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Stability of solutions of Caputo fractional stochastic differential equations*

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Abstract. In this paper, we study the stability of Caputo-type fractional stochastic differential equations. Stochastic stability and stochastic asymptotical stability are shown by stopping time technique. Almost surly exponential stability and pth moment exponentially stability are derived by a new established Itô's formula of Caputo version. Numerical examples are given to illustrate the main results.

Keywords: Caputo fractional derivative, stochastic differential equations, stability.

1 Introduction

Recently, fractional derivative is used to study the properties of memory and genetic for complex systems in different fields of application; see [2, 6]. Wu et al. [23] introduced a new result for Mittag-Leffler stability analysis of fractional discrete-time neural networks via fixed point technique. Huang et al. [10] gave the analysis for variable-order fractional discrete-time recurrent neural networks. For more details on fractional differential equations, we refer to the monographs [12, 25].

Stochastic is one of the essential properties of the world, and the stability is a top priority to the system application in practice. Stability analysis of stochastic systems is very necessary. Lyapunov direct method and frequency analysis method have been popular among scholars for their intuitive concept, general method, clear physical meaning and rigorous theory and have become the main tools to study the stability of differential

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systems. At the same time, Burton et al. [3–5] presented the application of fixed point technology to stability analysis and fractional differential equations. Existence, uniqueness and stability of solutions to stochastic partial differential equations have been studied by many researchers; see [2, 8, 15, 17, 20]. Many new achievements have been used in dealing with stochastic integral and differential equations; see [1, 9, 11, 18, 19, 21, 24]. For more details regarding functional analysis and mathematical analysis, we refer to [13, 14, 22].

Recently, Doan et al. [8] studied asymptotic separation between solutions of the following Caputo fractional stochastic differential equations:

$$\begin{aligned} {}^{C}D^{\alpha}_{0+}X(t) &= f\bigl(t, X(t)\bigr) + g\bigl(t, X(t)\bigr)\frac{\mathrm{d}W(t)}{\mathrm{d}t}, \quad t > 0,\ \alpha \in (0, 1], \\ X(0) &= X_0, \end{aligned} \tag{1}$$

where ${}^{C}D^{\alpha}_{0+}$ denotes the Caputo fractional derivative, $f, g : [0, +\infty) \times \mathbb{R} \to \mathbb{R}$ are measurable functions, and $\{W(t), t \in [0, +\infty)\}$ is a standard scalar Brownian motion on an underlying complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} := \{\mathcal{F}_t\}_{t\in[0,+\infty)}, \mathbf{P})$.

For each $t \in [0, +\infty)$, $\mathcal{X}_t := \mathbb{L}^2(\Omega, \mathcal{F}_t, \mathbf{P})$ denotes the space of all $\mathcal{F}_t$-measurable, mean-squared integrable functions $u : \Omega \to \mathbb{R}$ with $\|u\|^2 = \mathbf{E}|u|^2$. A process $X : [0, +\infty) \to \mathbb{L}^2(\Omega, \mathcal{F}, \mathbf{P})$ is called $\mathcal{F}$-adopted if $X(t) \in \mathcal{X}_t$, $t \in [0, +\infty)$.

Doan et al. [8, Thm. 1] applied contraction mapping principle to derive the existence and uniqueness result of (1) by imposing the Lispschitz condition on f and g.

Note that asymptotic behavior and exponential stability of fractional stochastic differential equations in the sense of expected have been studied in [6, 16]. However, fractional Itô formula of Caputo version, stochastic stability and stochastic asymptotic stability of (1) in a probabilistic sense have not been established.

In this paper, we present stochastic stability, stochastic asymptotical stability, almost surly exponential stability and pth moment exponential stability for (1). In Section 3, we establish a new established fractional Itô's formula of Caputo version, and in Section 4 we present the main results. Numerical simulation illustrates our theoretical results in the final section.

2 Preliminaries

Definition 1. (See [12].) Let $\alpha \in (0, 1]$, $T > 0$, $f : [0, T] \to \mathbb{R}$ be a differentiable function. The Caputo fractional derivative of f is defined as

$${}^{C}D^{\alpha}_{0+}f(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{f'(\tau)}{(t-\tau)^{\alpha}}\,\mathrm{d}\tau, \quad t \in [0, T],$$

where $\Gamma(\alpha) := \int_0^{\infty} \tau^{\alpha-1}\mathrm{e}^{-\tau}\,\mathrm{d}\tau$ is Gamma function.

