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Asymptotics for ultimate ruin probability in a by-claim risk model*

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Abstract. This paper considers a by-claim risk model with constant interest rate in which the main claim and by-claim random vectors form a sequence of independent and identically distributed random pairs with each pair obeying some certain dependence or arbitrary dependence structure. Under the assumption of heavy-tailed claims, we derive some asymptotic formulas for ultimate ruin probability. Some simulation studies are also performed to check the accuracy of the obtained theoretical results via the crude Monte Carlo method.

Keywords: ultimate ruin probability, asymptotic behavior, main claim and by-claim, bivariate regular variation.

1 Introduction

Consider a by-claim risk model in which every severe accident causes a main claim accompanied with a secondary claim occurring after a period of delay. In such a model, the claims $\{(X_i, Y_i);\ i \in \mathbb{N}\}$ form a sequence of independent and identically distributed (i.i.d.) nonnegative random vectors with a generic random vector (X, Y). Here, for each $i \in \mathbb{N}$, X_i and Y_i represent the ith main claim (original claim) and its corresponding by-claim (secondary claim), respectively, and they are highly dependent due to their being caused by the same accident. The main claims X_i's arrive at times τ_i, $i \in \mathbb{N}$, which constitute a renewal counting process $N_t = \sup\{n \in \mathbb{N}\colon \tau_n \leqslant t\}$ for some $t \geqslant 0$ with mean function $\lambda(t) = \mathbf{E}N_t$. Denote the inter-arrival times by $\theta_i = \tau_i - \tau_{i-1}$, $i \in \mathbb{N}$, which are i.i.d. nonnegative and nondegenerate at zero random variables. Let

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$\{D_i;\ i \in \mathbb{N}\}$ be the delay times of the by-claims, which also form a sequence of i.i.d. nonnegative, but possibly degenerate at zero, random variables with common distribution H. Assume, as usual, that the three sequences $\{(X_i, Y_i);\ i \in \mathbb{N}\}$, $\{\tau_i;\ i \in \mathbb{N}\}$, and $\{D_i;\ i \in \mathbb{N}\}$ are mutually independent. Denote by $x > 0$ the initial value of the insurer, by $c \geqslant 0$ the constant premium rate, and by $\delta > 0$ the constant interest rate. In this setting, the discounted surplus process of an insurer at time $t \geqslant 0$ is

$$U_t = x + c\int_0^t \mathrm{e}^{-\delta s}\,\mathrm{d}s - \sum_{i=1}^{N_t} X_i\mathrm{e}^{-\delta\tau_i} - \sum_{i=1}^{\infty} Y_i\mathrm{e}^{-\delta(\tau_i+D_i)}\mathbf{1}_{\{\tau_i+D_i\leqslant t\}}, \tag{1}$$

where $\mathbf{1}_A$ is the indicator function of a set A. In this way, the finite-time and ultimate ruin probabilities of model (1) can be defined, respectively, by

$$\psi(x;T) = \mathbf{P}\Big(\inf_{0\leqslant t\leqslant T} U_t < 0 \;\Big|\; U_0 = x\Big) \quad \text{for some } T > 0$$

and

$$\psi(x;\infty) = \lim_{T\to\infty}\psi(x;T) = \mathbf{P}\Big(\inf_{t\geqslant 0} U_t < 0 \;\Big|\; U_0 = x\Big). \tag{2}$$

In insurance risk management, this kind of risk model may be of practical use. For instance, a serious motor accident may cause two different kinds of claims, such as car damage and passenger injuries even death. The former can be dealt with immediately, while the latter needs an uncertain period of time to be settled. Hence, the claims for car damage can be regarded as the main claims, while the claims for passenger injuries as the by-claims.

