Nonlinear Analysis: Modelling and Control, Vol. 26, No. 1, 169–185
https://doi.org/10.15388/namc.2021.26.21413

Vilnius University Press

A variational principle, coupled fixed points and market equilibrium

Stanimir Kabaivanov[1], **Boyan Zlatanov**

University of Plovdiv "Paisii Hilendarski",
24, Tsar Assen str., 4000 Plovdiv, Bulgaria
stanimir.kabaivanov@gmail.com; bzlatanov@gmail.com

Received: August 12, 2019 / **Revised:** November 23, 2020 / **Published online:** January 1, 2021

Abstract. We present a possible kind of generalization of the notion of ordered pairs of cyclic maps and coupled fixed points and its application in modelling of equilibrium in oligopoly markets. We have obtained sufficient conditions for the existence and uniqueness of coupled fixed in complete metric spaces. We illustrate one possible application of the results by building a pragmatic model on competition in oligopoly markets. To achieve this goal, we use an approach based on studying the response functions of each market participant, thus making it possible to address both Cournot and Bertrand industrial structures with unified formal method. We show that whenever the response functions of the two players are identical, then the equilibrium will be attained at equal levels of production and equal prices. The response functions approach makes it also possible to take into consideration different barriers to entry. By fitting to the response functions rather than the profit maximization of the payoff functions problem we alter the classical optimization problem to a problem of coupled fixed points, which has the benefit that considering corner optimum, corner equilibrium and convexity condition of the payoff function can be skipped.

Keywords: variational principle, coupled fixed points, oligopoly market, market equilibrium.

1 Introduction

Ekeland proved a variational principle in 1972. In a series of articles, he enriches the results. Later he presented a more concise proof [15], which technique we will use. In the same article [15], various applications of the variational principle in different fields of mathematics are presented. Ekeland's variational principle has many applications and generalizations [3, 6, 10, 26, 27]. It is well known that fixed point theorems and variational principles are closely related [7, 15]. A very recent publication, which presents probably the best known generalization of Caristi's results, is [30]. Many well-known results are obtained as consequences of the main results from [30].

Fixed point theorems initiated by Banach's contraction principle has proved to be a powerful tool in nonlinear analysis. We cannot mention all kinds of generalizations

[1]The author is partially supported by grant No. KP–06 K2–2.

© 2021 Authors. Published by Vilnius University Press
This is an Open Access article distributed under the terms of the Creative Commons Attribution Licence, which permits unrestricted use, distribution, and reproduction in any medium, provided the original author and source are credited.

of Banach's contraction principle. One direction for generalization of it is the notion of coupled fixed points [17], where mixed monotone maps in partially ordered by a cone Banach spaces are investigated. Later this idea was developed for mixed monotone maps in partially ordered metric spaces [18]. It is impossible to summarize all generalizations of the ideas of coupled fixed points for mixed monotone maps in partially ordered metric spaces. The investigation on the subject continuous as seen [1,19,22,29], which is far from exhausting the most recent results. Another kind of maps considered in partially ordered complete metric spaces are for monotone maps without the mixed monotone property [13, 20].

Let us mention that Ekeland's variational principle holds for any l.s.c. maps $T: X \times X \to \mathbb{R}$, provided that X is a partially ordered complete metric space. Unfortunately, when investigating contraction type of maps $F : X \times X \to X$ satisfying the mixed monotone property in a partially ordered complete metric space $X \times X$ the contraction conditions holds only for part of the points $(x, y), (u, v) \in X \times X$. Thus we cannot apply Ekeland's variational principle as it is done in [15]. A similar approach was used in [27], where variational principles in partially ordered metric spaces were obtained and used to investigated problems, otherwise impossible to solve with the known variational principles.

In [33], the problem for generalizing Ekeland's variational principle on classes of subsets of partially ordered complete metric space $X \times X$, which need not to be compact or even closed is obtained. Number of applications of the main result are presented in [33], where existence and uniqueness are proven for well-known results [4, 18] with the help of the generalized variational principle, and some new theorems are obtained.

The considered maps are of the kind $F : X \times X \to X$, and we are searching for a coupled fixed point, i.e. $(x, y) \in X \times X$ be such that $x = F(x, y)$ and $y = F(y, x)$. As it is shown in a number of articles, there holds $x = y$. A generalization of coupled fixed or best proximity points (x, y) when $x \neq y$ is presented in [14].

Following [17, 18], let X be a set, and let $\preccurlyeq$ be a partial order in X, then $(X, \preccurlyeq)$ is called a partially ordered set. We call two elements $x, y \in X$ comparable if either $x \preccurlyeq y$ or $y \preccurlyeq x$. We denote $x \succcurlyeq y$ if $y \preccurlyeq x$. We say that $x \prec$ if $x \preccurlyeq y$, but $x \neq y$. Let (X, ρ) be a metric space with a partial order $\preccurlyeq$, then the triple $(X, \rho, \preccurlyeq)$ is called a partially ordered metric space. Ran and Reurings in [28] initiate the fixed point theory in partially ordered metric spaces.

Definition 1. Let $(Z, \preccurlyeq)$ be a partially ordered and $X, Y \subseteq Z$, let $F : X \times Y \to X$ and $f : X \times Y \to Y$. The ordered couple (F, f) is said to have the mixed monotone property if

(i) for any $x_1, x_2, y \in X$ such that $x_1 \preccurlyeq x_2$, there holds $F(x_1, y) \preccurlyeq F(x_2, y)$, and
(ii) for any $y_1, y_2, x \in X$ such that $y_1 \preccurlyeq y_2$, there holds $f(x, y_1) \succcurlyeq f(x, y_2)$.

If $X \equiv Y$, and $F \equiv f$, we get the definition from [17, 18].

Definition 2. Let $(Z, \preccurlyeq)$ be a partially ordered and $X, Y \subseteq Z$. Let $F : X \times Y \to X$ and $f : X \times Y \to Y$. An ordered pair $(x, y) \in X \times X$ is called coupled fixed point of (F, f) if $x = F(x, y)$ and $y = f(x, y)$.

If $X \equiv Y$ and $F \equiv f$, we get the definition from [17, 18].

Let $(X, \rho, \preccurlyeq)$ be a partially ordered complete metric space. We endow the product space $X \times X$ with the following partial order $(u, v) \preccurlyeq (x, y)$, provided that $x \succcurlyeq u$ and $y \preccurlyeq v$ holds simultaneously and with the following metric $d((x, y), (u, v)) = \rho(x, u) + \rho(y, v)$ for $(x, y), (u, v) \in X \times X$.

Every where for a partially ordered metric space $(X, \rho, \preccurlyeq)$, we will consider the product space $(X \times X, d, \preccurlyeq)$ endowed with the mentioned above partial order and metric.

