag{19}$$

On the probability space $(\underline{\Omega}^r, \mathcal{B}(\underline{\Omega}^r), m_H^r)$, define the random variable

$$\theta(\underline{\omega}) = \begin{cases} 1 & \text{if } \underline{\zeta}(s, \underline{\omega}; \underline{\mathfrak{a}}) \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, the expectation of $\theta(\underline{\omega})$ is

$$\mathbf{E}\theta = \int_{\underline{\Omega}^r} \theta \, \mathrm{d}m_H^r = m_H^r\bigl\{\underline{\omega} \in \underline{\Omega}^r\colon \underline{\zeta}(s, \underline{\omega}; \underline{\mathfrak{a}}) \in A\bigr\} = P_{\underline{\zeta}}(A). \tag{20}$$

In view of Lemma 11, the random process $\theta(\Phi_\tau^1(\underline{\omega}))$ is ergodic. Therefore, by the Birkhoff–Khintchine ergodic theorem [5], for almost all $\underline{\omega} \in \underline{\Omega}^r$,

$$\lim_{T\to\infty} \frac{1}{T-T_0} \int_{T_0}^{T} \theta\bigl(\Phi_\tau^1(\underline{\omega})\bigr) \, \mathrm{d}\tau = \mathbf{E}\theta. \tag{21}$$

On the other hand, by the definitions of θ and Φ_τ^1,

$$\frac{1}{T-T_0} \int_{T_0}^{T} \theta\bigl(\Phi_\tau^1(\underline{\omega})\bigr) \, \mathrm{d}\tau = \frac{1}{T-T_0} \operatorname{meas}\bigl\{\tau \in [T_0, T]\colon \underline{\zeta}\bigl(s + \mathrm{i}\underline{a}\gamma(\tau), \underline{\omega}; \underline{\mathfrak{a}}\bigr) \in A\bigr\}.$$

Thus, in virtue of (20) and (21),

$$\lim_{T\to\infty} \frac{1}{T-T_0} \operatorname{meas}\bigl\{\tau \in [T_0, T]\colon \underline{\zeta}\bigl(s + \mathrm{i}\underline{a}\gamma(\tau), \underline{\omega}; \underline{\mathfrak{a}}\bigr) \in A\bigr\} = P_{\underline{\zeta}}(A).$$

This, together with (19), implies the equality $P^1(A) = P_{\underline{\zeta}}(A)$ for all continuity sets A of P^1. Hence, $P^1(A) = P_{\underline{\zeta}}(A)$ for all $A \in \mathcal{B}(H^r(D))$. The theorem is proved. □

7 Support

For the proof of universality theorems, supports of limit measures in the space of analytic functions play the crucial role. Recall that the support of a probability measure P on

$(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ is a minimal closed set S_P such that $P(S_P) = 1$. The set S_P consists of all elements $x \in \mathbb{X}$ such that, for every open neighbourhood G of x, the inequality $P(G) > 0$ is satisfied.

Theorem 7. *Suppose that the sequences* $\mathfrak{a}_1, \dots, \mathfrak{a}_r$ *are multiplicative. Then the support of the measure* $P_{\underline{\zeta}}$ *is the set*

$$\big(\big\{g \in H(D)\colon\ g(s) \neq 0 \text{ or } g(s) \equiv 0\big\}\big)^r.$$

Proof. Denote by m_{jH} the probability Haar measure on $(\Omega_j, \mathcal{B}(\Omega_j))$. Then m_H^r is the product of the measures $m_{1H}, \dots, m_{rH}$, i.e., for $A = A_1 \times \dots \times A_r \in \mathcal{B}(H^r(D))$ with $A_j \in \mathcal{B}(H(D))$,

$$m_H^r(A) = m_{1H}(A_1) \cdots m_{rH}(A_r).$$

The space $H^r(D)$ is separable, therefore [4],

$$\mathcal{B}\big(H^r(D)\big) = \underbrace{\mathcal{B}\big(H(D)\big) \times \dots \times \mathcal{B}\big(H(D)\big)}_{r}.$$

Thus, it suffices to consider $P_{\underline{\zeta}}$ on the sets $A = A_1 \times \dots \times A_r$, $A_1, \dots, A_r \in \mathcal{B}(H(D))$. It is known [20] that the supports of the measures

$$P_{\zeta_j}(A) = m_{jH}\big\{\omega_j \in \Omega_j\colon\ \zeta(s, \omega_j; \mathfrak{a}_j) \in A_j\big\}, \quad A_j \in \mathcal{B}\big(H(D)\big),\ j = 1, \dots, r,$$

is the set $\{g \in H(D)\colon\ g(s) \neq 0 \text{ or } g(s) \equiv 0\}$. Moreover, by the above remarks,

$$\begin{aligned} P_{\underline{\zeta}}(A) &= m_{jH}\big\{\underline{\omega} \in \underline{\Omega}^r\colon\ \underline{\zeta}(s, \underline{\omega}; \underline{\mathfrak{a}}) \in A\big\} \\ &= m_{1H}\big\{\omega_1 \in \Omega_1\colon\ \zeta(s, \omega_1; \mathfrak{a}_1) \in A_1\big\} \cdots m_{rH}\big\{\omega_r \in \Omega_r\colon\ \zeta(s, \omega_r; \mathfrak{a}_r) \in A_r\big\} \\ &= P_{\zeta_1}(A_1) \cdots P_{\zeta_r}(A_r). \end{aligned}$$

