Nonlinear Analysis: Modelling and Control, Vol. 26, No. 1, 113–129
https://doi.org/10.15388/namc.2021.26.21415

Best proximity point theorems for cyclic p-contractions with some consequences and applications

Mustafa Aslantas[a], **Hakan Sahin**[b], **Ishak Altun**[c]

[a]Department of Mathematics, Faculty of Science,
Çankırı Karatekin University,
18100 Çankırı, Turkey
maslantas@karatekin.edu.tr

[b]Department of Mathematics, Faculty of Science and Arts,
Amasya University,
5220 Amasya, Turkey
hakan.sahin@amasya.edu.tr

[c]Department of Mathematics, Faculty of Science and Arts,
Kirikkale University,
71450 Yahsihan, Kirikkale, Turkey
ialtun@kku.edu.tr; ishakaltun@yahoo.com

Received: November 15, 2019 / **Revised:** May 7, 2020 / **Published online:** January 1, 2021

Abstract. In this paper, we introduce the concept of cyclic p-contraction pair for single-valued mappings. Then we present some best proximity point results for such mappings defined on proximally complete pair of subsets of a metric space. Also, we provide some illustrative examples that compared our results with some earliest. Finally, by taking into account a fixed point consequence of our main result we give an existence and uniqueness result for a common solution of a system of second order boundary value problems.

Keywords: best proximity point, cyclic p-contractions, fixed point, boundary value problem.

1 Introduction

In 1922, Banach [5] proved a very fundamental theorem, which is known as Banach contraction principle on complete metric spaces. According to this theorem, every self mapping T on complete metric space (X, d) satisfying

$$d(Tx, Ty) \leqslant kd(x, y)$$

for all $x, y \in X$, where $k \in (0, 1)$, has a unique fixed point in X.

This result has been extended and generalized by many authors in different ways. For example, by introducing the concept of cyclic mappings Kirk et al. [16] have obtained a nice and important generalization of it. Although the mapping T has to be continuous in Banach contraction principle, it is not necessary to be continuous in the following.

Theorem 1. (*See* [16].) *Let (X, d) be a complete metric space, A, B be two closed subsets of X and $T : A \cup B \to A \cup B$ be a mapping. Assume that T is a cyclic mapping, that is, $T(A) \subseteq B$ and $T(B) \subseteq A$. If there exists $k \in (0, 1)$ such that $d(Tx, Ty) \leqslant kd(x, y)$ for all $x \in A$ and $y \in B$, then T has a fixed point in $A \cap B$.*

Also, the following common fixed point result has been obtained by taking into account a similar approach.

Theorem 2. (*See* [16].) *Let A and B be two nonempty closed subsets of a complete metric space (X, d). Let $T : A \to B$ and $S : B \to A$ be two functions such that*

$$d(Tx, Sy) \leqslant kd(x, y)$$

for all $x \in A$ and $y \in B$, where $k \in (0, 1)$. Then there exists a unique $z \in A \cap B$ such that $Tz = Sz = z$.

On the other hand, some other extensions of Banach contraction principle have been presented by considering the concept of best proximity point [6]. Now, we recall some fundamental concepts and properties of best proximity points. Let (X, d) be a metric space, A, B be two subsets of X and $T : A \to B$ be a mapping. In this context, the mapping T does not have a fixed point in case of $A \cap B = \emptyset$. It is reasonable to research a point x, which is closest to Tx, that is, $d(x, Tx)$ is minimum. Since $d(x, Tx) \geqslant d(A, B)$ for all $x \in A$, then $d(x, Tx)$ is at least $d(A, B)$. Best proximity point theorems are related to find a point satisfying $d(x, Tx) = d(A, B)$, which is called a best proximity point. There are many results about this subject in literature [2, 11, 15, 18–20, 22].

Note that $A \cap B$ has to be nonempty in Theorem 4. Hence, by combining the ideas of both cyclic mapping and best proximity point Eldred and Veeremani [8] introduced the concept of cyclic contraction mapping and then obtained a result, which includes the situation $A \cap B = \emptyset$.

Definition 1. Let A and B be nonempty subsets of a metric space (X, d). A mapping $T : A \cup B \to A \cup B$ is called cyclic contraction if it satisfies $T(A) \subseteq B$ and $T(B) \subseteq A$ and the following condition:

$$d(Tx, Ty) \leqslant kd(x, y) + (1 - k)d(A, B)$$

for all $x \in A$ and $y \in B$, where $k \in (0, 1)$.

Theorem 3. *Let A and B be nonempty closed subsets of a metric space (X, d), and let $T : A \cup B \to A \cup B$ be a cyclic contraction. If either A or B is boundedly compact, then there exists x in $A \cup B$ with $d(x, Tx) = d(A, B)$.*

After that, Karapınar [12] defined some generalizations of cyclic contraction mappings such as Ciric type and Kannan type cyclic contraction mappings and then presented some nice results for existence of a best proximity point of such mappings in a uniformly convex Banach space. Until now, in almost all results about to guarantee the existence of best proximity points of a cyclic contraction mapping, it has been used some certain properties such as UC-property or boundedly compactness [4, 14]. Very recently, the notion of proximally completeness has been defined in [9], and so the UC-property and boundedly compactness have been neglected. Then many authors have studied to show the existence of best proximity points of a cyclic contraction self mapping of $A\cup B$, where (A, B) is a proximally complete pair [1, 7, 17, 23, 25].

Lately, Popescu [21] introduced a new type contractive condition named as p-contraction, which expands Banach contraction as follows: Let (X, d) be a metric space and $T : X \to X$ be a self mapping. If there exists a k in $[0, 1)$ such that

$$d(Tx, Ty) \leqslant k\big[d(x, y) + \big|d(x, Tx) - d(y, Ty)\big|\big]$$

for all $x, y \in X$, then T is said to be a p-contraction mapping. Then he proved that every p-contraction self mapping of a complete metric space has a unique fixed point.

In this paper, by unifying the concepts of p-contraction and cyclic contraction we introduce cyclic p-contraction pair and present some best proximity point results for such mappings. Therefore, our results extend, unify and generalize many fixed point and best proximity point theorems in the literature as properly. Also, we provide some illustrative and comparative examples to show the significance of our results. Finally, we provide the sufficient conditions to guarantee the existence and uniqueness of a common solution of a system of second order boundary value problems.

2 Best proximity results

We begin this section by introducing the cyclic p-contraction pair.

