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Fourth-order elliptic problems with critical nonlinearities by a sublinear perturbation

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Abstract. In this paper, we get the existence of two positive solutions for a fourth-order problem with Navier boundary condition. Our nonlinearity has a critical growth, and the method is a local minimum theorem obtained by Bonanno.

Keywords: critical growth, biharmonic operator, variational methods, local minimum.

1 Introduction and main result

In this paper, we consider the following fourth-order problem:

$$\begin{aligned} \Delta^2 u &= \lambda\big(|u|^{2^*-2}u + \mu|u|^{q-2}u\big) \quad \text{in } \Omega, \\ u &= \Delta u = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{P_λ}$$

where Ω is a nonempty bounded open subset of the Euclidean space $(\mathbb{R}^N, |\cdot|)$, $N \geqslant 5$, with sufficient smooth boundary, $2^* = 2N/(N-4)$, $1 < q < 2$, λ and μ are positive parameters.

Bernis, Garcia-Azorero and Peral [3] study a fourth-order problem with a critical growth, which presents several difficulties. Indeed, the Palais–Smale condition, as well as the weak lower semi-continuity of the associated functional, may fail because the Sobolev embedding is not compact. To be precise, consider the problem

$$\begin{aligned} \Delta^2 u &= |u|^{2^*-2}u + \mu|u|^{s-2}u \quad \text{in } \Omega, \\ u &= \Delta u = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{D}$$

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where $\mu > 0$ is a parameter. Bernis, Garcia-Azorero and Peral [3] study this problem following the idea of Ambrosetti, Brezis and Cerami [2]. They proved the following result.

Theorem 1. (*See* [3].) *Fix* $1 < s < 2$. *Then there is* $\Lambda > 0$ *such that for each* $\mu \in]0, \Lambda[$, *problem* (D) *admits at least two positive solutions.*

Moreover, they also proved that if $\mu > \Lambda$, the previous problem admits no solution (see [3, Thm. 2.1]). Their proof is combination of topological and variational methods. Precisely, they determine the existence of a first solution by using the method of sub- and super-solutions and then prove that this solution is the minimum of a suitable functional and apply the mountain pass theorem so ensuring the existence of a second solution. For other result of fourth-order problem and variational problem, we refer the reader to [1, 5, 8, 10–16] and references therein.

In this paper, we investigate a fourth-order problem with critical growth (P_λ). Our approach is due to Bonanno [4, 6]. Using the variational method, we will ensure that problem (P_λ) has one positive solution when the parameters λ and μ are in a suitable interval. Furthermore, when $\lambda = 1$, we can get another positive solution, where μ is in a suitable interval, and give the estimate of the parameter μ.

At first, we give the variational framework of this problem. As usual, put $X := H_0^1(\Omega) \cap H^2(\Omega)$ endowed with the norm

$$\|u\| = \left(\int_\Omega |\Delta u(x)|^2 \,\mathrm{d}x\right)^{1/2}$$

and

$$\Phi(u) = \frac{\|u\|^2}{2}, \qquad \Psi(u) = \int_\Omega \left(\frac{1}{2^*}|u(x)|^{2^*} + \mu\frac{1}{q}|u(x)|^q\right) \mathrm{d}x \tag{1}$$

for all $u \in X$. Obviously, $|\xi|^{2^*}/2^* + \mu|\xi|^q/q \geqslant 0$ for all $\xi \in \mathbb{R}$.

By the Sobolev embedding,

$$\|u\|_{L^s(\Omega)} \leqslant c_s\|u\|, \quad u \in X,\ s \in [1, 2^*],$$

and by Talenti [17] we obtain

$$c_{2^*} = \frac{1}{\sqrt{2}N^2\pi}\left(\frac{\Gamma^2(\frac{N}{2})}{\Gamma(\frac{N}{22^*})\Gamma((\frac{N}{2}) - (\frac{N}{22^*}))}\right)^{2/N}. \tag{2}$$

Due to (2), by the Hölder inequality it follows that

$$c_s \leqslant \frac{|\Omega|^{(2^*-s)/2^*s}}{\sqrt{2}N^2\pi}\left(\frac{\Gamma^2(\frac{N}{2})}{\Gamma(\frac{N}{22^*})\Gamma((\frac{N}{2}) - (\frac{N}{22^*}))}\right)^{2/N}, \tag{3}$$

where "$|\Omega|$" denotes the Lebesgue measure of the set Ω and that the embedding $X \hookrightarrow L^s(\Omega)$ is not compact if $s = 2^*$.