Definition 2. (See [12].) Let $\alpha \in (0,1]$, $f : [0,+\infty) \to \mathbb{R}$. The Caputo fractional integral of f is defined as

$$\mathbf{I}_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\,\mathrm{d}\tau, \quad t \in [0,T].$$

Definition 3. (See [8].) For each $X_0 \in \mathcal{X}_0$, a $\mathcal{F}$-adopted X is called a solution of (1) if the following holds for $t \in [0,+\infty)$:

$$\begin{aligned} X(t) &:= X(t;X_0) \\ &= X_0 + \frac{1}{\Gamma(\alpha)} \left(\int_0^t (t-\tau)^{\alpha-1} f\bigl(\tau, X(\tau)\bigr)\,\mathrm{d}\tau \right. \\ &\quad \left. + \int_0^t (t-\tau)^{\alpha-1} g\bigl(\tau, X(\tau)\bigr)\,\mathrm{d}W(\tau) \right). \end{aligned} \tag{2}$$

We introduce the following assumptions:

(H1) There exists a constant $L > 0$ such that for all $X, \hat{X} \in \mathbb{R}$, $t \in [0,+\infty)$,

$$\bigl|f(t,X) - f(t,\hat{X})\bigr| + \bigl|g(t,X) - g(t,\hat{X})\bigr| \leqslant L|X - \hat{X}|.$$

(H2) $g(\cdot,0)$ is essentially bounded, i.e.,

$$\bigl\|g(t,0)\bigr\|_\infty := \operatorname*{ess\,sup}_{t\in[0,+\infty)} \bigl|g(t,0)\bigr| < +\infty,$$

and $g(\cdot,0)$ is $\mathbb{L}^2$ integrable, i.e., $\int_0^{+\infty} |g(t,0)|^2 \mathrm{d}t < +\infty$.

Lemma 1. (*See* [8, *Thm.* 1].) *Suppose that* (H1) *and* (H2) *hold. Then for* $\alpha \in (1/2, 1)$*,* (1) *has a unique solution* $X(\cdot) \in \mathcal{X}_t := \mathbb{L}^2(\Omega, \mathcal{F}_t, \mathbf{P})$ *given by* (2)*.*

Definition 4. (See [15, Def. 2.1].)

(i) The trivial solution of (1) is said to be stochastically stable or stable in probability if for every pair of $\varepsilon \in (0,1)$ and $r > 0$, there exists a $\delta = \delta(\varepsilon, r, 0) > 0$ such that $\mathbf{P}\{|X(t)| < r\} \geqslant 1 - \varepsilon$, $t \geqslant 0$, whenever $|X_0| < \delta$. Otherwise, it is said to be stochastically unstable.

(ii) The trivial solution of (1) is said to be stochastically asymptotically stable if it is stochastically stable and, moreover, for every $\varepsilon \in (0,1)$, there exists a $\delta_0 = \delta_0(\varepsilon) > 0$ such that $\mathbf{P}\{\lim_{t\to+\infty} X(t) = \mathbf{0}\} \geqslant 1 - \varepsilon$ whenever $|X_0| < \delta_0$ and $\mathbf{0}$ denotes n-dimensional zero vector.

Definition 5. (See [15, Def. 3.1].) The trivial solution of (1) is said to be almost surely exponentially stable if

$$\lim_{t\to+\infty} \sup \frac{1}{t} \ln\bigl|X(t)\bigr| < 0 \quad \text{a.s.}$$

for all $X_0 \in \mathbb{R}^n$, where "a.s." means "almost surely".

Definition 6. (See [15, Def. 4.1].) The trivial solution of (1) is said to be pth moment exponentially stable if there is a pair of positive constants λ and C such that $\mathbf{E}(|X(t)|^p) \leqslant C|X_0|^p \mathrm{e}^{-\lambda t}$ for all $t \geqslant 0$, $X_0 \in \mathbb{R}$. If $p = 2$, then it reduces to exponentially stable in mean square.

Lemma 2. (*See* [15, *Lemma* 2.4].) *Let $\mathcal{F}$ is a $\sigma-$algebra, and let $\{A_k\} \subset \mathcal{F}$ and $\sum_{k=1}^{\infty} \mathbf{P}(A_k) < +\infty$. Then $\mathbf{P}\{\lim_{k\to+\infty} \sup A_k\} = 0$.*

3 Itô's formula of Caputo version

In this part, we introduce the Itô's formula of Caputo fractional version. It pointed out the rules for differentiating a function of Caputo fractional stochastic process.

Let $W(t)$, $t \geqslant 0$, be a standard scalar Brownian motion, and let $Y \in C^{2,1}(\mathbb{R}\times\mathbb{R}^+, \mathbb{R})$ denotes the family of all real-valued functions $Y(z(\cdot), \cdot)$ defined on $\mathbb{R}\times\mathbb{R}^+$ such that they are continuously twice differentiable in z and once in t.

Let $Z(t)$, $t \geqslant 0$, be an Itô process for $\mathrm{d}Z(t) = \tilde{f}(t)\mathrm{d}t + \tilde{g}(t)\mathrm{d}W(t)$, where $\tilde{f} \in \mathbb{L}^1(\mathbb{R}^+, \mathbb{R})$ and $\tilde{g} \in \mathbb{L}^2(\mathbb{R}^+, \mathbb{R})$.

We recall the standard one-dimensional Itô formula.