Definition 2. Let (X, d) be a metric space and A, B be nonempty subsets of X. Let $T : A \to B$ and $S : B \to A$ be two mappings. Then (T, S) is called a cyclic p-contraction pair if there exists $k \in (0, 1)$ such that

$$d(Tx, Sy) \leqslant k\big\{d(x, y) + \big|d(x, Tx) - d(y, Sy)\big|\big\} + (1 - k)d(A, B)$$

for all $x \in A, y \in B$. Note that if (T, S) is a cyclic p-contraction pair, then (S, T) is also a cyclic p-contraction pair.

Definition 3. (See [26].) Let (X, d) be a metric space and A, B be nonempty subsets of X. Let $\{x_n\}$, $\{z_n\}$ be two sequences in A and $\{y_n\}$ be a sequence in B. If $d(x_n, y_n) \to d(A, B)$ and $d(z_n, y_n) \to d(A, B)$ imply $d(x_n, z_n) \to 0$, then (A, B) is called to satisfy the UC-property.

It is well known that every pair of nonempty subsets A, B of a metric space (X, d) such that $d(A, B) = 0$ and also every pair of nonempty subsets A, B of a uniform convex Banach space X such that A is convex satisfy the UC-property.

The following lemma includes an important information about the UC-property.

Lemma 1. *(See [26].) Let A and B be subsets of a metric space (X,d). Assume that (A,B) has the UC-property. Let $\{x_n\}$ and $\{y_n\}$ be sequences in A and B, respectively, such that either of the following holds*:

$$\lim_{m\to\infty}\sup_{n\geqslant m} d(x_m,y_n)=d(A,B) \quad \textit{or} \quad \lim_{n\to\infty}\sup_{m\geqslant n} d(x_m,y_n)=d(A,B),$$

then $\{x_n\}$ is a Cauchy sequence.

Definition 4. (See [13].) Let X be a metric space and A, B be nonempty subsets of X. A sequence $\{x_n\}$ in $A\cup B$ with $\{x_{2n}\}\subset A$ and $\{x_{2n+1}\}\subset B$ is said to be a cyclically Cauchy sequence if for every $\varepsilon>0$, there exists an $N\in\mathbb{N}$ such that

$$d(x_n,x_m)<d(A,B)+\varepsilon$$

when n is even, m is odd and $n,m\geqslant N$.

Definition 5. (See [9].) A pair (A,B) of subsets of a metric space is said to be proximally complete if for every cyclically Cauchy sequence $\{x_n\}$ in $A\cup B$, the sequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ have convergent subsequences in A and B.

The following propositions, which show the relation between the UC-property (or boundedly compactness) and the notion of proximally completeness, play an important role to get some corollaries from our main result.

Proposition 1. (*See* [9].) *Let A and B be subsets of a metric space (X,d). Then the following hold*:

(i) *If A and B are complete subsets of X and (A,B) satisfies the UC-property, then (A,B) is proximally complete pair.*
(ii) *If A and B are boundedly compact, then (A,B) is proximally complete pair.*

Proposition 2. *Let A, B be nonempty subsets of a metric space (X,d). Let $T:A\to B$ and $S:B\to A$ be two mappings such that (T,S) is a cyclic p-contraction pair, and let $x_0\in A$. Consider the sequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ constructed as $x_{2n+1}=Tx_{2n}$, $x_{2n+2}=Sx_{2n+1}$ for all $n\in\mathbb{N}$. If there exists $n_0\in\mathbb{N}$ such that*

$$d(x_{2n_0},x_{2n_0+1})\leqslant d(x_{2n_0+1},x_{2n_0+2}),$$

then the mappings T and S have best proximity points.

Proof. Since (T,S) is cyclic p-contraction pair, then for all $n\in\mathbb{N}$, we have

$$\begin{aligned} d(Tx_{2n},Sx_{2n+1}) \leqslant{}& k\bigl\{d(x_{2n},x_{2n+1})+\bigl|d(x_{2n},Tx_{2n})-d(x_{2n+1},Sx_{2n+1})\bigr|\bigr\}\\ &+(1-k)d(A,B). \end{aligned} \tag{1}$$

Now, if there exists $n_0\in\mathbb{N}$ such that

$$d(x_{2n_0},x_{2n_0+1})\leqslant d(x_{2n_0+1},x_{2n_0+2}),$$

then from (1) we get

$$\begin{aligned}
&d(x_{2n_0+1}, x_{2n_0+2}) \\
&\quad = d(Tx_{2n_0}, Sx_{2n_0+1}) \\
&\quad \leqslant k\{d(x_{2n_0}, x_{2n_0+1}) + |d(x_{2n_0}, Tx_{2n_0}) - d(x_{2n_0+1}, Sx_{2n_0+1})|\} + (1-k)d(A,B) \\
&\quad = k\{d(x_{2n_0}, x_{2n_0+1}) + |d(x_{2n_0}, x_{2n_0+1}) - d(x_{2n_0+1}, x_{2n_0+2})|\} + (1-k)d(A,B) \\
&\quad = kd(x_{2n_0}, x_{2n_0+1}) - kd(x_{2n_0}, x_{2n_0+1}) + kd(x_{2n_0+1}, x_{2n_0+2}) + (1-k)d(A,B) \\
&\quad = kd(x_{2n_0+1}, x_{2n_0+2}) + (1-k)d(A,B)
\end{aligned}$$

and so

$$d(x_{2n_0+1}, x_{2n_0+2}) \leqslant d(A,B).$$

Then we have

$$\begin{aligned}
d(A,B) &\leqslant d(x_{2n_0}, Tx_{2n_0}) = d(x_{2n_0}, x_{2n_0+1}) \\
&\leqslant d(x_{2n_0+1}, x_{2n_0+2}) = d(x_{2n_0+1}, Sx_{2n_0+1}) \\
&\leqslant d(A,B).
\end{aligned}$$

Hence, x_{2n_0} is a best proximity point of T, and x_{2n_0+1} is a best proximity point of S. □

Remark 1. Note that if the sequence $\{x_n\}$, which mentioned in Proposition 2, satisfies $d(x_{2n_0}, x_{2n_0+1}) \leqslant d(x_{2n_0+1}, x_{2n_0+2})$ for some $n_0 \in \mathbb{N}$, then the cyclic p-contraction pair (T,S) has a best proximity point. Therefore, for the same sequence, we will investigate the condition $d(x_{2n+1}, x_{2n+2}) \leqslant d(x_{2n}, x_{2n+1})$ for all $n \in \mathbb{N}$ in the following two propositions.

