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On a singular Riemann–Liouville fractional boundary value problem with parameters

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Abstract. We investigate the existence of positive solutions for a nonlinear Riemann–Liouville fractional differential equation with a positive parameter subject to nonlocal boundary conditions, which contain fractional derivatives and Riemann–Stieltjes integrals. The nonlinearity of the equation is nonnegative, and it may have singularities at its variables. In the proof of the main results, we use the fixed point index theory and the principal characteristic value of an associated linear operator. A related semipositone problem is also studied by using the Guo–Krasnosel'skii fixed point theorem.

Keywords: Riemann–Liouville fractional differential equation, nonlocal boundary conditions, positive parameter, singularities, positive solutions, semipositone problem.

1 Introduction

We consider the nonlinear fractional differential equation

$$D_{0+}^{\alpha}u(t) + \lambda h(t) f\bigl(t, u(t)\bigr) = 0, \quad t \in (0,1), \tag{1}$$

with the nonlocal boundary conditions

$$u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \qquad D_{0+}^{\beta_0}u(1) = \sum_{i=1}^{m} \int_0^1 D_{0+}^{\beta_i}u(t)\,\mathrm{d}H_i(t), \tag{2}$$

where $\alpha \in R$, $\alpha \in (n-1, n]$, $n, m \in N$, $n \geqslant 3$, $\beta_i \in R$ for all $i = 0, \dots, m$, $0 \leqslant \beta_1 < \beta_2 < \cdots < \beta_m \leqslant \beta_0 < \alpha - 1$, $\beta_0 \geqslant 1$, λ is a positive parameter, and D_{0+}^{k} denotes the Riemann–Liouville derivative of order k (for $k = \alpha, \beta_0, \beta_1, \dots, \beta_m$).

The integrals from the boundary conditions (2) are Riemann–Stieltjes integrals with H_i, $i = 1, \ldots, m$, functions of bounded variation, the nonnegative function $f(t, u)$ may have singularity at $u = 0$, and the nonnegative function $h(t)$ may be singular at $t = 0$ and/or $t = 1$.

Under some assumptions for the functions h and f, we establish intervals for the parameter λ such that problem (1), (2) has positive solutions ($u(t) > 0$ for all $t \in (0, 1]$). These intervals for λ are expressed by using the principal characteristic value of an associated linear operator. In the proof of the main theorems, we use the fixed point index theory. In the case in which $h \equiv 1$ and f is a function which changes sign and has singularities at $t = 0$ and/or $t = 1$, we present two existence results for the positive solutions of this problem. In the proof of these results, we apply the Guo–Krasnosel'skii fixed point theorem. The boundary conditions (2) cover various cases, such as multi-point boundary conditions when the functions H_i are step functions, or classical integral boundary conditions, or a combination of them.

We present below some papers, which investigate particular cases of our boundary value problem (1), (2) and other problems related to (1), (2). Equation (1) with $h(t) \equiv 1$ subject to the boundary conditions

$$u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \qquad D^p_{0+}u(1) = \sum_{i=1}^{m} a_i D^q_{0+}u(\xi_i),$$

where $\xi_i \in R$, $i = 1, \ldots, m$, $0 < \xi_1 < \cdots < \xi_m < 1$, $p, q \in R$, $p \in [1, n-2]$, $q \in [0, p]$, was investigated in [11]. In paper [11], the nonlinearity f changes sign, and it is singular only at $t = 0$ and/or $t = 1$. The authors of [11] apply the Guo–Krasnosel'skii fixed point theorem to prove the existence of positive solutions when the parameter belongs to various intervals. Equation (1) with $\lambda = 1$ and $h(t) \equiv 1$ supplemented with the boundary conditions (2) with $m = 1$, where f may change sign and may be singular at the points $t = 0$, $t = 1$ and/or $u = 0$ has been studied in [20]. In the paper [20], the author presents some conditions for f, which contain height functions defined on special bounded sets under which he proves the existence and multiplicity of positive solutions. The existence of multiple positive solutions for equation (1) with $\lambda = 1$ and $h(t) \equiv 1$ subject to the boundary conditions (2) was investigated in the recent paper [1]. The authors use in [1] various height functions of the nonlinearity defined on special bounded sets and two theorems from the fixed point index theory. In the paper [35], the authors prove the existence of at least three positive solutions for equation (1) with $\lambda = 1$ and $h(t) \equiv 1$ with the boundary conditions

$$u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \qquad D^\beta_{0+}u(1) = \lambda \int_0^\eta \widetilde{h}(t) D^\beta_{0+}u(t)\,\mathrm{d}t, \tag{3}$$

where $\beta \geqslant 1$, $\alpha - \beta - 1 > 0$, $0 < \eta \leqslant 1$, $0 \leqslant \lambda \int_0^\eta \widetilde{h}(t) t^{\alpha-\beta-1}\,\mathrm{d}t < 1$, $\widetilde{h} \in L^1[0, 1]$ is nonnegative and may be singular at $t = 0$ and $t = 1$, and the function f is nonnegative and may be singular at the points $t = 0$, $t = 1$ and $u = 0$. Our boundary conditions (2) are more general than the above boundary conditions (3). Indeed, the last relation from (3) can be written as $D_{0+}^{\beta}u(1) = \int_0^1 D_{0+}^{\beta}u(t)\,\mathrm{d}H(t)$ with $H(t) = \{\lambda\int_0^t \widetilde{h}(s)\,\mathrm{d}s,\ t \in [0,\eta];\ \lambda\int_0^{\eta}\widetilde{h}(s)\,\mathrm{d}s,\ t \in (\eta,1]\}$, and in the right-hand side of the last condition in (2), we have a sum of Riemann–Stieltjes integrals from Riemann–Liouville derivatives of various orders. In the paper [35], the authors use different height functions of the nonlinear term on special bounded sets, the Krasnosel'skii theorem and the Leggett–Williams fixed point index theorem. We also mention the paper [33], where the authors prove the existence of positive solutions of fractional differential equation (1) supplemented with the boundary conditions

$$u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \qquad D_{0+}^{\beta}u(1) = \sum_{i=1}^{\infty}\alpha_i D_{0+}^{\gamma}u(\xi_i), \tag{4}$$

where $\beta \in [1, n-2]$, $\gamma \in [0,\beta]$, $\alpha_i \geqslant 0$, $i - 1, 2, \dots$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{i-1} < \xi_i < \cdots < 1$ and $\Gamma(\alpha-\gamma) > \Gamma(\alpha-\beta)\sum_{i=1}^{\infty}\alpha_i\xi_i^{\alpha-\gamma-1}$. The last condition of the boundary conditions (4) can be written as $D_{0+}^{\beta}u(1) = \int_0^1 D_{0+}^{\gamma}u(t)\,\mathrm{d}H(t)$, where H is the step function defined by $H(t) = \{0,\ t \in [0,\xi_1];\ \alpha_1,\ t \in (\xi_1,\xi_2];\ \alpha_1+\alpha_2,\ t \in (\xi_2,\xi_3];\ \dots;\ \sum_{i=1}^{n}\alpha_i,\ t \in (\xi_n,\xi_{n+1}];\ \dots\}$, so this condition is a particular case of our condition from (2). We mention that condition (I3) (see below, in Section 3) used in our results was first introduced in the paper [18], where the authors proved the existence of at least one positive solution for a fourth-order nonlinear singular Sturm–Liouville eigenvalue problem.