[20] considered a discrete-time risk model allowing for the delay in claim settlements called by-claims and used martingale techniques to derive some upper bounds for ruin probabilities. Since then, many researchers have paid their attention to by-claim risk models. To name a few, [21, 22, 30] investigated some independent by-claim risk models, that is, the main claim and by-claim sequences $\{X_i;\ i \in \mathbb{N}\}$ and $\{Y_i;\ i \in \mathbb{N}\}$, respectively, consist of i.i.d. random variables, and they are mutually independent, too. However, it is worth saying that the independence assumption between each main claim and its corresponding by-claim makes the model unrealistic. For example, in the above motor accident, the two corresponding claims for car damage and passenger injuries should be highly dependent. In this direction, [14] studied a by-claim risk model with no interest rate under the setting that each pair of the main claim and by-claim follow an asymptotic independence structure or possess a bivariate regularly varying tail (hence, are asymptotically dependent). Further, [27] generalized Li's result by extending the distributions of the main claims and by-claims from the regular variation to the consistent variation in the case that the two types of claims are asymptotically independent. They also complemented another case that each pair of main claim and by-claim are arbitrarily dependent, but the former dominates the latter. In the study of dependent by-claim risk models with positive interest rate, [13] considered the case that all main claims and by-claims are pairwise quasi-asymptotically independent and established an asymptotic formula for the ultimate

ruin probability. Based on [8], the paper [13] further studied a dependent renewal risk model with stochastic returns by allowing an insurer to invest its surplus into a portfolio consisting of risk-free and risky assets. For more recent advances in dependent (by-claim) risk models with interest rate, one can be referred to [2, 4, 9, 12, 15, 23–26, 28], among others.

Motivated by [13] and [27], in this paper, we continue to study a dependent by-claim risk model with interest rate in which the main claim and by-claim vectors $\{(X_i, Y_i);\ i \in \mathbb{N}\}$ are i.i.d., but each pair possesses some certain strong dependence or arbitrary dependence structure. In such a model, we aim to establish some asymptotic formulas for ultimate ruin probability.

In the rest of this paper, Section 2 presents the main results of this paper after preparing some preliminaries on some heavy-tailed distributions and dependence structures. Section 3 proves our results, and Section 4 performs some simulation studies to check the accuracy of our obtained theoretical results.

2 Preliminaries and main results

Throughout this paper, all limit relationships hold as $x \to \infty$ unless stated otherwise. For two positive functions f and g, we write $f(x) \lesssim g(x)$ if $\limsup f(x)/g(x) \leqslant 1$, write $f(x) \sim g(x)$ if both $f(x) \lesssim g(x)$ and $g(x) \lesssim f(x)$, and write $f(x) = o(g(x))$ if $\lim f(x)/g(x) = 0$. For two real numbers x and y, denote by $x \vee y = \max\{x, y\}$.

When modeling extremal events, heavy-tailed risks (claims) have played an important role in insurance and finance due to their ability to describe large claims efficiently. We now introduce some commonly-used heavy-tailed distributions. A distribution V on $\mathbb{R}$ is said to be consistently varying tailed, denoted by $V \in \mathcal{C}$, if $\overline{V}(x) = 1 - V(x) > 0$ for all x and

$$\lim_{y \downarrow 1} \liminf_{x \to \infty} \frac{\overline{V}(xy)}{\overline{V}(x)} = 1 \quad \text{or} \quad \lim_{y \uparrow 1} \limsup_{x \to \infty} \frac{\overline{V}(xy)}{\overline{V}(x)} = 1.$$

Particularly, a distribution V on $\mathbb{R}$ is said to be regularly varying tailed with index $-\alpha$, denoted by $V \in \mathcal{R}_{-\alpha}$, if

$$\lim_{x \to \infty} \frac{\overline{V}(xy)}{\overline{V}(x)} = y^{-\alpha}$$

for any $y > 0$ and some $\alpha > 0$. It should be mentioned that many popular distributions, such as the Pareto, Burr, Loggamma, and t-distributions, are all regularly varying tailed.

For any distribution V on $\mathbb{R}$, define

$$J_V^+ = -\lim_{y \to \infty} \frac{\log \overline{V}_*(y)}{\log y} \quad \text{and} \quad J_V^- = -\lim_{y \to \infty} \frac{\log \overline{V}^*(y)}{\log y}$$

with $\overline{V}_*(y) = \liminf \overline{V}(xy)/\overline{V}(x)$ and $\overline{V}^*(y) = \limsup \overline{V}(xy)/\overline{V}(x)$. Clearly, if $V \in \mathcal{C}$, then $\lim_{y \downarrow 1} \overline{V}_*(y) = 1$; and if $V \in \mathcal{R}_{-\alpha}$ for some $\alpha > 0$, then $J_V^+ = J_V^- = \alpha$. For more discussions on heavy-tailed distributions and their applications to insurance and finance, one can be referred to [1] and [6].