Fix $r > 0$ and put

$$\lambda_r^* = \frac{r}{\frac{\mu}{q} c_q^q (2r)^{q/2} + \frac{(2r)^{2^*/2}}{2^*} c_{2^*}^{2^*}}, \qquad \tilde{\lambda}_r = \frac{1}{c_{2^*}^{2^*} (rN)^{4/(N-4)}},$$
$$\bar{\lambda}_r = \min\{\lambda_r^*, \tilde{\lambda}_r\},$$

where c_{2^*}, c_q are given by (2) and (3).

Now, we give the first result of this paper.

Theorem 2. *Fix $q \in]1, 2[$. Then there exists $\mu^* > 0$, where*

$$mu^* = \left(\frac{q}{c_q^q} \frac{1}{2^{(q+2)/2}}\right) \left(\min\left\{ \left(\frac{2^*}{2^{(2^*+2)/2} c_{2^*}^{2^*}}\right)^{2/(2^*-2)}; \frac{2}{3N} \left(\frac{1}{c_{2^*}^{2^*}}\right)^{(N-4)/4} \right\}\right)^{(2-q)/2},$$

and c_q, c_{2^*} are given by (3) *and* (2) *such that for each $\lambda \in]0, \bar{\lambda}_r[$ and $\mu \in]0, \mu^*[$, problem* (P_λ) *admits at least one positive weak solution. Let $\lambda = 1$ and u_μ be the positive solution. Then*

$$\|u_\mu\| < \left(\frac{2^*}{c_{2^*}^{2^*}}\right)^{1/(2^*-2)}.$$

Moreover, the mapping

$$\mu \mapsto \frac{1}{2} \int\limits_\Omega |\Delta u_\lambda|^2 - \frac{1}{2^*} \int\limits_\Omega |u_\mu|^{2^*} \,\mathrm{d}x - \frac{\mu}{q} \int\limits_\Omega |u_\mu|^q \,\mathrm{d}x$$

is negative and strictly decreasing in $]0, \mu^*[$.

Next, we obtain the following existence result of two solutions. At the same time, an estimate of parameters is also obtained.

Theorem 3. *Fix $q \in]1, 2[$. Then there exists $\mu^* > 0$, where*

$$\mu^* = \left(\frac{q}{c_q^q} \frac{1}{2^{(q+2)/2}}\right) \left(\min\left\{ \left(\frac{2^*}{2^{(2^*+2)/2} c_{2^*}^{2^*}}\right)^{2/(2^*-2)}; \frac{2}{3N} \left(\frac{1}{c_{2^*}^{2^*}}\right)^{(N-4)/4} \right\}\right)^{(2-q)/2},$$

and c_q, c_{2^*} are given by (3) *and* (2) *such that for each $\mu \in]0, \mu^*[$, problem*

$$\begin{aligned} \Delta^2 u &= |u|^{2^*-2} u + \mu |u|^{q-2} u \quad \text{in } \Omega, \\ u &= \Delta u = 0 \quad \text{on } \partial\Omega \end{aligned} \tag{P_μ}$$

admits at least two positive solutions u_μ and w_μ such that $\|u_\mu\| < (2^*/c_{2^*}^{2^*})^{1/(2^*-2)}$ and $w_\mu > u_\mu$.

We observe that the solution obtained in Theorem 2 is a local minimum for the considered functional. To obtain the second solution, we will use the mountain pass theorem of Ambrosetti and Rabinowitiz. This argument is the same in the part of [3, Thm. 1.1].

Example 1. Fix $N = 5$ and let $\Omega = \{x \in \mathbb{R}^5\colon |x| < 1\}$. Then the problem

$$\Delta^2 u = u^9 + 2u^{1/2} \quad \text{in } \Omega,$$
$$u = \Delta u = 0 \quad \text{on } \partial\Omega$$

admits at least two positive solutions u_μ and w_μ such that $w_\mu > u_\mu$. In fact, it is enough to apply Theorem 3 by choosing $q = 3/2$ and taking into account that $c_{2_*}^{2^*} \leqslant 3^4/(2^{21} \cdot 5^{20}\pi^6)$, $c_q^q \leqslant 3^{3/5}/(2^{63/20} \cdot 5^3\pi^{9/10})$ for which $\mu^* \geqslant 3^{7/80} \cdot 2^{71/40} \cdot 5^{59/16}\pi^{51/40} > 2$.

2 Preliminaries

We present some definitions on differentiability of functionals and refer the reader to [4, Sect. 2]. Let X be a real Banach space. We denote the dual space of X by X^*, while $\langle\cdot,\cdot\rangle$ stands for the duality pairing between X^* and X. A functional $I : X \to \mathbb{R}$ is called Gâteaux differentiable at $u \in X$ if there is $\varphi \in X^*$ (denoted by $I'(u)$) such that

$$\lim_{t\to 0^+} \frac{I(u+tv) - I(u)}{t} = I'(u)(v) \quad \forall v \in X.$$

It is called continuously Gâteaux differentiable if it is Gâteaux differentiable for any $u \in X$ and the functional $u \mapsto I(u)$ is a continuous map from X to its dual X^*.