For some recent results on the existence, nonexistence and multiplicity of positive solutions for fractional differential equations and systems of fractional differential equations with various boundary conditions, we refer the reader to the monographs [10, 36] and the papers [2,3,8,12,13,17,19,25,28,30,31,34]. We also mention the books [14,15,24,26,27] and the papers [5–7, 21–23, 29] for applications of the fractional differential equations in various disciplines.

The paper is organized as follows. In Section 2, we present the solution of a linear fractional differential equation associated to equation (1) subject to the boundary conditions (2) and the properties of the corresponding Green functions. Some theorems from the fixed point index theory, the Guo–Krasnosel'skii fixed point theorem and an application of the Krein–Rutman theorem in the space $C[0,1]$ are recalled in Section 2, and they will be used in the next sections. In Section 3, we give and prove the main theorems for the existence of at least one positive solution for problem (1), (2). In Section 4, we present two existence results for the positive solutions of problem (1), (2) with $h \equiv 1$, where the nonlinearity changes sign, and it is singular at $t = 0$ and/or $t = 1$. Section 5 contains some examples, which illustrate the obtained results, and in Section 6, we give the conclusions for our fractional boundary value problems.

2 Auxiliary results

In this section, we present some auxiliary results from [1] that we will use in the proof of the main theorems. We consider the fractional differential equation

$$D_{0+}^{\alpha}u(t) + x(t) = 0, \quad t \in (0,1), \tag{5}$$

with the boundary conditions (2), where $x \in C(0,1) \cap L^1(0,1)$. We denote

$$\varDelta = \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta_0)} - \sum_{i=1}^{m} \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta_i)} \int_0^1 s^{\alpha-\beta_i-1}\,\mathrm{d}H_i(s).$$

Lemma 1. (*See* [1].) *If* $\varDelta \neq 0$, *then the unique solution* $u \in C[0,1]$ *of problem* (5), (2) *is given by*

$$u(t) = \int_0^1 \mathcal{G}(t,s)x(s)\,\mathrm{d}s, \quad t \in [0,1], \tag{6}$$

where

$$\mathcal{G}(t,s) = g_1(t,s) + \frac{t^{\alpha-1}}{\varDelta} \sum_{i=1}^{m} \int_0^1 g_{2i}(\tau,s)\,\mathrm{d}H_i(\tau) \tag{7}$$

and

$$\begin{aligned} g_1(t,s) &= \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1}(1-s)^{\alpha-\beta_0-1} - (t-s)^{\alpha-1}, & 0 \leqslant s \leqslant t \leqslant 1, \\ t^{\alpha-1}(1-s)^{\alpha-\beta_0-1}, & 0 \leqslant t \leqslant s \leqslant 1, \end{cases} \\ g_{2i}(t,s) &= \frac{1}{\Gamma(\alpha-\beta_i)} \begin{cases} t^{\alpha-\beta_i-1}(1-s)^{\alpha-\beta_0-1} - (t-s)^{\alpha-\beta_i-1}, & 0 \leqslant s \leqslant t \leqslant 1, \\ t^{\alpha-\beta_i-1}(1-s)^{\alpha-\beta_0-1}, & 0 \leqslant t \leqslant s \leqslant 1, \end{cases} \end{aligned} \tag{8}$$

for all $(t,s) \in [0,1] \times [0,1]$, $i = 1,\dots,m$.

Based on some properties of functions g_1 and g_{2i}, $i = 1,\dots,m$, given by (8) (see [11]), we have the following lemma.

Lemma 2. (*See* [1].) *We suppose that* $\varDelta > 0$. *Then the Green function* $\mathcal{G}$ *given by* (7) *is a continuous function on* $[0,1] \times [0,1]$ *and satisfies the inequalities*:

(i) $\mathcal{G}(t,s) \leqslant \mathcal{J}(s)$ *for all* $t,s \in [0,1]$, *where*

$$\mathcal{J}(s) = h_1(s) + \frac{1}{\varDelta} \sum_{i=1}^{m} \int_0^1 g_{2i}(\tau,s)\,\mathrm{d}H_i(\tau), \quad s \in [0,1],$$

$$h_1(s) = \frac{1}{\Gamma(\alpha)}(1-s)^{\alpha-\beta_0-1}\big(1-(1-s)^{\beta_0}\big), \quad s \in [0,1];$$

(ii) $\mathcal{G}(t,s) \geqslant t^{\alpha-1}\mathcal{J}(s)$ *for all* $t,s \in [0,1]$;

(iii) $\mathcal{G}(t,s) \leqslant \sigma t^{\alpha-1}$ *for all* $t,s \in [0,1]$, *where*

$$\sigma = \frac{1}{\Gamma(\alpha)} + \frac{1}{\varDelta} \sum_{i=1}^{m} \frac{1}{\Gamma(\alpha-\beta_i)} \int_0^1 \tau^{\alpha-\beta_i-1}\,\mathrm{d}H_i(\tau).$$

Lemma 3. (*See* [1].) *We suppose that* $\Delta > 0$, $x \in C(0,1) \cap L^1(0,1)$ *and* $x(t) \geqslant 0$ *for all* $t \in (0,1)$. *Then the solution* u *of problem* (5), (2) *given by* (6) *satisfies the inequality* $u(t) \geqslant t^{\alpha-1}\|u\|$ *for all* $t \in [0,1]$, *where* $\|u\| = \sup_{t\in[0,1]} |u(t)|$, *and so* $u(t) \geqslant 0$ *for all* $t \in [0,1]$.

We recall now some theorems concerning the fixed point index theory and the Guo–Krasnosel'skii fixed point theorem. Let X be a real Banach space with the norm $\|\cdot\|$, $C \subset X$ a cone, "$\leqslant$" the partial ordering defined by C and θ the zero element in X. For $\varrho > 0$, let $B_\varrho = \{u \in X\colon \|u\| < \varrho\}$ be the open ball of radius ϱ centered at θ, its closure $\overline{B}_\varrho = \{u \in X\colon \|u\| \leqslant \varrho\}$ and its boundary $\partial B_\varrho = \{u \in X\colon \|u\| = \varrho\}$. The proofs of our results are based on the following fixed point index theorems.

Theorem 1. (*See* [4].) *Let* $\mathcal{A} : \overline{B}_\varrho \cap C \to C$ *be a completely continuous operator. If there exists* $u_0 \in C \setminus \{\theta\}$ *such that* $u - \mathcal{A}u \neq \lambda u_0$ *for all* $\lambda \geqslant 0$ *and* $u \in \partial B_\varrho \cap C$, *then* $i(\mathcal{A}, B_\varrho \cap C, C) = 0$.

Theorem 2. (*See* [4].) *Let* $\mathcal{A} : \overline{B}_\varrho \cap C \to C$ *be a completely continuous operator. If* $\mathcal{A}u \neq \mu u$ *for all* $u \in \partial B_\varrho \cap C$ *and* $\mu \geqslant 1$, *then* $i(\mathcal{A}, B_\varrho \cap C, C) = 1$.

