Nonlinear Analysis: Modelling and Control, Vol. 26, No. 2, 334–348
https://doi.org/10.15388/namc.2021.26.22357

Vilnius University Press

Existence theorem for integral inclusions by a fixed point theorem for multivalued implicit-type contractive mappings

Muhammad Usman Ali[a], **Ariana Pitea**[b]

[a]Department of Mathematics,
COMSATS University Islamabad, Attock Campus,
Attock, Pakistan
muh_usman_ali@yahoo.com

[b]Department of Mathematics and Informatics,
University Politehnica of Bucharest, Romania
arianapitea@yahoo.com

Received: January 26, 2020 / **Revised:** August 13, 2020 / **Published online:** March 1, 2021

Abstract. In this article, we introduce fixed point theorems for multivalued mappings satisfying implicit-type contractive conditions based on a special form of simulation functions. We also provide an application of our result in integral inclusions. Our outcomes generalize/extend many existing fixed point results.

Keywords: fixed point, implicit simulation function, integral inclusions, multivalued mappings.

1 Introduction

The contraction mapping principle proposed by Banach is an important and significant inception in functional analysis due to its applicability in other areas of mathematics and applied sciences. With the passage of time, this result has become a solid base for metric fixed point theory. Thus, several researchers generalized it by stating fixed point, common fixed point, coincidence point, couple fixed point theorems regarding mappings satisfying certain type of contractive conditions on metric spaces or on various abstract spaces. In [31], nonlinear contractions in ordered metric spaces are studied in their cyclic form. In [13], metric spaces endowed with partial order proved to be a suitable setting to develop fixed point theorems for adequate mappings and also an application regarding the existence and uniqueness of a solution to a periodic boundary value problem. Generalized weak Berinde contractions on partial metric spaces are studied in [30]. In [15], a quasicontractivity-type condition, which entails the conclusions from Banach principle, is presented. In [34], a Banach-type condition is studied in connection with the completeness of the underlying metric space. [16] and [32] refer to some Prešić-type generalization of the Banach principle. Fixed point properties in the context of $\alpha{-}\psi$-contractions are developed in [25]. In [8], closed multivalued mappings are in view with respect to $\alpha{-}\phi$-contractive conditions. In [9], weakly compatible mappings, which satisfy an implicit relation, are studied. In [12] and [14], it is stated a generalization of some weak contraction principle. In [10], b-metric-like spaces feature common fixed point properties. In [29], cone b-metric spaces are an adequate setting to develop generalized Hardy–Rogers-type contractions. [20] focuses on Mizoguchi–Takahashi-type fixed point theorems. In [23], approximate fixed point theorems are stated in the context of α-contractive mappings. The α-admissibility was used to prove interesting fixed point results in [2, 26, 28] or [27]. In [35], implicit contraction mappings are studied. Later on, the combination of metric fixed point theory and optimization theory enabled discussions on best proximity points of nonself mappings satisfying certain types of proximal contractive conditions on metric spaces or on abstract spaces. Browder theorems are extended in [17]. In [1], multivalued operators are studied from the point of view of their best proximity points. [3] has in view Kakutani multimappings, while [4] equilibrium pairs for finite families of multivalued mappings are presented. Work [7] is dedicated to the study of proximal contractions by means of suitable simulation functions. Controlled contractions are used in [5] in order to obtain best proximity properties. [6] is devoted to best proximity results for Prešić-type operators. Generalized proximal contractive mappings are developed in [11], while [18] refers to global optimal solutions. The existence of best proximity points for generalized classes of contractions is performed in [21]. Applications of best proximity points associated with $\alpha{-}\psi$-proximal contractions are presented in [19]. Hyperconvex spaces proved to be an adequate framework to develop best proximity theorems for mappings endowed with suitable continuity properties in [22].

2 Preliminaries

In this article, we will use the following type of implicit functions presented in [7]. Here κ_ψ represents the set of functions $\kappa : (\mathbb{R}^+)^4 \to \mathbb{R}^+ = [0, \infty)$ endowed with the following properties:

(K1) κ is continuous and nondecreasing in each coordinate;
(K2) If $l \geqslant j$ and $l \leqslant \kappa(l, j, l, l)$, then $l = 0$;
(K3) If $l < j$ and $l \leqslant \kappa(j, j, l, j)$, then $l \leqslant \psi(j)$, where $\psi : [0, \infty) \to [0, \infty)$ is a nondecreasing mapping with $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t \geqslant 0$.

Definition 1. (See [7].) A mapping $\chi : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is known as an implicit simulation function with respect to κ_ψ if the following conditions hold:

(C1) $\chi(al, \kappa(m, n, o, p)) \leqslant \kappa(m, n, o, p) - al$ for any $a, m, n, o, p \geqslant 0$;
(C2) $\chi(j, \kappa(0, 0, j, j/2)) \geqslant a$ if and only if $\kappa(0, 0, j, j/2) - j \geqslant a$ for any real number a;
(C3) $\chi(j, \kappa(0, 0, j, j/2)) \geqslant 0$ implies $j = 0$.

The next example illustrates such a kind of mappings.

Example 1. (See [7].) Define the mappings $\chi : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ and $\kappa : (\mathbb{R}^+)^4 \to \mathbb{R}^+ = [0, \infty)$,

$$\chi(l, j) = j - l \quad \text{and} \quad \kappa(m, n, o, p) = \zeta \max\{m, n, o, p\},$$

where $\zeta \in [0, 1)$. One can check that the above defined χ is an implicit simulation function with respect to the above defined function κ.

The following result was presented in [7] (along the paper, $\mathbb{N} = \{1, 2, \dots\}$).

Theorem 1. (*See* [7].) *Let T be a mapping from a complete metric space (X, d_m) into itself, which satisfies*

$$\chi\biggl(\alpha(j,l)d_m(Tj,Tl), \kappa\biggl(d_m(j,l), d_m(j,Tj), d_m(l,Tl), \frac{d_m(l,Tj)+d_m(j,Tl)}{2}\biggr)\biggr) \geqslant 0$$

for all $j, l \in X$, where χ is an implicit simulation function with respect to κ_ψ. Further, assume that the following conditions hold:

(i) *T is α-admissible, that is, for $j, l \in X$, $\alpha(j,l) \geqslant 1$ implies $\alpha(Tj,Tl) \geqslant 1$;*
(ii) *There exists $j_1 \in X$ satisfying $\alpha(j_1, Tj_1) \geqslant 1$;*
(iii) *For all sequences $\{j_n\}$ in X with $\alpha(j_n, j_{n+1}) \geqslant 1$, $n \in \mathbb{N}$ and $j_n \to j$, we have $\alpha(j_n, j) \geqslant 1$ for each $n \in \mathbb{N}$.*

Then T has a fixed point.