Bivariate regular variation is a natural extension of the univariate one in the two-dimensional case, which was firstly introduced by [5]. It provides an integrated framework for modelling extreme risks (claims) with both heavy tails and asymptotic (in)dependence. Recent works in this direction include [3, 7, 17, 18], among others.

A random vector (ξ, η) taking values in $[0, \infty)^2$ is said to follow a distribution with a bivariate regularly varying (BRV) tail if there exist a distribution V and a nondegenerate (i.e. not identically 0) limit measure ν such that

$$\frac{1}{\overline{V}(x)}\mathbf{P}\left(\frac{1}{x}(\xi, \eta) \in \cdot\right) \xrightarrow{\mathrm{v}} \nu(\cdot) \quad \text{on } [0, \infty]^2 \setminus \{(0, 0)\}. \tag{3}$$

In (3), the notation $\xrightarrow{\mathrm{v}}$ denotes vague convergence meaning that the relation

$$\lim \frac{1}{\overline{V}(x)}\mathbf{P}\left(\frac{1}{x}(\xi, \eta) \in B\right) = \nu(B)$$

holds for every Borel set $B \subset [0, \infty]^2$ that is away from $(0, 0)$ and ν-continuous (i.e. its boundary ∂B has ν-measure 0). Related discussions on vague convergence can be found in [16, Sect. 3.3.5]. Necessarily, the reference distribution $V \in \mathcal{R}_{-\alpha}$ for some $\alpha > 0$, for which case we write $(\xi, \eta) \in \mathrm{BRV}_{-\alpha}(\nu, V)$. By definition, for a random vector $(\xi, \eta) \in \mathrm{BRV}_{-\alpha}(\nu, V)$, its marginal distributions satisfy

$$\lim \frac{\overline{F_\xi}(x)}{\overline{V}(x)} = \nu\big((1, \infty] \times (0, \infty]\big) \quad \text{and} \quad \lim \frac{\overline{F_\eta}(x)}{\overline{V}(x)} = \nu\big((0, \infty] \times (1, \infty]\big).$$

Recall the by-claim risk model (1) in which $\{(X_i, Y_i);\ i \in \mathbb{N}\}$ is a sequence of i.i.d. nonnegative random pairs with generic random vector (X, Y) having marginal distributions F and G, respectively; $\{D_i;\ i \in \mathbb{N}\}$ is a sequence of i.i.d. nonnegative random variables with generic random variable D and distribution H; N_t is a renewal counting process with mean function $\lambda(t)$; and $\{(X_i, Y_i);\ i \in \mathbb{N}\}$, $\{D_i;\ i \in \mathbb{N}\}$, and $\{N_t;\ t \geqslant 0\}$ are mutually independent.

Now we are ready to state our main results regarding ultimate ruin probability. The first result considers the case that each pair of main claim and its corresponding by-claim follow a joint distribution with a bivariate regularly varying tail satisfying $\nu((1, \infty]^2) > 0$, hence, they are highly dependent on each other.

Theorem 1. *Consider the by-claim risk model* (1). *If* $(X, Y) \in \mathrm{BRV}_{-\alpha}(\nu, F_0)$ *for some reference distribution* F_0 *and some* $\alpha > 0$ *with* $\nu((1, \infty]^2) > 0$, *then*

$$\psi(x; \infty) \sim \overline{F_0}(x) \int_{0-}^{\infty} \nu(A_s)\, H(\mathrm{d}s) \cdot \int_{0-}^{\infty} \mathrm{e}^{-\delta\alpha t}\, \lambda(\mathrm{d}t), \tag{4}$$

where $A_s = \{(u, v) \in [0, \infty]^2\colon\ u + v\mathrm{e}^{-\delta s} > 1\}$ *for any* $s \geqslant 0$.