Proposition 3. *Let A, B be nonempty subsets of a metric space (X,d). Let $T: A \to B$ and $S: B \to A$ be two mappings such that (T,S) is a cyclic p-contraction pair, and let $x_0 \in A$. Then for the sequence, which is constructed as in Proposition 2, we have $d(x_n, x_{n+1}) \to d(A,B)$ as $n \to \infty$.*

Proof. Since (T,S) is cyclic p-contraction pair, by considering Remark 1 we have

$$\begin{aligned}
&d(x_{2n+1}, x_{2n+2}) \\
&\quad = d(Tx_{2n}, Sx_{2n+1}) \\
&\quad \leqslant k\{d(x_{2n}, x_{2n+1}) + |d(x_{2n}, Tx_{2n}) - d(x_{2n+1}, Sx_{2n+1})|\} + (1-k)d(A,B) \\
&\quad = k\{d(x_{2n}, x_{2n+1}) + |d(x_{2n}, x_{2n+1}) - d(x_{2n+1}, x_{2n+2})|\} + (1-k)d(A,B) \\
&\quad = k\{d(x_{2n}, x_{2n+1}) + d(x_{2n}, x_{2n+1}) - d(x_{2n+1}, x_{2n+2})\} + (1-k)d(A,B) \\
&\quad = kd(x_{2n}, x_{2n+1}) + kd(x_{2n}, x_{2n+1}) - kd(x_{2n+1}, x_{2n+2}) + (1-k)d(A,B) \\
&\quad = 2kd(x_{2n}, x_{2n+1}) - kd(x_{2n+1}, x_{2n+2}) + (1-k)d(A,B)
\end{aligned}$$

and so

$$d(x_{2n+1}, x_{2n+2}) \leqslant \lambda d(x_{2n}, x_{2n+1}) + \mu d(A,B)$$

for all $n \in \mathbb{N}$, where $\lambda = 2k/(1+k) < 1$ and $\mu = (1-k)/(1+k)$. By using last inequality we have

$$\begin{aligned} d(A,B) \leqslant d(x_{2n+1}, x_{2n+2}) &\leqslant \lambda d(x_{2n}, x_{2n+1}) + \mu d(A,B) \\ &\leqslant \lambda\bigl(\lambda d(x_{2n-1}, x_{2n}) + \mu d(A,B)\bigr) + \mu d(A,B) \\ &= \lambda^2 d(x_{2n-1}, x_{2n}) + \mu d(A,B)\bigl(1+\lambda\bigr) \\ &\dots \\ &\leqslant \lambda^{2n+1} d(x_0, x_1) + \mu d(A,B)\bigl(1+\lambda+\dots+\lambda^{2n}\bigr) \\ &= \lambda^{2n+1} d(x_0, x_1) + \mu d(A,B)\left(\frac{1-\lambda^{2n+1}}{1-\lambda}\right) \\ &= \lambda^{2n+1} d(x_0, x_1) + d(A,B)\bigl(1-\lambda^{2n+1}\bigr) \end{aligned}$$

for all $n \in \mathbb{N}$. Therefore, we conclude that

$$\lim_{n\to\infty} d(x_{2n+1}, x_{2n+2}) = d(A,B).$$

Similarly, we can show that

$$\lim_{n\to\infty} d(x_{2n}, x_{2n+1}) = d(A,B).$$

Hence, we have

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = d(A,B).$$ □

The following proposition is crucial for our main result. Recall that a sequence $\{x_n\}$ in a metric space (X,d) is bounded if there exists $M > 0$ such that $d(x_n, x_m) \leqslant M$ for all $n, m \in \mathbb{N}$.

Proposition 4. *Let A, B be nonempty subsets of a metric space X. Let $T : A \to B$ and $S : B \to A$ be two mappings such that (T,S) is a cyclic p-contraction pair, and let $x_0 \in A$. Then the subsequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ of $\{x_n\}$, which is constructed as in Proposition 2, are bounded.*

Proof. Since $d(x_{2n}, x_{2n+1})$ converges to $d(A,B)$, it is sufficient to show that the sequence $\{x_{2n}\}$ is bounded. Now, we assume the contrary. Let M be a real constant with

$$M > \max\left\{\frac{k(3k+1)}{1-k^2} d(x_1, x_2) + d(A,B), d(x_1, x_2)\right\}, \tag{2}$$

where k is the contractive constant of cyclic p-contraction pair. Then since $\{x_{2n}\}$ is not bounded, there exists a natural number $n_0 > 1$ such that

$$d(x_3, x_{2n_0}) > M \quad \text{and} \quad d(x_3, x_{2n_0-2}) \leqslant M.$$

Because of cyclic p-contraction pair of (T,S), we have

$$\begin{aligned}
&M-d(A,B)+k^2d(A,B)\\
&\quad< d(x_3,x_{2n_0})-d(A,B)+k^2d(A,B)\\
&\quad= d(Tx_2,Sx_{2n_0-1})-d(A,B)+k^2d(A,B)\\
&\quad\leqslant k\bigl\{d(x_2,x_{2n_0-1})+\bigl|d(x_2,x_3)-d(x_{2n_0-1},x_{2n_0})\bigr|\bigr\}-kd(A,B)+k^2d(A,B)\\
&\quad\leqslant kd(x_2,x_{2n_0-1})+kd(x_2,x_3)-kd(A,B)+k^2d(A,B)\\
&\quad= kd(Sx_1,Tx_{2n_0-2})+kd(x_2,x_3)-kd(A,B)+k^2d(A,B)\\
&\quad\leqslant k^2d(x_1,x_{2n_0-2})+kd(x_2,x_3)+k^2d(x_1,x_2)\\
&\quad< k^2d(x_1,x_3)+k^2d(x_3,x_{2n_0-2})+kd(x_1,x_2)+k^2d(x_1,x_2)\\
&\quad\leqslant k^2d(x_1,x_2)+k^2d(x_2,x_3)+k^2d(x_3,x_{2n_0-2})+kd(x_1,x_2)+k^2d(x_1,x_2)\\
&\quad\leqslant k^2d(x_1,x_2)+k^2d(x_1,x_2)+k^2d(x_3,x_{2n_0-2})+kd(x_1,x_2)+k^2d(x_1,x_2)\\
&\quad\leqslant k^2M+\bigl(3k^3+k\bigr)d(x_1,x_2).
\end{aligned}$$

This contradicts (2). Thus, $\{x_{2n}\}$ is bounded. □

Theorem 4. *Let A, B be nonempty subsets of a metric space (X,d) such that (A,B) is a proximally complete pair. Let $T: A\to B$ and $S: B\to A$ be a cyclic p-contraction pair. Then T and S have best proximity points.*