Consider (X, d_m) a metric space and $\mathrm{CL}(X)$ the collection of all nonempty closed subsets of X. For $J \in \mathrm{CL}(X)$ and $l \in X$, $d_m(l, J) = \inf\{d_m(l, j),\ j \in J\}$. For each $J, L \in \mathrm{CL}(X)$, let

$$H_m(J, L) = \begin{cases} \max\{\sup_{j\in J} d_m(j, L), \sup_{l \in L} d_m(l, J)\} & \text{if the maximum exists;} \\ \infty & \text{otherwise.} \end{cases}$$

The function H_m is known as the generalized Hausdorff metric induced by d_m.

3 Main results

Here Γ represents the set of all functions $\eta : (\mathbb{R}^+)^5 \to \mathbb{R}^+ = [0, \infty)$ having the following properties:

(E1) $\eta(m, n, o, p, q) = 0$ if and only if at least one of m, n, o, p, q is zero;
(E2) η is continuous.

We now present the first result of this section.

Theorem 2. *Let X be a nonempty set and d_m a distance on it. Assume that T is a mapping from X into $\mathrm{CL}(X)$ that satisfies*

$$\chi\biggl(\alpha(j,l)H_m(Tj,Tl), \kappa\biggl(d_m(j,l), d_m(j,Tj), d_m(l,Tl), \frac{d_m(l,Tj)+d_m(j,Tl)}{2}\biggr)\biggr) + L\eta\bigl(d_m(j,l), d_m(j,Tj), d_m(l,Tl), d_m(j,Tl), d_m(l,Tj)\bigr) > 0 \tag{1}$$

for all $j, l \in X$ *with* $j \neq l$, *where* χ *is an implicit simulation function with respect to* κ_ψ, $\eta \in \Gamma$ *and* $L \geqslant 0$. *Further, assume that the following conditions hold*:

(i) T *is* α*-admissible, that is, for* $j, l \in X$, $\alpha(j, l) \geqslant 1$ *implies* $\alpha(a, b) \geqslant 1$ *for all* $a \in Tj$, $b \in Tl$;

(ii) *There exist* $j_1 \in X$ *and* $j_2 \in Tj_1$ *satisfying* $\alpha(j_1, j_2) \geqslant 1$;

(iii) *For all sequences* $\{j_n\}$ *in* X *with* $\alpha(j_n, j_{n+1}) \geqslant 1$, $n \in \mathbb{N}$ *and* $j_n \to j$, *we have* $\alpha(j_n, j) \geqslant 1$ *for each* $n \in \mathbb{N}$.

Then T *has a fixed point.*

Proof. Hypothesis (ii) ensures the existence of two elements $j_1, j_2 \in X$ with $\alpha(j_1, j_2) \geqslant 1$ and $j_2 \in Tj_1$. Without loss of generality, we may presume that $j_1 \neq j_2$. Inequality (1) and condition (C1) imply that

$$\begin{aligned}
&\kappa\left(d_m(j_1, j_2), d_m(j_1, Tj_1), d_m(j_2, Tj_2), \frac{d_m(j_2, Tj_1) + d_m(j_1, Tj_2)}{2}\right) \\
&\quad - \alpha(j_1, j_2) H_m(Tj_1, Tj_2) \\
&\quad + L\eta\big(d_m(j_1, j_2), d_m(j_1, Tj_1), d_m(j_2, Tj_2), d_m(j_1, Tj_2), d_m(j_2, Tj_1)\big) > 0. \qquad (2)
\end{aligned}$$

As $d_m(j_2, Tj_1) = 0$, we get by the use of (E1) that

$$\eta\big(d_m(j_1, j_2), d_m(j_1, Tj_1), d_m(j_2, Tj_2), d_m(j_1, Tj_2), d_m(j_2, Tj_1)\big) = 0.$$

Thus, inequality (2) implies

$$\begin{aligned}
&H_m(Tj_1, Tj_2) \\
&\quad \leqslant \alpha(j_1, j_2) H_m(Tj_1, Tj_2) \\
&\quad < \kappa\left(d_m(j_1, j_2), d_m(j_1, Tj_1), d_m(j_2, Tj_2), \frac{d_m(j_2, Tj_1) + d_m(j_1, Tj_2)}{2}\right).
\end{aligned}$$

There is $\varepsilon_1 > 0$ such that

$$\begin{aligned}
&H_m(Tj_1, Tj_2) + \varepsilon_1 \\
&\quad \leqslant \kappa\left(d_m(j_1, j_2), d_m(j_1, Tj_1), d_m(j_2, Tj_2), \frac{d_m(j_2, Tj_1) + d_m(j_1, Tj_2)}{2}\right). \qquad (3)
\end{aligned}$$

Since $\varepsilon_1 > 0$, then we have an element $j_3 \in Tj_2$ satisfying

$$d_m(j_2, j_3) \leqslant H_m(Tj_1, Tj_2) + \varepsilon_1. \qquad (4)$$

If $j_2 = j_3$, there is nothing left to be proved. Therefore, we may consider $j_2 \neq j_3$. By using relations (3), (4) and the nondecreasing behavior of κ we get

$$d_m(j_2, j_3) \leqslant \kappa\left(d_m(j_1, j_2), d_m(j_1, j_2), d_m(j_2, j_3), \frac{d_m(j_1, j_2) + d_m(j_2, j_3)}{2}\right). \qquad (5)$$

We now claim that $d_m(j_2, j_3) < d_m(j_1, j_2)$. Suppose this inequality does not hold; then we have $d_m(j_2, j_3) \geqslant d_m(j_1, j_2)$. By using this in inequality (5) it follows

$$d_m(j_2, j_3) \leqslant \kappa\big(d_m(j_2, j_3), d_m(j_1, j_2), d_m(j_2, j_3), d_m(j_2, j_3)\big).$$

From this inequality and condition (K2) we get $d_m(j_2, j_3) = 0$, which is not possible due to our assumption. Thus, $d_m(j_2, j_3) < d_m(j_1, j_2)$. Hence, inequality (5) yields

$$d_m(j_2, j_3) \leqslant \kappa\bigl(d_m(j_1, j_2), d_m(j_1, j_2), d_m(j_2, j_3), d_m(j_1, j_2)\bigr).$$

This relation, together with condition (K3), compels

$$d_m(j_2, j_3) \leqslant \psi\bigl(d_m(j_1, j_2)\bigr). \tag{6}$$

Note that hypothesis (i) compels that $\alpha(j_2, j_3) \geqslant 1$.