Theorem 3. (*See* [9].) *Let* X *be a Banach space, and let* $C \subset X$ *be a cone in* X. *Assume* Ω_1 *and* Ω_2 *are bounded open subsets of* X *with* $\theta \in \Omega_1$, $\overline{\Omega_1} \subset \Omega_2$, *and let* $\mathcal{A} : C \cap (\overline{\Omega_2} \setminus \Omega_1) \to C$ *be a completely continuous operator such that either*

(i) $\|\mathcal{A}u\| \leqslant \|u\|$, $u \in C \cap \partial\Omega_1$, *and* $\|\mathcal{A}u\| \geqslant \|u\|$, $u \in C \cap \partial\Omega_2$, *or*
(ii) $\|\mathcal{A}u\| \geqslant \|u\|$, $u \in C \cap \partial\Omega_1$, *and* $\|\mathcal{A}u\| \leqslant \|u\|$, $u \in C \cap \partial\Omega_2$.

Then $\mathcal{A}$ *has a fixed point in* $C \cap (\overline{\Omega_2} \setminus \Omega_1)$.

Let the space $C[0,1]$ and the cone $P = \{u \in C[0,1]\colon u(t) \geqslant 0\ \forall t \in [0,1]\}$. We present next an application of the Krein–Rutman theorem in the space $C[0,1]$.

Theorem 4. (*See* [16, 32].) *Suppose that* $A : C[0,1] \to C[0,1]$ *is a completely continuous linear operator and* $A(P) \subset P$. *If there exist* $v \in C[0,1] \setminus (-P)$ *and a constant* $c > 0$ *such that* $cAv \geqslant v$, *then the spectral radius* $r(A) \neq 0$ *and* A *has an eigenvector* $u_0 \in P \setminus \{\theta\}$ *corresponding to its principal characteristic value* $\lambda_1 = (r(A))^{-1}$, *that is* $\lambda_1 A u_0 = u_0$ *or* $Au_0 = r(A)u_0$, *and so* $r(A) > 0$.

3 Main results

In this section, we present intervals for the parameter λ such that our problem (1), (2) has at least one positive solution. We consider the Banach space $X = C[0,1]$ with the supremum norm $\|u\| = \sup_{t\in[0,1]} |u(t)|$, and we define the cones

$$P = \big\{u \in X\colon u(t) \geqslant 0\ \forall t \in [0,1]\big\},$$
$$Q = \big\{u \in X\colon u(t) \geqslant t^{\alpha-1}\|u\|\ \forall t \in [0,1]\big\} \subset P.$$

We define the operator $\mathcal{A} : P \to P$ and the linear operator $\mathcal{T} : X \to X$ by

$$\mathcal{A}u(t) = \lambda \int_0^1 \mathcal{G}(t,s)h(s)f\big(s,u(s)\big)\,\mathrm{d}s, \quad t \in [0,1],\ u \in P,$$

$$\mathcal{T}u(t) = \int_0^1 \mathcal{G}(t,s)h(s)u(s)\,\mathrm{d}s, \quad t \in [0,1],\ u \in X.$$

We see that u is a solution of problem (1), (2) if and only if u is a fixed point of operator $\mathcal{A}$. For $r > 0$, we denote $Q_r = B_r \cap Q$ and $\overline{Q}_r = \overline{B}_r \cap Q$.

We introduce now the assumptions that we will use in what follows.

(I1) $\alpha \in R$, $\alpha \in (n-1, n]$, $n, m \in N$, $n \geqslant 3$, $\beta_i \in R$ for all $i = 0, \dots, m$, $0 \leqslant \beta_1 < \beta_2 < \cdots < \beta_m \leqslant \beta_0 < \alpha - 1$, $\beta_0 \geqslant 1$, $H_i : [0,1] \to R$, $i = 1, \dots, m$, are nondecreasing functions, $\lambda > 0$, and

$$\Delta = \frac{\Gamma(\alpha)}{\Gamma(\alpha - \beta_0)} - \sum_{i=1}^{m} \frac{\Gamma(\alpha)}{\Gamma(\alpha - \beta_i)} \int_0^1 s^{\alpha - \beta_i - 1}\,\mathrm{d}H_i(s) > 0.$$

(I2) The function $h \in C((0,1), [0,\infty))$, and $\int_0^1 \mathcal{J}(s)h(s)\,\mathrm{d}s < \infty$.

(I3) The function $f \in C([0,1] \times (0,\infty), [0,\infty))$, and for any $0 < r < R$, we have

$$\lim_{n\to\infty} \sup_{u \in \overline{Q}_R \setminus Q_r} \int_{A_n} h(s)f\big(s,u(s)\big)\,\mathrm{d}s = 0,$$

where $A_n = [0, 1/n] \cup [(n-1)/n, 1]$.

Lemma 4. *Assume that assumptions* (I1)–(I3) *hold. Then, for any* $0 < r < R$*, the operator* $\mathcal{A} : \overline{Q}_R \setminus Q_r \to Q$ *is completely continuous.*

Proof. By (I3) we deduce that there exists a natural number $n_1 \geqslant 3$ such that

$$\sup_{u \in \overline{Q}_R \setminus Q_r} \int_{A_{n_1}} h(s)f\big(s,u(s)\big)\,\mathrm{d}s < 1.$$

For $u \in \overline{Q}_R \setminus Q_r$, there exists $r_1 \in [r, R]$ such that $\|u\| = r_1$, and then

$$t^{\alpha-1}r \leqslant t^{\alpha-1}r_1 \leqslant u(t) \leqslant r_1 \leqslant R \quad \forall t \in [0,1].$$

Let $L_1 = \max\{f(t,x),\ t \in [1/n_1, (n_1-1)/n_1],\ x \in [r/n_1^{\alpha-1}, R]\}$. By Lemma 2, (I2) and (I3) we find

$$\sup_{u \in \overline{Q}_R \setminus Q_r} \lambda \int_0^1 \mathcal{G}(t,s)h(s)f\big(s,u(s)\big)\,\mathrm{d}s \leqslant \sup_{u \in \overline{Q}_R \setminus Q_r} \lambda \int_0^1 \mathcal{J}(s)h(s)f\big(s,u(s)\big)\,\mathrm{d}s,$$

$$\begin{aligned}
&\sup_{u\in\overline{Q}_R\setminus Q_r} \lambda \int_0^1 \mathcal{J}(s)h(s)f\big(s,u(s)\big)\,\mathrm{d}s \\
&\quad\leqslant \sup_{u\in\overline{Q}_R\setminus Q_r} \lambda \int_{A_{n_1}} \mathcal{J}(s)h(s)f\big(s,u(s)\big)\,\mathrm{d}s + \sup_{u\in\overline{Q}_R\setminus Q_r} \lambda \int_{1/n_1}^{(n_1-1)/n_1} \mathcal{J}(s)h(s)f\big(s,u(s)\big)\,\mathrm{d}s \\
&\quad\leqslant \lambda J_0 + \lambda L_1 \int_{1/n_1}^{(n_1-1)/n_1} \mathcal{J}(s)h(s)\,\mathrm{d}s \leqslant \lambda J_0 + \lambda L_1 \int_0^1 \mathcal{J}(s)h(s)\,\mathrm{d}s \quad < \infty,
\end{aligned}$$

where $J_0 = \max_{t\in[0,1]} \mathcal{J}(t)$. This implies that the operator $\mathcal{A}$ is well defined.