Let Φ, $\Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals and put

$$I = \Phi - \Psi.$$

Fix $r_1, r_2 \in [-\infty, +\infty]$ with $r_1 < r_2$. We say that the functional I verifies the Palais–Smale condition cut off lower at r_1 and upper at r_2 (in short $(\mathrm{PS})_{[r_1]}^{[r_2]}$-condition) if any sequence (u_n) such that

(i) $(I(u_n))$ is bounded,
(ii) $\lim_{n\to+\infty} \|I'(u_n)\|_{X^*} = 0$,
(iii) $r_1 < \Phi(u_n) < r_2$ for all $n \in \mathbb{N}$

has a convergent subsequence.

When we fix $r_1 = -\infty$, that is, $\Phi(u_n) < r_2$ for all $n \in \mathbb{N}$, we denote this type of Palais–Smale condition with $(\mathrm{PS})^{[r_2]}$. When, in addition, $r_2 = +\infty$, it is the classical Palais–Smale condition.

Now, we recall the following local minimum theorem.

Theorem 4. (*See* [6, *Thm.* 3.3].) *Let X be a real Banach space, and let Φ, $\Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that* $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$. *Assume that there are $r \in \mathbb{R}$ and $\tilde{u} \in X$ with $0 < \Phi(\tilde{u}) < r$ such that*

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}{r} < \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})}$$

and, for each $\lambda \in]\Phi(\tilde{u})/\Psi(\tilde{u}), r/\sup_{u\in\Phi^{-1}(]-\infty,r[)}\Psi(u)[$*, the functional* $I_\lambda = \Phi - \lambda\Psi$ *satisfies* $(\mathrm{PS})^{[r]}$*-condition. Then, for each* $\lambda \in]\Phi(\tilde{u})/\Psi(\tilde{u}), r/\sup_{u\in\Phi^{-1}(]-\infty,r[)}\Psi(u)[$*, there is* $u_\lambda \in \Phi^{-1}(]0,r[)$ *(hence,* $u_\lambda \neq 0$*) such that* $I_\lambda(u_\lambda) \leqslant I_\lambda(u)$ *for all* $u \in \Phi^{-1}(]0,r[)$ *and* $I'(u_\lambda) = 0$*.*

3 Proof of the main results

Firstly, we establish the following result.

Lemma 1. *Let* Φ *and* Ψ *be the functional defined in* (1) *and fix* $r > 0$*. Then, for each* $\lambda \in]0, \bar{\lambda}_r[$*, the functional* $I_\lambda = \Phi - \lambda\Psi$ *satisfies the* $(\mathrm{PS})^{[r]}$*-condition.*

Proof. Let $(u_n) \subseteq X$ be a$(\mathrm{PS})^{[r]}$ sequence, that is,

(i) $(I_\lambda(u_n))$ is bounded,
(ii) $\lim_{n\to+\infty}\|I'_\lambda(u_n)\|_{X^*} = 0$,
(iii) $\Phi(u_n) < r$ for all $n \in \mathbb{N}$.

From $\Phi(u_n) < r$, for all $n \in \mathbb{N}$, (u_n) is bounded in X. Going to a subsequence if necessary, we can assume

$$u_n \rightharpoonup u_0 \quad \text{in } X, \qquad u_n \to u_0 \quad \text{in } L^q(\Omega),$$
$$u_n \to u_0 \quad \text{a.e. on } \Omega.$$

Taking (i) into account, for a constant c, $\lim_{n\to\infty} I_\lambda(u_n) = c$. Moreover, (u_n) is bounded in $L^{2^*}(\Omega)$. Now, we proof our result by many steps.

Step 1. u_0 is a weak solution of problem (P_λ). Since (u_n) is bounded in $L^{2^*}(\Omega)$, we get that $(u_n^{2^*-1})$ is bounded in $L^{2^*/(2^*-1)}(\Omega)$. Indeed, we have

$$\int\limits_\Omega \left|u_n^{2^*-1}\right|^{2^*/(2^*-1)} \mathrm{d}x = \int\limits_\Omega |u_n|^{2^*} \mathrm{d}x.$$

Therefore, we get that

$$u_n^{2^*-1} \rightharpoonup u_0^{2^*-1} \quad \text{in } L^{2^*/(2^*-1)}.$$

In fact, since $u_n \to u_0$ a.e. $x \in \Omega$, we obtain $u_n^{2^*-1} \to u_0^{2^*-1}$ a.e. $x \in \Omega$, and that, together with the boundedness of $(u_n^{2^*-1})$ in $L^{2^*/(2^*-1)}$, ensures the weak convergence of $u_n^{2^*-1}$ to $u_0^{2^*-1}$ in $L^{2^*/(2^*-1)}$ (see [7, Rem. (iii)]).