The second result relaxes the dependence between each pair of the two types of claims as well as the common distribution of the main claims, but requires that the by-claims are dominated by the main claims.

Theorem 2. *Consider the by-claim risk model* (1). *If X and Y are arbitrarily dependent and $F \in \mathcal{C}$, $\overline{G}(x) = o(\overline{F}(x))$, then*

$$\psi(x;\infty) \sim \int_{0-}^{\infty} \overline{F}\big(x\mathrm{e}^{\delta t}\big)\,\lambda(\mathrm{d}t). \tag{5}$$

In particular, if $F \in \mathcal{R}_{-\alpha}$ for some $\alpha > 0$, then

$$\psi(x;\infty) \sim \overline{F}(x)\int_{0-}^{\infty} \mathrm{e}^{-\delta\alpha t}\,\lambda(\mathrm{d}t). \tag{6}$$

3 Proofs of main results

We shall adopt the recent method on the asymptotic tail behavior for infinite randomly weighted sums to prove Theorems 1 and 2. The first lemma considers the infinite randomly weighted sums of consistently varying tailed random variables, which can be found in [10].

Lemma 1. *Let $\{\xi_i;\, i \in \mathbb{N}\}$ be a sequence of i.i.d. real-valued random variables with common distribution $V \in \mathcal{C}$ and $V(-x) = o(\overline{V}(x))$, and let $\{\Theta_i;\, i \in \mathbb{N}\}$ be another sequence of nonnegative random variables independent of $\{\xi_i;\, i \in \mathbb{N}\}$. Assume that there exists a small $\delta > 0$ such that*

$$\sum_{i=1}^{\infty} \mathbf{E}\big(\Theta_i^{J_V^+ + \delta}\big) \vee \sum_{i=1}^{\infty} \mathbf{E}\big(\Theta_i^{J_V^- - \delta}\big) < \infty \quad \text{if } 0 < J_V^+ < 1,$$

$$\sum_{i=1}^{\infty} \big(\mathbf{E}\big(\Theta_i^{J_V^+ + \delta}\big)\big)^{1/(J_V^+ + \delta)} \vee \sum_{i=1}^{\infty} \big(\mathbf{E}\big(\Theta_i^{J_V^- - \delta}\big)\big)^{1/(J_V^+ + \delta)} < \infty \quad \text{if } 1 \leqslant J_V^+ < \infty.$$

Then,

$$\mathbf{P}\left(\sum_{i=1}^{\infty} \Theta_i \xi_i > x\right) \sim \sum_{i=1}^{\infty} \mathbf{P}(\Theta_i \xi_i > x).$$

Proof of Theorem 1. On the one hand,

$$U_t \geqslant x - \sum_{i=1}^{\infty} \big(X_i + Y_i \mathrm{e}^{-\delta D_i}\big)\mathrm{e}^{-\delta \tau_i},$$

which implies

$$\psi(x;\infty) \leqslant \mathbf{P}\left(\sum_{i=1}^{\infty} \big(X_i + Y_i \mathrm{e}^{-\delta D_i}\big)\mathrm{e}^{-\delta \tau_i} > x\right) =: \psi^*(x;\infty). \tag{7}$$

Since $(X,Y)\in \mathrm{BRV}_{-\alpha}(\nu,F_0)$ and is independent of D, we have

$$\begin{aligned}
&\mathbf{P}\bigl(X+Y\mathrm{e}^{-\delta D}>x\bigr)\\
&\quad=\int_{0-}^{\infty}\mathbf{P}\bigl(X+Y\mathrm{e}^{-\delta s}>x\bigr)\,H(\mathrm{d}s)\sim\overline{F_0}(x)\int_{0-}^{\infty}\nu(A_s)\,H(\mathrm{d}s),
\end{aligned}\tag{8}$$

where we used the dominated convergence theorem in the second step. Indeed, for any $s\geqslant 0$, by $(X,Y)\in \mathrm{BRV}_{-\alpha}(\nu,F_0)$,