Proof. Let x_0 be an arbitrary point in A, and let $\{x_n\}$ be a sequences constructed as $x_{2n+1}=Tx_{2n}$, $x_{2n+2}=Sx_{2n+1}$ for all $n\in$. If there exists $n_0\in\mathbb{N}$ such that

$$d(x_{2n_0},x_{2n_0+1})\leqslant d(x_{2n_0+1},x_{2n_0+2}),$$

then by Proposition 2 the mappings T and S have best proximity points. Now, assume

$$d(x_{2n+1},x_{2n+2})\leqslant d(x_{2n},x_{2n+1})$$

for all $n\in\mathbb{N}$. Then from Proposition 3 we know that

$$\lim_{n\to\infty} d(x_{2n},x_{2n+1})=d(A,B)$$

and

$$\lim_{n\to\infty} d(x_{2n+1},x_{2n+2})=d(A,B).$$

Now, we want to show that $\{x_n\}$ is a cyclically Cauchy sequence. For this, it is sufficient to show $\lim_{n,m\to\infty} d(x_{2n},x_{2m+1})=d(A,B)$. Let $n\geqslant m$. Since (T,S) is a cyclic p-contraction pair, we have

$$\begin{aligned}
&d(x_{2n+2},x_{2m+1})\\
&\quad= d(Tx_{2m},Sx_{2n+1})\\
&\quad\leqslant k\bigl\{d(x_{2m},x_{2n+1})+d(x_{2m},x_{2m+1})-d(x_{2n+1},x_{2n+2})\bigr\}+(1-k)d(A,B)\\
&\quad\leqslant k\bigl\{d(x_{2m},x_{2n+1})+d(x_{2m},x_{2m+1})-d(A,B)\bigr\}+(1-k)d(A,B)\\
&\quad= k\bigl\{d(x_{2m},x_{2n+1})+d(x_{2m},x_{2m+1})\bigr\}+(1-2k)d(A,B)
\end{aligned}$$

for all $n, m \in \mathbb{N}$ with $n \geqslant m$ and (as in the proof of Proposition 3) for all $m \in \mathbb{N}$,

$$d(x_{2m}, x_{2m+1}) \leqslant \lambda d(x_{2m-1}, x_{2m}) + \mu d(A, B),$$

where $\lambda = 2k/(1+k) < 1$ and $\mu = (1-k)/(1+k)$. Therefore, we have

$$\begin{aligned}
d(A, B) &\leqslant d(x_{2n+2}, x_{2m+1}) = d(Tx_{2m}, Sx_{2n+1}) \\
&\leqslant kd(x_{2m}, x_{2n+1}) + kd(x_{2m}, x_{2m+1}) + (1-2k)d(A, B) \\
&\leqslant k\big\{kd(x_{2m-1}, x_{2n}) + kd(x_{2m-1}, x_{2m}) + (1-2k)d(A, B)\big\} \\
&\quad + k\big\{\lambda d(x_{2m-1}, x_{2m}) + \mu d(A, B)\big\} + (1-2k)d(A, B) \\
&= k^2 d(x_{2m-1}, x_{2n}) + \big(k^2 + \lambda k\big)d(x_{2m-1}, x_{2m}) + \mu k d(A, B) \\
&\quad + (1+k)(1-2k)d(A, B) \\
&\leqslant k^3 d(x_{2m-2}, x_{2n-1}) + \big(k^3 + \lambda k^2 + \lambda^2 k\big)d(x_{2m-2}, x_{2m-1}) \\
&\quad + \mu k d(A, B) + \big(k^2 + \lambda k\big)\mu d(A, B) + \big(1 + k + k^2\big)(1-2k)d(A, B) \\
&\leqslant k^4 d(x_{2m-3}, x_{2n-2}) + \big(k^4 + \lambda k^3 + \lambda^2 k^2 + \lambda^3 k\big)d(x_{2m-3}, x_{2m-2}) \\
&\quad + \mu k d(A, B) + \big(k^2 + \lambda k\big)\mu d(A, B) + \mu\big(k^3 + \lambda k^2 + \lambda^2 k\big)d(A, B) \\
&\quad + \big(1 + k + k^2 + k^3\big)(1-2k)d(A, B).
\end{aligned}$$

Continuing this process, we have

$$\begin{aligned}
d(A, B) &\leqslant d(x_{2n+2}, x_{2m+1}) \\
&\leqslant k^{2m+1} d(x_0, x_{2n-2m+1}) \\
&\quad + k^{2m+1}\left\{1 + \frac{\lambda}{k} + \left(\frac{\lambda}{k}\right)^2 + \left(\frac{\lambda}{k}\right)^3 + \cdots + \left(\frac{\lambda}{k}\right)^{2m}\right\}d(x_0, x_1) \\
&\quad + \mu\big\{\big(k + k^2 + k^3 + \cdots + k^{2m}\big) + \lambda\big(k + k^2 + k^3 + \cdots + k^{2m-1}\big) \\
&\quad + \lambda^2\big(k + k^2 + k^3 + \cdots + k^{2m-2}\big) + \cdots + \lambda^{2m-1}k\big\}d(A, B) \\
&\quad + (1-2k)\big\{1 + k + k^2 + \cdots + k^{2m}\big\}d(A, B) \\
&\leqslant k^{2m+1} d(x_0, x_{2n-2m+1}) + k^{2m+1}\sum_{i=0}^{2m}\left(\frac{\lambda}{k}\right)^i d(x_0, x_1) \\
&\quad + \mu\left\{\sum_{i=1}^{\infty} k^i + \lambda\sum_{i=1}^{\infty} k^i + \lambda^2\sum_{i=1}^{\infty} k^i + \cdots + \lambda^{2m-1}\sum_{i=1}^{\infty} k^i\right\}d(A, B) \\
&\quad + (1-2k)\sum_{i=0}^{2m} k^i d(A, B) \\
&= k^{2m+1} d(x_0, x_{2n-2m+1}) + k^{2m+1}\frac{1 - \left(\frac{\lambda}{k}\right)^{2m+1}}{1 - \frac{\lambda}{k}}d(x_0, x_1)
\end{aligned}$$

$$
\begin{aligned}
&+\mu\frac{k}{1-k}\left\{1+\lambda+\lambda^2+\dots+\lambda^{2m-1}\right\}d(A,B)\\
&+(1-2k)\frac{1-k^{2m+1}}{1-k}d(A,B)\\
\leqslant{}& k^{2m+1}d(x_0,x_{2n-2m+1})+\frac{k^{2m+1}-\lambda^{2m+1}}{1-\frac{\lambda}{k}}d(x_0,x_1)+\frac{k}{1-k}d(A,B)\\
&+(1-2k)\frac{1-k^{2m+1}}{1-k}d(A,B).
\end{aligned}
$$

