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Radial symmetry for a generalized nonlinear fractional p-Laplacian problem*

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Abstract. This paper first introduces a generalized fractional p-Laplacian operator $(-\Delta)^s_{\mathcal{F},p}$. By using the direct method of moving planes, with the help of two lemmas, namely decay at infinity and narrow region principle involving the generalized fractional p-Laplacian, we study the monotonicity and radial symmetry of positive solutions of a generalized fractional p-Laplacian equation with negative power. In addition, a similar conclusion is also given for a generalized Hénon-type nonlinear fractional p-Laplacian equation.

Keywords: generalized fractional p-Laplacian, method of moving planes, negative powers, radial symmetry and monotonicity.

1 Introduction

Fractional-order differential equations are very suitable for describing materials and processes with memory and heritability, and their description of complex systems has the advantages of simple modeling, clear physical meaning of parameters and accurate description. Examples include a fractional differential model for the free dynamic response of viscoelastic single degree of freedom systems [18], a new noninteger model for

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convective straight fins with temperature-dependent thermal conductivity associated with Caputo–Fabrizio fractional derivative [20] and so on.

In this paper, we are concerned with a generalized nonlinear fractional p-Laplacian equation with negative power

$$(-\Delta)^s_{\mathcal{F},p}\phi(x) + \phi^{-\gamma}(x) = 0, \quad x \in \mathbb{R}^n, \tag{1}$$

where

$$(-\Delta)^s_{\mathcal{F},p}\phi(x) = C_{n,sp}\mathrm{PV}\int\limits_{\mathbb{R}^n} \mathcal{F}\left(\frac{|\phi(x)-\phi(y)|^{p-2}[\phi(x)-\phi(y)]}{|x-y|^{n+sp}}\right)\mathrm{d}y.$$

Here $0 < s < 1$, $2 < p < \infty$, PV means the Cauchy principal value and $\mathcal{F}$ is a continuous function. For the purpose of making the integral meaningful, we need that

$$\phi \in C^{1,1}_{\mathrm{loc}} \cap l_{sp}$$

with

$$l_{sp} = \left\{\phi \in l^{p-1}_{\mathrm{loc}} \;\Big|\; \int\limits_{\mathbb{R}^n} \frac{|1+\phi(x)|^{p-1}}{1+|x|^{n+sp}}\,\mathrm{d}x < \infty\right\}.$$

The operator $(-\Delta)^s_{\mathcal{F},p}$ introduced in this paper includes some special cases. When $\mathcal{F}(\cdot)$ is an identity map, $(-\Delta)^s_{\mathcal{F},p}$ becomes the fractional p-Laplacian $(-\Delta)^s_p$. Based on this, $(-\Delta)^s_p$ will become fractional Laplacian$(-\Delta)^s$ if $p = 2$, this is well known. In order to surmount the nonlocality of fractional Laplacian, Caffarelli and Silvestre [4] introduced the extension method that reduced this nonlocal problem into a local one in higher dimensions. This method is briefly described below. Given a function $g : \mathbb{R}^n \to \mathbb{R}$, let the extension $G : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}^n$ that meets the following condition:

$$\mathrm{div}\big(y^{1-\delta}\nabla G\big) = 0, \quad (x,y) \in \mathbb{R}^n \times [0,\infty), \qquad G(x,0) = g(x).$$

They concluded that

$$(-\Delta)^{\delta/2}g(x) = -C_{n,\delta}\lim_{y\to 0^+} y^{1-\delta}\frac{\partial G}{\partial y}, \quad x \in \mathbb{R}^n.$$

The extension method mentioned above has been utilized to discuss equations involving fractional Laplacian; see [3,5,15]. Another way to overcome the nonlocality is the integral equations method. Applications of this method can be found in [7, 12, 13, 23, 25, 26, 36].

However, there are still some operators that cannot be solved by the above methods; see [5]. To overcome the difficulty, a direct method of moving planes is introduced in [11]. Gradually, it is used to tackle a series of problems involving kinds of nonlinear operators. For example, the relevant properties of solutions for nonlinear elliptic equations are obtained, besides, it has been highly applied in studying the properties of fractional Laplacian equations and systems; see [8, 14, 24, 28, 31, 32]. Furthermore, there are some excellent results by using this method to study the radial symmetry and monotonicity of the solutions of fractional p-Laplacian equations and systems; see [9, 16, 22, 27, 33–35]. For fully nonlinear nonlocal operators, for example,

$$K_\alpha\big(r(x)\big) \equiv C_{n,\alpha} \lim_{\epsilon \to 0} \int\limits_{\mathbb{R}^n \setminus B_\epsilon(x)} \frac{G(r(x) - r(z))}{|x - z|^{n+\alpha}}\, \mathrm{d}z = f(x, r),$$

this direct method has been further developed by Chen and Li in [10].

Recently, the problems involving negative powers are studied in many fields, for example, in MEMS device, singular minimal surface equations, and described curvature equations in conformal geometry; see [1, 2, 6, 17, 19, 21, 29, 30].