$$\begin{aligned}
\frac{\mathbf{P}(X+Y\mathrm{e}^{-\delta s}>x)}{\overline{F_0}(x)}&\leqslant\frac{\mathbf{P}(X+Y>x)}{\overline{F_0}(x)}\\
&\to\nu\bigl(\bigl\{(u,v)\in[0,\infty]^2\colon u+v>1\bigr\}\bigr),
\end{aligned}$$

which is integrable with respect to $H(\mathrm{d}s)$. By (8), $F_0\in\mathcal{R}_{-\alpha}$, and $\nu((1,\infty]^2)>0$ we have that $\{X_i+Y_i\mathrm{e}^{-\delta D_i};\ i\in\mathbb{N}\}$ constitutes a sequence of i.i.d. random variables with regularly varying tails. Note that for any $p>0$ and $q>0$,

$$\sum_{i=1}^{\infty}\bigl(\mathbf{E}\bigl(\mathrm{e}^{-\delta\tau_i}\bigr)^p\bigr)^q=\sum_{i=1}^{\infty}\bigl(\mathbf{E}\bigl(\mathrm{e}^{-\delta p\theta_1}\bigr)\bigr)^{iq}<\infty.$$

Then by using Lemma 1 and (8) we obtain

$$\begin{aligned}
\psi^*(x;\infty)&\sim\sum_{i=1}^{\infty}\mathbf{P}\bigl(\bigl(X_i+Y_i\mathrm{e}^{-\delta D_i}\bigr)\mathrm{e}^{-\delta\tau_i}>x\bigr)\\
&=\sum_{i=1}^{\infty}\int_{0-}^{\infty}\mathbf{P}\bigl(X+Y\mathrm{e}^{-\delta D}>x\mathrm{e}^{\delta t}\bigr)\,\mathbf{P}(\tau_i\in\mathrm{d}t)\\
&\sim\int_{0-}^{\infty}\nu(A_s)\,H(\mathrm{d}s)\int_{0-}^{\infty}\overline{F_0}\bigl(x\mathrm{e}^{\delta t}\bigr)\,\lambda(\mathrm{d}t)\\
&\sim\overline{F_0}(x)\int_{0-}^{\infty}\nu(A_s)\,H(\mathrm{d}s)\int_{0-}^{\infty}\mathrm{e}^{-\delta\alpha t}\,\lambda(\mathrm{d}t).
\end{aligned}\tag{9}$$

Thus, we can derive the upper bound in (4) from (7) and (9).

On the other hand, by (2) and (9) we have

$$\begin{aligned}
\psi(x;\infty)&\geqslant\mathbf{P}\left(\sum_{i=1}^{\infty}\bigl(X_i+Y_i\mathrm{e}^{-\delta D_i}\bigr)\mathrm{e}^{-\delta\tau_i}>x+\frac{c}{\delta}\right)\\
&=\psi^*\left(x+\frac{c}{\delta};\infty\right)&&(10)\\
&\sim\overline{F_0}\left(x+\frac{c}{\delta}\right)\int_{0-}^{\infty}\nu(A_s)\,H(\mathrm{d}s)\int_{0-}^{\infty}\mathrm{e}^{-\delta\alpha t}\,\lambda(\mathrm{d}t).&&(11)
\end{aligned}$$

By $F_0 \in \mathcal{R}_{-\alpha}$ we have that for any $0 < \varepsilon < 1$ and sufficiently large x,

$$\overline{F_0}\left(x + \frac{c}{\delta}\right) \geqslant \overline{F_0}\big((1+\varepsilon)x\big) \sim (1+\varepsilon)^{-\alpha}\overline{F_0}(xt).$$

This, by (11) and the arbitrariness of $\varepsilon > 0$, leads to the lower bound in (4). □

The following lemma will be used in the proof of Theorem 2, which is due to [29].