Let (X, ρ) be a metric space. Following [6], an extended real valued function $T : X \to (-\infty, +\infty]$ on X is called lower semicontinuous (for short l.s.c.) if $\{x \in X\colon f(x) > a\}$ is an open set for each $a \in (-\infty, +\infty]$. Equivalently, T is l.s.c. if and only if, at any point $x_0 \in X$, there holds $\liminf_{x \to x_0} f(x) \geqslant f(x_0)$. A function T is called to be proper function, provided that $T \not\equiv +\infty$.

2 Variational principle

Just to fit some of the formulas in the text field, we will use the notation $u = (u^{(1)}, u^{(2)}) \in Z \times Z$, and for any $u \in Z \times Z$, let us denote $\overline{u} = (u^{(2)}, u^{(1)})$.

Theorem 1. *Let $(Z, \rho, \preccurlyeq)$ be a partially ordered complete metric space, $(Z \times Z, d, \preccurlyeq)$, $X, Y \subseteq Z$ and $F : X \times Y \to X$ and $f : X \times Y \to Y$ be a continuous maps with the mixed monotone property. Let*

$$V \times U = \{x = (x^{(1)}, x^{(2)}) \in X \times Y\colon x^{(1)} \preccurlyeq F(x) \text{ and } x^{(2)} \succcurlyeq f(x)\} \neq \emptyset.$$

Let $T : X \times Y \to \mathbb{R} \cup \{+\infty\}$ be a proper, l.s.c., bounded from below function. Let $\varepsilon > 0$ be arbitrary chosen and fixed, and let $u_0 \in V \times U$ be an ordered pair such that the inequality

$$T(u_0) \leqslant \inf_{V \times U} T(v) + \varepsilon \tag{1}$$

holds. Then there exists an ordered pair $x \in V \times U$ such that

(i) $T(x) \leqslant \inf_{u \in V \times U} T(u)$;
(ii) $d(x, u_0) \leqslant 1$;
(iii) *For every $w \in V \times U$ different from $x \in V \times U$, the following inequality holds*:

$$T(w) > T(x) - \varepsilon d(w, v).$$

Proof. Let us define inductively a sequence of ordered pairs $\{u_n\}_{n=0}^{\infty} \subset X \times Y$ starting with the pair $u_0 \in V \times U$ that satisfies (1).

Suppose that we have already chosen $u_n \in V \times U$. There holds either:

(a) For every ordered pair $w \neq u_n$, $w \in V \times U$, holds the inequality $T(w) > T(u_n) - \varepsilon d(w, u_n)$; or
(b) There exists $w \neq u_n$, $w \in V \times U$, so that there holds the inequality

$$T(w) \leqslant T(u_n) - \varepsilon d(w, u_n). \tag{2}$$

If case (a) holds, we choose $u_{n+1} = u_n$. In case of (b), let us denote by $S_n \subset V \times U$ the set of all ordered pairs $w \in V \times U$, which satisfy inequality (2). We choose $u_{n+1} \in S_n$ so that

$$T(u_{n+1}) \leqslant \frac{T(u_n)}{2} + \frac{\inf_{v \in S_n} T(v)}{2}. \tag{3}$$

We claim that, in both cases, $\{u_n\}_{n=0}^{\infty}$ is a Cauchy sequence.

Indeed, if case (a) ever occurs, the sequence is stationary starting from some index n. If case (a) does not occur for any index $n \in \mathbb{N}$, then it should be case (b) for all indexes $n \in \mathbb{N}$. Therefore, by (2) we have the inequalities

$$d(u_k, u_{k+1}) \leqslant T(u_k) - T(u_{k+1})$$

for $k = 0, 1, 2, \ldots$. Summing up the above inequalities for k from n to $p-1$, we get

$$\begin{aligned} \varepsilon d(u_n, u_p) &\leqslant \sum_{k=n}^{p-1} \varepsilon d(u_k, u_{k+1}) \leqslant \sum_{k=n}^{p-1} \bigl(T(u_k) - T(u_{k+1})\bigr) \\ &= T(u_n) - T(u_p). \end{aligned} \tag{4}$$

From the inequality

$$T(u_{n+1}) \leqslant T(u_n) - \varepsilon d(u_n, u_{n+1}) < T(u_n)$$

it follows that the sequence $\{T(u_n)\}_{n=0}^{\infty}$ is a decreasing one and bounded from below (by $\inf_{v \in V \times U} T(v)$). Hence it is convergent. So the right-hand side in (4) goes to zero when n and p go to infinity simultaneously. Consequently, $\{u_n\}_{n=0}^{\infty}$ is a Cauchy sequence. Since $(Z \times Z, d)$ is a complete metric space (because (Z, ρ) is complete), it follows that the sequence $\{u_n\}_{n=0}^{\infty} = \{(u_n^{(1)}, u_n^{(2)})\}_{n=0}^{\infty}$ converges to some $x = (x^{(1)}, x^{(2)}) \in Z \times Z$.

We claim that $(x^{(1)}, x^{(2)}) \in V \times U$ and satisfies (i), (ii) and (iii).

Indeed from the continuity of F and f and the choice of $u_n = (u_n^{(1)}, u_n^{(2)}) \in V \times U$ we have

$$x^{(1)} = \lim_{n \to +\infty} u_n^{(1)} \preccurlyeq \lim_{n \to +\infty} F(u_n) = F(x)$$

and

$$x^{(2)} = \lim_{n \to +\infty} u_n^{(2)} \succcurlyeq \lim_{n \to +\infty} f(u_n) = f(x).$$

(i) By construction the sequence $\{T(u_n)\}_{n=0}^{\infty}$ is monotonously decreasing and consequently using the l.s.c. of T we get $T(x) \leqslant \lim_{n \to +\infty} T(u_n) \leqslant T(u_0)$, and consequently (i) holds.

(ii) Let us put $n = 0$ in (4), i.e.

$$\varepsilon d(u_0, u_p) \leqslant T(u_0) - T(u_p) \leqslant T(u_0) - \inf_{v \in V \times U} T(v) \leqslant \varepsilon.$$

Letting p to infinity in the last inequality, we get $\varepsilon d(u_0, x) = \lim_{p \to +\infty} \varepsilon d(u_0, u_p) \leqslant \varepsilon$, i.e. $d(x, u) \leqslant 1$.