Moreover, since $u_n \to u_0$ in $L^q(\Omega)$, taking into account [18, Thm. A.2], one has that

$$u_n^{q-1} \to u_0^{q-1} \quad \text{in } L^{q/(q-1)}(\Omega).$$

In particular,

$$u_n^{q-1} \rightharpoonup u_0^{q-1} \quad \text{in } L^{q/(q-1)}(\Omega).$$

One has

$$\lim_{n\to\infty}\Bigg(\int\limits_{\Omega}\Delta u_n(x)\Delta v(x)\,\mathrm{d}x-\lambda\int\limits_{\Omega}u_n(x)^{2^*-1}v(x)\,\mathrm{d}x-\lambda\mu\int\limits_{\Omega}u_n(x)^{q-1}v(x)\,\mathrm{d}x\Bigg)$$
$$=\int\limits_{\Omega}\Delta u_0(x)\Delta v(x)\,\mathrm{d}x-\lambda\int\limits_{\Omega}u_0(x)^{2^*-1}v(x)\,\mathrm{d}x-\lambda\mu\int\limits_{\Omega}u_0(x)^{q-1}v(x)\,\mathrm{d}x$$

for all $v\in X$. Therefore, due to (ii), we obtain that

$$0=\int\limits_{\Omega}\Delta u_0(x)\Delta v(x)\,\mathrm{d}x-\lambda\int\limits_{\Omega}u_0^{2^*-1}v(x)\,\mathrm{d}x-\lambda\mu\int\limits_{\Omega}u_0^{q-1}v(x)\,\mathrm{d}x$$

for all $v\in X$, that is, u_0 is a weak solution of (P_λ).

Step 2. We prove that

$$I_\lambda(u_0)>-r. \tag{4}$$

Let us consider the nonlinear term

$$\begin{aligned}\Psi(u)&=\int\limits_{\Omega}\bigg(\frac{1}{2^*}|u(x)|^{2^*}+\mu\frac{1}{q}|u(x)|^q\bigg)\,\mathrm{d}x=\frac{\mu}{q}\|u\|^q_{L^q(\Omega)}+\frac{1}{2^*}\|u\|^{2^*}_{L^{2^*}(\Omega)}\\&\leqslant\frac{\mu}{q}c_q^q\|u\|^q+\frac{1}{2^*}c_{2^*}^{2^*}\|u\|^{2^*}.\end{aligned}$$

So,

$$\Psi(u)\leqslant\frac{\mu}{q}c_q^q\|u\|^q+\frac{1}{2^*}c_{2^*}^{2^*}\|u\|^{2^*}\quad\forall u\in X.$$

It follows that for all $u\in X$, $\|u\|\leqslant(2r)^{1/2}$, we obtained

$$\begin{aligned}I_\lambda(u)=\Phi(u)-\lambda\Psi(u)&\geqslant\frac{\|u\|^2}{2}-\lambda\bigg(\frac{\mu}{q}c_q^q\|u\|^q+\frac{1}{2^*}c_{2^*}^{2^*}\|u\|^{2^*}\bigg)\\&\geqslant-\lambda\bigg(\frac{\mu}{q}c_q^q(2r)^{q/2}+\frac{1}{2^*}c_{2^*}^{2^*}(2r)^{2^*/2}\bigg)=-\lambda\frac{r}{\lambda_r^*}>-r.\end{aligned}$$

Noting (iii) and Φ is sequentially weakly lower semicontinuous, we have

$$\|u_0\|\leqslant\liminf_{n\to\infty}\|u_n\|\leqslant\sqrt{2r}.$$

That is,

$$I_\lambda(u_0)>-r.$$

Step 3. Let $v_n=u_n-u_0$. We get that

$$c=\Phi(u_0)-\lambda\Psi(u_0)+\lim_{n\to\infty}\bigg(\frac{1}{2}\|v_n\|^2-\lambda\int\limits_{\Omega}\frac{1}{2^*}|v_n|^{2^*}\,\mathrm{d}x\bigg). \tag{5}$$

In fact, $\|u_n\|^2 = \|v_n + u_0\|^2 = \|v_n\|^2 + \|u_0\|^2 + 2\langle v_n, u_0\rangle$, so, we obtained

$$\|u_n\|^2 = \|v_n\|^2 + \|u_0\|^2 + o(1).$$

Moreover, by the Brezis–Lieb lemma (see [7, Thm. 1]) one has

$$\int\limits_{\Omega} |u_n|^{2^*}\,\mathrm{d}x = \int\limits_{\Omega} |v_n|^{2^*}\,\mathrm{d}x + \int\limits_{\Omega} |u_0|^{2^*}\,\mathrm{d}x + o(1).$$