We show next that $\mathcal{A} : \overline{Q}_R \setminus Q_r \to Q$. Indeed, for any $u \in \overline{Q}_R \setminus Q_r$ and $t \in [0,1]$, we have

$$(\mathcal{A}u)(t) = \lambda \int_0^1 \mathcal{G}(t,s)h(s)f\big(s,u(s)\big)\,\mathrm{d}s \leqslant \lambda \int_0^1 \mathcal{J}(s)h(s)f\big(s,u(s)\big)\,\mathrm{d}s,$$

and then

$$\|\mathcal{A}u\| \leqslant \lambda \int_0^1 \mathcal{J}(s)h(s)f\big(s,u(s)\big)\,\mathrm{d}s.$$

On the other hand, by Lemma 2 we obtain

$$(\mathcal{A}u)(t) \geqslant \lambda t^{\alpha-1} \int_0^1 \mathcal{J}(s)h(s)f\big(s,u(s)\big)\,\mathrm{d}s \geqslant t^{\alpha-1}\|\mathcal{A}u\| \quad \forall t \in [0,1],$$

so $\mathcal{A}u \in Q$. Therefore $\mathcal{A}(\overline{Q}_R \setminus Q_r) \subset Q$.

We prove now that $\mathcal{A} : \overline{Q}_R \setminus Q_r \to Q$ is completely continuous. We assume that $E \subset \overline{Q}_R \setminus Q_r$ is an arbitrary bounded set. From the first part of the proof we know that $\mathcal{A}(E)$ is uniformly bounded. Then we show that $\mathcal{A}(E)$ is equicontinuous. Indeed, for $\varepsilon > 0$, there exists a natural number $n_2 \geqslant 3$ such that

$$\sup_{u\in\overline{Q}_R\setminus Q_r} \int_{A_{n_2}} h(s)f\big(s,u(s)\big)\,\mathrm{d}s < \frac{\varepsilon}{4\lambda J_0}.$$

Since $\mathcal{G}(t,s)$ is uniformly continuous on $[0,1]\times[0,1]$, for the above $\varepsilon > 0$, there exists $\delta > 0$ such that, for any $t_1, t_2 \in [0,1]$ with $|t_1 - t_2| < \delta$ and $s \in [1/n_2, (n_2-1)/n_2]$, we have

$$\big|\mathcal{G}(t_1,s) - \mathcal{G}(t_2,s)\big| < \frac{\varepsilon}{2\lambda\bar{h}L_2},$$

where $L_2 = \max\{1, \max\{f(t,x),\ t \in [1/n_2, (n_2-1)/n_2],\ x \in [r/n_2^{\alpha-1}, R]\}\}$ and $\bar{h} = \max\{1, \max\{h(t),\ t \in [1/n_2, (n_2-1)/n_2]\}\}$.

Then, for any $u \in E$, $t_1, t_2 \in [0,1]$ with $|t_1 - t_2| < \delta$, we deduce

$$\begin{aligned}
&\big|(\mathcal{A}u)(t_1) - (\mathcal{A}u)(t_2)\big| \\
&\quad = \lambda \left| \int_0^1 \big(\mathcal{G}(t_1,s) - \mathcal{G}(t_2,s)\big) h(s) f\big(s,u(s)\big)\,\mathrm{d}s \right| \\
&\quad \leqslant 2\lambda \int_{A_{n_2}} \mathcal{J}(s) h(s) f\big(s,u(s)\big)\,\mathrm{d}s \\
&\qquad + \lambda \sup_{u\in E} \int_{1/n_2}^{(n_2-1)/n_2} \big|\mathcal{G}(t_1,s) - \mathcal{G}(t_2,s)\big| h(s) f\big(s,u(s)\big)\,\mathrm{d}s \\
&\quad \leqslant 2\lambda J_0 \frac{\varepsilon}{4\lambda J_0} + \frac{\varepsilon\lambda}{2\lambda \bar{h} L_2} \left(\int_{1/n_2}^{(n_2-1)/n_2} h(s)\,\mathrm{d}s \right) L_2 \\
&\quad \leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

This gives us that $\mathcal{A}(E)$ is equicontinuous. By the Arzelà–Ascoli theorem we conclude that $\mathcal{A} : \overline{Q}_R \setminus Q_r \to Q$ is compact.

Finally, we prove that $\mathcal{A} : \overline{Q}_R \setminus Q_r \to Q$ is continuous. We suppose that $u_n, u_0 \in \overline{Q}_R \setminus Q_r$ for all $n \geqslant 1$ and $\|u_n - u_0\| \to 0$ as $n \to \infty$. Then $r \leqslant \|u_n\| \leqslant R$ for all $n \geqslant 0$. By (I3), for $\varepsilon > 0$, there exists a natural number $n_3 \geqslant 3$ such that

$$\sup_{u\in\overline{Q}_R\setminus Q_r} \int_{A_{n_3}} h(s) f\big(s,u(s)\big)\,\mathrm{d}s < \frac{\varepsilon}{4\lambda J_0}. \tag{9}$$

Because $f(t,x)$ is uniformly continuous in $[1/n_3, (n_3-1)/n_3] \times [r/n_3^{\alpha-1}, R]$, we obtain

$$\lim_{n\to\infty} \big|f\big(s,u(s)\big) - f\big(s,u_0(s)\big)\big| = 0$$

uniformly for $s \in [1/n_3, (n_3-1)/n_3]$. Then the Lebesgue dominated convergence theorem gives us

$$\int_{1/n_3}^{(n_3-1)/n_3} h(s)\big|f\big(s,u_n(s)\big) - f\big(s,u_0(s)\big)\big|\,\mathrm{d}s \to 0 \quad \text{as } n \to \infty.$$

Thus, for the above $\varepsilon > 0$, there exists a natural number N such that, for $n > N$, we have

$$\int_{1/n_3}^{(n_3-1)/n_3} h(s)\big|f\big(s,u_n(s)\big) - f\big(s,u_0(s)\big)\big|\,\mathrm{d}s < \frac{\varepsilon}{2\lambda J_0}. \tag{10}$$

By (9) and (10) we conclude that

$$\begin{aligned}
&\|\mathcal{A}u_n - \mathcal{A}u_0\| \\
&\quad\leqslant \sup_{u\in\overline{Q}_R\setminus Q_r} \lambda \int\limits_{A_{n_3}} \mathcal{J}(s)h(s)\bigl|f\bigl(s,u_n(s)\bigr) - f\bigl(s,u_0(s)\bigr)\bigr|\,\mathrm{d}s \\
&\qquad + \sup_{u\in\overline{Q}_R\setminus Q_r} \lambda \int\limits_{1/n_3}^{(n_3-1)/n_3} \mathcal{J}(s)h(s)\bigl|f\bigl(s,u_n(s)\bigr) - f\bigl(s,u_0(s)\bigr)\bigr|\,\mathrm{d}s \\
&\quad\leqslant \lambda J_0\frac{\varepsilon}{4\lambda J_0} + \lambda J_0\frac{\varepsilon}{4\lambda J_0} + \frac{\varepsilon}{2\lambda J_0}\lambda J_0 = \varepsilon.
\end{aligned}$$

This implies that $\mathcal{A} : \overline{Q}_R\setminus Q_r \to Q$ is continuous. Hence $\mathcal{A} : \overline{Q}_R\setminus Q_r \to Q$ is completely continuous. □

Under assumptions (I1)–(I3), by the extension theorem the operator $\mathcal{A}$ has a completely continuous extension (also denoted by $\mathcal{A}$) from Q to Q.