In [17], Davila, Wang and Wei proved sharp Hölder continuity and an estimate of rupture sets for sequences of solutions of the following nonlinear problem with negative exponent:

$$\Delta u = \frac{1}{u^p} \quad \text{in } \Omega,\ p > 1.$$

The above problem arises in modeling an electrostatic microelectromechanical system (MEMS) device. The solution of the singularity in the equation, namely $u \approx 0$ in some region, represents the rupture in the device in the physical model.

In [19], Jiang and Ni studied the singular elliptic equation

$$\Delta h = \frac{1}{\alpha} h^{-\alpha} - p \quad \text{in } \Omega,$$

where $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, is a bounded smooth domain and $\alpha > 1$. When $n = 2$ and $\alpha = 3$, the above equation is used to model steady states of van der Waals force driven thin films of viscous fluids. They also considered the physical problem when total volume of the fluid is prescribed. Singular elliptic equations modeling steady states of van der Waals force driven thin films have been mathematically rigorously studied with no flux Neumann boundary condition. They gave a complete description of all continuous radially symmetric solutions. In particular, they constructed both nontrivial smooth solutions and singular solutions.

In [6], Ma and Cai studied the following nonlinear fractional Laplacian equation with negative powers by using the direct method of moving planes:

$$(-\Delta)^{\alpha/2} v(x) + v^{-\beta}(x) = 0, \quad x \in \mathbb{R}^n.$$

The above results of all encourage us to further study a generalized nonlinear fractional p-Laplacian equation with negative powers by the direct method of moving planes. As far as authors know, up to now, this is a new attempt to study a class of equations (1) combining a generalized fractional p-Laplacian with negative powers. Interestingly, this method can also be analogously applied to a generalized Hénon-type nonlinear fractional p-Laplacian equation with negative power

$$(-\Delta)^s_{\mathcal{F},p} \phi(x) + |x|^\sigma \phi^{-\gamma}(x) = 0, \quad x \in \mathbb{R}^n \setminus \{0\}, \tag{2}$$

where $\sigma < 0$ and $\gamma > 0$ are constants.

The paper is structured as follows. In Section 2, we mainly present some lemmas used in the following part. In Section 3, we study radial symmetry and monotonicity of two generalized fractional p-Laplacian equations with negative powers by applying the direct method of moving planes.

2 Auxiliary lemmas

Let $P_\kappa = \{x \in \mathbb{R}^n \mid x_1 = \kappa$ for some $\kappa \in \mathbb{R}\}$ be the moving planes, $\Sigma_\kappa = \{x \in \mathbb{R}^n \mid x_1 < \kappa\}$ be the area to the left of P_κ, and $x^\kappa = (2\kappa - x_1, x_2, \dots, x_n)$ be the reflection of x about P_κ. Meanwhile, we denote

$$\phi_\kappa(x) = \phi\bigl(x^\kappa\bigr), \qquad d_\kappa(x) = \phi(x) - \phi_\kappa(x),$$
$$\tilde{\Sigma}_\kappa = \bigl\{x \bigm| x^\kappa \in \Sigma_\kappa\bigr\}, \qquad \Sigma_\kappa^- = \bigl\{x \in \Sigma_\kappa \bigm| d_\kappa(x) < 0\bigr\}.$$

Throughout the next section, we assume that there exists a constant $L > 0$ such that $\mathcal{F}(x) - \mathcal{F}(y) \leqslant L(x - y)$ and $\mathcal{F}(0) = 0$.

Lemma 1 [A simple maximum principle]. *Let $\aleph$ be bounded area in $\mathbb{R}^n$. Presume that $u \in l_sp \cap C^{1,1}_{\mathrm{loc}}(\aleph)$ is lower semicontinuous on $\bar{\aleph}$ and satisfies*

$$(-\Delta)^s_{\mathcal{F},p}\phi(x) \geqslant 0 \quad \text{in } \aleph, \qquad \phi(x) \geqslant 0 \quad \text{in } \aleph^c. \tag{3}$$

Then

$$\phi(x) \geqslant 0, \quad x \in \aleph. \tag{4}$$

If $\phi(x) = 0$ at some point $x \in \aleph$, then $\phi(x) = 0$ holds for almost all points x in $\mathbb{R}^n$. When $\aleph$ is unbounded area, we need further assume that $\varliminf_{|x|\to\infty} \phi(x) \geqslant 0$, then the same conclusion still holds.