Finally, since $u \mapsto \int_{\Omega}(1/q)|u|^q\,\mathrm{d}x$ is locally Lipschitz in $L^q(\Omega)$ (see, for example, [9, Thm. 7.2.1]) and $u_n \to u_0$ in $L^q(\Omega)$, we obtained

$$\int\limits_{\Omega} |u_n|^q\,\mathrm{d}x = \int\limits_{\Omega} |u_0|^q\,\mathrm{d}x + o(1).$$

Hence,

$$c = \lim_{n\to\infty}\bigl(\Phi(u_n) - \lambda\Psi(u_n)\bigr),$$

that is,

$$\begin{aligned}
c &= \Phi(u_n) - \lambda\Psi(u_n) + o(1) \\
&= \frac{1}{2}\|u_n\|^2 - \lambda\frac{1}{2^*}\int\limits_{\Omega} |u_n|^{2^*}\,\mathrm{d}x - \lambda\mu\frac{1}{q}\int\limits_{\Omega} |u_n|^q\,\mathrm{d}x + o(1) \\
&= \frac{1}{2}\|v_n\|^2 + \frac{1}{2}\|u_0\|^2 - \lambda\frac{1}{2^*}\int\limits_{\Omega} |v_n|^{2^*}\,\mathrm{d}x - \lambda\frac{1}{2^*}\int\limits_{\Omega} |u_0|^{2^*}\,\mathrm{d}x \\
&\quad - \lambda\mu\frac{1}{q}\int\limits_{\Omega} |u_0|^q\,\mathrm{d}x + o(1) \\
&= \Phi(u_0) - \lambda\Psi(u_0) + \frac{1}{2}\|v_n\|^2 - \lambda\frac{1}{2^*}\int\limits_{\Omega} |v_n|^{2^*}\,\mathrm{d}x + o(1).
\end{aligned}$$

We get (5).

Step 4. The following equality is satisfied:

$$\lim_{n\to\infty}\Biggl(\|v_n\|^2 - \lambda\int\limits_{\Omega} |v_n|^{2^*}\,\mathrm{d}x\Biggr) = 0. \tag{6}$$

From (ii) we have $\lim_{n\to\infty} I'(u_n)(u_n) = 0$. We get

$$\int\limits_{\Omega} \Delta u_n \Delta u_n\,\mathrm{d}x - \lambda\int\limits_{\Omega} |u_n|^{2^*-1}u_n\,\mathrm{d}x - \lambda\mu\int\limits_{\Omega} |u_n|^{q-1}u_n\,\mathrm{d}x = o(1).$$

Therefore, seen in the proof of (5) and

$$\int_\Omega |u_n|^{q-1}u_n\,\mathrm{d}x = \int_\Omega |u_0|^{q-1}u_0\,\mathrm{d}x + o(1),$$

we get that $|u_n|^{q-1} \to |u_0|^{q-1}$ in $L^{q/(q-1)}(\Omega)$ (see the first step) and $u_n \to u_0$ in $L^q(\Omega)$. One has

$$\|v_n\|^2 + \|u_0\|^2 - \lambda\int_\Omega |v_n|^{2^*}\,\mathrm{d}x - \lambda\int_\Omega |u_0|^{2^*}\,\mathrm{d}x - \lambda\mu\int_\Omega |u_0|^q\,\mathrm{d}x = o(1),$$

that is,

$$\|v_n\|^2 - \lambda\int_\Omega |v_n|^{2^*}\,\mathrm{d}x = -\|u_0\|^2 + \lambda\int_\Omega |u_0|^{2^*}\,\mathrm{d}x + \lambda\mu\int_\Omega |u_0|^q\,\mathrm{d}x + o(1).$$

Since u_0 is a weak solution of (P_λ), one has

$$\|u_0\|^2 - \lambda\int_\Omega |u_0|^{2^*}\,\mathrm{d}x - \lambda\mu\int_\Omega |u_0|^q\,\mathrm{d}x = 0.$$

We get,

$$\|v_n\|^2 - \lambda\int_\Omega |v_n|^{2^*}\,\mathrm{d}x = o(1),$$

that is, (6).

Conclusion. Finally, we observe that $\|v_n\|^2$ is bounded in $\mathbb{R}$. Thus, there is a subsequence, still denoted by $\|v_n\|^2$, which converges to $b \in \mathbb{R}$. That is, $\lim_{n\to\infty}\|v_n\|^2 = b$. If $b = 0$, we have proved the lemma. In this situation, we have $\lim_{n\to\infty}\|u_n - u_0\| = 0$.