(iii) Let us suppose that (iii) were not true for all $w \in V \times U$. Therefore we can choose $w \neq x$, $w \in V \times U$, so that

$$T(w) \leqslant T(x) - \varepsilon d(w,x) < T(x). \tag{5}$$

Letting $p \to +\infty$ in (4), we obtain

$$\varepsilon d(u_n, x) \leqslant T(u_n) - T(x). \tag{6}$$

From (5) and (6) we get the chain of inequalities

$$\begin{aligned} T(w) &\leqslant T(x) - \varepsilon d(w,x) \leqslant T(u_n) - \varepsilon d(u_n,x) - \varepsilon d(x,w) \\ &= T(u_n) - \varepsilon\bigl(d(u_n,x) + d(x,w)\bigr) \leqslant T(u_n) - \varepsilon d(u_n,w), \end{aligned}$$

and thus $w \in S_n$ for all $n \in \mathbb{N}$. From (3) we have

$$2T(u_{n+1}) - T(u_n) \leqslant \inf_{S_n} F \leqslant T(w) \tag{7}$$

because $w \in \cap_{n=0}^{\infty} S_n$. From the existence of $\lim_{n\to+\infty} F(u_n) = l$ and (7) it follows that

$$\lim_{n\to+\infty} \bigl(2T(u_{n+1}) - T(u_n)\bigr) = \lim_{n\to+\infty} T(u_n) = l \leqslant T(w). \tag{8}$$

Since T is l.s.c., we have the inequality

$$T(x) \leqslant \lim_{n\to+\infty} T(u_n) = l, \tag{9}$$

and thus (8) and (9) imply that $T(x) \leqslant T(w)$, a contradiction with (5). $\square$

3 Coupled fixed points

We will need the next observation that is used in [4, 11, 18], but not stated as a proposition.

Proposition 1. *Let $(Z, \preccurlyeq)$ be a partially ordered set, $X, Y \subseteq Z$ and $F : X \times Y \to X$, $f : X \times Y \to Y$ be an ordered pair of map with the mixed monotone property. Let $(x,y) \in X \times Y$ satisfies the inequalities $x \preccurlyeq F(x,y)$, $y \succcurlyeq f(x,y)$, and let us put $u = F(x,y)$ and $v = f(x,y)$. Then there hold $u \preccurlyeq F(u,v)$, $v \succcurlyeq f(u,v)$, $u \succcurlyeq x$ and $v \preccurlyeq y$.*

Proof. By the definition of $(u,v) \in X \times Y$ there hold $x \preccurlyeq F(x,y) = u$ and $y \succcurlyeq f(x,y) = v$. From the assumption that (F,f) satisfies the mixed monotone property we get the inequalities

$$F(u,v) \succcurlyeq F(x,v) \succcurlyeq F(x,y) = u$$

and

$$f(u,v) \preccurlyeq f(u,y) \preccurlyeq f(x,y) = v. \qquad \square$$

Let $(Z, \preccurlyeq)$ be a partially ordered set, $X, Y \subseteq Z$ and $F : X \times Y \to X$, $f : X \times Y \to Y$. Following [18], for any $(\xi_0, \eta_0) \in X \times Y$, we will consider the sequence $\{\xi_n, \eta_n\}_{n=0}^{\infty}$ defined by $\xi_n = F(\xi_{n-1}, \eta_{n-1}) = F^n(\xi_0, \eta_0)$ and $\eta_n = f(\xi_{n-1}, \eta_{n-1}) = f^n(\xi_0, \eta_0)$ for $n \in \mathbb{N}$.

Proposition 2. *Let $(Z, \preccurlyeq)$ be a partially ordered set, $X, Y \subseteq Z$ and $F : X \times Y \to X$, $f : X \times Y \to Y$ be an ordered pair of maps with the mixed monotone property. Let $(x, y) \in X \times Y$ be a coupled fixed point, i.e. $x = F(x, y)$, $y = f(x, y)$, and let (ξ_0, η_0) be comparable with (x, y). Then (ξ_n, η_n) is comparable with $(x, y) = (F(x, y), f(x, y))$ and (η_n, ξ_n) is comparable with $(y, x) = (f(x, y), F(x, y))$.*

Proof. If (ξ_0, η_0) is comparable with (x, y), then there holds either $\xi_0 \preccurlyeq x$ and $\eta_0 \succcurlyeq y$ or $\xi_0 \succcurlyeq x$ and $\eta_0 \preccurlyeq y$. Let us assume that there holds the second one (i.e. $\xi_0 \succcurlyeq x$ and $\eta_0 \preccurlyeq y$). Using the mixed monotone property, we get

$$\xi_1 = F(\xi_0, \eta_0) \succcurlyeq F(x, \eta_0) \succcurlyeq F(x, y) = x$$

and

$$\eta_1 = f(\xi_0, \eta_0) \preccurlyeq f(\xi_0, y) \preccurlyeq f(x, y) = y.$$

Therefore $(\xi_1, \eta_1) \succcurlyeq (x, y) = (F(x, y), f(x, y))$. We can get by induction that

$$\xi_n = F(\xi_{n-1}, \eta_{n-1}) \succcurlyeq F(x, \eta_{n-1}) \succcurlyeq F(x, y) = x$$

and

$$\eta_n = f(\xi_{n-1}, \eta_{n-1}) \preccurlyeq f(\xi_{n-1}, y) \preccurlyeq f(x, y) = y.$$

Consequently, (ξ_n, η_n) is comparable with (x, y) and

$$(\xi_n, \eta_n) \succcurlyeq (x, y) = \bigl(F(x, y), f(x, y)\bigr).$$

If there holds the first case (i.e. $\xi_0 \preccurlyeq x$ and $\eta_0 \succcurlyeq y$), we can get in a similar fashion that there hold $\xi_n \preccurlyeq x$ and $\eta_n \succcurlyeq y$ and thus

$$\bigl(F(x, y), f(x, y)\bigr) = (x, y) \succcurlyeq (\xi_n, \eta_n).$$

Therefore (ξ_n, η_n) is comparable with $(x, y) = (F(x, y), F(y, x))$ in both cases. $\square$

We will need the result from [25] that, in a partially ordered space, any element has an lower or an upper bound is equivalent to, for every two element, there exists an element, which is comparable with both of them.

Theorem 2. *Let $(Z, \rho, \preccurlyeq)$ be a partially ordered complete metric space, $(Z \times Z, d, \preccurlyeq)$, $X, Y \subseteq Z$ and $F : X \times Y \to X$ and $f : X \times Y \to Y$ be a continuous map with the mixed monotone property. Let there exists $\alpha \in [0, 1)$, so that the inequality*

$$\rho\bigl(F(x, y), F(u, v)\bigr) + \rho\bigl(f(x, y), f(u, v)\bigr) \leqslant \alpha\rho(x, u) + \alpha\rho(y, v) \tag{10}$$

holds for all $x \succcurlyeq u$ and $y \preccurlyeq v$. If there exists at least one ordered pair $(x, y) \in X \times Y$ such that $x \preccurlyeq F(x, y)$ and $y \succcurlyeq f(x, y)$, then there exists a coupled fixed points (x, y) of (F, f).

(i) *If, in addition, every pair of elements in $X \times Y$ has a lower or an upper bound, then the coupled fixed point is unique.*

(ii) *If, in addition, every element in Z has a lower or an upper bound and $f(x, y) = F(y, x)$, then the coupled fixed point $(x, y))$ satisfies $x = y$.*

Remark. If $X = Y = Z$, in Theorem 2, we get the results from [4]. We will justify in the application section that the generalization, which we have made, is interesting when we try to apply the theory of coupled fixed points in oligopoly markets.