Continuing the proof after the above pattern, we get a sequence $\{j_n\}$ satisfying $j_{n+1} \in Tj_n$, $\alpha(j_n, j_{n+1}) \geqslant 1$ and $d_m(j_{n+1}, j_{n+2}) < \psi(d_m(j_n, j_{n+1})) < d_m(j_n, j_{n+1})$ for $n \in \mathbb{N}$. Having in mind the monotone behavior of ψ and inequality (6), we are led to

$$d_m(j_{n+1}, j_{n+2}) \leqslant \psi^n\bigl(d_m(j_1, j_2)\bigr) \quad \text{for each } n \in \mathbb{N}. \tag{7}$$

Next, we will prove that $\{j_n\}$ is a Cauchy sequence. Consider the natural numbers q, p, $q > p$. By using the triangular inequality and relation (7) we obtain

$$\begin{aligned} d_m(j_q, j_p) &\leqslant \sum_{e=p}^{q-1} d_m\bigl(j_e, j_{e+1}\bigr) \leqslant \sum_{e=p}^{q-1} \psi^e\bigl(d_m(j_1, j_2)\bigr) \\ &\leqslant \sum_{e=p}^{\infty} \psi^e\bigl(d_m(j_1, j_2)\bigr) \to 0. \end{aligned}$$

This shows that $\{j_n\}$ is a Cauchy sequence in X. By the completeness of X there is an element $j^* \in X$ so that $j_n \to j^*$. Hypothesis (iii) implies $\alpha(j_n, j^*) \geqslant 1$ for each $n \in \mathbb{N}$. Without loss of generality, we may assume that $j_n \neq j^*$, $n \in \mathbb{N}$. From inequality (1), for all $n \in \mathbb{N}$, it follows that

$$\begin{aligned} &\chi\Biggl(\alpha(j_n, j^*)H_m(Tj_n, Tj^*), \kappa\Biggl(d_m(j_n, j^*), d_m(j_n, Tj_n), d_m(j^*, Tj^*), \\ &\quad \frac{d_m(j^*, Tj_n) + d_m(j_n, Tj^*)}{2}\Biggr)\Biggr) \\ &\qquad + L\eta\bigl(d_m(j_n, j^*), d_m(j_n, Tj_n), d_m(j^*, Tj^*), d_m(j_n, Tj^*), d_m(j^*, Tj_n)\bigr) > 0. \end{aligned}$$

By using hypothesis (C1) and the fact that $\alpha(j_n, j^*) \geqslant 1$ in the above inequality we get

$$\begin{aligned} &d_m(j_{n+1}, Tj^*) \\ &\quad \leqslant \alpha(j_n, j^*)H_m(Tj_n, Tj^*) \\ &\quad < \kappa\Biggl(d_m(j_n, j^*), d_m(j_n, Tj_n), d_m(j^*, Tj^*), \frac{d_m(j^*, Tj_n) + d_m(j_n, Tj^*)}{2}\Biggr) \\ &\qquad + L\eta\bigl(d_m(j_n, j^*), d_m(j_n, Tj_n), d_m(j^*, Tj^*), d_m(j_n, Tj^*), d_m(j^*, Tj_n)\bigr) \\ &\quad \leqslant \kappa\Biggl(d_m(j_n, j^*), d_m(j_n, j_{n+1}), d_m(j^*, Tj^*), \frac{d_m(j^*, j_{n+1}) + d_m(j_n, Tj^*)}{2}\Biggr) \\ &\qquad + L\eta\bigl(d_m(j_n, j^*), d_m(j_n, Tj_n), d_m(j^*, Tj^*), d_m(j_n, Tj^*), d_m(j^*, Tj_n)\bigr). \end{aligned}$$

Taking the limit $n \to \infty$ leads us to

$$d_m(j^*, Tj^*) \leqslant \kappa\left(0, 0, d_m(j^*, Tj^*), \frac{d_m(j^*, Tj^*)}{2}\right).$$

From the above inequality and (C2) we get

$$\chi\left(d_m(j^*, Tj^*), \kappa\left(0, 0, d_m(j^*, Tj^*), \frac{d_m(j^*, Tj^*)}{2}\right)\right) \geqslant 0.$$

By using condition (C3) it follows $d_m(j^*, Tj^*) = 0$, that is, $j^* \in Tj^*$. $\square$

The next result can be obtained from the above one by considering $\alpha(j, l) = 1$ for all $j, l \in X$.

Corollary 1. *Let T be a mapping from a complete metric space (X, d_m) into $\mathrm{CL}(X)$, which satisfies*

$$\chi\left(H_m(Tj, Tl), \kappa\left(d_m(j, l), d_m(j, Tj), d_m(l, Tl), \frac{d_m(l, Tj) + d_m(j, Tl)}{2}\right)\right) + L\eta\bigl(d_m(j, l), d_m(j, Tj), d_m(l, Tl), d_m(j, Tl), d_m(l, Tj)\bigr) > 0 \tag{8}$$

for all $j, l \in X$ with $j \neq l$, where χ is an implicit simulation function with respect to κ_ψ, $\eta \in \Gamma$ and $L \geqslant 0$. Then T has a fixed point.

Example 2. Let $X = C[0, 1]$ be the collection of all continuous functions from $[0, 1]$ into $\mathbb{R}$ with the metric $d(u, l) = \max_{p \in [0,1]} |u(p) - l(p)|$. Define an operator T as

$$Tu = \left\{ \int_0^1 (1 - p) u(p)\, \mathrm{d}p \right\} \quad \text{for all } u \in X.$$

° Then we have

$$\begin{aligned} H_m(Tu, Tl) &= \max_{p \in [0,1]} \left| \int_0^1 (1 - p) u(p)\, \mathrm{d}p - \int_0^1 (1 - p) l(p)\, \mathrm{d}p \right| \\ &\leqslant \max_{p \in [0,1]} \left| \int_0^1 (1 - p)\, \mathrm{d}p \right| \max_{p \in [0,1]} \bigl| u(p) - l(p) \bigr| \\ &= \frac{1}{2} d(u, l) < \frac{2}{3} d(u, l), \quad u \neq l. \end{aligned}$$

Thus, by taking $\chi(l, j) = j - l$, $\kappa(m, n, o, p) = (2/3)m$ and $L = 0$ in inequality (8) we get the above inequality. Thus, by Theorem 2 and Corollary 1 the mapping T has a fixed point.