Lemma 5. *Assume that assumptions* (I1), (I2) *hold. Then the spectral radius* $r(\mathcal{T}) \neq 0$*, and* $\mathcal{T}$ *has an eigenfunction* $\psi_1 \in P\setminus\{\theta\}$ *corresponding to the principal eigenvalue* $r(\mathcal{T})$*, that is* $\mathcal{T}\psi_1 = r(\mathcal{T})\psi_1$*. So* $r(\mathcal{T}) > 0$.

Proof. The operator $\mathcal{T} : X \to X$ is a linear completely continuous operator. By Lemma 2 we know that $\mathcal{G}(t,s) > 0$ for all $t,s \in (0,1)$. By (I2) we deduce that there exists an interval $[c,d] \subset (0,1)$ $(0 < c < d < 1)$ such that $h(t) > 0$ for all $t \in [c,d]$. We consider a function $\varphi \in C[0,1]$ satisfying the conditions $\varphi(t) > 0$ for $t \in (c,d)$ and $\varphi(t) = 0$ for $t \notin (c,d)$. Then, for all $t \in [c,d]$, we have

$$(\mathcal{T}\varphi)(t) = \int\limits_0^1 \mathcal{G}(t,s)h(s)\varphi(s)\,\mathrm{d}s \geqslant \int\limits_c^d \mathcal{G}(t,s)h(s)\varphi(s)\,\mathrm{d}s > 0 \quad \forall t \in [c,d].$$

Hence there exists a constant $a > 0$ ($a = \max_{t\in[c,d]}\varphi(t)/\min_{t\in[c,d]}(\mathcal{T}\varphi)(t)$), which satisfies the inequality $a(\mathcal{T}\varphi)(t) \geqslant \varphi(t)$ for all $t \in [0,1]$. By Theorem 4 we conclude that the spectral radius $r(\mathcal{T}) \neq 0$ and $\mathcal{T}$ has an eigenfunction $\psi_1 \in P\setminus\{\theta\}$ corresponding to its principal characteristic value $\lambda_1 = (r(\mathcal{T}))^{-1}$ such that $\mathcal{T}\psi_1 = r(\mathcal{T})\psi_1$, and so $r(\mathcal{T}) > 0$. □

Using a similar argument as that used in the proof of Lemma 4 for operator $\mathcal{A}$, we obtain that $\mathcal{T}(Q) \subset Q$.

Theorem 5. *Assume that assumptions* (I1)–(I3) *hold. If*

$$0 \leqslant f_\infty^s := \limsup_{u\to\infty} \max_{t\in[0,1]} \frac{f(t,u)}{u} < f_0^i := \liminf_{u\to 0+} \min_{t\in[0,1]} \frac{f(t,u)}{u} \leqslant \infty,$$

then, for any $\lambda \in (1/(f_0^i r(\mathcal{T})), 1/(f_\infty^s r(\mathcal{T})))$*, problem* (1), (2) *has at least one positive solution* $u(t)$*,* $t \in [0,1]$ (*with the conventions* $1/0_+ = \infty$ and $1/\infty = 0_+$).

Proof. We consider $\lambda \in (1/(f_0^i r(\mathcal{T})), 1/(f_\infty^s r(\mathcal{T})))$. For f_0^i, we have the cases: $f_0^i \in (0,\infty)$ with $f_0^i > 1/(\lambda r(\mathcal{T}))$ and $f_0^i = \infty$. In the first case, $f_0^i \in (0,\infty)$ with $f_0^i > 1/(\lambda r(\mathcal{T}))$, we obtain

$$\forall \varepsilon > 0,\ \exists\, \delta(\varepsilon) > 0 \quad \text{s.t.} \quad \frac{f(t,u)}{u} \geqslant f_0^i - \varepsilon \quad \forall t \in [0,1],\ u \in \bigl(0, \delta(\varepsilon)\bigr].$$

By taking $\varepsilon = f_0^i - 1/(\lambda r(\mathcal{T}))$ we deduce that there exists $r_1' > 0$ such that $f(t,u)/u \geqslant 1/(\lambda r(\mathcal{T}))$ for all $t \in [0,1]$ and $u \in (0, r_1']$, and so $f(t,u) \geqslant u/(\lambda r(\mathcal{T}))$ for all $t \in [0,1]$ and $u \in [0, r_1']$.

In the case $f_0^i = \infty$, we have

$$\forall \varepsilon > 0,\ \exists\, \delta(\varepsilon) > 0 \quad \text{s.t.} \quad \frac{f(t,u)}{u} \geqslant \varepsilon \quad \forall t \in [0,1],\ u \in \bigl(0, \delta(\varepsilon)\bigr].$$

So for $\varepsilon = 1/(\lambda r(\mathcal{T}))$, we deduce that there exists $r_1'' > 0$ such that $f(t,u) \geqslant u/(\lambda r(\mathcal{T}))$ for all $t \in [0,1]$ and $u \in [0, r_1'']$.

Hence, in the above both cases, we conclude that there exists $r_1 > 0$ such that $f(t,u) \geqslant u/(\lambda r(\mathcal{T}))$ for all $t \in [0,1]$ and $u \in [0, r_1]$.

Then, for any $u \in \partial Q_{r_1}$, we find

$$\begin{aligned}
\mathcal{A}u(t) &= \lambda \int_0^1 \mathcal{G}(t,s) h(s) f\bigl(s, u(s)\bigr)\, \mathrm{d}s \\
&\geqslant \frac{1}{r(\mathcal{T})} \int_0^1 \mathcal{G}(t,s) h(s) u(s)\, \mathrm{d}s = \frac{1}{r(\mathcal{T})} \mathcal{T} u(t) \quad \forall t \in [0,1].
\end{aligned}$$

We assume that $\mathcal{A}$ has no fixed point on ∂Q_{r_1}, (otherwise the proof is finished). We will prove that

$$u - \mathcal{A}u \neq \mu \psi_1 \quad \forall u \in \partial Q_{r_1},\ \mu \geqslant 0, \tag{11}$$

where ψ_1 is given in Lemma 5. We suppose that there exist $u_1 \in \partial Q_{r_1}$ and $\mu_1 \geqslant 0$ such that $u_1 - \mathcal{A}u_1 = \mu_1 \psi_1$. Then $\mu_1 > 0$ and $u_1 = \mathcal{A}u_1 + \mu_1 \psi_1 \geqslant \mu_1 \psi_1$. We denote $\mu_0 = \sup\{\mu\colon\ u_1 \geqslant \mu \psi_1\}$. Then $\mu_0 \geqslant \mu_1$, $u_1 \geqslant \mu_0 \psi_1$ and

$$\mathcal{A}u_1 \geqslant \frac{1}{r(\mathcal{T})} \mathcal{T} u_1 \geqslant \frac{1}{r(\mathcal{T})} \mu_0 \mathcal{T} \psi_1 = \mu_0 \psi_1.$$