Lemma 2. *Let (X, Y) be a nonnegative random vector with marginal distributions F and G, respectively. If $F \in \mathcal{C}$ and $\overline{G}(x) = o(\overline{F}(x))$, then, regardless of arbitrary dependence between X and Y,*

$$\mathbf{P}(X + Y > x) \sim \overline{F}(x).$$

Proof of Theorem 2. The proof is much similar to that of Theorem 1 with some slight modification. Note that by Lemma 2, for any $s \geqslant 0$,

$$\mathbf{P}\big(X + Y\mathrm{e}^{-\delta s} > x\big) \leqslant \mathbf{P}(X + Y > x) \sim \overline{F}(x),$$

which is integrable with respect to $H(\mathrm{d}s)$. Then, by the dominated convergence theorem, relation (8) can be rewritten as

$$\mathbf{P}\big(X + Y\mathrm{e}^{-\delta D} > x\big) \sim \overline{F}(x)$$

implying that $X + Y\mathrm{e}^{-\delta D}$ has also a consistently varying tail. Again by using Lemma 1 we have

$$\psi^*(x;\infty) \sim \int_{0-}^{\infty} \overline{F}\big(x\mathrm{e}^{\delta t}\big)\,\lambda(\mathrm{d}t). \tag{12}$$

Thus, the upper bound in (5) is derived from (7) and (12).

As for the lower bound of (5), for any $\varepsilon > 0$ and all $x \geqslant c/(\delta\varepsilon)$, by (12) and $F \in \mathcal{C}$,

$$\begin{aligned}
&\psi^*\left(x + \frac{c}{\delta}; \infty\right) \\
&\quad \sim \int_{0-}^{\infty} \overline{F}\left(\left(x + \frac{c}{\delta}\right)\mathrm{e}^{\delta t}\right)\lambda(\mathrm{d}t) \geqslant \int_{0-}^{\infty} \frac{\overline{F}((1+\varepsilon)x\mathrm{e}^{\delta t})}{\overline{F}(x\mathrm{e}^{\delta t})}\overline{F}\big(x\mathrm{e}^{\delta t}\big)\,\lambda(\mathrm{d}t) \\
&\quad \gtrsim \overline{F}_*(1+\varepsilon)\int_{0-}^{\infty} \overline{F}\big(x\mathrm{e}^{\delta t}\big)\,\lambda(\mathrm{d}t) \sim \int_{0-}^{\infty} \overline{F}\big(x\mathrm{e}^{\delta t}\big)\,\lambda(\mathrm{d}t),
\end{aligned} \tag{13}$$

by letting firstly $x \to \infty$ then $\varepsilon \downarrow 0$. Therefore, the desired lower bound in (5) can be obtained from (10) and (13).

If further $F \in \mathcal{R}_{-\alpha}$, relation (6) follows from (5) immediately. □

4 Simulation studies

In this section, we use some numerical simulations to verify the accuracy of the asymptotic results for $\psi(x;\infty)$ in Theorems 1 and 2. To this end, we adopt the crude Monte Carlo (CMC) method to compare the simulated ruin probability $\psi(x;\infty)$ in (2) with the asymptotic one on the right hand side of (4) or (6).

Throughout this section, we specify the renewal counting process N_t in (1) to a homogeneous Poisson process with intensity $\lambda_1 > 0$, and we suppose the delay time D also follows the exponential distribution with parameter $\lambda_2 > 0$. Although we estimate the ultimate ruin probability $\psi(x;\infty)$, when simulating it, we choose $\psi(x;T)$ as the replacement for large T but fixed due to (2): $\psi(x;\infty) = \lim_{T\to\infty}\psi(x;T)$.

As for Theorem 1, assume that the random pair (X,Y) possesses the Gumbel copula of the form

$$C(u,v) = \exp\bigl\{-\bigl((-\ln u)^\gamma + (-\ln v)^\gamma\bigr)\bigr\}^{1/\gamma}, \quad (u,v)\in[0,1]^2, \tag{14}$$

with parameter $\gamma \geqslant 1$.