In the following theorem, we will use b-metric spaces, where the triangle inequality of metric spaces is replaced by one in which the right-hand side of the classical triangle inequality is multiplied by a constant s, $s \geqslant 1$. Here we denote by κ_{ψ_s} the collection of functions $\kappa : (\mathbb{R}^+)^4 \to \mathbb{R}^+ = [0, \infty)$, which satisfy (K1), (K2) and (K3) with a nondecreasing mapping $\psi : [0, \infty) \to [0, \infty)$ such that $\sum_{n=1}^{\infty} s^n \psi^n(t) < \infty$ for all $t \geqslant 0$, where $s \geqslant 1$.

Theorem 3. *Let X be a nonempty set and b_m a continuous b-metric so that (X, b_m) is complete. Let T be a mapping from X into $\mathrm{CL}(X)$ that satisfies*

$$\chi\biggl(\alpha(j,l)H_{b_m}(Tj,Tl), \kappa\biggl(b_m(j,l), b_m(j,Tj), b_m(l,Tl), \frac{b_m(l,Tj)+b_m(j,Tl)}{2s}\biggr)\biggr) \\ + L\eta\bigl(b_m(j,l), b_m(j,Tj), b_m(l,Tl), b_m(j,Tl), b_m(l,Tj)\bigr) \geqslant 0 \tag{9}$$

for all $j, l \in X$, where χ is an implicit simulation function with respect to κ_{ψ_s}, $\eta \in \Gamma$ and $L \geqslant 0$. Further, assume that the following conditions hold:

(i) *T is α_s-admissible, that is, for $j, l \in X$, $\alpha(j,l) \geqslant s$ implies $\alpha(a,b) \geqslant s$ for each $a \in Tj$, $b \in Tl$*;
(ii) *There exist $j_1 \in X$ and $j_2 \in Tj_1$ satisfying $\alpha(j_1, j_2) \geqslant s$*;
(iii) *For each sequence $\{j_n\}$ in X with $\alpha(j_n, j_{n+1}) \geqslant s$, $n \in \mathbb{N}$ and $j_n \to j$, we have $\alpha(j_n, j) \geqslant s$ for all $n \in \mathbb{N}$.*

Then T has a fixed point.

Proof. Hypothesis (ii) ensures that there are two elements j_1, j_2 in X with $\alpha(j_1, j_2) \geqslant s$ and $j_2 \in Tj_1$. Without loss of generality, we may assume that $j_1 \neq j_2$. Inequality (9) and condition (C1) imply that

$$\kappa\biggl(b_m(j_1,j_2), b_m(j_1,Tj_1), b_m(j_2,Tj_2), \frac{b_m(j_2,Tj_1)+b_m(j_1,Tj_2)}{2s}\biggr) \\ - \alpha(j_1,j_2)H_{b_m}(Tj_1,Tj_2) \\ + L\eta\bigl(b_m(j_1,j_2), b_m(j_1,Tj_1), b_m(j_2,Tj_2), b_m(j_1,Tj_2), b_m(j_2,Tj_1)\bigr) \geqslant 0. \tag{10}$$

Taking account of property (E1) of the functions from Γ, as $b_m(j_2, Tj_1) = 0$, we get that

$$\eta\bigl(b_m(j_1,j_2), b_m(j_1,Tj_1), b_m(j_2,Tj_2), b_m(j_1,Tj_2), b_m(j_2,Tj_1)\bigr) = 0.$$

Thus, relation (10) compels

$$sH_{b_m}(Tj_1,Tj_2) \\ \leqslant \alpha(j_1,j_2)H_{b_m}(Tj_1,Tj_2) \\ \leqslant \kappa\biggl(b_m(j_1,j_2), b_m(j_1,Tj_1), b_m(j_2,Tj_2), \frac{b_m(j_2,Tj_1)+b_m(j_1,Tj_2)}{2s}\biggr). \tag{11}$$

As $s \geqslant 1$, there exists an element $j_3 \in Tj_2$ such that

$$b_m(j_2, j_3) \leqslant sH_{b_m}(Tj_1, Tj_2). \tag{12}$$

Again, we may assume $j_2 \neq j_3$. By taking advantage of relations (11), (12) and the nondecreasing behavior of κ it follows

$$b_m(j_2, j_3) \leqslant \kappa\left(b_m(j_1, j_2), b_m(j_1, j_2), b_m(j_2, j_3), \frac{b_m(j_1, j_2) + b_m(j_2, j_3)}{2}\right). \tag{13}$$

We now claim $b_m(j_2, j_3) < b_m(j_1, j_2)$. By reductio ad absurdum we presume that $b_m(j_2, j_3) \geqslant b_m(j_1, j_2)$. Having in mind also inequality (13), we obtain

$$b_m(j_2, j_3) \leqslant \kappa\big(b_m(j_2, j_3), b_m(j_1, j_2), b_m(j_2, j_3), b_m(j_2, j_3)\big).$$

This inequality, jointly with condition (K2), imposes $b_m(j_2, j_3) = 0$, which is a contradiction to the assumption that $j_2 \neq j_3$. Thus, $b_m(j_2, j_3) < b_m(j_1, j_2)$. Relation (13) yields

$$b_m(j_2, j_3) \leqslant \kappa\big(b_m(j_1, j_2), b_m(j_1, j_2), b_m(j_2, j_3), b_m(j_1, j_2)\big).$$

This inequality combined with property (K3) compel

$$b_m(j_2, j_3) \leqslant \psi(b_m(j_1, j_2)).$$

Note that hypothesis (i) ensures that $\alpha(j_2, j_3) \geqslant s$.