Since $\{x_n\}$ is a bounded sequence, then there exists a real number $M>0$ such that $d(x_0,x_{2n-2m+1})\leqslant M$ for all $n,m\in\mathbb{N}$ with $n\geqslant m$, and so, we get

$$
\begin{aligned}
d(A,B)&\leqslant d(x_{2n+2},x_{2m+1})\\
&\leqslant k^{2m+1}M+\frac{k^{2m+1}-\lambda^{2m+1}}{1-\frac{\lambda}{k}}d(x_0,x_1)+\frac{k}{1-k}d(A,B)\\
&\quad+(1-2k)\frac{1-k^{2m+1}}{1-k}d(A,B).
\end{aligned}
$$

Hence, we have

$$\lim_{n,m\to\infty}d(x_{2n+2},x_{2m+1})=d(A,B).$$

Now, since (A,B) is a proximally complete pair, there exist subsequences $\{x_{2n_i}\}\subset\{x_{2n}\}$ and $\{x_{2n_j+1}\}\subset\{x_{2n+1}\}$ such that $x_{2n_i}\to x^*\in A$ and $x_{2n_j+1}\to y^*\in B$. Moreover,

$$d(A,B)\leqslant d(x^*,x_{2n_i-1})\leqslant d(x^*,x_{2n_i})+d(x_{2n_i},x_{2n_i-1}).$$

Since $\lim_{i\to\infty}d(x_{2n_i},x_{2n_i-1})=d(A,B)$, we get $\lim_{i\to\infty}d(x^*,x_{2n_i-1})=d(A,B)$. Also, we have

$$
\begin{aligned}
&d(x_{2n_i},Tx^*)\\
&\quad\leqslant d(Sx_{2n_i-1},Tx^*)\\
&\quad\leqslant k\left\{d(x_{2n_i-1},x^*)+\left|d(x^*,Tx^*)-d(x_{2n_i-1},Sx_{2n_i-1})\right|\right\}+(1-k)d(A,B)\\
&\quad=k\left\{d(x_{2n_i-1},x^*)+\left|d(x^*,Tx^*)-d(x_{2n_i-1},x_{2n_i})\right|\right\}+(1-k)d(A,B)
\end{aligned}
$$

and so

$$
\begin{aligned}
d(x_{2n_i},Tx^*)\leqslant{}& k\left\{d(x_{2n_i-1},x^*)+\left|d(x^*,Tx^*)-d(x_{2n_i-1},x_{2n_i})\right|\right\}\\
&+(1-k)d(A,B).
\end{aligned}
$$

From the last inequality we get

$$
\begin{aligned}
d(x^*,Tx^*)&=\lim_{i\to\infty}d(x_{2n_i},Tx^*)\\
&\leqslant kd(A,B)+k\left|d(x^*,Tx^*)-d(A,B)\right|+(1-k)d(A,B)\\
&=kd(A,B)+kd(x^*,Tx^*)-kd(A,B)+(1-k)d(A,B).
\end{aligned}
$$

Therefore, we have $d(x^*, Tx^*) \leqslant d(A,B)$ and so $d(x^*, Tx^*) = d(A,B)$. That is, x^* is a best proximity point of T. Similarly, it can be shown that y^* is a best proximity point of S. □

By taking into account Proposition 1 we can immediately obtain the following corollaries.

Corollary 1. *Let A, B be nonempty closed subsets of complete metric space (X,d) such that (A,B) satisfies the UC-property. Assume that $T: A \to B$ and $S: B \to A$ are two mappings such that (T,S) is a cyclic p-contraction pair. Then T and S have best proximity points.*

Proof. By Proposition 1(i) we get that (A,B) is a proximally complete pair. Thus, by using Theorem 4 we say that T and S have best proximity points. □

Corollary 2. *Let A and B be nonempty closed subsets of a metric space (X,d). Assume that $T: A \to B$ and $S: B \to A$ are mappings such that (T,S) is a cyclic p-contraction pair. If A and B are boundedly compact, then the mappings T and S have best proximity points.*

Proof. From Proposition 1(ii) we know that every boundedly compact subsets pair (A,B) of a metric space is proximally complete. By using Theorem 4 we say that T and S have best proximity points. □

The following example is important to show that the best proximity point results including the conditions boundedly compactness or the UC-property in the literature cannot be applied to any cyclic mappings even if they are a kind of cyclic contraction.

Example 1. Let $X = \mathbb{R}^2$ be endowed with taxi-cab metric d, that is, for $x = (x_1, x_2)$, $y = (y_1, y_2) \in X$,

$$d(x,y) = |x_1 - y_1| + |x_2 - y_2|.$$

Let

$$A_1 = \{(a,1)\colon\ 0 \leqslant a \leqslant 1\}, \qquad A_2 = \{(a,-1)\colon\ -1 < a \leqslant 0\},$$
$$A = A_1 \cup A_2, \quad \text{and} \quad B = \big\{(b,0),(b,2)\colon\ 0 \leqslant b \leqslant 1\big\},$$

then $d(A,B) = 1$. It is clear that A is not boundedly compact. Also, the pair (A,B) and (B,A) do not satisfy the UC-property. Indeed, if we choose the sequences $x_n = (1/n, 1)$ and $z_n = (-1/n, -1)$ in A and $y_n = (0,0)$ in B for all $n \in \mathbb{N}$, then we have

$$d(x_n, y_n) \to d(A,B) \quad \text{and} \quad d(z_n, y_n) \to d(A,B),$$

but $d(x_n, z_n) \to 2$. That is, (A,B) does not satisfy the UC-property. Also, if we choose the sequences $x_n = (1/n, 2)$ and $z_n = (1/n, 0)$ in B and $y_n = (1/n, 1)$ in A for all $n \in \mathbb{N}$, then we have

$$d(x_n, y_n) \to d(B,A) \quad \text{and} \quad d(z_n, y_n) \to d(B,A),$$

but $d(x_n, z_n) \to 2$. That is, (B,A) does not satisfy the UC-property.