We assume that $b \neq 0$, arguing by contradiction. From (6) we obtain

$$\lim_{n\to\infty}\lambda\int_\Omega |v_n|^{2^*}\,\mathrm{d}x = b.$$

By the Sobolev embedding, $\|v_n\|_{L^{2^*}(\Omega)} \leqslant c_{2^*}\|v_n\|$, and passing to the limit, we obtained $b/\lambda \leqslant c_{2^*}^{2^*}b^{2^*/2}$. Since $b \neq 0$, we get

$$b \geqslant \left(\frac{1}{\lambda}\right)^{(N-4)/4}\left(\frac{1}{c_{2^*}}\right)^{N/2}.$$

Due to (4) and (5), one has

$$c = \Phi(u_0) - \lambda\Psi(u_0) + \frac{1}{2}b - \frac{1}{2^*}b > -r + \left(\frac{1}{2} - \frac{1}{2^*}\right)b = -r + \frac{2}{N}b,$$

that is, $c > -r + 2b/N$. On the other hand, since

$$\frac{1}{2^*}|\xi|^{2^*} + \mu\frac{1}{q}|\xi|^q \geqslant 0$$

for all $\xi \in \mathbb{R}$, we obtained

$$\Phi(u_n) - \lambda\Psi(u_n) < r$$

for all $n \in \mathbb{N}$. That is, $c \leqslant r$. Thus, $-r + 2b/N < c \leqslant r$. It follows that $2b/N < 2r$, that is, $b < rN$. Therefore, one has

$$\left(\frac{1}{\lambda}\right)^{(N-4)/4}\left(\frac{1}{c_{2^*}}\right)^{N/2} \leqslant b < rN,$$

so, it follows that $1/\lambda < (rNc_{2^*}^{N/2})^{4/(N-4)}$. Hence, we get

$$\lambda > \frac{1}{(rN)^{4/(N-4)}}\frac{1}{c_{2^*}^{2^*}} = \tilde{\lambda}_r,$$

and this is a contradiction. □

Now, we give the proof of Theorem 2.

Proof of Theorem 2. Let

$$r = \min\left\{\left(\frac{2^*}{2^{(2^*+2)/2}c_{2^*}^{2^*}}\right)^{2/(2^*-2)}; \frac{2}{3N}\left(\frac{1}{c_{2^*}^{2^*}}\right)^{(N-4)/4}\right\}$$

and

$$\mu^* = \left(\frac{q}{c_q^q}\frac{1}{2^{(q+2)/2}}\right)r^{(2-q)/2}.$$

Fix $0 < \mu < \mu^*$, and one has $\bar{\lambda}_r > 1$. Indeed,

$$\begin{aligned}\tilde{\lambda}_r &= \frac{1}{c_{2^*}^{2^*}(rN)^{4/(N-4)}} \geqslant \frac{1}{c_{2^*}^{2^*}(N)^{4/(N-4)}[\frac{2}{3N}(\frac{1}{c_{2^*}^{2^*}})^{(N-4)/4}]^{4/(N-4)}} \\ &= \left(\frac{3}{2}\right)^{4/(N-4)} > 1\end{aligned}$$

and

$$\begin{aligned}\lambda_r^* &= \frac{1}{\frac{\mu}{q}c_q^q 2^{q/2} r^{\frac{q-2}{2}} + \frac{2^{2^*/2}}{2^*}c_{2^*}^{2^*}r^{(2^*-2)/2}} \\ &\geqslant \frac{1}{\frac{\mu}{q}c_q^q 2^{q/2} r^{\frac{q-2}{2}} + \frac{2^{2^*/2}}{2^*}c_{2^*}^{2^*}[(\frac{2^*}{2^{(2^*+2)/2}c_{2^*}^{2^*}})^{2/(2^*-2)}]^{(2^*-2)/2}} \\ &> \frac{1}{\frac{\mu}{q}c_q^q 2^{q/2} r^{\frac{q-2}{2}} + \frac{1}{2}} = 1.\end{aligned}$$

Therefore, from Lemma 1 the functional $I_\lambda = \Phi - \lambda\Psi$ satisfies the $(\mathrm{PS})^{[r]}$-condition for all $\lambda \in]0, \bar{\lambda}_r[$.