Continuing the above pattern, we get a sequence $\{j_n\}$ endowed with the properties $j_{n+1} \in Tj_n$ and $\alpha(j_n, j_{n+1}) \geqslant s$, $b_m(j_{n+1}, j_{n+2}) < b_m(j_n, j_{n+1})$ and

$$b_m(j_{n+1}, j_{n+2}) \leqslant \psi^n\big(b_m(j_1, j_2)\big) \quad \text{for all } n \in \mathbb{N}. \tag{14}$$

Next, we will prove that $\{j_n\}$ is a Cauchy sequence. Consider the natural numbers q, p, $q > p$. By using the triangular inequality, relation (14), and the fact that $s \geqslant 1$ we get

$$\begin{aligned} b_m(j_q, j_p) &\leqslant \sum_{e=p}^{q-1} s^e b_m(j_e, j_{e+1}) \leqslant \sum_{e=p}^{q-1} s^e \psi^e\big(b_m(j_1, j_2)\big) \\ &\leqslant \sum_{e=p}^{\infty} s^e \psi^e\big(b_m(j_1, j_2)\big) \to 0. \end{aligned}$$

This shows that $\{j_n\}$ is a Cauchy sequence in X, whose completeness ensures the existence of an element $j^* \in X$ so that $j_n \to j^*$. Hypothesis (iii) implies $\alpha(j_n, j^*) \geqslant s$ for each $n \in \mathbb{N}$ since $\alpha(j_n, j_{n+1}) \geqslant s$ for all $n \in \mathbb{N}$. Without loss of generality, it can be presumed that $j_n \neq j^*$ for all $n \in \mathbb{N}$. From (9) we get

$$\begin{aligned} \chi\bigg(&\alpha(j_n, j^*) H_{b_m}(Tj_n, Tj^*), \\ &\kappa\left(b_m(j_n, j^*), b_m(j_n, Tj_n), b_m(j^*, Tj^*), \frac{b_m(j^*, Tj_n) + b_m(j_n, Tj^*)}{2s}\right)\bigg) \\ &+ L\eta\big(b_m(j_n, j^*), b_m(j_n, Tj_n), b_m(j^*, Tj^*), b_m(j_n, Tj^*), b_m(j^*, Tj_n)\big) \geqslant 0. \end{aligned}$$

By the use of condition (C1) and the relation $\alpha(j_n, j^*) \geqslant s$ in the above inequality, for $n \in \mathbb{N}$, we get

$$\begin{aligned}
&sb_m(j_{n+1}, Tj^*)\\
&\quad\leqslant \alpha(j_n, j^*)H_{b_m}(Tj_n, Tj^*)\\
&\quad\leqslant \kappa\biggl(b_m(j_n, j^*), b_m(j_n, Tj_n), b_m(j^*, Tj^*), \frac{b_m(j^*, Tj_n) + b_m(j_n, Tj^*)}{2s}\biggr)\\
&\qquad + L\eta\bigl(b_m(j_n, j^*), b_m(j_n, Tj_n), b_m(j^*, Tj^*), b_m(j_n, Tj^*), b_m(j^*, Tj_n)\bigr)\\
&\quad\leqslant \kappa\biggl(b_m(j_n, j^*), b_m(j_n, j_{n+1}), b_m(j^*, Tj^*),\\
&\qquad\qquad \frac{b_m(j^*, j_{n+1}) + b_m(j_n, j^*) + b_m(j^*, Tj^*)}{2}\biggr)\\
&\qquad + L\eta\bigl(b_m(j_n, j^*), b_m(j_n, Tj_n), b_m(j^*, Tj^*), b_m(j_n, Tj^*), b_m(j^*, Tj_n)\bigr).
\end{aligned}$$

By taking the limit $n \to \infty$, using the continuity of b_m, κ and η and the fact that $sb_m(j_{n+1}, Tj^*) \geqslant b_m(j^*, Tj^*) - sb_m(j_{n+1}, j^*)$, it follows that

$$b_m(j^*, Tj^*) \leqslant \kappa\biggl(0, 0, b_m(j^*, Tj^*), \frac{b_m(j^*, Tj^*)}{2}\biggr).$$

From the above inequality and condition (C2) we get

$$\chi\biggl(b_m(j^*, Tj^*), \kappa\biggl(0, 0, b_m(j^*, Tj^*), \frac{b_m(j^*, Tj^*)}{2}\biggr)\biggr) \geqslant 0.$$

By using (C3) we obtain $b_m(j^*, Tj^*) = 0$. That is, $j^* \in Tj^*$. □

By defining $\alpha : X \times X \to [0, \infty)$, $\alpha(j, k) = s$ for each $j, k \in X$, we get the following corollary.

Corollary 2. *Consider X a nonempty set endowed with a continuous b-metric b_m so that (X, b_m) is complete. Let T be a mapping from X into $\mathrm{CL}(X)$ that satisfies*

$$\begin{aligned}
&\chi\biggl(sH_{b_m}(Tj, Tl), \kappa\biggl(b_m(j, l), b_m(j, Tj), b_m(l, Tl), \frac{b_m(l, Tj) + b_m(j, Tl)}{2s}\biggr)\biggr)\\
&\qquad + L\eta\bigl(b_m(j, l), b_m(j, Tj), b_m(l, Tl), b_m(j, Tl), b_m(l, Tj)\bigr) \geqslant 0
\end{aligned}$$

for all $j, l \in X$, where χ is an implicit simulation function with respect to κ_{ψ_s}, $\eta \in \Gamma$ and $L \geqslant 0$. Then T has a fixed point.

The following result can be obtained directly from Theorem 3 by considering $\chi(l, j) = j - l$ for all $l, j \in \mathbb{R}^+$, $\kappa(m, n, o, p) = q \max\{m, n, o, p\}$ and $\eta(m, n, o, p, q) = mnopq$ for all $m, n, o, p, q \in \mathbb{R}^+$.

Corollary 3. *Consider X a nonempty set endowed with a continuous b-metric b_m so that (X, b_m) is complete. Let T be a mapping from X into $\mathrm{CL}(X)$ that satisfies*

$$\begin{aligned}&\alpha(j,l)H_{b_m}(Tj,Tl)\\&\quad\leqslant q\max\left\{b_m(j,l), b_m(j,Tj), b_m(l,Tl), \frac{b_m(l,Tj)+b_m(j,Tl)}{2}\right\}\\&\qquad + Lb_m(j,l)b_m(j,Tj)b_m(l,Tl)b_m(j,Tl)b_m(l,Tj), \quad j,l \in X,\end{aligned}$$

where $q \in [0,1)$ and $L \geqslant 0$. Further, assume that the following conditions hold:

(i) *T is α_s-admissible, that is, for $j, l \in X$, $\alpha(j,l) \geqslant s$ implies $\alpha(a,b) \geqslant s$ for all $a \in Tj$, $b \in Tl$*;
(ii) *There exist $j_1 \in X$ and $j_2 \in Tj_1$ satisfying $\alpha(j_1, j_2) \geqslant s$*;
(iii) *For all $\{j_n\}$ in X with $\alpha(j_n, j_{n+1}) \geqslant s$, $n \in \mathbb{N}$ and $j_n \to j$, we have $\alpha(j_n, j) \geqslant s$ for all $n \in \mathbb{N}$.*

Then T has a fixed point.