Hence $u_1 = \mathcal{A}u_1 + \mu_1 \psi_1 \geqslant \mu_0 \psi_1 + \mu_1 \psi_1 = (\mu_0 + \mu_1) \psi_1$, which contradicts the definition of μ_0. So relation (11) holds, and by Theorem 1 we deduce that

$$i(\mathcal{A}, Q_{r_1}, Q) = 0. \tag{12}$$

For f_∞^s, we have also two cases: $f_\infty^s \in (0,\infty)$ with $f_\infty^s < 1/(\lambda r(\mathcal{T}))$ and $f_\infty^s = 0$. In the first case, $f_\infty^s \in (0,\infty)$ with $f_\infty^s < 1/(\lambda r(\mathcal{T}))$, we obtain

$$\forall \varepsilon > 0,\ \exists\, \delta(\varepsilon) > 0 \quad \text{s.t.} \quad \frac{f(t,u)}{u} \leqslant f_\infty^s + \varepsilon \quad \forall t \in [0,1],\ u \geqslant \delta(\varepsilon).$$

By taking $\varepsilon = 1/(2\lambda r(\mathcal{T})) - f_\infty^s/2$ we deduce that there exists $r_2' > r_1$ such that $f(t,u) \leqslant \theta_1/(\lambda r(\mathcal{T}))u$ for all $t \in [0,1]$ and $u \in [r_2', \infty)$, where $\theta_1 = 1/2 + f_\infty^s \lambda r(\mathcal{T})/2 \in (0,1)$.

In the case $f_\infty^s = 0$, we have

$$\forall \varepsilon > 0,\ \exists\, \delta(\varepsilon) > 0 \quad \text{s.t.} \quad \frac{f(t,u)}{u} \leqslant \varepsilon \quad \forall t \in [0,1],\ u \geqslant \delta(\varepsilon).$$

So for $\varepsilon = 1/(2\lambda r(\mathcal{T}))$, we deduce that there exists $r_2'' > r_1$ such that $f(t,u) \leqslant 1/(2\lambda r(\mathcal{T}))u$ for all $t \in [0,1]$ and $u \in [r_2'', \infty)$.

Therefore, in the above both cases, we conclude that there exist $\theta \in (0,1)$ and $r_2 > r_1$ such that $f(t,u) \leqslant \theta 1/(\lambda r(\mathcal{T}))u$ for all $t \in [0,1]$ and $u \in [r_2, \infty)$.

We define now the operator $\mathcal{T}_1 : X \to X$ by

$$\mathcal{T}_1 u = \theta \frac{1}{r(\mathcal{T})} \mathcal{T} u = \frac{\theta}{r(\mathcal{T})} \int_0^1 \mathcal{G}(t,s) h(s) u(s)\, \mathrm{d}s, \quad t \in [0,1],\ u \in X.$$

The operator $\mathcal{T}_1$ is linear and bounded, and $\mathcal{T}_1(Q) \subset Q$. Because $\theta \in (0,1)$, we obtain $r(\mathcal{T}_1) = \theta < 1$. We consider the set

$$Z = \{u \in Q \setminus B_{r_1}\colon\ \mu u = \mathcal{A} u \text{ with } \mu \geqslant 1\}.$$

For $u \in Q$, we denote $D(u) = \{t \in [0,1]\colon\ u(t) \geqslant r_2\}$. Then, for $u \in Q$, we have $u(t) \geqslant r_2$ for all $t \in D(u)$, and so

$$f\bigl(t, u(t)\bigr) \leqslant \theta \frac{1}{\lambda r(\mathcal{T})} u(t) \quad \forall t \in D(u). \tag{13}$$

By (13) and the definition of operator $\mathcal{T}$, for any $u \in Z$, $\mu \geqslant 1$ and $t \in [0,1]$, we deduce

$$\begin{aligned} u(t) &\leqslant \mu u(t) = (\mathcal{A}u)(t) = \lambda \int_0^1 \mathcal{G}(t,s) h(s) f\bigl(s, u(s)\bigr)\, \mathrm{d}s \\ &= \lambda \int_{D(u)} \mathcal{G}(t,s) h(s) f\bigl(s, u(s)\bigr)\, \mathrm{d}s + \lambda \int_{[0,1]\setminus D(u)} \mathcal{G}(t,s) h(s) f\bigl(s, u(s)\bigr)\, \mathrm{d}s \\ &\leqslant \frac{\theta}{r(\mathcal{T})} \int_{D(u)} \mathcal{G}(t,s) h(s) u(s)\, \mathrm{d}s + \lambda \int_0^1 \mathcal{J}(s) h(s) f\bigl(s, \widetilde{u}(s)\bigr)\, \mathrm{d}s \\ &\leqslant \frac{\theta}{r(\mathcal{T})} \int_0^1 \mathcal{G}(t,s) h(s) u(s)\, \mathrm{d}s + \lambda J_0 M_1 = (\mathcal{T}_1 u)(t) + \lambda J_0 M_1, \end{aligned} \tag{14}$$

where $\widetilde{u}(t) = \min\{u(t), r_2\}$ for all $t \in [0,1]$ (which satisfies $r_1 t^{\alpha-1} \leqslant \widetilde{u}(t) \leqslant r_2$ for all $t \in [0,1]$), $J_0 = \sup_{s\in[0,1]} \mathcal{J}(s)$, and $M_1 = \sup_{u\in \overline{Q}_{r_2}\setminus Q_{r_1}} \int_0^1 h(s) f(s,u(s))\,\mathrm{d}s$ (as in the proof of Lemma 4, we obtain that $M_1 < \infty$). By the Gelfand formula we know that $(I-\mathcal{T}_1)^{-1}$ exists and $(I-\mathcal{T}_1)^{-1} = \sum_{i=1}^{\infty} \mathcal{T}_1^i$, which implies $(I-\mathcal{T}_1)^{-1}(Q) \subset Q$. This, together with (14), gives us $u(t) \leqslant (I-\mathcal{T}_1)^{-1}(\lambda J_0 M_1)$, and so $u(t) \leqslant \lambda J_0 M_1 \times \|(I-\mathcal{T}_1)^{-1}\|$ for all $t \in [0,1]$, which means that Z is bounded. Now we choose $R > \max\{r_2, \sup\{\|u\|, u \in Z\}\}$. Then we obtain that $\mu u \neq \mathcal{A}u$ for all $u \in \partial Q_R$ and $\mu \geqslant 1$. By Theorem 2 we conclude that

$$i(\mathcal{A}, Q_R, Q) = 1. \tag{15}$$

By (12), (15) and the additivity property of the fixed point index we deduce that

$$i(\mathcal{A}, Q_R \setminus \overline{Q}_{r_1}, Q) = i(\mathcal{A}, Q_R, Q) - i(\mathcal{A}, Q_{r_1}, Q) = 1.$$

So operator $\mathcal{A}$ has at least one fixed point on $Q_R \setminus \overline{Q}_{r_1}$, which is a positive solution of problem (1), (2). □

By using a similar approach as that used in the proof of Theorem 5, we obtain the following result.