Fix $\lambda < \bar{\lambda}_r$. We claim that there is a $v_0 \in X$, with $0 < \Phi(v_0) < r$, such that

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}{r} < \frac{\Psi(v_0)}{\Phi(v_0)}.$$

Consider $\|u\|_{L^s(\Omega)} \leqslant c_s\|u\|$, $u \in X$, we get

$$\begin{aligned}\frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}{r} &\leqslant \frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)}(\frac{\mu}{q}\|u\|^q_{L^q(\Omega)} + \frac{1}{2^*}\|u\|^{2^*}_{L^{2^*}(\Omega)})}{r}\\ &\leqslant \frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)}(\frac{\mu}{q}c_q^q\|u\|^q + \frac{1}{2^*}c_{2^*}^{2^*}\|u\|^{2^*})}{r}\\ &\leqslant \frac{\frac{\mu}{q}c_q^q(2r)^q + \frac{1}{2^*}c_{2^*}^{2^*}(2r)^{2^*}}{r}.\end{aligned}$$

Hence, we get

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}{r} \leqslant \frac{1}{\lambda_r^*} < \frac{1}{\lambda}.$$

Let $R = \sup_{x\in\Omega} d(x, \partial\Omega)$, and let $x_0 \in \Omega$ such that $B(x, R) \subseteq \Omega$. Moreover, put

$$v_\delta(x) := \begin{cases} 0 & \text{if } x \in \Omega \setminus B(x_0, R),\\ 16\frac{l^2}{R^4}(R-l)^2\delta & \text{if } x \in B(x_0, R) \setminus B(x_0, R/2),\\ \delta & \text{if } x \in B(x_0, R/2),\end{cases}$$

where $l := \sqrt{\sum_{i=1}^N (x^i - x_0^i)^2}$. Clearly, $v_\delta \in X$, and since

$$\sum_{i=1}^N \frac{\partial^2 v_\delta(x)}{\partial^2 x^i} = 32\delta\,\frac{2(N+2)l^2 - 3R(N+1)l + NR^2}{R^4}$$

for every $x \in B(x_0, R) \setminus B(x_0, R/2)$, we get

$$\begin{aligned}\Phi(v_\delta) &= \frac{1}{2}\int_\Omega \left|\Delta v_\delta(x)\right|^2 \mathrm{d}x\\ &= \frac{2^{10}\pi^{N/2}\delta^2}{R^8\Gamma(N/2)}\int_{R/2}^{R} \left|2(N+2)s^2 - 3(N+1)Rs + NR^2\right|^2 s^{N-1}\,\mathrm{d}s,\end{aligned}$$

where Γ is the gamma function. Moreover, we get

$$\begin{aligned}\Psi(v_\delta) &= \int_\Omega \left(\frac{1}{2^*}\left|v_\delta(x)\right|^{2^*} + \mu\frac{1}{q}\left|v_\delta(x)\right|^q\right)\mathrm{d}x \geqslant \int_{B(x_0,R/2)} \left(\frac{1}{2^*}|\delta|^{2^*} + \mu\frac{1}{q}|\delta|^q\right)\mathrm{d}x\\ &\geqslant \left(\frac{1}{2^*}|\delta|^{2^*} + \mu\frac{1}{q}|\delta|^q\right)\frac{\pi^{N/2}}{\Gamma(1+N/2)}\frac{R^N}{2^N},\end{aligned}$$

and so,

$$\frac{\Psi(v_\delta)}{\Phi(v_\delta)} \geqslant \frac{R^2}{2(2^N-1)} \frac{\frac{1}{2^*}|\delta|^{2^*} + \mu\frac{1}{q}|\delta|^q}{\delta^2}.$$

From $\lim_{t\to 0^+} |t|^q/t^2 = +\infty$ we get that

$$\limsup_{t\to 0^+} \frac{\frac{1}{2^*}|t|^{2^*} + \mu\frac{1}{q}|t|^q}{t^2} = +\infty.$$

So, by

$$\Phi(v_\delta) = \frac{2^{10}\pi^{N/2}\delta^2}{R^8\Gamma(\frac{N}{2})} \int\limits_{R/2}^{R} \left|2(N+2)s^2 - 3(N+1)Rs + NR^2\right|^2 s^{N-1}\,\mathrm{d}s$$

there is a $\bar{\delta} > 0$ such that

$$\frac{R^2}{2(2^N-1)} \frac{\frac{1}{2^*}|\bar{\delta}|^{2^*} + \mu\frac{1}{q}|\bar{\delta}|^q}{\bar{\delta}^2} > \frac{1}{\lambda}$$

and $\Phi(v_{\bar{\delta}}) < r$. Therefore,

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)}\Psi(u)}{r} < \frac{1}{\lambda} < \frac{R^2}{2(2^N-1)} \frac{\frac{1}{2^*}|\bar{\delta}|^{2^*} + \mu\frac{1}{q}|\bar{\delta}|^q}{\bar{\delta}^2} \leqslant \frac{\Psi(v_{\bar{\delta}})}{\Phi(v_{\bar{\delta}})}$$

with $0 < \Phi(v_{\bar{\delta}}) < r$. Hence, the claim is proved.