Now, we show that the pair (A, B) is a proximally complete pair. Indeed, let $\{x_n\}$ be a cyclically Cauchy sequence in $A \cup B$, then $\{x_{2n}\} \subset A$, $\{x_{2n+1}\} \subset B$, and for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that

$$d(x_n, x_m) < d(A, B) + \varepsilon \tag{3}$$

when n is even, m is odd and $n, m \geqslant N$. We have to show that $\{x_{2n}\}$ and $\{x_{2n+1}\}$ have convergent subsequences in A and B, respectively. First, note that since B is a compact subset of X, then $\{x_{2n+1}\}$ has a such subsequence. Now, if $\{x_{2n}\}$ has a subsequence in A_1, then $\{x_{2n}\}$ has a convergent subsequence because A_1 is a compact subset of X. Assume $\{x_{2n}\}$ has not a subsequence in A_1, then there exists $n_0 \in \mathbb{N}$ such that $x_{2n} \in A_2$ for all $n \geqslant n_0$. In this case, from equation (3) we can say that the sequence $x_{2n} \to (0, -1) \in A_2$ as $n \to \infty$.

Now, we define two mappings $T : A \to B$ and $S : B \to A$ by

$$Tx = \begin{cases} (\frac{a}{2}, 0), & x = (a, 1) \in A_1, \\ (0, 0), & x = (a, -1) \in A_2, \end{cases}$$

and

$$Sy = \begin{cases} (0, -1), & y = (b, 2), \\ (\frac{b}{2}, 1), & y = (b, 0), \end{cases}$$

for all $x \in A$, $y \in B$. It can be shown that (T, S) is a cyclic p-contraction pair with $k = 1/2$. Therefore, since all the hypotheses of Theorem 4 are satisfied, T and S have best proximity points in A and B, respectively. Besides, if we define the mapping $F : A \cup B \to A \cup B$ as

$$Fx = \begin{cases} Tx, & x \in A, \\ Sx, & x \in B, \end{cases}$$

for all $x \in A \cup B$, then F is not cyclic contraction mapping. Actually, let $x = (1/2, 1)$ and $y = (1/2, 2)$. So that, we have

$$d(Fx, Fy) = \frac{5}{4} > 1 = kd(x, y) + (1 - k)d(A, B)$$

for each $k \in (0, 1)$.

Taking into account similar contractive inequality, the following best proximity point result can be obtained for cyclic self mapping of $A \cup B$.

Corollary 3. *Let A, B be nonempty subsets of a metric space (X, d) such that (A, B) is a proximally complete pair, and let $F : A \cup B \to A \cup B$ be a cyclic mapping. Assume that there exists $k \in (0, 1)$ such that*

$$d(Fx, Fy) \leqslant k\bigl\{d(x, y) + \bigl|d(x, Fx) - d(y, Fy)\bigr|\bigr\} + (1 - k)d(A, B) \tag{4}$$

for all $x \in A$ and $y \in B$. Then there exist $z \in A$ and $w \in B$, which are best proximity points of F.

Proof. Define two mappings T and S by $Tx = Fx$ for all $x \in A$ and $Sy = Fy$ for all $y \in B$. Since F is a cyclic mapping satisfying (4), then (T, S) is a cyclic p-contraction pair. Therefore, from Theorem 4 there exist $z \in A$ and $w \in B$, which are best proximity points of F. □

Now, we present an example showing that Corollary 3 can be applied, but some existing similar results cannot.

Example 2. Let $X = \mathbb{R}$ be endowed with usual metric d, $A = [0, 1/3]$ and $B = [2/3, 1]$. Then (A, B) is a proximally complete pair, and $d(A, B) = 1/3$. Now, we define a mapping $F : A \cup B \to A \cup B$ as $Fx = 1 - x$ for all $x \in A \cup B$. Then it is clear that $F(A) \subseteq B$ and $F(B) \subseteq A$, hence, F is a cyclic mapping. Besides, we can see that inequality (4) holds for $k = 1/2$. Indeed, we have

$$\begin{aligned} d(Fx, Fy) &= d(1-x, 1-y) = \big|(1-x) - (1-y)\big| = \big|x - y\big| \\ &\leqslant \frac{1}{2}\big|x-y\big| + \frac{1}{2}2\big|x-y\big| + \frac{1}{2}d(A, B) \\ &= \frac{1}{2}d(x, y) + \frac{1}{2}\big\{\big|d(x, Fx) - d(y, Fy)\big|\big\} + \frac{1}{2}d(A, B) \\ &= kd(x, y) + k\big\{\big|d(x, Fx) - d(y, Fy)\big|\big\} + (1-k)d(A, B) \end{aligned}$$

for all $x \in A$, $y \in B$. Therefore, since all the hypotheses of Corollary 3 are satisfied, then F has at least two best proximity points in $A \cup B$. Actually, $x = 1/3$ and $y = 2/3$ are best proximity points of F.

Now, we show that some earlier best proximity point results cannot be applied to this example. For $x = 0$ and $y = 1$, we have

$$d(Fx, Fy) = d(x, y) = d(x, Fx) = d(y, Fy) = 1 \quad \text{and} \quad d(A, B) = \frac{1}{3}.$$

Therefore, we cannot find $k \in (0, 1)$ satisfying

$$d(Fx, Fy) \leqslant \frac{k}{3}\big\{d(x, y) + d(x, Fx) + d(y, Fy)\big\} + (1-k)d(A, B)$$

or

$$d(Fx, Fy) \leqslant k \max\left\{d(x, y), \frac{1}{2}\big[d(x, Fx) + d(y, Fy)\big]\right\} + (1-k)d(A, B).$$

Hence, Theorems 10 and 12, which are main results of [12], cannot be applied to this example.

3 Applications

Let (X, d) be a metric space and A, B be two subsets of X. If $A \cap B \neq \emptyset$, then $d(A, B) = 0$. In this case, a best proximity result turns to a fixed point result. In this context, we can obtain the following:

Corollary 4. *Let (X,d) be a complete metric space, A, B be two closed subsets of X and $T : A \to B$ and $S : B \to A$ be two mappings. If there exists $k \in (0,1)$ such that*

$$d(Tx, Sy) \leqslant k\big\{d(x,y) + \big|d(x,Tx) - d(y,Sy)\big|\big\}$$

for all $x \in A$ and $y \in B$, then T and S have a unique common fixed point in $A \cap B$.

Corollary 5. *Let (X,d) be a complete metric space and $T,S : X \to X$ be two mappings. If there exists $k \in (0,1)$ such that*

$$d(Tx, Sy) \leqslant k\big\{d(x,y) + \big|d(x,Tx) - d(y,Sy)\big|\big\}$$

for all $x,y \in X$, then T and S have a unique common fixed point in X.