It can be verified from [19, Lemma 5.2] that if (X,Y) possesses a bivariate Gumbel copula (14) with $\gamma > 1$ and the marginal distributions $F = G \in \mathcal{R}_{-\alpha}$ for some $\alpha > 0$, then $(X,Y) \in \mathrm{BRV}_{-\alpha}(\nu, F_0)$ for some nondegenerate limit measure ν and some reference distribution F_0. Furthermore, according the discussions in [17, Sect. 4], it can be calculated that for any Borel set $B \subset [0,\infty]^2$,

$$\nu(B) = \alpha^2(\gamma-1)\iint\limits_B \bigl(s^{-\alpha\gamma} + t^{-\alpha\gamma}\bigr)^{1/\gamma-2} s^{-\alpha\gamma-1} t^{-\alpha\gamma-1}\,\mathrm{d}s\,\mathrm{d}t,$$

which implies $\nu((1,\infty]\times(0,\infty]) = \nu((0,\infty]\times(1,\infty]) = 1$. Thus, the reference distribution F_0 can be chosen as $F = G$. Let F_0 be the Pareto distribution of the form

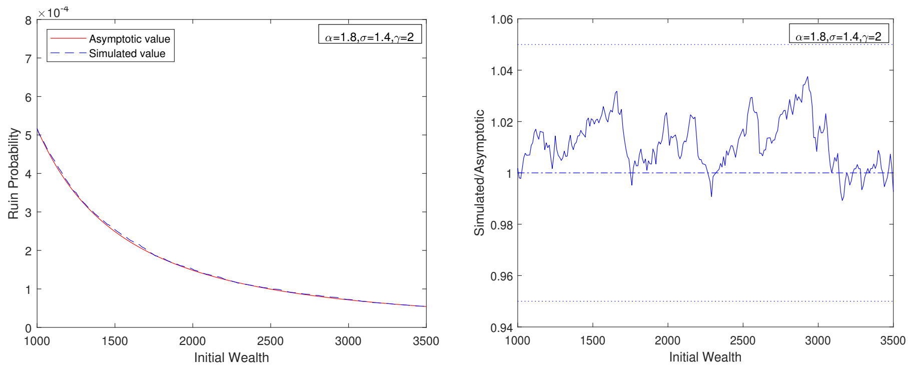
$$F_0(x) = 1 - \left(1 + \frac{x}{\sigma}\right)^{-\alpha}, \quad x \geqslant 0, \tag{15}$$

with parameters $\alpha > 0$ and $\sigma > 0$, which implies $F_0 \in \mathcal{R}_{-\alpha}$.

The various parameters are set to:

- $c = 1$, $\delta = 0.005$, $T = 1000$;
- $\lambda_1 = 0.2$, $\lambda_2 = 0.25$;
- $\gamma = 2$, $\alpha = 1.8$, $\sigma = 1.4$.

For the simulated estimation $\hat{\psi}(x;T)$ of the ultimate ruin probability $\psi(x;\infty)$, we first divide the time interval $[0,T]$ into n parts, and for the given $t_k = kT/n$, $k = 1,\dots,n$, we generate m samples $N_{t_k}^{(j)}$, $j = 1,\dots,m$. Then, for the jth sample $N_{t_k}^{(j)}$, generate $N_{t_k}^{(j)}$ pairs of $(X_i^{(j)}, Y_i^{(j)})$ following the Gumbel copula of the form (14) with marginal Pareto distributions $F = G$ of the form (15), and generate the delay times $D_i^{(j)}$, $i = 1,\dots,N_{t_k}^{(j)}$. Thus, the discounted surplus process $U_{t_k}^{(j)}$ can be calculated according to (1). In this way,



(a) (b)

Figure 1. Comparison of the simulated estimate and asymptotic value for ultimate ruin probability (left) and their ratio (right) in Theorem 1.

the ultimate ruin probability $\psi(x;\infty)$ can be estimated by

$$\hat{\psi}(x;T)=\frac{1}{m}\sum_{j=1}^{m}\mathbf{1}_{\{\min_{k=1,\dots,n}U_{t_k}^{(j)}<0\}}. \tag{16}$$

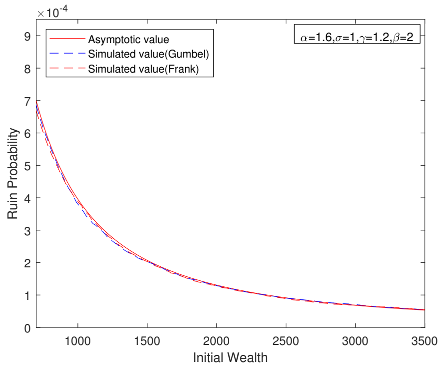
The asymptotic value on the right-hand side of (4) is computed by numerical integration with $\int_0^\infty \nu(A_s)\lambda_2\mathrm{e}^{-\lambda_2 s}\,\mathrm{d}s\approx 3.190531$ and $\int$ _0$^\infty \mathrm{e}^{-\delta\alpha t}\lambda_1\,\mathrm{d}t=\lambda_1/(\delta\alpha)\approx 22.222$.