Proof. Suppose (4) is not true, then there is an x^0 such that $\phi(x^0) = \min_\aleph \phi < 0$. According to the second inequality in (3),

$$(-\Delta)^s_{\mathcal{F},p}\phi\bigl(x^0\bigr) = C_{n,sp}\,\mathrm{PV}\int\limits_{\mathbb{R}^n} \mathcal{F}\biggl(\frac{|\phi(x^0) - \phi(z)|^{p-2}[\phi(x^0) - \phi(z)]}{|x^0 - z|^{n+sp}}\biggr)\,\mathrm{d}z < 0.$$

This is a direct contradiction to the first inequality in (3), hence (4) holds. When $\phi(x^0) = 0$ at some point $x^0 \in \aleph$, then

$$(-\Delta)^s_{\mathcal{F},p}\phi\bigl(x^0\bigr) = C_{n,sp}\,\mathrm{PV}\int\limits_{\mathbb{R}^n} \mathcal{F}\biggl(\frac{|\phi(z)|^{p-2}[-\phi(z)]}{|x^0 - z|^{n+sp}}\biggr)\,\mathrm{d}z \leqslant 0.$$

On the other hand, from (3) we have

$$\mathrm{PV}\int\limits_{\mathbb{R}^n} \mathcal{F}\biggl(\frac{|\phi(z)|^{p-2}[-\phi(z)]}{|x^0 - z|^{n+sp}}\biggr)\,\mathrm{d}z \geqslant 0,$$

so, the integral result is 0. Because u is nonnegative, one could get that $\phi(x) = 0$ almost everywhere in $\mathbb{R}^n$. This completes the lemma. $\square$

Lemma 2 [Maximum principle for antisymmetric functions]. *Let $\aleph$ be bounded area in Σ and $\phi \in l_sp \cap C^{1,1}_{\mathrm{loc}}(\aleph)$. If*

$$(-\Delta)^s_{\mathcal{F},p}\phi(x) - (-\Delta)^s_{\mathcal{F},p}\phi_\kappa(x) \geqslant 0 \quad \text{in } \aleph, \qquad d_\kappa(x) \geqslant 0 \quad \text{in } \Sigma \setminus \aleph, \tag{5}$$

then

$$d_\kappa(x) \geqslant 0 \quad \text{in } \aleph.$$

If $d_\kappa(x) = 0$ at some point in $\aleph$, then $d_\kappa(x) = 0$ almost everywhere in $\mathbb{R}^n$. When $\aleph$ is unbounded area, we need further presume that

$$\varliminf_{|x|\to\infty} d_\kappa(x) \geqslant 0,$$

then the same conclusions still hold.

Proof. Suppose that $d_\kappa(x) \geqslant 0$ in $\aleph$ is not true, then there is a point $\hat{x}$ in $\aleph$ such that

$$d_\kappa(\hat{x}) = \min_{\aleph} d_\kappa(\hat{x}) < 0.$$

To simplify writing, let $Q(m) = |m|^{p-2}m$, then $Q'(m) = (p-1)|m|^{p-2} \geqslant 0$.

$$\begin{aligned}
&(-\Delta)^s_{\mathcal{F},p}\phi(x) - (-\Delta)^s_{\mathcal{F},p}\phi_\kappa(\hat{x}) \\
&\quad= C_{n,sp}\,\mathrm{PV}\int_{\mathbb{R}^n} \mathcal{F}\biggl(\frac{Q(\phi(\hat{x}) - \phi(y))}{|\hat{x}-y|^{n+sp}}\biggr)\,\mathrm{d}y \\
&\qquad- C_{n,sp}\,\mathrm{PV}\int_{\mathbb{R}^n} \mathcal{F}\biggl(\frac{Q(\phi_\kappa(\hat{x}) - \phi_\kappa(y))}{|\hat{x}-y|^{n+sp}}\biggr)\,\mathrm{d}y \\
&\quad\leqslant C_{n,sp}\int_{\mathbb{R}^n} L\frac{Q(\phi(x) - \phi(y)) - Q(\phi_\kappa(\hat{x}) - \phi_\kappa(y))}{|\hat{x}-y|^{n+sp}}\,\mathrm{d}y \\
&\quad= C_{n,sp}\int_{\Sigma} L\frac{Q(\phi(\hat{x}) - \phi(y)) - Q(\phi_\kappa(\hat{x}) - \phi_\kappa(y))}{|\hat{x}-y|^{n+sp}}\,\mathrm{d}y \\
&\qquad+ C_{n,sp}\int_{\Sigma} L\frac{Q(\phi(\hat{x}) - \phi_\kappa(y)) - Q(\phi_\kappa(\hat{x}) - \phi(y))}{|\hat{x}-y^\kappa|^{n+sp}}\,\mathrm{d}y \\
&\quad= C_{n,sp}L\biggl\{\int_{\Sigma}\biggl(\frac{1}{|\hat{x}-y|^{n+sp}} - \frac{1}{|\hat{x}-y^\kappa|^{n+sp}}\biggr) \\
&\qquad\qquad\times\bigl(Q\bigl(\phi(\hat{x}) - \phi(y)\bigr) - Q\bigl(\phi_\kappa(\hat{x}) - \phi_\kappa(y)\bigr)\bigr)\,\mathrm{d}y \\
&\qquad+ \int_{\Sigma}\frac{1}{|\hat{x}-y^\kappa|^{n+sp}}\bigl(Q\bigl(\phi(\hat{x}) - \phi(y)\bigr) - Q\bigl(\phi_\kappa(\hat{x}) - \phi_\kappa(y)\bigr) \\
&\qquad\qquad+ Q\bigl(\phi(\hat{x}) - \phi_\kappa(y)\bigr) - Q\bigl(\phi_\kappa(\hat{x}) - \phi(y)\bigr)\bigr)\,\mathrm{d}y\biggr\} \\
&\quad= C_{n,sp}L\{J_1 + J_2\},
\end{aligned} \tag{6}$$