Finally, from Theorem 4 then functional $\Phi - \lambda\Psi$ admits a critical point $u_{\lambda,\mu}$ such that $\|u_{\lambda,\mu}\|^2/2 > 0$, which is a positive weak solution for problem (P_λ). In particular, by choosing $\lambda = 1$ a positive weak solution u_μ for problem (P_λ) is obtained. Moreover, one has $\|u_\mu\|^2/2 < r$ from which $\|u_\mu\|^2/2 < (2^*/(2^{(2^*+2)/2}c_{2^*}^{2^*}))^{2/(2^*-2)}$, that is,

$$\|u_\mu\| < \left(\frac{2^*}{c_{2^*}^{2^*}}\right)^{1/(2^*-2)}.$$

Since u_μ is a global minimum for I_1 in $\Phi^{-1}(]0,r[)$ again from Theorem 4, and $v_{\bar{\delta}} \in \Phi^{-1}(]0,r[)$, one has $I_1(u_\mu) \leqslant I_1(v_{\bar{\delta}})$. So, by $\Psi(v_{\bar{\delta}})/\Phi(v_{\bar{\delta}}) > 1/\lambda > 1$ we get

$$I_1(u_\mu) \leqslant I_1(v_{\bar{\delta}}) < 0.$$

Next, fix $0 < \mu_1 < \mu_2$. We get

$$\begin{aligned} I_1(u_{\mu_1}) &= \min_{u\in\Phi^{-1}(]0,r[)} \left(\frac{1}{2}\int\limits_\Omega |\Delta u|^2\,\mathrm{d}x - \frac{1}{2^*}\int\limits_\Omega |u|^{2^*}\,\mathrm{d}x - \mu_1\frac{1}{q}\int\limits_\Omega |u|^q\,\mathrm{d}x\right) \\ &> \min_{u\in\Phi^{-1}(]0,r[)} \left(\frac{1}{2}\int\limits_\Omega |\Delta u|^2\,\mathrm{d}x - \frac{1}{2^*}\int\limits_\Omega |u|^{2^*}\,\mathrm{d}x - \mu_2\frac{1}{q}\int\limits_\Omega |u|^q\,\mathrm{d}x\right) \\ &= I_1(u_{\mu_2}), \end{aligned}$$

and the conclusion is achieved. □

Proof of Theorem 3. Fix $\mu \in]0, \mu^*[$. From Theorem 2 there exists a positive solution u_μ of (P_λ) such that u_μ is a local minimum for the functional

$$I(u) = \Phi(u) - \Psi(u) = \frac{\|u\|^2}{2} - \int\limits_{\Omega} F\big(u(x)\big)\,\mathrm{d}x,$$

where F is the primitive of $f(t)$, and

$$f(t) = \begin{cases} t^{2^*-1} + \mu t^{q-1} & \text{if } t \geqslant 0, \\ 0 & \text{if } t < 0. \end{cases}$$

We consider a new problem

$$\begin{aligned} &\Delta^2 v = (u_\mu + v)^{2^*-1} - u_\mu^{2^*-1} + \mu(u_\mu + v)^{q-1} - \mu u_\mu^{q-1} \quad \text{in } \Omega, \\ &\quad v = \Delta v = 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{7}$$

Clearly, if v_μ is a positive weak solution to (7), then $w_\mu = u_\mu + v_\mu$ is a weak solution of (P_λ) such that $w_\mu > u_\mu > 0$. Now, our aim is to prove that (7) admits at least one positive weak solution. Consider the functional J defined as

$$J(v) = \frac{\|v\|^2}{2} - \int\limits_{\Omega} L\big(x, v(x)\big)\,\mathrm{d}x, \quad L(x,\xi) = \int\limits_0^{\xi} l(x,t)\,\mathrm{d}t,$$

and

$$l(x,t) = \begin{cases} (u(x) + t)^{2^*-1} - [u_\mu(x)]^{2^*-1} & \\ \quad +\mu(u_\mu(x)+t)^{q-1} - \mu[u_\mu(x)]^{q-1} & \text{if } t \geqslant 0, \\ 0 & \text{if } t < 0. \end{cases}$$

Clearly, nonzero critical points of J are positive weak solutions of (7). Since u_μ is a local minimum of I, one has

$$I(u_\mu + v) - I(u_\mu) \geqslant 0$$

for all $v \in X$ such that $\|v\| < \delta$ for some $\delta > 0$. So, taking into account that

$$J(v) = \frac{1}{2}\|v^-\|^2 + I\big(u_\mu + v^+\big) - I(u_\mu) \geqslant 0$$

for all $v \in X$ (see [3]), we get $J(v) \geqslant 0$ for all $v \in X$ such that $\|v\| < \delta$. That is, 0 is a local minimum of J.

By using the same proof in [3], the functional J admits a positive critical point v_μ for which $w_\mu = u_\mu + v_\mu$ is the second weak solution of (7), and the proof is complete. □

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