Theorem 6. *Assume that assumptions* (I1)–(I3) *hold. If*

$$0 \leqslant f_0^s := \limsup_{u\to 0+} \max_{t\in[0,1]} \frac{f(t,u)}{u} < f_\infty^i := \liminf_{u\to\infty} \min_{t\in[0,1]} \frac{f(t,u)}{u} \leqslant \infty,$$

then, for any $\lambda \in (1/(f_\infty^i r(\mathcal{T})), 1/(f_0^s r(\mathcal{T})))$, *problem* (1), (2) *has at least one positive solution* $u(t)$, $t \in [0,1]$.

4 Some remarks on a related semipositone problem

In this section, we present two existence results for a semipositone problem associated to problem (1), (2). More precisely, we consider the fractional differential equation

$$D_{0+}^{\alpha} u(t) + \lambda \widetilde{f}\big(t, u(t)\big) = 0, \quad t \in (0,1), \tag{16}$$

subject to the boundary conditions (2). We suppose that assumption (I1) holds and $\widetilde{f}$ satisfies the conditions

- (I2′) The function $\widetilde{f} \in C((0,1)\times[0,\infty), R)$ may be singular at $t = 0$ and/or $t = 1$, and there exist the functions $p, q \in C((0,1),[0,\infty))$, $g \in C([0,1]\times[0,\infty),[0,\infty))$ such that $-p(t) \leqslant \widetilde{f}(t,u) \leqslant q(t)g(t,u)$ for all $t \in (0,1)$ and $u \in [0,\infty)$ with $0 < \int_0^1 p(t)\,\mathrm{d}t < \infty$, $0 < \int_0^1 q(t)\,\mathrm{d}t < \infty$.
- (I3′) There exists $\zeta \in (0,1/2)$ such that $\lim_{u\to\infty} \min_{t\in[\zeta,1-\zeta]} \widetilde{f}(t,u)/u = \infty$.

By using the Guo–Krasnosel'skii fixed point theorem (Theorem 3) and similar arguments as those used in [11] (Theorems 3.1 and 3.2) we obtain the following results for problem (16), (2).

Theorem 7. *Assume that* (I1), (I2′) *and* (I3′) *hold. Then there exists* $\lambda^* > 0$ *such that, for any* $\lambda \in (0, \lambda^*]$, *the boundary value problem* (16), (2) *has at least one positive solution.*

In the proof of Theorem 7, we consider $R_1 > \sigma \int_0^1 p(t)\,\mathrm{d}t > 0$, and we define

$$\lambda^* = \min\left\{1, R_1\left(M_2 \int_0^1 \mathcal{J}(s)\big(q(s) + p(s)\big)\,\mathrm{d}s\right)^{-1}\right\}$$

with $M_2 = \max\{\max_{t\in[0,1],\,u\in[0,R_1]} g(t,u), 1\}$. The solution $u(t)$, $t \in [0,1]$, satisfies the condition $u(t) \geqslant \Lambda_1 t^{\alpha-1}$ for all $t \in [0,1]$, where $\Lambda_1 = R_1 - \sigma \int_0^1 p(s)\,\mathrm{d}s > 0$.

Theorem 8. *Assume that* (I1), (I2′) *and*

(I4) *There exists* $\zeta \in (0, 1/2)$ *such that the following hold*:

$$\lim_{u\to\infty} \min_{t\in[\zeta,1-\zeta]} \widetilde{f}(t,u) = \infty \quad \textit{and} \quad \lim_{u\to\infty} \max_{t\in[0,1]} \frac{g(t,u)}{u} = 0.$$

Then there exists $\lambda_* > 0$ *such that, for any* $\lambda \geqslant \lambda_*$, *the boundary value problem* (16), (2) *has at least one positive solution.*

By (I4) we know that for $\zeta \in (0, 1/2)$ and for a fixed number $L_0 > 0$, there exists $M_3 > 0$ such that $\widetilde{f}(t,u) \geqslant L_0$ for all $t \in [\zeta, 1-\zeta]$ and $u \geqslant M_3$. In the proof of Theorem 8, we define $\lambda_* = M_3(\zeta^{\alpha-1}\sigma \int_0^1 p(s)\,\mathrm{d}s)^{-1}$. The solution $u(t)$, $t \in [0,1]$, satisfies the condition $u(t) \geqslant \widetilde{\Lambda}_1 t^{\alpha-1}$ for all $t \in [0,1]$, where $\widetilde{\Lambda}_1 = M_3/\zeta^{\alpha-1}$.

5 Examples

Let $\alpha = 10/3$, $n = 4$, $\beta_0 = 11/5$, $m = 2$, $\beta_1 = 1/2$, $\beta_2 = 5/4$, $H_1(t) = t$ for all $t \in [0,1]$, $H_2(t) = \{0 \text{ for } t \in [0,1/2);\ 1 \text{ for } t \in [1/2,1]\}$.

We consider the fractional differential equations

$$D_{0+}^{10/3} u(t) + \lambda h(t) f\big(t, u(t)\big) = 0, \quad t \in (0,1), \tag{17}$$

$$D_{0+}^{10/3} u(t) + \lambda \widetilde{f}\big(t, u(t)\big) = 0, \quad t \in (0,1), \tag{18}$$

subject to the boundary conditions

$$u(0) = u'(0) = u''(0) = 0, \qquad D_{0+}^{11/5} u(1) = \int_0^1 D_{0+}^{1/2} u(t)\,\mathrm{d}t + D_{0+}^{5/4} u\left(\frac{1}{2}\right). \tag{19}$$

We have $\varDelta \approx 1.12792427 > 0$ and $\sigma \approx 0.94443688$. So assumption (I1) is satisfied. In addition, we obtain

$$g_{21}(t,s) = \frac{1}{\Gamma(17/6)} \begin{cases} t^{11/6}(1-s)^{2/15} - (t-s)^{11/6}, & 0 \leqslant s \leqslant t \leqslant 1, \\ t^{11/6}(1-s)^{2/15}, & 0 \leqslant t \leqslant s \leqslant 1, \end{cases}$$

$$g_{22}(t,s) = \frac{1}{\Gamma(25/12)} \begin{cases} t^{13/12}(1-s)^{2/15} - (t-s)^{13/12}, & 0 \leqslant s \leqslant t \leqslant 1, \\ t^{13/12}(1-s)^{2/15}, & 0 \leqslant t \leqslant s \leqslant 1, \end{cases}$$

$$h_1(s) = \frac{1}{\Gamma(10/3)}(1-s)^{2/15}\bigl(1-(1-s)^{11/5}\bigr), \quad s \in [0,1],$$

$$\mathcal{J}(s) = \begin{cases} h_1(s) + \frac{1}{\varDelta}\{\frac{1}{\Gamma(23/6)}(1-s)^{2/15} - \frac{1}{\Gamma(23/6)}(1-s)^{17/6} \\ \quad + \frac{1}{\Gamma(25/12)}[(\frac{1}{2})^{13/12}(1-s)^{2/15} - (\frac{1}{2}-s)^{13/12}]\}, & 0 \leqslant s \leqslant \frac{1}{2}, \\ h_1(s) + \frac{1}{\varDelta}\{\frac{1}{\Gamma(23/6)}(1-s)^{2/15} - \frac{1}{\Gamma(23/6)}(1-s)^{17/6} \\ \quad + \frac{1}{\Gamma(25/12)}(\frac{1}{2})^{13/12}(1-s)^{2/15}\}, & \frac{1}{2} < s \leqslant 1. \end{cases}$$