where

$$J_1 = \int\limits_{\Sigma} \left(\frac{1}{|\hat{x} - y|^{n+sp}} - \frac{1}{|\hat{x} - y^\kappa|^{n+sp}} \right) \\ \times \left(Q\bigl(\phi(\hat{x}) - \phi(y)\bigr) - Q\bigl(\phi_\kappa(\hat{x}) - \phi_\kappa(y)\bigr) \right) \mathrm{d}y,$$

$$J_2 = \int\limits_{\Sigma} \frac{1}{|\hat{x} - y^\kappa|^{n+sp}} \left(Q\bigl(\phi(\hat{x}) - \phi(y)\bigr) - Q\bigl(\phi_\kappa(\hat{x}) - \phi_\kappa(y)\bigr) \right. \\ \left. + Q\bigl(\phi(\hat{x}) - \phi_\kappa(y)\bigr) - Q\bigl(\phi_\kappa(\hat{x}) - \phi(y)\bigr) \right) \mathrm{d}y.$$

To estimate J_1, we notice the fact

$$\frac{1}{|x - y|^{n+sp}} > \frac{1}{|x - y^\kappa|^{n+sp}} \quad \forall x, y \in \Sigma_\kappa.$$

Due to

$$\bigl[\phi(\hat{x}) - \phi(y)\bigr] - \bigl[\phi_\kappa(\hat{x}) - \phi_\kappa(y)\bigr] = d_\kappa(\hat{x}) - d_\kappa(y) \leqslant 0, \text{ but } \not\equiv 0,$$

on the basis of strict monotonicity of Q, we have

$$Q\bigl(\phi(\hat{x}) - \phi(y)\bigr) - Q\bigl(\phi_\kappa(\hat{x}) - \phi_\kappa(y)\bigr) \leqslant 0, \text{ but } \not\equiv 0.$$

Therefore,

$$J_1 < 0. \tag{7}$$

To evaluate J_2, by using the mean value theorem we obtain

$$\begin{aligned} J_2 &= \int\limits_{\Sigma} \frac{L}{|\hat{x} - y^\kappa|^{n+sp}} \left(Q\bigl(\phi(\hat{x}) - \phi(y)\bigr) - Q\bigl(\phi_\kappa(\hat{x}) - \phi(y)\bigr) \right. \\ &\quad \left. + Q\bigl(\phi(\hat{x}) - \phi_\kappa(y)\bigr) - Q\bigl(\phi_\kappa(\hat{x}) - \phi_\kappa(y)\bigr) \right) \mathrm{d}y \\ &= d_\kappa(\hat{x}) \int\limits_{\Sigma} \frac{L}{|\hat{x} - y^\kappa|^{n+sp}} \bigl(Q'\bigl(\xi(y)\bigr) + Q'\bigl(\eta(y)\bigr) \bigr) \mathrm{d}y \leqslant 0. \end{aligned} \tag{8}$$

Combining (6), (7) and (8), one can deduce

$$(-\Delta)^s_{\mathcal{F},p} \phi(\hat{x}) - (-\Delta)^s_{\mathcal{F},p} \phi_\kappa(\hat{x}) < 0.$$

This inequality contradicts the first condition in (5), thus $d_\kappa(\hat{x}) \geqslant 0$.

If $d_\kappa(x) = 0$ at some point $x \in \aleph$, equivalently, x is a minimum of d_κ in $\aleph$, so, $J_2 = 0$. Now, according to the first inequality in (5), we get $J_1 \geqslant 0$, which means

$$Q\bigl(\phi(x) - \phi(y)\bigr) - Q\bigl(\phi_\kappa(x) - \phi_\kappa(y)\bigr) \geqslant 0.$$

Considering the monotonicity of Q, for almost all $y \in \Sigma$,

$$\bigl[\phi(x) - \phi(y)\bigr] - \bigl[\phi_\kappa(x) - \phi_\kappa(y)\bigr] = d_\kappa(x) - d_\kappa(y) = -d_\kappa(y) \geqslant 0.$$

Consequently, $d_\kappa(y) = 0$ almost everywhere in Σ. Besides, in the light of the antisymmetry of d_κ, we receive $d_\kappa(y) = 0$ almost everywhere in $\mathbb{R}^n$. If $\aleph$ is unbounded area, under this circumstance, in view of assumption $\underline{\lim}_{|x|\to\infty} d_\kappa(x) \geqslant 0$, suppose that $d_\kappa(x) \geqslant 0$, $x \in \Sigma$, is false, then a negative minimum of d_κ is obtained at some point $x \in \Sigma$. Being similar to the above argument, one can find a contradiction. The proof is completed. □