Corollary 6. *Let (X,d) be a complete metric space and $T : X \to X$ be a self mapping. If there exists $k \in (0,1)$ such that*

$$d(Tx, Ty) \leqslant k\big\{d(x,y) + \big|d(x,Tx) - d(y,Ty)\big|\big\}$$

for all $x,y \in X$. Then the mapping T has a unique fixed point in X.

Corollary 7. *Let (X,d) be a complete metric space, A, B be two closed subsets of X and $F : A \cup B \to A \cup B$ be a mapping. Assume that F is a cyclic mapping, that is, $F(A) \subseteq B$ and $F(B) \subseteq A$. If there exists $k \in (0,1)$ such that*

$$d(Fx, Fy) \leqslant k\big\{d(x,y) + \big|d(x,Fx) - d(y,Fy)\big|\big\}$$

for all $x \in A$ and $y \in B$, then F has a unique fixed point in $A \cap B$.

Now, by considering the Corollary 5 we present an existence and uniqueness theorem for common solution of a system of second order two point boundary value problem as follows:

$$-\frac{\mathrm{d}^2u}{\mathrm{d}t^2} = f\big(t, u(t)\big), \quad t \in [0,1], \qquad u(0) = u(1) = 0, \tag{5}$$

$$-\frac{\mathrm{d}^2u}{\mathrm{d}t^2} = g\big(t, u(t)\big), \quad t \in [0,1], \qquad u(0) = u(1) = 0, \tag{6}$$

where $f,g : [0,1] \times \mathbb{R} \to \mathbb{R}$ are continuous functions. By considering some certain conditions on the function f many existence results provided for problem (5) in the literature (see [3,10,24,27]). Here, we will consider a common type condition on f and g, we provide a new existence and uniqueness result for common solution of the system.

By considering the Green's function defined as

$$G(t,s) = \begin{cases} t(1-s), & 0 \leqslant t \leqslant s \leqslant 1, \\ s(1-t), & 0 \leqslant s \leqslant t \leqslant 1, \end{cases}$$

we can see that problems (5) and (6) are equivalent to the integral equations

$$u(t) = \int_0^1 G(t,s) f\bigl(s, u(s)\bigr)\,\mathrm{d}s, \quad t \in [0,1], \tag{7}$$

and

$$u(t) = \int_0^1 G(t,s) g\bigl(s, u(s)\bigr)\,\mathrm{d}s, \quad t \in [0,1], \tag{8}$$

respectively. Therefore, $u \in C^2[0,1]$ is a common solution of (5) and (6) if and only if it is a common solution of (7) and (8). Hence, the existence of common solution of (5) and (6) can be considered as the existence of common fixed point of the operators T and S defined on $X = C[0,1]$ by

$$Tu(t) = \int_0^1 G(t,s) f\bigl(s, u(s)\bigr)\,\mathrm{d}s, \qquad Su(t) = \int_0^1 G(t,s) g\bigl(s, u(s)\bigr)\,\mathrm{d}s, \tag{9}$$

where $X = C[0,1]$ is the space of all continuous real-valued functions defined on $[0,1]$. It is clear that $\int_0^1 G(t,s)\,\mathrm{d}s = t(1-t)/2$ and thus $\sup_{t\in[0,1]} \int_0^1 G(t,s)\,\mathrm{d}s = 1/8$.

Consider the uniform metric d_∞ on X, that is,

$$d_\infty(u,v) = \|u - v\| = \sup\bigl\{\bigl|u(t) - v(t)\bigr|:\ t \in [0,1]\bigr\},$$

then it is well known that the space (X, d_∞) is complete.

Theorem 5. *Suppose the following conditions hold*:

(i) *there exists a continuous function* $p : [0,1] \to [0,\infty)$ *satisfying*

$$\bigl|f\bigl(t,u(t)\bigr) - g\bigl(t,v(t)\bigr)\bigr| \leqslant p(t)\bigl\{\bigl|u(t) - v(t)\bigr| + \bigl|\|u - Tu\| - \|v - Sv\|\bigr|\bigr\},$$

(ii) *there exists* $k < 1$ *such that* $\int_0^1 G(t,s)p(s)\,\mathrm{d}s \leqslant k$.

Then the system of second order two point boundary value problem given by (5) *and* (6) *has a unique common solution.*

Remark 2. Note that if $\max_{s\in[0,1]} p(s) \leqslant 8k$ for $k < 1$, we have $\int_0^1 G(t,s)p(s)\,\mathrm{d}s \leqslant k$.

Proof of Theorem 5. Consider the complete metric space (X, d_∞) and the operators T and S on X given by (9). Then for any $u, v \in X$ and $t \in [0,1]$, we have

$$\begin{aligned}&\bigl|Tu(t) - Sv(t)\bigr| \\ &\quad = \left|\int_0^1 G(t,s) f\bigl(s,u(s)\bigr)\,\mathrm{d}s - \int_0^1 G(t,s) g\bigl(s,v(s)\bigr)\,\mathrm{d}s\right|\end{aligned}$$

$$\leqslant \int_0^1 G(t,s)\bigl|f(s,u(s)) - g\bigl(s,v(s)\bigr)\bigr|\,\mathrm{d}s$$
$$\leqslant \int_0^1 G(t,s)p(s)\bigl\{\bigl|u(s)-v(s)\bigr| + \bigl|\|u-Tu\| - \|v-Sv\|\bigr|\bigr\}\,\mathrm{d}s$$
$$\leqslant \bigl\{\|u-v\| + \bigl|\|u-Tu\| - \|v-Sv\|\bigr|\bigr\}\int_0^1 G(t,s)p(s)\,\mathrm{d}s$$
$$\leqslant k\bigl\{\|u-v\| + \bigl|\|u-Tu\| - \|v-Sv\|\bigr|\bigr\}.$$

Hence, we have

$$\|Tu-Tv\| \leqslant k\bigl\{\|u-v\| + \bigl|\|u-Tu\| - \|v-Sv\|\bigr|\bigr\}$$

or, equivalently,

$$d_\infty(Tu,Tv) \leqslant k\bigl\{d_\infty(u,v) + \bigl|d_\infty(u,Tu) - d_\infty(v,Sv)\bigr|\bigr\}.$$

Therefore, the contractive condition of Corollary 5 is satisfied. Hence, the operators T and S have a unique common fixed point, and thus the system of second order two point boundary value problem given by (5) and (6) has a unique common solution. □

4 Conclusion

Unifying the concepts of p-contraction and cyclic contraction, we introduce cyclic p-contraction pair and present some best proximity point results for such mappings. We also provide the sufficient conditions to guarantee the existence and uniqueness of a common solution of a system of second order boundary value problems by taking into account our theoretical results.