Proof of Theorem 2. Let us consider the function $T : X \times Y \to \mathbb{R}$ defined by

$$T(z) = d\bigl(z, \bigl(F(z), f(z)\bigr)\bigr) = \rho\bigl(x, F(x, y)\bigr) + \rho\bigl(y, f(x, y)\bigr),$$

where $z = (x, y) \in X \times Y$ and $\overline{z} = (y, x)$. The map T satisfies the conditions of Theorem 1 as far as T is continuous, proper function, bounded from below, and the set of all $z \in X \times X$ such that $x \preccurlyeq F(z)$ and $y \succcurlyeq f(z)$ is not empty. Let us choose $\varepsilon \in (0, 1 - \alpha)$. By Theorem 1 there exists $(x, y) \in X \times Y$, satisfying $x \preccurlyeq F(x, y)$ and $y \succcurlyeq f(x, y)$, such that there holds the inequality

$$T(x, y) \leqslant T(u, v) + \varepsilon d\bigl((x, y), (u, v)\bigr) \tag{11}$$

for every $u \preccurlyeq F(u, v)$ and $v \succcurlyeq f(u, v)$.

Let us put $u = F(x, y)$, $v = f(x, y)$ and $w = (u, v)$. By Proposition 1 it follows that $u \preccurlyeq F(u, v)$, $v \succcurlyeq f(u, v)$, $u \succcurlyeq x$ and $v \preccurlyeq y$. From (10), using the symmetry of the metrics ρ, we obtain

$$\begin{aligned}
T(w) &= d\bigl(w, \bigl(F(w), f(w)\bigr)\bigr) \\
&= \rho\bigl(F(x, y), F\bigl(F(x, y), f(x, y)\bigr)\bigr) + \rho\bigl(f(x, y), f\bigl(F(x, y), f(x, y)\bigr)\bigr) \\
&= \rho\bigl(F\bigl(F(x, y), f(x, y)\bigr), F(x, y)\bigr) + \rho\bigl(f\bigl(F(x, y), f(x, y)\bigr), F(y, x)\bigr) \\
&\leqslant \alpha\bigl(\rho\bigl(F(x, y), x\bigr) + \rho\bigl(f(x, y), y\bigr)\bigr) = \alpha T(x, y).
\end{aligned} \tag{12}$$

Consequently, using (12), from (11) we get

$$T(x, y) \leqslant T(w) + \varepsilon d\bigl((x, y), w\bigr) \leqslant \alpha T(x, y) + \varepsilon T(x, y) = \bigl(\alpha + \varepsilon\bigr)T(x, y).$$

From the choice of $\varepsilon \in (0, 1 - \alpha)$ we obtain $T(x, y) < T(x, y)$. From the last inequality it follows that $T(x, y) = d((x, y), (F(x, y), f(x, y))) = 0$, i.e. $\rho(x, F(x, y)) + \rho(y, f(x, y)) = 0$. Therefore (x, y) is a coupled fixed points of (F, f).

(i) Let there are two coupled fixed points $(x, y), (u, v) \in X \times Y$, then $x = F(x, y)$, $y = f(x, y)$, $u = F(u, v)$ and $v = f(u, v)$. By the assumption that any element has an lower or an upper bound it follows from [25] that there exists (ξ_0, η_0) comparable with (x, y) and (u, v). From Proposition 2 it follows that (ξ_n, η_n) is comparable with both $(x, y) = (F(x, y), f(x, y))$ and $(u, v) = (F(u, v), f(u, v))$, and (η_n, ξ_n) is comparable with both (y, x) and (v, u).

We will apply inequality (10). If $(\xi_n, \eta_n) \succcurlyeq (x, y)$, then it satisfies the assumptions of the theorem.

If $(\xi_n, \eta_n) \preccurlyeq (x, y)$, using the symmetry of the metrics ρ, we get

$$\begin{aligned}
S_1 &= \rho\bigl(F(\xi_n, \eta_n), F(x, y)\bigr) + \rho\bigl(f(\xi_n, \eta_n), f(x, y)\bigr) \\
&= \rho\bigl(F(x, y), F(\xi_n, \eta_n)\bigr) + \rho\bigl(f(x, y), f(\xi_n, \eta_n)\bigr) \\
&\leqslant \alpha\rho(x, \xi_n) + \alpha\rho(y, \eta_n).
\end{aligned}$$

Thus we can apply (10) when (ξ_n, η_n) is comparable with $(F(x, y), f(x, y))$.

There exists $n_0 \in \mathbb{N}$ such that $\alpha^{n_0} < (\rho(x, u) + \rho(y, v))/(\rho(\xi_0, x) + \rho(\eta_0, y) + \rho(\xi_0, u) + \rho(\eta_0, v))$.

For any arbitrary coupled fixed point (x, y) and the sequence (ξ_n, η_n) is comparable with (x, y), there holds

$$\begin{aligned}
I_n &= \rho(\xi_n, x) + \rho(\eta_n, y) \\
&= \rho\bigl(F(\xi_{n-1}, \eta_{n-1}), F(x, y)\bigr) + \rho\bigl(f(\xi_{n-1}, \eta_{n-1}), f(x, y)\bigr) \\
&\leqslant \alpha\rho(\xi_{n-1}, x) + \alpha\rho(\eta_{n-1}, y) = \alpha\bigl(\rho(\xi_{n-1}, x) + \rho(\eta_{n-1}, y)\bigr) \\
&\dots \\
&= \alpha^n\bigl(\rho(\xi_0, x) + \rho(\eta_0, y)\bigr).
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
\rho(x, u) + \rho(y, v) &\leqslant \rho(x, \xi_{n_0}) + \rho(\xi_{n_0}, u) + \rho(y, \eta_{n_0}) + \rho(\eta_{n_0}, v) \\
&= \rho(x, \xi_{n_0}) + \rho(\eta_{n_0}, y) + \rho(\xi_{n_0}, u) + \rho(\eta_{n_0}, v) \\
&\leqslant \alpha^{n_0}\bigl(\rho(\xi_0, x) + \rho(\eta_0, y) + \rho(\xi_0, u) + \rho(\eta_0, v)\bigr) \\
&< \rho(x, u) + \rho(y, v),
\end{aligned}$$

which is a contradiction, and that $(x, y) = (u, v)$.

(ii) Let us put $f(x, y) = F(y, x)$ and $u = y$ and $v = x$ in (10). We get

$$\begin{aligned}
2\rho\bigl(F(x, y), F(y, x)\bigr) &= \rho\bigl(F(x, y), F(y, x)\bigr) + \rho\bigl(F(y, x), F(x, y)\bigr) \\
&= \rho\bigl(F(x, y), F(y, x)\bigr) + \rho\bigl(f(x, y), f(y, x)\bigr) \\
&\leqslant 2\alpha\rho(x, y)
\end{aligned} \tag{13}$$

and

$$\begin{aligned}
S_2 &= \rho\bigl(F^n(x, y), F^n(x, z)\bigr) + \rho\bigl(F^n(y, x), F^n(z, x)\bigr) \\
&= \rho\bigl(F^n(x, y), F^n(x, z)\bigr) + \rho\bigl(f^n(x, y), f^n(x, z)\bigr) \\
&\leqslant \alpha^n\rho(y, z).
\end{aligned} \tag{14}$$

Consequently, from (13) we get that $\rho(F^n(x, y), F^n(y, x)) \leqslant \alpha^n\rho(x, y)$.