4 Application to integral inclusions

Here we apply our result to prove the existence of a solution to the integral inclusion having the following form:

$$j(u) \in \int\limits_{c(u)}^{v(u)} W\big(u,p,j(p)\big)\,\mathrm{d}p + l(u), \quad u \in [a,b], \tag{15}$$

where $l, c, v : [a,b] \to \mathbb{R}$ are continuous functions, $c(u) \leqslant v(u)$ for all $u \in [a,b]$, and $W : [a,b] \times [a,b] \times \mathbb{R} \to P_{\mathrm{cv}}(\mathbb{R})$, $P_{\mathrm{cv}}(\mathbb{R})$ is a collection of nonempty, convex and compact subsets of $\mathbb{R}$ such that $W(\cdot,\cdot,j)$ is a lower semicontinuous operator for each $j \in C([a,b],\mathbb{R})$, where $C([a,b],\mathbb{R})$ represents the space of all continuous functions from $[a,b]$ into $\mathbb{R}$.

Consider $X = C([a,b],\mathbb{R})$; this space is a complete b-metric space with $d_m(j,l) = \sup_{u\in[a,b]} |j(u) - l(u)|^2$ and $s = 2$.

We now define an operator $T : C([a,b],\mathbb{R}) \to \mathrm{CL}(C([a,b],\mathbb{R}))$ for the integral inclusion (15) as

$$Tj(u) = \left\{e \in C\big([a,b],\mathbb{R}\big)\colon\ e \in \int\limits_{c(u)}^{v(u)} W\big(u,p,j(p)\big)\,\mathrm{d}p + l(u),\ u \in [a,b]\right\}.$$

We denote by $W_j(u,p) := W(u,p,j(p))$ for all $u, p \in [a,b]$, $j \in C([a,b],\mathbb{R})$. The Michael's selection theorem [24] ensures that for $W_j : [a,b] \times [a,b] \to P_{\mathrm{cv}}(\mathbb{R})$, there exists a continuous operator $w_j : [a,b] \times [a,b] \to \mathbb{R}$ with $w_j(u,p) \in W_j(u,p)$ for each

$u, p \in [a, b]$. By this fact we get $\int_{c(u)}^{v(u)} w_j(u,p)\,\mathrm{d}p + l(u) \in Tj(u)$. Thus, Tj is nonempty. Moreover, the arguments provided in [33] confirm that Tj is also a closed operator.

We now state and prove an existence theorem for the integral inclusion (15).

Theorem 4. *Let $X = C([a,b],\mathbb{R})$, and let the operator $T : X \to \mathrm{CL}(X)$,*

$$Tj = \left\{ e \in C\big([a,b],\mathbb{R}\big)\colon\ e(u) \in \int\limits_{c(u)}^{v(u)} W\big(u,p,j(p)\big)\,\mathrm{d}p + l(u),\ u \in [a,b] \right\},$$

where $l, c, v, j : [a,b] \to \mathbb{R}$ are continuous functions, $c(u) \leqslant v(u)$ for all $u \in [a,b]$, and $W : [a,b] \times [a,b] \times \mathbb{R} \to P_{\mathrm{cv}}(\mathbb{R})$ is such that $W(\cdot,\cdot,j)$ is a lower semicontinuous operator for all j. Further, assume that the following conditions hold:

(i) *There exists a mapping $\alpha : X \times X \to (0,\infty)$ satisfying the following*:
  - (i-a) *There exist $j_1 \in X$ and $j_2 \in Tj_1$ with $\alpha(j_1, j_2) \geqslant 2$;*
  - (i-b) *For $j, k \in X$ with $\alpha(j,k) \geqslant 2$, we have $\alpha(a,b) \geqslant 2$ for each $a \in Tj$, $b \in Tk$;*
  - (i-c) *For all $\{j_n\}$ in X with $\alpha(j_n, j_{n+1}) \geqslant 2$, $n \in \mathbb{N}$ and $j_n \to j$, we have $\alpha(j_n, j) \geqslant 2$ for all $n \in \mathbb{N}$.*

(ii) *There exists a continuous mapping $q : X \times X \to [0,\infty)$ such that*

$$H_{b_m}\big(W\big(u,p,j(p)\big), W\big(u,p,k(p)\big)\big) \leqslant q\big(j(p),k(p)\big)\sqrt{\psi\big(\big|j(p)-k(p)\big|^2\big)}$$

for each $u, p \in [a,b]$ and $j, k \in X$, where $\psi : [0,\infty) \to [0,\infty)$ is a nondecreasing continuous mapping such that $\sum_{n=1}^{\infty} 2^n \psi^n(t) < \infty$ for all $t \geqslant 0$. Moreover, the mapping $q : X \times X \to [0,\infty)$ satisfies

$$\int\limits_{c(u)}^{v(u)} q\big(j(p),k(p)\big)\,\mathrm{d}p \leqslant \sqrt{\frac{1}{\alpha(j,k)}}, \quad u \in [a,b].$$

Then the integral inclusion (15) *has a solution.*

Proof. Our aim is to prove the existence of a fixed point for the above defined operator T by using Theorem 3. For this, we focus on relation (9). Let $j, k \in X$ and $e \in Tj$. Let $w_j(u,p) \in W_j(u,p)$ for $u, p \in [a,b]$ with $e(u) = \int_{c(u)}^{v(u)} w_j(u,p)\,\mathrm{d}p + l(u)$, $u \in [a,b]$. By using hypothesis (ii) we have $r(u,p) \in W_k(u,p)$ such that

$$\big|w_j(u,p) - r(u,p)\big| \leqslant q\big(j(p),k(p)\big)\sqrt{\psi\big(\big|j(p)-k(p)\big|^2\big)} \quad \text{for all } u, p \in [a,b].$$