Example 1. We consider the functions

$$h(t) = \frac{1}{\sqrt[3]{t(1-t)^2}}, \quad t \in (0,1); \qquad f(t,u) = \sqrt{u} + t + \frac{1}{\sqrt[4]{u}}, \quad t \in [0,1],\ u > 0.$$

The cone Q from Section 3 is here $Q = \{u \in C[0,1]\colon\ u(t) \geqslant t^{7/3}\|u\|\ \ \forall t \in [0,1]\}$. For $0 < r < R$ and $u \in \overline{Q}_R \setminus Q_r$, we deduce

$$f\bigl(t,u(t)\bigr) \leqslant \sqrt{R} + 1 + \frac{1}{\sqrt[4]{t^{7/3}r}} \quad \forall t \in (0,1].$$

Besides, we obtain $\int_0^1 \mathcal{J}(s)h(s)\,\mathrm{d}s \leqslant J_0\Gamma(2/3)\Gamma(1/3) < \infty$, $J_0 = \max_{s\in[0,1]}\mathcal{J}(s) \approx 0.781$. Hence assumption (I2) is satisfied.

For $u \in \overline{Q}_R \setminus Q_r$ and $A_n = [0,1/n] \cup [(n-1)/n,\ 1]$, we find

$$\begin{aligned} C_n &= \int\limits_{A_n} h(s)f\bigl(s,u(s)\bigr)\,\mathrm{d}s = \int\limits_{A_n} \frac{1}{\sqrt[3]{s(1-s)^2}}\left(\sqrt{u(s)} + s + \frac{1}{\sqrt[4]{u(s)}}\right)\mathrm{d}s \\ &\leqslant \int\limits_{A_n} \frac{1}{\sqrt[3]{s(1-s)^2}}\left(\sqrt{R} + 1 + \frac{1}{\sqrt[4]{s^{7/3}r}}\right)\mathrm{d}s \\ &= (\sqrt{R}+1)\int\limits_{A_n} \frac{\mathrm{d}s}{\sqrt[3]{s(1-s)^2}} + \frac{1}{\sqrt[4]{r}}\int\limits_{A_n} \frac{1}{s^{11/12}(1-s)^{2/3}}\,\mathrm{d}s, \end{aligned}$$

and then $\lim_{n\to\infty}\sup_{u\in\overline{Q}_R\setminus Q_r} C_n = 0$ because $f_1(s) = 1/(\sqrt[3]{s(1-s)^2}) \in L^1(0,1)$ and $f_2(s) = 1/(s^{11/12}(1-s)^{2/3}) \in L^1(0,1)$. Hence assumption (I3) is satisfied. We also have $f_\infty^s = 0$ and $f_0^i = \infty$. Then by using Theorem 5 we deduce that, for any $\lambda \in (0,\infty)$, problem (17), (19) has at least one positive solution $u(t)$, $t \in [0,1]$, which satisfies the condition $u(t) \geqslant t^{7/3}\|u\|$ for all $t \in [0,1]$.

Example 2. We consider the function

$$\widetilde{f}(t,u) = \frac{u^3+u+1}{\sqrt[4]{t(1-t)^3}} + \ln t, \quad t \in (0,1),\ u \geqslant 0.$$

For this example, we have $p(t) = -\ln t$ and $q(t) = 1/(\sqrt[4]{t(1-t)^3})$ for all $t \in (0,1)$, $g(t,u) = u^3+u+1$ for all $t \in [0,1]$ and $u \geqslant 0$, $\int_0^1 p(t)\,\mathrm{d}t = 1$, $\int_0^1 q(t)\,\mathrm{d}t = \Gamma(3/4)\Gamma(1/4) \approx 4.44288$. Then assumption (I2′) is satisfied. In addition, for $\zeta \in (0,1/2)$ fixed, assumption (I3′) is also satisfied. By some computations we obtain that $\int_0^1 \mathcal{J}(s)(q(s)+p(s))\,\mathrm{d}s \approx 2.71742073$. We choose $R_1 = 2$, which satisfies the condition $R_1 > \sigma \int_0^1 p(t)\,\mathrm{d}t \approx 0.944$, and then we deduce $M_2 = 11$ and $\lambda^* \approx 0.0669084$. By Theorem 7 we conclude that, for any $\lambda \in (0,\lambda^*]$, problem (18), (19) has at least one positive solution $u(t)$, $t \in [0,1]$, which satisfies the condition $u(t) \geqslant \varLambda_1 t^{7/3}$ for all $t \in [0,1]$, where $\varLambda_1 \approx 1.05556$.

Example 3. We consider the function

$$\widetilde{f}(t,u) = \frac{\sqrt{u+1/3}}{\sqrt[5]{t^3(1-t)^2}} - \frac{1}{\sqrt[3]{t}}, \quad t \in (0,1),\ u \geqslant 0.$$

Here we have $p(t) = 1/\sqrt[3]{t}$ and $q(t) = 1/\sqrt[5]{t^3(1-t)^2}$ for all $t \in (0,1)$, $g(t,u) = \sqrt{u+1/3}$ for all $t \in [0,1]$ and $u \geqslant 0$. Because $\int_0^1 p(t)\,\mathrm{d}t = 3/2$, $\int_0^1 q(t)\,\mathrm{d}t \approx 3.30327$, assumption (I2′) is satisfied. In addition, for $\zeta \in (0,1/2)$, we obtain that $\lim_{u\to\infty} \min_{t\in[\zeta,1-\zeta]} \widetilde{f}(t,u) = \infty$ and $\lim_{u\to\infty} \max_{t\in[0,1]} g(t,u)/u = 0$, and then assumption (I4) is also satisfied. We choose $\zeta = 1/4$ and $L_0 = 100$, and then we find $M_3 = 5805$ and $\lambda_* \approx 104075$. Then by Theorem 8 we deduce that, for any $\lambda \geqslant \lambda_*$, problem (18), (19) has at least one positive solution $u(t)$, $t \in [0,1]$, which satisfies the inequality $u(t) \geqslant \widetilde{\varLambda}_1 t^{7/3}$ for all $t \in [0,1]$, where $\widetilde{\varLambda}_1 \approx 147438$.

6 Conclusion

In this paper, we study the existence of positive solutions for the nonlinear Riemann–Liouville fractional boundary value problem (1), (2), where λ is a positive parameter. The function f is nonnegative, and it may be singular at the second variable, and the function h is also nonnegative, and it may have singularities at $t=0$ and/or $t=1$. We present conditions for f and h and intervals for λ, which are expressed in term of the principal characteristic value of an associated linear operator. In the proof of the existence theorems, we use two results from the fixed point index theory. We also investigate a related semipositone problem, namely, equation (1) with $h \equiv 1$ and f a sign-changing function with singularities at $t=0$ and/or $t=1$ subject to the nonlocal boundary conditions (2). For this problem, we give two existence results for the positive solutions when λ belongs to various intervals. Three examples, which illustrate the obtained existence theorems, are finally presented.

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