This, the supports of the measures P_{ζ_j} and the minimality of the support prove the theorem. □

8 Proof of universality

Theorems 2 and 3 easily follows from Theorems 6 and 7 as well as the Mergelyan theorem [22] on the approximation of analytic functions by polynomials. For convenience, we recall the latter beautiful theorem.

Lemma 12. *Suppose that* $K \subset \mathbb{C}$ *is a compact set with connected complements, and* $g(s)$ *is a continuous function on* K *and analytic in the interior of* K*. Then, for every* $\varepsilon > 0$*, there exists a polynomial* $p(s)$ *such that* $\sup_{s \in K} |g(s) - p(s)| < \varepsilon$.

Proof of Theorem 2.

1. Lemma 12 implies the existence of polynomials $p_1(s), \dots, p_r(s)$ such that

$$\sup_{1 \leqslant j \leqslant r} \sup_{s \in K_j} \big|f_j(s) - \mathrm{e}^{p_j(s)}\big| < \frac{\varepsilon}{2}. \tag{22}$$

Consider the set

$$G_\varepsilon = \left\{ (g_1, \dots, g_r) \in H^r(D)\colon \sup_{1 \leqslant j \leqslant r} \sup_{s \in K_j} \left| g_j(s) - \mathrm{e}^{p_j(s)} \right| < \frac{\varepsilon}{2} \right\}.$$

By Theorem 7, the set G_ε is an open neighbourhood of the element $(\mathrm{e}^{p_1(s)}, \dots, \mathrm{e}^{p_r(s)})$ of the support of the measure $P_{\underline{\zeta}}$. Thus, by a property of the support,

$$P_{\underline{\zeta}}(G_\varepsilon) > 0. \tag{23}$$

Therefore, Theorem 6, together with equivalent of weak convergence of probability measures in terms of open sets [4, Thm. 2.1], gives

$$\liminf_{T \to \infty} P_T^1(G_\varepsilon) \geqslant P_{\underline{\zeta}}(G_\varepsilon) > 0.$$

This, the definitions of P_T^1 and G_ε, and (22) prove the first part of the theorem.

2. Define one more set

$$\hat{G}_\varepsilon = \left\{ (g_1, \dots, g_r) \in H^r(D)\colon \sup_{1 \leqslant j \leqslant r} \sup_{s \in K_j} \left| g_j(s) - f_j(s) \right| < \varepsilon \right\}.$$

The boundary $\partial \hat{G}_\varepsilon$ of $\hat{G}_\varepsilon$ lies in the set

$$\left\{ (g_1, \dots, g_r) \in H^r(D)\colon \sup_{1 \leqslant j \leqslant r} \sup_{s \in K_j} \left| g_j(s) - f_j(s) \right| = \varepsilon \right\},$$

therefore, $\partial \hat{G}_{\varepsilon_1} \cap \partial \hat{G}_{\varepsilon_2} = \emptyset$ for different positive ε_1 and ε_2. From this we have that $P_{\underline{\zeta}}(\partial \hat{G}_\varepsilon) = 0$, i.e., the set $\hat{G}_\varepsilon$ is a continuity set of the measure $P_{\underline{\zeta}}$ for all but at most countably many $\varepsilon > 0$. Therefore, Theorem 6, together with equivalent of weak convergence of probability measures in terms of continuity sets [4, Thm. 2.1], shows that

$$\lim_{T \to \infty} P_T^1(\hat{G}_\varepsilon) = P_{\underline{\zeta}}(\hat{G}_\varepsilon) \tag{24}$$

for all but at most countably many $\varepsilon > 0$. In view of (22), the inclusion $G_\varepsilon^r \subset \hat{G}_\varepsilon^r$ follows. Thus, by (23), we have $P_{\underline{\zeta}}(\hat{G}_\varepsilon) > 0$. This, the definitions of P_T^1 and $\hat{G}_\varepsilon$, and (24) prove the second assertion of the theorem. □

Proof of Theorem 3. We repeat the proof of Theorem 2 with P_T^r in place of P_T^1. □

References

1. B. Bagchi, *The Statistical Behaviour and Universality Properties of the Riemann zeta-function and allied Dirichlet Series*, PhD thesis, Indian Statistical Institute, Calcutta, 1981.
2. B. Bagchi, Joint universality theorem for Dirichlet L-functions, *Math. Z.*, **181**(3):319–334, 1982.