Now, consider the operator S,

$$\begin{aligned} S(u,p) &= W_k(u,p) \\ &\qquad \cap \big\{ m \in \mathbb{R}\colon\ \big|w_j(u,p) - m\big| \leqslant q\big(j(p),k(p)\big)\sqrt{\psi\big(\big|j(p)-k(p)\big|^2\big)} \big\}, \end{aligned}$$

where $u, p \in [a, b]$. The lower semicontinuity of the operator S yields that there exists $w_k : [a, b] \times [a, b] \to \mathbb{R}$ with $w_k(u, p) \in S(u, p)$ for each $u, p \in [a, b]$. Thus, we get

$$h(u) = \int\limits_{c(u)}^{v(u)} w_k(u, p) \,\mathrm{d}p + l(u) \in \int\limits_{c(u)}^{v(u)} W\bigl(u, p, k(p)\bigr) \,\mathrm{d}p + l(u), \quad u \in [a, b],$$

and, for each $u \in [a, b]$, we have

$$\begin{aligned}
\bigl|e(u) - h(u)\bigr|^2 &\leqslant \Biggl(\int\limits_{c(u)}^{v(u)} \bigl|w_j(u, p) - w_k(u, p)\bigr| \,\mathrm{d}p\Biggr)^2 \\
&\leqslant \Biggl(\int\limits_{c(u)}^{v(u)} q\bigl(j(p), k(p)\bigr) \sqrt{\psi\bigl(\bigl|j(p) - k(p)\bigr|^2\bigr)} \,\mathrm{d}p\Biggr)^2 \\
&\leqslant \Biggl(\sqrt{\psi\Bigl(\sup_{p \in [a,b]} \bigl|j(p) - k(p)\bigr|^2\Bigr)} \int\limits_{c(u)}^{v(u)} q\bigl(j(p), k(p)\bigr) \,\mathrm{d}p\Biggr)^2 \\
&= \psi\bigl(d_m(j, k)\bigr) \Biggl(\int\limits_{c(u)}^{v(u)} q\bigl(j(p), k(p)\bigr) \,\mathrm{d}p\Biggr)^2 \leqslant \frac{1}{\alpha(j, k)} \psi\bigl(d_m(j, k)\bigr).
\end{aligned}$$

Consequently, it follows that

$$\alpha(j, k) d_m(e, h) \leqslant \psi\bigl(d_m(j, k)\bigr).$$

By replacing the role of j and k we conclude that

$$\alpha(j, k) H_{b_m}(Tj, Tk) \leqslant \psi\bigl(d_m(j, k)\bigr) \quad \text{for each } j, k \in X.$$

By taking $\chi(l, j) = j - l$ for all $l, j \in \mathbb{R}^+$, $\kappa(m, n, o, p) = \psi(m)$, $\eta(m, n, o, p, q) = mnopq$ for all $m, n, o, p, q \in \mathbb{R}^+$ and $L = 0$, inequality (9) reduces to the above inequality. Moreover, hypotheses (i-a), (i-b), (i-c) of the result imply hypotheses (i), (ii) and (iii) of Theorem 3. Thus, Theorem 3 ensures that a fixed point of the operator T does exist, that is, the integral inclusion (15) has a solution. $\square$

The following existence theorem is obtained by defining $\alpha : X \times X \to [0, \infty)$, $\alpha(j, k) = 2$ for each $j, k \in X$ in Theorem 4. Also, note that the following result can be proved by using Corollary 2.

Theorem 5. *Let $X = C([a, b], \mathbb{R})$, and let the operator $T : X \to \mathrm{CL}(X)$,*

$$Tj(u) = \Biggl\{e \in C\bigl([a, b], \mathbb{R}\bigr) \colon e \in \int\limits_{c(u)}^{v(u)} W\bigl(u, p, j(p)\bigr) \,\mathrm{d}p + l(u),\ u \in [a, b]\Biggr\},$$

where $l, c, v, j : [a,b] \to \mathbb{R}$ *are continuous functions,* $c(u) \leqslant v(u)$ *for all* $u \in [a,b]$*, and* $W : [a,b] \times [a,b] \times \mathbb{R} \to P_{\mathrm{cv}}(\mathbb{R})$ *is such that* $W(\cdot,\cdot,j)$ *is a lower semicontinuous operator for any* j*. Further, assume that there exists a continuous mapping* $q : X \times X \to [0,\infty)$ *such that*

$$H_{b_m}\big(W\big(u,p,j(p)\big), W\big(u,p,k(p)\big)\big) \leqslant q\big(j(p),k(p)\big)\sqrt{\psi\big(\big|j(p)-k(p)\big|^2\big)},$$

$u, p \in [a,b]$*,* $j, k \in X$*, where* $\psi : [0,\infty) \to [0,\infty)$ *is a nondecreasing continuous mapping such that* $\sum_{n=1}^{\infty} 2^n \psi^n(t) < \infty$ *for all* $t \geqslant 0$*. Moreover, the mapping* $q : X \times X \to [0,\infty)$ *satisfies*

$$\int\limits_{c(u)}^{v(u)} q\big(j(p),k(p)\big)\,\mathrm{d}p \leqslant \frac{1}{2}, \quad u \in [a,b].$$

Then the integral inclusion (15) *has a solution.*

5 Conclusion

Fixed point results with regard to multivalued mappings endowed with implicit-type contractive conditions of a special form of simulation functions are stated and proved. An application of our result in integral inclusions is presented. As further development, we intend to design numerical schemes based on our outcomes.

References

1. A. Abkar, M. Gabeleh, The existence of best proximity points for multivalued non-self mappings, *Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat., RACSAM*, **107**(2):319–325, 2013, https://doi.org/10.1007/s13398-012-0074-6.
2. A. Al-Rawashdeh, H. Aydi, F. Abdelbasset, S. Sahmim, W. Shatanawi, On common fixed points for $\alpha{-}F$-contractions and applications, *J. Nonlinear Sci. Appl.*, **9**(5):3445–3458, 2016, https://doi.org/10.22436/jnsa.009.05.128.
3. M. A. Al-Thagafi, N. Shahzad, Best proximity pairs and equilibrium pairs for Kakutani multimaps, *Nonlinear Anal., Theory Methods Appl., Ser. A*, **70**(3):1209–1216, 2009, https://doi.org/10.1016/j.na.2008.02.004.
4. M.A. Al-Thagafi, N. Shahzad, Best proximity sets and equilibrium pairs for a finite family of multimaps, *Fixed Point Theory Appl.*, **2008**:457069, 2008, https://doi.org/10.1155/2008/457069.
5. M.U. Ali, B. Alqahtani, T. Kamran, E. Karapınar, Best proximity point results for γ-controlled proximal contraction, *IEEE Access*, **7**:128009–128013, 2019, https://doi.org/10.1109/ACCESS.2019.2937160.
6. M.U. Ali, M. Farheen, T. Kamran, G. Maniu, Prešić type nonself operators and related best proximity results, *Mathematics*, **7**(5):394, 2019, https://doi.org/10.3390/math7050394.