Following the technique from [4], we need to consider the two cases when x ad y are comparable and when they are not comparable.

Let x and y be comparable. Using the assumption that (x, y) is a coupled fixed point, we get

$$\rho(x, y) = \rho\bigl(F(x, y), F(y, x)\bigr) \leqslant \alpha\rho(x, y) < \rho(x, y),$$

which can hold only if $x = y$.

Let x and y be not comparable, then there exists $z \in Z$, which is comparable to x and y. Suppose that $x \preccurlyeq z$ and $y \preccurlyeq z$. By the partial ordering of Z^2 it follows $(x,y) \succcurlyeq (x,z)$, $(x,z) \preccurlyeq (z,x)$, $(z,x) \succcurlyeq (y,x)$. From Proposition 2 it follows that, for any (x,y) and any comparable with it (ξ_0,η_0), the iterative sequence (ξ_n,η_n) is comparable with (x,y), and (η_n,ξ_n) is comparable with (y,x). Thus form (13) and (14) we get the inequalities

$$\begin{aligned}\rho(x,y) &= \rho\bigl(F^n(x,y),F^n(y,x)\bigr)\\ &\leqslant \rho\bigl(F^n(x,z),F^n(z,x)\bigr)+\rho\bigl(F^n(z,x),F^n(y,x)\bigr)\\ &\quad+\rho\bigl(F^n(x,y),F^n(x,z)\bigr)\\ &\leqslant \alpha^n\rho(x,z)+\alpha^n\rho(y,z)=\alpha^n\bigl(\rho(x,z)\rho(y,z)\bigr).\end{aligned}$$

After taking a limit in the above chain of inequalities when $n \to +\infty$, we get that

$$0 \leqslant \rho(x,y) \leqslant \lim_{n\to+\infty} \alpha^n\bigl(\rho(x,z)\rho(y,z)\bigr) = 0. \qquad \square$$

4 Oligopoly (duopoly) markets

Cournot in 1838 was the first to build a complete model of a market, where few players exercise control over prices and supply [12]. The original model introduces four assumptions:

- (i) there are two players each with sufficient market power to affect prices;
- (ii) there is no product differentiation;
- (iii) decisions on output are taken simultaneously;
- (iv) players are rational, seeking to maximize their own profit and do not work in cooperation.

As in [32], solution for the equilibrium will differ if any of the parties is not acting in a rational way.

Cournot's approach is a static one since players hold a naive view that others will keep their production and price fixed at least for a given period of time. I.e. player i assumes that, in time t, other participants produce the quantities that they have produced in time $t-1$. In the dynamic case, each participant tries to guess what output others will have at time t [8].

A generalization of Cournot's model is the Stackelberg duopoly [2] having a leading player and a follower. It is applicable when participants choose their output sequentially and not simultaneously. Cournot's model and equilibrium are in fact the direct predecessor of Nash's equilibrium point. Bertrand has introduced another kind of a duopoly model, where firms compete on prices rather than on outputs.

Contemporary markets can be subject to different regulations and barriers. These constraints influence the stability of market equilibrium and time required to reach it, for example:

- (i) The number of market participants;

  There are various market conditions with various number of players and leader-follower roles. Thus oligopoly models should be flexible enough to account for these characteristics.

(ii) The interdependence, availability and access to information;
Rational players working in an environment with high concentration ratios are not naive to make decisions in a completely isolated way.

(iii) The price and non-price competition terms;
There are competitive advantages other than price. Product differentiation may not be strong, but loyalty schemes or aggressive advertisement campaigns may still affect the equilibrium in indirect way.

(iv) Consistency of behavior and time dependence of market conditions;
Companies evolve and adapt to market changes as in [9], so this flexibility has to be accounted for.

(v) Entry and exit barriers;
Entry and exit barriers can directly influence the number of companies operating on the oligopoly market. They also play important role in shaping the decisions of each participant as barriers can be considered as additional limiting/boundary conditions.

(vi) Profit maximization and strategic goals;
Profit maximization is extremely important. But there may be periods of time asking for other goals (as in [21]). We assume that, in a long term companies, focus on profit maximization as its vital for their prosperity.

(vii) Linear and nonlinear changes in market conditions and firm behavior.

Changes in market conditions and environment mean that players evolve and adapt their views on the economy and competitors. To be able to explain these changes, a model needs to allow for some flexibility in describing player reactions and goals.

4.1 Modeling real-world oligopoly markets

Let us consider a simple case of two companies with identical products. While the assumption for having homogeneous goods is quite restrictive, it helps to start with a simple model and then extend it by adding nonprice competition and brand loyalty. It should be noted that oligopoly markets with heterogeneous goods can be analyzed in a similar way, although with a slightly more complicated response functions.

The static Cournot's oligopoly is a fully rational game based on the assumptions:

(i) each company, in taking its optimal production decision rationally, must know before hand all its rival's production and both firms should take their decisions simultaneously;

(ii) each firm has a perfect knowledge of the market demand function.

The dynamic model is a game, where restrictive assumption (i) is replaced by an of expectation on the rivals' outputs. While the simplest way is to use naive view that production will remain at its most recent level, it is also possible to impose more realistic views as in [24, 31]. Lets first consider a situation in which there are two players "A" and

"B" producing at moment $n+1$ output $F(x_n, y_n)$ and $f(x_n, y_n)$ given that, at moment n, they have produced x_n and y_n, respectively. Depending on the functions $F(x_n, y_n)$ and $f(x_n, y_n)$, the model can be static or dynamic, as well as symmetric or asymmetric.

To be able to reach market equilibrium, the pair (x, y) should satisfy the equations $x = F(x, y)$ and $y = f(x, y)$.

Thus we search for sufficient conditions, depending only on the response functions, that ensure the existence and uniqueness of equilibrium pair. Compared to the classical approaches in oligopoly markets, this way has several important advantages, making it possible to:

- to account for protective capacity in contemporary production environments, which allows to have zero marginal costs within some output ranges;
- to assess whether marker can reach equilibrium, regardless of the initial position;
- and, finally, to assess time necessary to reach equilibrium and whether it can hold on.

An extensive study on the oligopoly markets can be found in [5, 16, 23].