Acknowledgment. We would like to thank editor and all anonymous reviewers for their comments, which help to improve the quality of this paper.

References

1. A. Abkar, N. Moezzifar, A. Azizi, N. Shahzad, Best proximity point theorems for cyclic generalized proximal contractions, *Fixed Point Theory Appl.*, **2016**(1):66, 2016, `https://doi.org/10.1186/s13663-016-0557-9`.
2. M.A. Alghamdi, N. Shahzad, F. Vetro, Best proximity points for some classes of proximal contractions, *Abstr. Appl. Anal.*, **2013**, 2013, `https://doi.org/10.1155/2013/713252`.
3. H. Aydi, E. Karapinar, Fixed point results for generalized α-ψ-contractions in metric-like spaces and applications, *Electron. J. Differ. Equ.*, **2015**:133, 2015.

4. H. Aydi, H. Lakzian, Z.D. Mitrović, S. Radenović, Best proximity points of MT-cyclic contractions with property UC, *Numer. Funct. Anal. Optim.*, **41**(7):871–882, 2020, https://doi.org/10.1080/01630563.2019.1708390.
5. S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, *Fundam. Math.*, **3**(1):133–181, 1922.
6. S.S. Basha, Extensions of Banach's contraction principle, *Numer. Funct. Anal. Optim.*, **31**(5): 569–576, 2010, https://doi.org/10.1080/01630563.2010.485713.
7. S.S. Basha, N. Shahzad, C. Vetro, Best proximity point theorems for proximal cyclic contractions, *J. Fixed Point Theory Appl.*, **19**(4):2647–2661, 2017, https://doi.org/10.1007/s11784-017-0447-8.
8. A.A. Eldred, P. Veeramani, Existence and convergence of best proximity points, *J. Math. Anal. Appl.*, **323**(2):1001–1006, 2006, https://doi.org/10.1016/j.jmaa.2005.10.081.
9. R. Espínola, G.S.R Kosuru, P. Veeramani, Pythagorean property and best-proximity point theorems, *J. Optim. Theory Appl.*, **164**(2):534–550, 2015, https://doi.org/10.1007/s10957-014-0583-x.
10. M. Gabeleh, S.P. Moshokoa, C. Vetro, Cyclic (noncyclic) ϕ-condensing operator and its application to a system of differential equations, *Nonlinear Anal. Model. Control*, **24**(6):985–1000, 2019, https://doi.org/10.15388/NA.2019.6.8.
11. M. Gabeleh, C. Vetro, A note on best proximity point theory using proximal contractions, *J. Fixed Point Theory Appl.*, **20**(4):149, 2018, https://doi.org/10.1007/s11784-018-0624-4.
12. E. Karapınar, Best proximity points of cyclic mappings, *Appl. Math. Lett.*, **25**(11):1761–1766, 2012, https://doi.org/10.1016/j.aml.2012.02.008.
13. S. Karpagam, S. Agrawal, Best proximity point theorems for p-cyclic Meir–Keeler contractions, *Fixed Point Theory Appl.*, **2009**(1):197308, 2009, https://doi.org/10.1155/2009/197308.
14. K. Khammahawong, P. Sa Ngiamsunthorn, P. Kumam, On best proximity points for multivalued cyclic F-contraction mappings, *Int. J. Nonlinear Anal. Appl.*, **7**(2):363–374, 2016, https://doi.org/10.22075/ijnaa.2017.2322.
15. W.K. Kim, K.H. Lee, Existence of best proximity pairs and equilibrium pairs, *J. Math. Anal. Appl.*, **316**(2):433–446, 2006, https://doi.org/10.1016/j.jmaa.2006.07.040.
16. W.A. Kirk, P.S. Srinivasan, P. Veeramani, Fixed points for mappings satisfying cyclical contractive conditions, *Fixed Point Theory*, **4**(1):79–89, 2003.
17. P. Kumam, C. Mongkolkeha, Global optimization for quasi-noncyclic relatively nonexpansive mappings with application to analytic complex functions, *Mathematics*, **7**(1):46, 2019, https://doi.org/10.3390/math7010046.
18. C. Mongkolkeha, Y.J. Cho, P. Kumam, Best proximity points for Geraghty's proximal contraction mappings, *Fixed Point Theory Appl.*, **2013**(1):180, 2013, https://doi.org/10.1186/1687-1812-2013-180.
19. C. Mongkolkeha, P. Kumam, Some common best proximity points for proximity commuting mappings, *Optim. Lett.*, **7**(8):1825–1836, 2013, https://doi.org/10.1007/s11590-012-0525-1.

20. H.K. Nashine, P. Kumam, C. Vetro, Best proximity point theorems for rational proximal contractions, *Fixed Point Theory Appl.*, **2013**(1):95, 2013, https://doi.org/10.1186/1687-1812-2013-95.
21. O. Popescu, A new type of contractive mappings in complete metric spaces, *Bull. Transilv. Univ. Braşov, Ser. III, Math. Inform. Phys.*, **1**(50):479–482, 2008.
22. V.S. Raj, A best proximity point theorem for weakly contractive non-self-mappings, *Nonlinear Anal., Theory Methods Appl.*, **74**(14):4804–4808, 2011, https://doi.org/10.1016/j.na.2011.04.052.
23. S. Sahmim, A. Felhi, H. Aydi, Convergence and best proximity points for generalized contraction pairs, *Mathematics*, **7**(2):176, 2019, https://doi.org/10.3390/math7020176.
24. P. Saipara, K. Khammahawong, P. Kumam, Fixed-point theorem for a generalized almost Hardy–Rogers-type F-contraction on metric-like spaces, *Math. Methods Appl. Sci.*, **42**(17):5898–5919, 2019, https://doi.org/10.1002/mma.5793.
25. S.A. Shukri, A.R. Khan, Best proximity points in partially ordered metric spaces, *Adv. Fixed Point Theory*, **8**(2):118–130, 2018.
26. T. Suzuki, M. Kikkawa, C. Vetro, The existence of best proximity points in metric spaces with the property UC, *Nonlinear Anal., Theory Methods Appl.*, **71**(7-8):2918–2926, 2009, https://doi.org/10.1016/j.na.2009.01.173.
27. F. Vetro, Fixed point for α-θ-ϕ-contractions and first-order periodic differential problem, *RACSAM, Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat.*, **113**(3):1823–1837, 2019, https://doi.org/10.1007/s13398-018-0586-9.