Lemma 3 [Narrow region principle]. *Let $\aleph$ be bounded narrow area in Σ such that it is contained in $\{x \mid \kappa - \delta < x_1 < \kappa\}$ with small δ. Presume that $c(x)$ is bounded from below in $\aleph$ and*

$$\mathcal{F}\big((-\Delta)_p^s\big)\phi(x) - \mathcal{F}\big((-\Delta)_p^s\big)\phi_\kappa(x) + c(x)d_\kappa(x) \geqslant 0 \quad \textit{in } \aleph,$$
$$d_\kappa(x) \geqslant 0 \quad \textit{in } \Sigma \setminus \aleph,$$

and there exists $y^0 \in \Sigma$ satisfying $d_\kappa(y^0) > 0$, then when δ is sufficiently small, one can get

$$d_\kappa(x) \geqslant 0 \quad \textit{in } \aleph.$$

Further, if $d_\kappa(x) = 0$ holds for some point in $\aleph$, then $d_\kappa(x) = 0$ holds for almost all points x in $\mathbb{R}^n$. Besides, if $\aleph$ is unbounded region, we need presume that

$$\underline{\lim}_{|x|\to\infty} d_\kappa(x) \geqslant 0,$$

then above conclusions still hold.

Proof. Suppose the contrary, then for any $\delta > 0$, there exists an $x_\delta \in \aleph_\delta$ such that

$$d_\kappa(x_\delta) = \min_{\aleph_\delta} d_\kappa(x) < 0.$$

Then for $\delta_k = 1/k$, $k = 1, 2, \dots$, there exists x_{δ_k} and $\aleph_{\delta_k}$, let us call them x_k and $\aleph_k$ such that

$$d_\kappa(x_k) = \min_{\aleph_k} d_\kappa(x) < 0.$$

By inequality (6) we deduce

$$\begin{aligned}
&(-\Delta)^s_{\mathcal{F},p}\phi(x_k) - (-\Delta)^s_{\mathcal{F},p}\phi_\lambda(x_k) \\
&\quad\leqslant C_{n,sp}L\bigg\{\int\limits_\Sigma \bigg(\frac{1}{|x_k - y|^{n+sp}} - \frac{1}{|x_k - y^\kappa|^{n+sp}}\bigg) \\
&\qquad\qquad \times \big(Q\big(\phi(x_k) - \phi(y)\big) - Q\big(\phi_\kappa(x_k) - \phi_\kappa(y)\big)\big)\,\mathrm{d}y \\
&\qquad + \int\limits_\Sigma \frac{1}{|x_k - y^\kappa|^{n+sp}}\big(Q\big(\phi(x_k) - \phi(y)\big) - Q\big(\phi_\kappa(x_k) - \phi_\kappa(y)\big)\big) \\
&\qquad\qquad + Q\big(\phi(x_k) - \phi_\kappa(y)\big) - Q\big(\phi_\kappa(x_k) - \phi(y)\big)\big)\,\mathrm{d}y\bigg\} \\
&\quad= C_{n,sp}L\{H_1 + H_2\}, \qquad (9)
\end{aligned}$$

where

$$H_1 = \int\limits_{\Sigma} \left(\frac{1}{|x_k - y|^{n+sp}} - \frac{1}{|x_k - y^\kappa|^{n+sp}} \right) \times \big(Q\big(\phi(x_k) - \phi(y)\big) - Q\big(\phi_\kappa(x_k) - \phi_\kappa(y)\big)\big)\,\mathrm{d}y,$$

$$H_2 = \int\limits_{\Sigma} \frac{1}{|x_k - y^\kappa|^{n+sp}} \big(Q\big(\phi(x_k) - \phi(y)\big) - Q\big(\phi_\kappa(x_k) - \phi_\kappa(y)\big) + Q\big(\phi(x_k) - \phi_\kappa(y)\big) - Q\big(\phi_\kappa(x_k) - \phi(y)\big)\big)\,\mathrm{d}y.$$

Similarly to (8), we can get $H_2 \leqslant 0$, from [33] we can get $H_1 \leqslant -(C_0/2)\delta_{x_k}$,

$$c(x_k)d_\kappa(x_k) \leqslant o(1)\delta_{x_k}. \tag{10}$$

Combining (9) with (10), one can get

$$\mathcal{F}\big((-\Delta)_p^s\big)\phi(x_k) - \mathcal{F}\big((-\Delta)_p^s\big)\phi_\kappa(x_k) + c(x_k)d_\kappa(x_k) \leqslant (-C + o(1))\delta_{x_k} < 0.$$

This contradicts with the equation, hence the proof is completed. □

Lemma 4 [Decay at infinity]. *Let $\aleph$ be unbound area in Σ, and let $\phi \in l_sp \cap C^{1,1}_{\mathrm{loc}}(\aleph)$ be a solution of*

$$(-\Delta)^s_{\mathcal{F},p}\phi(x) - (-\Delta)^s_{\mathcal{F},p}\phi_\kappa(x) + c(x)d_\kappa(x) \geqslant 0 \quad \textit{in } \aleph,$$
$$d_\kappa(x) \geqslant 0 \quad \textit{in } \Sigma \setminus \aleph$$

with

$$\varliminf_{|x|\to\infty} |x|^{sp}c(x) \geqslant 0.$$

Then there exists a positive constant R_0 (depending on $c(x)$ and independent of $\phi(x)$ and $\phi_\kappa(x)$) such that if $d_\kappa(x^0) = \min_\aleph d_\kappa(x) < 0$, then $|x^0| \leqslant R_0$.