4.2 The basic model

Assuming, we have two companies competing over the same customers [16] and striving to meet the demand with overall production of $Z = x + y$. The market price is defined as $P(Z) = P(x+y)$, which is the inverse of the demand function. Cost functions are $c_1(x)$ and $c_2(y)$, respectively. Assuming that both companies are rational, the profit functions are $\Pi_1(x, y) = xP(x + y) - c_1(x)$ and $\Pi_2(x, y) = yP(x + y) - c_2(y)$. The goal of each player is maximizing profits, i.e. $\max\{\Pi_1(x, y)\colon x,$ assuming that y is fixed$\}$ and $\max\{\Pi_2(x, y)\colon y,$ assuming that x is fixed$\}$. Provided that functions P and c_i, $i = 1, 2$, are differentiable, we get the equations

$$\begin{aligned}\frac{\partial \Pi_1(x, y)}{\partial x} &= P(x + y) + xP'(x + y) - c_1'(x) = 0,\\ \frac{\partial \Pi_2(x, y)}{\partial y} &= P(x + y) + yP'(x + y) - c_2'(y) = 0.\end{aligned} \tag{15}$$

The solution of (15) presents the equilibrium pair of production [16]. Often equations (15) have solutions in the form of $x = b_1(y)$ and $y = b_2(x)$, which are called response functions [16].

It may turn out difficult or impossible to solve (15), thus it is often advised to search for an approximate solution. An important drawback however is that it be not stable. Fortunately, we can find an implicit formula for the response function in (15) i.e. $x = (c_1'(x) - P(x+y))/P'(x+y) = F(x, y)$ and $y = (c_2'(y) - P(x+y))/P'(x+y) = f(x, y)$.

We may end up with response functions that do not lead to maximization of the profit Π. It is often assumed, each participant's response depends its own output as well as that of others. E.g. if, at a moment n, the output quantities are (x_n, y_n) and the first player changes its productions to $x_{n+1} = F(x_n, y_n)$, then the second one will also change its output to $y_{n+1} = f(x_n, y_n)$. We reach an equilibrium if there are x and y such that

$x = F(x, y)$ and $y = f(x, y)$. The functions Π_i are called payoff functions. To ensure that the solutions of (15) will present a maximization of the payoff functions, a sufficient condition is that Π_i be concave functions [23]. By using of response function we alter the maximization problem into a coupled fixed point one, thus all assumptions of concavity and differentiability can be skipped. The problem of solving the equations $x = F(x, y)$ and $y = f(x, y)$ is the problem of finding of coupled fixed points for an ordered pair of maps (F, f) [18]. Yet an important limitation may be that players cannot change output too fast, and thus the player may not perform maximize their profits.

Focusing on response functions, allows to put together Cournot and Bertand models. Indeed, let the first company have reaction be $F(X, Y)$, and the second one $f(X, Y)$, where $X = (x, p)$ and $Y = (y, q)$. Here x and y denote the output quantity, and (p, q) are the prices set by players. In this case, companies can compete in terms of both price and quantity.

5 Application of the main results in duopoly markets

Quite often oligopolies are studied from market participant's perspective. While this offers a very good possibility to analyze what should be done, adequacy of such approach depends on the existence and stability of a number of assumptions: information available to market participants; short-term goals, which almost universally point toward profit maximization; stability of participant's behavior and strategies.

Such assumptions can simplify the analysis, but they are also quite restrictive and limit our flexibility to describe different real-world scenarios. The approach suggested in this paper can help in reducing the number of restrictive assumptions, while still being compliant with rational economic behavior. As a result, a number of special duopoly market cases can be explained:

Empty intersection of production sets – Such a case may seem extreme, but it is not impossible. For example, if one of the companies is working at a very large scale, it may simply be impractical to sustain a low level of output [14]. On the other hand, if the company is too small to undertake large projects, it may also happen that expanding its production beyond certain limit is not feasible. Therefore, it is possible that long term contracts or technical issues prohibit certain type of actions and impose additional restrictions;

Profit maximization over a longer period – Provided that profit maximization goals are based on a long-term planning, it may involve special cases that aim at first increasing the market share or preventing entry;

Different types of strategic behavior – Different types of behavior can help investigate market behavior, where participants choose their production and pricing levels from a portfolio of strategies. Depending on the reaction of other players, these choices can be switched to match different strategy. Complex response functions are suitable for handling such cases, while reducing the complexity of the complete model;

Leader-follower relations – In addition to sequential games of Stackelberg, the approach discussed here can help in resolving situations, where followers do not always react in the same manner, or can only partially replicate the expected price following steps.

Now we can restate Theorem 2 in terms of oligopoly.

Theorem 3. *Let us assume that two companies are offering products that are perfect substitutes. The first one can produce qualities from the set X, and the second firm can produce qualities from the set Y, where X and Y be nonempty subsets of a partially ordered complete metric space $(Z, \rho, \preccurlyeq)$. Let $F : X \times Y \to X$, $f : X \times Y \to Y$ be the respective response functions. Let there exists $\alpha \in (0, 1)$, such that*

$$\rho(F(x,y), F(u,v)) + \rho\big(f(x,y), f(u,v)\big) \leqslant \alpha\rho(x,u) + \alpha\rho(y,v) \tag{16}$$

holds for all $x \succcurlyeq u$ and $y \preccurlyeq v$. If there exists at least one ordered pair $(x, y) \in X \times Y$ such that $x \preccurlyeq F(x, y)$ and $y \succcurlyeq f(x, y)$, then there exists a market equilibrium point (x, y), which is a coupled fixed points of (F, f).

If, in addition, every pair of elements in $X \times Y$ has an lower or an upper bound, then the coupled fixed point is unique.

The conditions imposed on the response functions states that we can say something only if when ever the production of firm one decreases i.e. $x \succcurlyeq u$, the production of firm two increases i.e. $y \preccurlyeq v$. One case where it can happen is if in a monopoly market enters a second firm. In this case, the first player will decrease its market share, and the second one will increase it.

Example 1 [*Cournot's model*]. Let there be two companies producing a pair of products, which are again perfect substitutes. Let us assume that the second player enters the market, so that outputs are (x_1, x_2) and (y_1, y_2). Then $(x_1, x_2) \succcurlyeq (y_1, y_2)$. Let endow the production set $\mathbb{R}$ with the Euclidean norm $\|\cdot\|_2$. Consider the response functions $F(x_1, x_2, y_1, y_2)$ and $f(x_1, x_2, y_1, y_2)$ defined by

$$F(x,y) = \begin{cases} \frac{x_1+y_1}{3} + 1, \\ \frac{x_2+y_2}{4} + 1, \end{cases} \qquad f(x,y) = \begin{cases} \frac{x_1+y_1}{3} + 1, \\ \frac{x_2+y_2}{2} + 1. \end{cases}$$

We will need the inequality $(|a| - |b|)^2 \leqslant a^2 + b^2$. We need to check that (16) for $(x_1, x_2) \succcurlyeq (u_1, u_2)$, i.e. $x_1 - u_1 \geqslant 0$ and $x_2 - u_2 \geqslant 0$ and for $(y_1, y_2) \preccurlyeq (v_1, v_2)$, i.e. $v_1 - y_1 \geqslant 0$ and $v_2 - y_2 \geqslant 0$. Thus $(x_1 - u_1 + y_1 - v_1)^2 + (x_2 - u_2 + y_2 - v_2)^2 = (|x_1 - u_1| - |y_1 - v_1|)^2 + (|x_2 - u_2| - |y_2 - v_2|)^2$.