3. A. Baker, The theory of linear forms in logarithms, in A. Baker, D.W. Masser (Eds.), *Transcendence Theory: Advances and Applications. Proceedings of a Conference Held in Cambridge in 1976*, Academic Press, Boston, MA, 1977, pp. 1–27.
4. P. Billingsley, *Convergence of Probability Measures*, Willey, New York, 1968.
5. H. Cramér, M. Leadbetter, *Stationary and Related Stochastic Processes*, Willey, New York, 1967.
6. S.M. Gonek, *Analytic Properties of Zeta and L-Functions*, PhD thesis, University of Michigan, Ann Arbor, 1979.
7. R. Kačinskaitė, K. Matsumoto, The mixed joint universality for a class of zeta-functions, *Math. Nachr.*, **288**(16):1900–1909, 2015.
8. R. Kačinskaitė, K. Matsumoto, Remarks on the mixed joint universality for a class of zeta-functions, *Bull. Aust. Math. Soc.*, **95**(2):187–198, 2017.
9. R. Kačinskaitė, K. Matsumoto, On mixed joint discrete universality for a class of zeta-functions. II, *Lith. Math. J.*, **59**(1):54–66, 2019.
10. J. Kaczorowski, Some remarks on the universality of periodic L-functions, in R. Steuding, J. Sreuding (Eds.), *New Directions in Value-Distribution Theory of Zeta and L-functions. Proceedings of the Würzburg Conference*, Shaker Verlag, Aachen, 2009, pp. 113–120.
11. A.A. Karatsuba, S.M. Voronin, *The Riemann Zeta-Function*, Walter de Gruyter, Berlin, 1992.
12. A. Laurinčikas, Joint discrete universality for periodic zeta-functions. II, *Quaest. Math.*, https://doi.org/10.2989/16073606.2019.1654554.
13. A. Laurinčikas, *Limit Theorems for the Riemann Zeta-Function*, Kluwer, Dordrecht, Boston, London, 1996.
14. A. Laurinčikas, Joint universality of zeta-functions with periodic coefficients, *Izv. Ross. Akad. Nauk, Ser. Mat.*, **74**:79–102, 2010.
15. A. Laurinčikas, On joint universality of Dirichlet L-functions, *Chebyshevskii Sb.*, **12**(1):124–139, 2011.
16. A. Laurinčikas, Extension of the universality of zeta-functions with periodic coefficients, *Sib. Math. J.*, **57**:330–339, 2016.
17. A. Laurinčikas, Joint discrete universality for periodic zeta-functions, *Quaest. Math.*, **42**(5): 687–699, 2019.
18. A. Laurinčikas, R. Macaitienė, On the joint universality for periodic zeta-functions, *Math. Notes*, **85**(1-2):51–60, 2009.
19. A. Laurinčikas, K. Matsumoto, The universality of zeta-functions attached to certain cusp forms, *Acta Arith.*, **98**:345–359, 2001.
20. A. Laurinčikas, D. Šiaučiūnas, Remarks on the universality of the periodic zeta-functions, *Math. Notes*, **80**(3-4):532–538, 2006.
21. K. Matsumoto, A survey on the theory of universality for zeta and L-functions, in M. Kaneko, Sh. Kanemitsu, J. Liu (Eds.), *Number Theory: Plowing and Starring Through High Wawe Forms. Proceedings of the 7th China–Japan Seminar, Fukuoka, Japan, 28 October–1 November, 2013*, Number Theory Appl., Vol. 11,, World Scientific, Singapore, 2015, pp. 95–144.

22. S.N. Mergelyan, Uniform approximations to functions of a complex variable, *Usp. Mat. Nauk*, **7**(2):31–122, 1952 (in Russian).
23. Ł. Pańkowski, Joint universality for dependent L-functions, *Ramanujan J.*, **45**:181–195, 2018.
24. J. Steuding, On Dirichlet series with periodic coefficients, *Ramanujan J.*, **6**:295–306, 2002.
25. J. Steuding, *Value-Distribution of L-Functions*, Lect. Notes Math., Vol. 1877, Springer, Berlin, Heidelberg, New York, 2007.
26. S.M. Voronin, On the functional independence of Dirichlet L-functions, *Acta Arith.*, **27**:493–503, 1975 (in Russian).
27. S.M. Voronin, Theorem on the "universality" of the Riemann zeta-function, *Izv. Akad. Nauk SSSR, Ser. Mat.*, **39**(3):475–486, 1975 (in Russian).