Proof. By inequality (6) we get

$$\begin{aligned}
&(-\Delta)^s_{\mathcal{F},p}\phi\big(x^0\big) - (-\Delta)^s_{\mathcal{F},p}\phi_\kappa\big(x^0\big) + c\big(x^0\big)d_\kappa\big(x^0\big) \\
&\quad < C_{n,sp} \int\limits_{\Sigma} \frac{L}{|x^0 - y^\kappa|^{n+sp}} \big(Q\big(\phi\big(x^0\big) - \phi(y)\big) - Q\big(\phi_\kappa\big(x^0\big) - \phi(y)\big) \\
&\qquad + Q\big(\phi\big(x^0\big) - \phi_\kappa(y)\big) - Q\big(\phi_\kappa\big(x^0\big) - \phi_\kappa(y)\big)\big)\,\mathrm{d}y \\
&\qquad + c\big(x^0\big)d_\kappa\big(x^0\big) \\
&\quad \leqslant d_\kappa\big(x^0\big)\left(C_{n,sp} \int\limits_{\Sigma} L\frac{Q'(\xi(y)) + Q'(\eta(y))}{|x^0 - y^\kappa|^{n+sp}}\,\mathrm{d}y + c\big(x^0\big)\right) \\
&\quad \leqslant d_\kappa\big(x^0\big)\left(C_{n,sp} \int\limits_{\Sigma} L\frac{M}{|x^0 - y^\kappa|^{n+sp}}\,\mathrm{d}y + c\big(x^0\big)\right),
\end{aligned}$$

where M is a constant. For each fixed κ, when $|x^0| \geqslant \kappa$, $x^1 = (3|x^0| + x_1^0, (x^0)')$, then $B_{|x^0|}(x^1) \subset \widetilde{\Sigma}$. Consequently,

$$\int_{\Sigma} \frac{1}{|x^0 - y^\kappa|^{n+sp}} \,\mathrm{d}y \geqslant \int_{B_{|x^0|}(x^1)} \frac{1}{|x^0 - y|^{n+sp}} \,\mathrm{d}y \geqslant \frac{w_n}{4^{n+sp}|x^0|^{sp}}.$$

Then we have

$$\begin{aligned} 0 &\leqslant (-\Delta)^s_{\mathcal{F},p}\phi(x^0) - (-\Delta)^s_{\mathcal{F},p}\phi_\kappa(x^0) + c(x^0)d_\kappa(x^0) \\ &< d_\kappa(x^0)\left[C_{n,sp}LM\frac{w_n}{4^{n+sp}|x^0|^{sp}} + c(x^0)\right], \end{aligned}$$

this contradicts with the condition of $c(x)$. This completes the proof. □

3 The generalized fractional p-Laplacian equations with negative powers

Theorem 1. *Assume that $\phi \in l_sp \cap C^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$ is the positive solution of equation* (1) *with*

$$\phi(x) = \varrho|x|^t + o(1) \quad \textit{as } |x| \to \infty,$$

where $sp/(\gamma+1) < t < 1$ and $\varrho > 0$ are constants. Then $\phi(x)$ must be radially symmetric and monotone increasing about some point in $\mathbb{R}^n$.

Proof. Step 1. In the first step, we indicate that when κ is sufficiently negative,

$$d_\kappa(x) \geqslant 0 \quad \forall x \in \Sigma_\kappa. \tag{11}$$

According to equation (1), for $x \in \Sigma_\kappa^-$, we can derive

$$\begin{aligned} (-\Delta)^s_{\mathcal{F},p}\phi(x) - (-\Delta)^s_{\mathcal{F},p}\phi_\kappa(x) &= \phi_\kappa^{-\gamma}(x) - \phi^{-\gamma}(x) = \gamma\xi_\kappa^{-\gamma-1}(x)d_\kappa(x) \\ &\geqslant \gamma\phi^{-\gamma-1}(x)d_\kappa(x), \end{aligned}$$

where $\xi_\kappa(x)$ values between $\phi_\kappa(x)$ and $\phi(x)$, that means

$$(-\Delta)^s_{\mathcal{F},p}\phi(x) - (-\Delta)^s_{\mathcal{F},p}\phi_\kappa(x) + c(x)d_\kappa(x) \geqslant 0,$$

here $c(x) = -\gamma\phi^{-\gamma-1}(x)$. Consider $\phi(x) = \varrho|x|^t + o(1)$ near infinity, so

$$\phi^{-\gamma-1}(x) = \frac{1}{|x|^{t(\gamma+1)}} + o\left(\frac{1}{|x|^{t(\gamma+1)}}\right)$$

near infinity. Therefore, $c(x)$ satisfies the condition of Lemma 4. Since $\phi(x)$ and $\phi_\kappa(x)$ satisfy