$$\begin{aligned} S_3 &= \big\|F(x_1, x_2, y_1, y_2) - F(u_1, u_2, v_1, v_2)\big\|_2 \\ &= \left\|\left(\frac{x_1+y_1}{3}, \frac{x_2+y_2}{4}\right) - \left(\frac{u_1+v_1}{3}, \frac{u_2+v_2}{4}\right)\right\|_2 \\ &\leqslant \frac{\sqrt{(x_1 - u_1 + y_1 - v_1)^2 + (x_2 - u_2 + y_2 - v_2)^2}}{3} \\ &= \frac{\sqrt{(|x_1 - u_1| - |y_1 - v_1|)^2 + (|x_2 - u_2| - |y_2 - v_2|)^2}}{3} \end{aligned}$$

$$
\begin{aligned}
&\leqslant \frac{\sqrt{|x_1-u_1|^2+|x_2-u_2|^2+|y_1-v_1|^2+|y_2-v_2|^2}}{3}\\
&= \frac{\sqrt{|x_1-u_1|-|y_1-v_1|)^2+(|x_2-u_2|-|y_2-v_2|)^2}}{3}\\
&\leqslant \frac{\sqrt{|x_1-u_1|^2+|x_2-u_2|^2}}{3}+\frac{\sqrt{|y_1-v_1|^2+|y_2-v_2|^2}}{3}\\
&= \frac{1}{3}\rho(x,u)+\frac{1}{3}\rho(y,v),
\end{aligned}
$$

$$
\begin{aligned}
S_4 &= \big\|f(x_1,x_2,y_1,y_2)-f(u_1,u_2,v_1,v_2)\big\|_2\\
&= \left\|\left(\frac{x_1+y_1}{3},\frac{x_2+y_2}{2}\right)-\left(\frac{u_1+v_1}{3},\frac{u_2+v_2}{2}\right)\right\|_2\\
&\leqslant \frac{\sqrt{(x_1-u_1+y_1-v_1)^2+(x_2-u_2+y_2-v_2)^2}}{2}\\
&= \frac{\sqrt{(|x_1-u_1|-|y_1-v_1|)^2+(|x_2-u_2|-|y_2-v_2|)^2}}{2}\\
&\leqslant \frac{\sqrt{|x_1-u_1|^2+|x_2-u_2|^2+|y_1-v_1|^2+|y_2-v_2|^2}}{2}\\
&= \frac{\sqrt{|x_1-u_1|-|y_1-v_1|)^2+(|x_2-u_2|-|y_2-v_2|)^2}}{2}\\
&\leqslant \frac{\sqrt{|x_1-u_1|^2+|x_2-u_2|^2}}{2}+\frac{\sqrt{|y_1-v_1|^2+|y_2-v_2|^2}}{2}\\
&= \frac{1}{2}\rho(x,u)+\frac{1}{2}\rho(y,v).
\end{aligned}
$$

If we denote by $X=(x_1,x_2)$, $Y=(y_1,y_2)$, $U=(u_1,u_2)$ and $V=(v_1,v_2)$, we get

$$
\begin{aligned}
&\big\|F(X,Y)-F(U,V)\big\|_2+\big\|f(X,Y)-f(U,V)\big\|_2\\
&\quad\leqslant \frac{5}{6}\|X-U\|+\frac{5}{6}\|Y-V\|.
\end{aligned}
$$

Therefore there exists a market equilibrium, where production is $x_1=3$, $x_2=3$ for the first player and $y_1=3$, $y_2=5$ for the second one.

For response functions F and f, if we try to apply the classical inequality for convex functions $((a+b)/2)^2\leqslant(a^2+b^2)/2$, we will not be able to prove that inequality (16) holds true. We will be able to just get

$$
\begin{aligned}
&\big\|F(X,Y)-F(U,V)\big\|_2+\big\|f(X,Y)-f(U,V)\big\|_2\\
&\quad\leqslant \frac{5\sqrt{2}}{6}\|X-U\|+\frac{5\sqrt{2}}{6}\|Y-V\|.
\end{aligned}
$$

Thus the consideration of a partially ordered metric space and that inequality (16) holds only for part of the elements of the space significantly increases the classes of oligopoly that can be investigated.

Example 2 [*Bertrand's model*]. Let us alter Example 1 by assuming a market with two competing companies each producing a single homogeneous product. The sole competitive advantage is the price. Let us assume that the second firm enters the market, i.e. if the productions are (x, p), x – quantity at a price of p and (y, q), y – quantity at a price of q of the first and the second firm, respectively, then $(x, p) \succcurlyeq (y, q)$, assuming that the second firm is smaller, can produce at a higher prices. Let endow the production set $\mathbb{R}$ with the Euclidean norm $\|\cdot\|_2$. Let us consider the response functions $F(x, p, y, q)$ and $f(x, p, y, q)$ defined in Example 1. Therefore there exists a market equilibrium. Actually, the equilibrium production is $x = 3$ at a price $p = 3$ of the first player and $y = 3$ at a price $q = 5$ for the second player.

Remark. Let the two players have one and same response function. That is, if player one has a production x and player two has a production y, then the first player reaction will be $F(x, y)$, and the second player reaction will be $f(x, y) = F(y, x)$. From Theorem 2(ii) it follows that the equilibrium pair (x, y) will satisfy $x = y$, i.e. both firms will have equal production. This means that if both firms have one and the same technology, one and the same knowledge on the market that will affect to one and the same response functions, then the equilibrium will be reached at the level of equal productions. If we consider the Bertrand's model with the same assumption of equal response functions, then not only the quantities will be equal but also and the prices.

Results about existence, uniqueness and stability in duopoly markets when the response functions do not satisfy the mixed monotone property are obtained in [14] and illustrated with number of different kinds of examples.

References

1. M. Abtahi, Z. Kadelburg, S. Radenović, Fixed points and coupled fixed points in partially ordered ν-generalized metric spaces, *Appl. Gen. Topol.*, **19**(2):189–201, 2018, https://doi.org/10.4995/agt.2018.7409.
2. S.P. Anderson, M. Engers, Stackelberg versus Cournot oligopoly equilibrium, *Int. J. Ind. Organ.*, **10**(1):127–135, 1992, https://doi.org/10.1016/0167-7187(92)90052-Z.
3. L. Bai, J.J. Nieto, Variational approach to differential equations with not instantaneous impulses, *Appl. Math. Lett.*, **73**:44–48, 2017, https://doi.org/10.1016/j.aml.2017.02.019.
4. V. Berinde, Generalized coupled fixed point theorems for mixed monotone mappings in partially ordered metric spaces, *Nonlinear Anal., Theory Methods Appl.*, **74**(18):7347–7355, 2011, https://doi.org/10.1016/j.na.2011.07.053.
5. G.I. Bischi, C. Chiarella, M. Kopel, F. Szidarovszky, *Nonlinear Oligopolies Stability and Bifurcations*, Springer, Berlin, Heidelberg, 2010, https://doi.org/10.1007/s11403-020-00298-y.