7. M.U. Ali, H. Houmani, T. Kamran, New type of proximal contractions via implicit simulation functions, *J. Nonlinear Convex Anal.*, **20**(3):435–445, 2019.
8. M.U. Ali, C. Vetro, Fixed point theorems for multivalued maps via new auxiliary function, *Nonlinear Anal. Model. Control*, **22**(1):84–98, 2017, `https://doi.org/10.15388/NA.2017.1.6`.
9. I. Altun, D. Turkoglu, Some fixed point theorems for weakly compatible mappings satisfying an implicit relation, *Taiwanese J. Math.*, **13**(4):1291–1304, 2009, `https://doi.org/10.11650/twjm/1500405509`.
10. H. Aydi, A. Felhi, S. Sahmim, Common fixed points via implicit contractions on b-metric-like spaces, *J. Nonlinear Sci. Appl.*, **10**(4):1524–1537, 2017, `https://doi.org/10.22436/jnsa.010.04.20`.
11. S.S. Basha, N. Shahzad, Best proximity point theorems for generalized proximal contractions, *Fixed Point Theory Appl.*, **2012**:42, 2012, `https://doi.org/10.1186/1687-1812-2012-42`.
12. A. Bejenaru, M. Postolache, On Suzuki mappings in modular spaces, *Symmetry*, **11**(3):319, 2019, `https://doi.org/10.3390/sym11030319`.
13. G. Bhaskar, V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, *Nonlinear Anal., Theory Methods Appl.*, **65**(7):1379–1393, 2006, `https://doi.org/10.1016/j.na.2005.10.017`.
14. B.S. Choudhury, N. Metiya, M. Postolache, A generalized weak contraction principle with applications to coupled coincidence point problems, *Fixed Point Theory Appl.*, **2013**:152, 2013, `https://doi.org/10.1186/1687-1812-2013-152`.
15. L.B. Ćirić, A generalization of Banach's contraction principle, *Proc. Am. Math. Soc.*, **45**:267–273, 1974, `https://doi.org/10.2307/2040075`.
16. L.B. Ćirić, S.B. Prešić, On Prešić type generalisation of Banach contraction principle, *Acta Math. Univ. Comen., New Ser.*, **LXXVI**(2):143–147, 2007.
17. K. Fan, Extensions of two fixed point theorems of F.E. Browder, *Math. Z.*, **112**:234–240, 1969, `https://doi.org/10.1007/BF01110225`.
18. M. Gabeleh, Global optimal solutions of non-self mappings, *Sci. Bull., Ser. A, Appl. Math. Phys., Politeh. Univ. Buchar.*, **75**(3):67–74, 2014.
19. M. Jleli, B. Samet, Best proximity point for $\alpha-\psi$-proximal contraction type mappings and applications, *Bull. Sci. Math.*, **137**(8):977–995, 2013, `https://doi.org/10.1016/j.bulsci.2013.02.003`.
20. T. Kamran, Mizoguchi–Takahashi's type fixed point theorem, *Comput. Math. Appl.*, **57**(3):507–511, 2009, `https://doi.org/10.1016/j.camwa.2008.10.075`.
21. Q. Kiran, M.U. Ali, T. Kamran, E. Karapınar, Existence of best proximity points for controlled proximal contraction, *Fixed Point Theory Appl.*, **2015**:207, 2015, `https://doi.org/10.1186/s13663-015-0455-6`.
22. J. Markin, N. Shahzad, Best proximity points for relatively u-continuous mappings in Banach and hyperconvex spaces, *Abstr. Appl. Anal.*, **2013**:680186, 2013, `https://doi.org/10.1155/2013/680186`.

23. M.A. Miandaragh, M. Postolache, S. Rezapour, Some approximate fixed point results for generalized α-contractive mappings, *Sci. Bull., Ser. A, Appl. Math. Phys., Politeh. Univ. Buchar.*, **75**(2):3–10, 2013.

24. E. Michael, Continuous selection I, *Annals Math.*, **63**(2):361–382, 1956.

25. B. Samet, C. Vetro, P. Vetro, Fixed point theorems for $\alpha{-}\psi$-contractive type mappings, *Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods*, **75**(4):2154–2165, 2012, https://doi.org/10.1016/j.na.2011.10.014.

26. W. Shatanawi, Common fixed points for mappings under contractive conditions of (α, β, ψ)-admissibility, *Mathematics*, **6**(11):261, 2018, https://doi.org/10.3390/math6110261.

27. W. Shatanawi, K. Abodayeh, Common fixed point for mappings under contractive condition based on almost perfect functions and α-admissibility, *Nonlinear Funct. Anal. Appl.*, **23**(2): 247–257, 2018.

28. W. Shatanawi, K. Abodayeh, Fixed point results for mapping of nonlinear contractive conditions of α-admissibility form, *IEEE Access*, **7**:50280–50286, 2019, https://doi.org/10.1109/ACCESS.2019.2910794.

29. W. Shatanawi, Z.D. Mitrović, N. Hussain, S. Radenović, On generalized Hardy–Rogers type α-admissible mappings in cone b-metric spaces over Banach algebras, *Symmetry-Basel*, **12**(1): 81, 2020, https://doi.org/10.3390/sym12010081.

30. W. Shatanawi, M. Postolache, Coincidence and fixed point results for generalized weak contractions in the sense of Berinde on partial metric spaces, *Fixed Point Theory Appl.*, **2013**:54, 2013, https://doi.org/10.1186/1687-1812-2013-54.

31. W. Shatanawi, M. Postolache, Common fixed point results of mappings for nonlinear contractions of cyclic form in ordered metric spaces, *Fixed Point Theory Appl.*, **2013**:60, 2013, https://doi.org/10.1186/1687-1812-2013-60.

32. S. Shukla, N. Shahzad, G-Prešić operators on metric spaces endowed with a graph and fixed point theorems, *Fixed Point Theory Appl.*, **2014**:127, 2014, https://doi.org/10.1186/1687-1812-2014-127.

33. A. Sîntămărian, Integral inclusions of Fredholm type relative to multivalued ϕ-contraction, *Semin. Fixed Point Theory Cluj-Napoca*, **3**:361–368, 2002.

34. T. Suzuki, A generalized banach contraction principle that characterizes metric completeness, *Proc. Am. Math. Soc.*, **136**(5):1861–1869, 2008, https://doi.org/10.1090/S0002-9939-07-09055-7.

35. M. Turinici, Implicit contractive maps in ordered metric spaces, in T. Rasias, L. Tóth (Eds.), *Topics in Mathematical Analysis and Applications*, Springer Optim. Appl., Vol. 94, Springer, Cham, 2014, pp. 715–746, https://doi.org/10.1007/978-3-319-06554-0_32.