$$\begin{aligned} &(-\Delta)^s_{\mathcal{F},p}\phi(x) - (-\Delta)^s_{\mathcal{F},p}\phi_\kappa(x) + c(x)d_\kappa(x) \geqslant 0, \quad x \in \Sigma_\kappa^-, \\ &d_\kappa(x) \geqslant 0, \quad x \in \Sigma_\kappa \setminus \Sigma_\kappa^-, \end{aligned}$$

with $c(x) \sim 1/|x|^{t(\gamma+1)}$ for $|x|$ large, using Lemma 4 to $d_\kappa(x)$, if $d_\kappa(x)$ acquires the negative minimum at point $\tilde{x}$ in Σ_κ, then $|\tilde{x}| \leqslant R_0$, that is, when κ is sufficiently negative, (11) holds.

Step 2. In this step, we will move P_κ to the limiting position, this moving process could give the symmetry about the positive solution $\phi(x)$. (11) means that there is an initial point to move P_κ, we could move P_κ so long as (11) holds. Let

$$\kappa_0 = \sup\bigl\{\kappa \bigm| d_\mu(x) \geqslant 0 \ \forall x \in \Sigma_\mu, \mu \leqslant \kappa\bigr\}.$$

We show

$$d_{\kappa_0}(x) \equiv 0, \quad x \in \Sigma_{\kappa_0}. \tag{12}$$

If (12) does not hold, then by Lemma 2 we get

$$d_{\kappa_0}(x) > 0 \quad \forall x \in \Sigma_{\kappa_0},$$

therefore, there exists a small $\delta > 0$ and a constant b_δ, which satisfy

$$d_{\kappa_0}(x) \geqslant b_\delta > 0 \quad \forall x \in \overline{\Sigma_{\kappa_0-\delta} \cap B_{R_0}(0)}.$$

According to the continuity of d_κ about κ, there exists $0 < \varpi < \delta$ such that

$$d_\kappa(x) \geqslant 0 \quad \forall x \in \overline{\Sigma_{\kappa_0-\delta} \cap B_{R_0}(0)},\ \kappa \in [\kappa_0,\ \kappa_0 + \varpi). \tag{13}$$

In Lemma 3, $\aleph = \Sigma_\kappa^- \setminus \Sigma_{\kappa_0-\delta}$ is the narrow region, then

$$d_\kappa(x) \geqslant 0 \quad \forall x \in \Sigma_\kappa \setminus \Sigma_{\kappa_0-\delta}.$$

This, combining with (13), indicates

$$d_\kappa(x) \geqslant 0 \quad \forall x \in \Sigma_\kappa,\ \kappa \in [\kappa_0,\ \kappa_0 + \varpi).$$

This violates the definition of κ_0. Thus, (12) holds. The proof is completed. □

Next, we discuss the radial symmetry of a generalized nonlinear Hénon-type fractional p-Laplacian equation with negative power.

Theorem 2. *Presume $\phi \in l_sp \cap C^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$ is a positive solution of* (2) *with*

$$\phi(x) = \varrho|x|^t + o(1) \quad \textit{for } |x| \textit{ large},$$

where γ, $\varrho > 0$ and $\sigma < 0$ are constants, $0 < t < 1$ and $t > (sp + \sigma)/(\gamma + 1)$. Then $\phi(x)$ is radially symmetric about origin.

Proof. Before we prove Theorem 2, we first consider the singularity of equation (2) at the origin. For $\kappa < 0$ and $x \in \Sigma_\kappa^- \setminus \{0^\kappa\}$, $\phi(x)$ and $\phi_\kappa(x)$ satisfy

$$\begin{aligned}
&(-\Delta)^s_{\mathcal{F},p}\phi(x) - (-\Delta)^s_{\mathcal{F},p}\phi_\kappa(x) \\
&\quad = \bigl|x^\kappa\bigr|^\sigma \phi_\kappa^{-\gamma}(x) - |x|^\sigma \phi^{-\gamma}(x) \\
&\quad = \bigl[\bigl|x^\kappa\bigr|^\sigma - |x|^\sigma\bigr]\phi_\kappa^{-\gamma}(x) + |x|^\sigma\bigl[\phi_\kappa^{-\gamma}(x) - \phi^{-\gamma}(x)\bigr] \\
&\quad \geqslant |x|^\sigma\bigl[\phi_\kappa^{-\gamma}(x) - \phi^{-\gamma}(x)\bigr] \geqslant \gamma|x|^\sigma \xi_\kappa^{-\gamma-1}(x) d_\kappa(x) \\
&\quad \geqslant \gamma|x|^\sigma \phi^{-\gamma-1}(x) d_\kappa(x),
\end{aligned}$$

where $\xi_\kappa(x)$ is between $\phi(x)$ and $\phi_\kappa(x)$. This implies,

$$(-\Delta)^s_{\mathcal{F},p}\phi(x) - (-\Delta)^s_{\mathcal{F},p}\phi_\kappa(x) + c(x)d_\kappa(x) \geqslant 0$$

with $c(x) = -\gamma|x|^\sigma\phi^{-\gamma-1}(x)$. Seeing that $\phi(x) = \varrho|x|^t + o(1)$ near infinity, so

$$c(x) \sim \frac{1}{|x|^{t(\gamma+1)-\sigma}} + o\left(\frac{1}{|x|^{t(\gamma+1)-\sigma}}\right).$$

By our assumption $t > (sp+\sigma)/(\gamma+1)$ we get $c(x) \sim o(1/|x|^{sp})$. Next, we still prove the conclusion of Theorem 2 in two steps.