6. J. Borwein, Q. Zhu, *Techniques of Variational Analysis*, CMS Books Math./Ouvrages Math. SMC, Springer, New York, 2005, https://doi.org/10.1007/0-387-28271-8.
7. J. Caristi, Fixed point theorems for mappings satisfying inwardness conditions, *Trans. Am. Math. Soc.*, **215**:241–251, 1976, https://doi.org/10.1090/S0002-9947-1976-0394329-4.
8. R. Cellini, L. Lambertini, Dynamic oligopoly with sticky prices: Closed-loop, feedback and open-loop solutions, *J. Dyn. Control Syst.*, **10**:303–314, 2004, https://doi.org/10.1016/j.jmaa.2003.10.019.
9. C.K. Chan, Y. Zhou, K.H. Wong, A dynamic equilibrium model of the oligopolistic closed-loop supply chain network under uncertain and time-dependent demands, in *Transportation Research Part E: Logistics and Transportation Review, Vol. 118(C)*, Elsevier, Amsterdam, 2018, pp. 325–354, https://doi.org/10.1016/j.tre.2018.07.008.
10. C.E. Chidume, M.O. Nnakwe, Convergence theorems of subgradient extragradient algorithm for solving variational inequalities and a convex feasibility problem, *Fixed Point Theory Appl.*, **2018**:16, 2018, https://doi.org/10.1186/s13663-018-0641-4.
11. B.S. Choudhury, P. Maity, Cyclic coupled fixed point result using Kannan type contractions, *J. Oper.*, **2014**:876749, 2014, https://doi.org/10.1155/2014/876749.
12. A.A. Cournot, *Researches Into the Mathematical Principles of the Theory of Wealth*, Macmillan, New York, 1897, https://doi.org/10.1090/S0002-9904-1929-04725-6.
13. D. Đorić, Z. Kadelburg, S. Radenović, Poom Kumam, A note on fixed point results without monotone property in partially ordered metric spaces, *Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat., RACSAM*, **108**(2):503–510, 2014, https://doi.org/10.1007/s13398-013-0121-y.
14. Y. Dzhabarova, S. Kabaivanov, M. Ruseva, B. Zlatanov, Existence, uniqueness and stability of market equilibrium in oligopoly markets, *Administrative Sciences*, **10**(3):70, 2020, https://doi.org/10.3390/admsci10030070.
15. I. Ekeland, Nonconvex minimization problems, *Bull. Am. Math. Soc.*, **1**(3):443–474, 1979, https://doi.org/10.1090/S0273-0979-1979-14595-6.
16. J.W. Friedman, *Oligopoly Theory*, Cambridge Univ. Press, Cambridge, 2007, https://doi.org/10.1017/CBO9780511571893.
17. D. Guo, V. Lakshmikantham, Coupled fixed points of nonlinear operators with application, *Nonlinear Anal., Theory Methods Appl.*, **11**(5):623–632, 1987, https://doi.org/10.1016/0362-546X(87)90077-0.
18. D. Guo, V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, *Nonlinear Anal., Theory Methods Appl.*, **65**(7):1379–1393, 2006, https://doi.org/10.1016/j.na.2005.10.017.
19. B. Hazarika, R. Arab, P. Kumam, Coupled fixed point theorems in partially ordered metric spaces via mixed g-monotone property, *J. Fixed Point Theory Appl.*, **21**(1):1, 2019, https://doi.org/10.1007/s11784-018-0638-y.
20. Z. Kadelburg, P. Kumam, S. Radenović, W. Sintunavarat, Common coupled fixed point theorems for Geraghty's type contraction mappings without mixed monotone property, *Fixed Point Theory Appl.*, **2015**:27, 2015, https://doi.org/10.1186/s13663-015-0278-5.

21. A. Klemm, Profit maximisation and alternatives in oligopolies, Technical report, Industrial Organization from University Library of Munich, Germany, 2004, https://EconPapers.repec.org/RePEc:wpa:wuwpio:0409003.

22. Z.H. Maibed, Some generalized n-tuplet coincidence point theorems for nonlinear contraction mappings, *J. Eng. Appl. Sci.*, **13**(24):10375–10379, 2018, https://doi.org/10.3923/jeasci.2018.10375.10379.

23. A. Matsumoto, F. Szidarovszky, *Dynamic Oligopolies with Time Delays*, Springer, Singapore, 2018, https://doi.org/10.1007/978-981-13-1786-6.

24. M. McManus, R.E. Quandt, Comments on the stability of the Cournot oligopoly model, *Rev. Econ. Stud.*, **27**:136–139, 1961, https://doi.org/10.2307/2295711.

25. J.J. Nieto, R. Rodríguez-López, Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations, *Order*, **22**(3):223–239, 2005, https://doi.org/10.1007/s11083-005-9018-5.

26. A. Petruşel, Fixed points vs. coupled fixed points, *J. Fixed Point Theory Appl.*, **20**(4):150, 2018, https://doi.org/10.1007/s11784-018-0630-6.

27. J.H. Qiu, Generalized Gerstewitz's functions and vector variational principle for ε-efficient solutions in the sense of Németh, *Acta Math. Sci., Ser. B, Engl. Ed.*, **35**(3):297–320, 2019, https://doi.org/10.1007/s10114-018-7159-x.

28. A.C.M. Ran, M.C.B. Reurings, A fixed point theorem in partially ordered sets and some application to matrix equations, *Proc. Am. Math. Soc.*, **132**(5):1435–1443, 2004, https://doi.org/10.1090/S0002-9939-03-07220-4.

29. Z. Sadeghi, S.M. Vaezpour, Fixed point theorems for multivalued and single-valued contractive mappings on Menger PM spaces with applications, *J. Fixed Point Theory Appl.*, **20**(3):114, 2018, https://doi.org/10.1007/s11784-018-0594-6.

30. N. Shahzad, O. Valero, On Bishop–Phelps partial order, variation mappings and Caristi's fixed point theorem in quasi-metric spaces, *Fixed Point Theory*, **21**(2):739–754, 2020, https://doi.org/10.24193/fpt-ro.2020.2.53.

31. R.D. Theocharis, On the stability of the Cournot solution of the oligopoly problem, *Rev. Econ. Stud.*, **27**:133–134, 1960, https://doi.org/10.2307/2296135.

32. M. Ueda, Effect of information asymmetry in Cournot duopoly game with bounded rationality, *Appl. Math. Comput.*, **362**:124535, 2019, https://doi.org/10.1016/j.amc.2019.06.049.

33. B. Zlatanov, A variational principle and coupled fixed points, *J. Fixed Point Theory Appl.*, **21**:19, 2019, https://doi.org/10.1007/s11784-019-0706-y.