In Fig. 1, we compare the CMC estimate $\hat{\psi}(x;T)$ in (16) with the asymptotic value given by (4) on the left, and we show their ratio on the right. The CMC simulation is conducted with the sample size $m=5\times10^6$, the time step size $T/n=10^{-4}$ with $n=10^7$, and the initial wealth x varying from 1000 to 3500. From Fig. 1(a) it can be seen that with the increase of the initial wealth x, both of the estimates decrease gradually and the two lines get closer. In Fig. 1(b), the ratios of the simulated and asymptotic values are close to 1. The fluctuation is due to the poor performance of the CMC method, which requires a sufficiently large sample size to meet the demands of high accuracy.

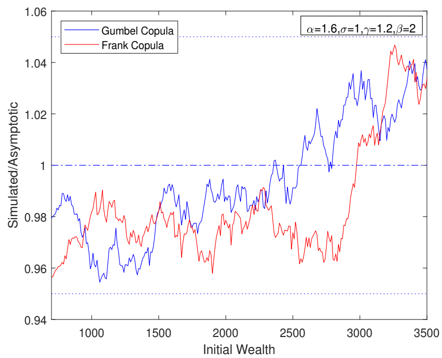
Next, we consider the situation of Theorem 2 in which, assume that X still follows the Pareto distribution of form (15) with parameters $\alpha>0$ and $\sigma>0$, but Y follows the standard lognormal distribution

$$G(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\ln x}\mathrm{e}^{-u^2/2}\,\mathrm{d}u,\quad x>0.$$

Clearly, $F\in\mathcal{R}_{-\alpha}$ and $\overline{G}(x)=o(\overline{F}(x))$. The following simulation aims to check the accuracy of relation (6) and the influence caused by different dependence structures between X and Y. For this purpose, the Gumbel copula (14) and the Frank copula are used to model the dependence between X and Y. Recall that a random pair (X,Y) possesses



(a)



(b)

Figure 2. Comparison of the two simulated estimates under the Gumbel and Frank copulas and asymptotic value for ultimate ruin probability (left) and their ratio (right) in Theorem 2.

the Frank copula of the form

$$C(u,v) = -\frac{1}{\beta}\ln\left(1 + \frac{(\mathrm{e}^{-\beta u}-1)(\mathrm{e}^{-\beta v}-1)}{\mathrm{e}^{-\beta}-1}\right), \quad (u,v) \in [0,1]^2,$$

for some parameter $\beta > 0$. In the terminology of [11], the Gumbel copula exhibits the asymptotic dependence between two random variables, whereas the Frank copula shows the asymptotic independence. Hence, the former reflects a type of strong dependence, but the latter is relatively weak.

The various parameters are set to:

- $c = 1$, $\delta = 0.005$, $T = 1000$;
- $\lambda_1 = 0.2$, $\lambda_2 = 0.25$;
- $\gamma = 1.2$, $\alpha = 1.6$, $\sigma = 1$, $\beta = 2$.

We continue to simulate the ruin probability through the CMC method. The procedure is similar to the previous case. We compare the two simulated estimates under the Gumbel and Frank copulas with the asymptotic value in Fig. 2(a), and we present the two ratios in Fig. 2(b). The two simulated estimates are obtained with a sample of size $m = 5 \times 10^6$, the time step size $T/n = 10^{-4}$ with $n = 10^7$, and x varying from 700 to 3500. From Fig. 2(a) it can be seen that the two simulated lines in different copulas almost coincide, so in the setting of Theorem 2, the ultimate ruin probability is insensitive to different dependence structures between each pair of the main claim and by-claim. Figure 2(b) indicates that the two convergences in different dependence structures are both robust. This confirms Theorem 2.

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