Step 1. We demonstrate that

$$d_\kappa(x) \geqslant 0 \quad \forall x \in \Sigma_\kappa \setminus \{0^\kappa\} \tag{14}$$

holds when κ is sufficiently negative.

Near the singular point 0^κ of $\phi(x)$ and $\phi_\kappa(x)$, we manifest that Σ_κ^- has no intersection with $B_\epsilon(0^\kappa)$ for certain small $\varepsilon > 0$. In fact, we consider that when κ is sufficiently negative, obviously, 0^κ is a sufficiently negative point, we have

$$\lim_{|x|\to 0^\kappa} d_\lambda(x) = \lim_{|x|\to 0^\kappa} \phi(x) - \phi(0) > 0.$$

Since $c(x) \sim o(1/|x|^{sp})$, by applying Lemma 4 we could work out that there is an $R_0 > 0$, when $d_\kappa(x)$ gets the negative minimum at x^* in Σ_κ, then the following relation is true:

$$|x^*| \leqslant R_0. \tag{15}$$

That is to say, (14) holds when κ sufficiently negative.

Step 2. So long as (14) holds, we could move P_κ from left to right up to its critical position. Let

$$\kappa_0 = \sup\{\kappa \mid d_\mu(x) \geqslant 0 \ \forall x \in \Sigma_\mu \setminus \{0^\mu\},\ \mu \leqslant \kappa\}.$$

We prove that

$$\kappa_0 = 0, \qquad d_{\kappa_0}(x) \equiv 0, \quad x \in \Sigma_{\kappa_0} \setminus \{0^{\kappa_0}\}.$$

Suppose $\kappa_0 < 0$, we can use Lemmas 3 and 4 to state that P_κ can be moved further right, this contradicts with κ_0. By condition of $u(x)$,

$$\lim_{|x|\to 0^{\kappa_0}} d_{\kappa_0}(x) = \lim_{|x|\to 0^{\kappa_0}} \phi(x) - \phi(0) > 0.$$

That is, there exist ε, $h_0 > 0$ such that

$$d_{\kappa_0}(x) \geqslant h_0 \quad \forall x \in B_\varepsilon(0^{\kappa_0}) \setminus \{0^{\kappa_0}\}.$$

From (15) the situation where the negative minimum of $d_{\kappa_0}(x)$ is obtained in $B^c_{R_0}(0)$ does not exist. We also show that it cannot be gained in the internal of $B_{R_0}(0)$. That is, when κ is close enough to κ_0,

$$d_\kappa(x) \geqslant 0, \quad x \in \left(\Sigma_\kappa \cap B_{R_0}(0)\right) \setminus \left\{0^\kappa\right\}.$$

When $\kappa_0 < 0$, by Lemma 2 one can get

$$d_{\kappa_0}(x) > 0 \quad \forall x \in \Sigma_{\kappa_0} \setminus \left\{0^{\kappa_0}\right\}.$$

There is a positive constant j_0, which satisfies

$$d_{\kappa_0}(x) \geqslant j_0, \quad x \in \overline{(\Sigma_{\kappa_0-\delta} \cap B_{R_0}(0)) \setminus \left\{0^{\kappa_0}\right\}}.$$

Since $d_\kappa(x)$ is continuous with respect to κ, there is $0 < \epsilon < \delta$,

$$d_\kappa(x) \geqslant 0, \quad x \in \overline{(\Sigma_{\kappa_0-\delta} \cap B_{R_0}(0)) \setminus \left\{0^\kappa\right\}}$$

holds for $\kappa \in (\kappa_0,\, \kappa_0 + \epsilon)$.

Considering that $c(x)$ is bounded from below, for narrow region $\Sigma_\kappa^- \setminus \Sigma_{\kappa_0-\delta}$, using Lemma 3 to $d_\kappa(x)$, we receive

$$d_\kappa(x) \geqslant 0, \quad x \in (\Sigma_\kappa \setminus \Sigma_{\kappa_0-\delta}) \setminus \left\{0^\kappa\right\}.$$

From all above we get

$$d_\kappa(x) \geqslant 0, \quad x \in \Sigma_\kappa \setminus \left\{0^\kappa\right\},\ \kappa \in (\kappa_0,\, \kappa_0 + \epsilon).$$

This goes against the definition of κ_0. Hence, $\kappa_0 = 0$, $d_{\kappa_0}(x) = 0$, $x \in \Sigma_{\kappa_0} \setminus \{0^{\kappa_0}\}$ hold. Therefore, the proof is completed. □

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