Lemma 3. (*See* [15, *p. 32, Thm.* 6.2].) *Let $Y(\cdot) := Y(Z(\cdot), \cdot) \in C^{2,1}(\mathbb{R}\times\mathbb{R}^+, \mathbb{R})$. Then $Y(t), t \geqslant 0$ is an Itô process given by*

$$\begin{aligned}\mathrm{d}Y(t) = {}& \left[Y_t\big(Z(t), t\big) + Y_Z\big(Z(t), t\big)\tilde{f}(t) + \frac{1}{2}Y_{ZZ}\big(Z(t), t\big)\tilde{g}^2(t)\right]\mathrm{d}t \\ & + Y_Z\big(Z(t), t\big)\tilde{g}(t)\mathrm{d}W(t) \quad \textit{a.s.}\end{aligned}$$

Now, set $T > 0$. Suppose that $\tilde{X}(t)$ is an Itô process for

$$ {}^C D_{0+}^{\alpha} \tilde{X}(t) = f(t) + g(t)\frac{\mathrm{d}W(t)}{\mathrm{d}t}, \quad t \in [0, T],\ \alpha \in \left(\frac{1}{2}, 1\right), \tag{3}$$

with the initial condition $\tilde{X}(0) = X_0$.

From Lemma 1, (3) has the following unique solution for $t \in [0, T]$:

$$\tilde{X}(t) = X_0 + \frac{1}{\Gamma(\alpha)}\left(\int_0^t (t-\tau)^{\alpha-1} f(\tau)\,\mathrm{d}\tau + \int_0^t (t-\tau)^{\alpha-1} g(\tau)\,\mathrm{d}W(\tau)\right).$$

Note that when $t \in [0, T]$, (3) has an equivalent form

$$\begin{aligned}\mathrm{d}\tilde{X}(t) &= \tilde{X}'(t)\mathrm{d}t \\ &= \frac{\alpha-1}{\Gamma(\alpha)}\left(\int_0^t (t-\tau)^{\alpha-2} f(\tau)\,\mathrm{d}\tau + \int_0^t (t-\tau)^{\alpha-2} g(\tau)\,\mathrm{d}W(\tau)\right)\mathrm{d}t,\end{aligned} \tag{4}$$

where $f(\cdot)(t-\cdot)^{\alpha-2} \in \mathbb{L}^1[0, T]$ and $g(\cdot)(t-\cdot)^{\alpha-2} \in \mathbb{L}^2[0, T]$.

Now, we are ready to introduce the following fractional Itô's formula of the Caputo version.

Theorem 1. *Let* $Y(\cdot) := Y(\tilde{X}(\cdot), \cdot) \in C^{2,1}(\mathbb{R} \times \mathbb{R}^+, \mathbb{R})$*. Then* $Y(\cdot)$ *is an Itô process given by*

$$\begin{aligned}
\mathrm{d}Y\big(\tilde{X}(t), t\big) &= Y_t\big(\tilde{X}(t), t\big)\mathrm{d}t \\
&\quad + \frac{\alpha - 1}{\Gamma(\alpha)} Y_{\tilde{X}}\big(\tilde{X}(t), t\big) \int_0^t (t-\tau)^{\alpha-2} f(\tau)\,\mathrm{d}\tau\,\mathrm{d}t \\
&\quad + \frac{\alpha - 1}{\Gamma(\alpha)} Y_{\tilde{X}}\big(\tilde{X}(t), t\big) \int_0^t (t-\tau)^{\alpha-2} g(\tau)\,\mathrm{d}W(\tau)\,\mathrm{d}t.
\end{aligned}$$

Proof. From Lemma 3 via (4) we can derive that

$$\begin{aligned}
\mathrm{d}Y\big(\tilde{X}(t), t\big) &= \frac{\partial Y(\tilde{X}(t), t)}{\partial t}\mathrm{d}t + \frac{\partial Y(\tilde{X}(t), t)}{\partial \tilde{X}}\,\mathrm{d}\tilde{X}(t) + \frac{1}{2}\frac{\partial Y^2(\tilde{X}(t), t)}{\partial \tilde{X}^2}\big(\mathrm{d}\tilde{X}(t)\big)^2 \\
&= Y_t\big(\tilde{X}(t), t\big)\mathrm{d}t + \frac{\alpha - 1}{\Gamma(\alpha)} Y_{\tilde{X}}\big(\tilde{X}(t), t\big) \int_0^t (t-\tau)^{\alpha-2} f(\tau)\,\mathrm{d}\tau\,\mathrm{d}t \\
&\quad + \frac{\alpha - 1}{\Gamma(\alpha)} Y_{\tilde{X}}\big(\tilde{X}(t), t\big) \int_0^t (t-\tau)^{\alpha-2} g(\tau)\,\mathrm{d}W(\tau)\,\mathrm{d}t.
\end{aligned}$$

The proof is completed. □

4 Stability results

Let $k > 0$ be arbitrary, denote $S_k := \{X(\cdot) \in \mathbb{R}\colon |X(\cdot)| < k\}$, $a \wedge b$ denotes the minimum of a and b, $a \vee b$ denotes the maximum of a and b, $\mathbf{I}_{\{\cdot\}}$ is indicative function.

(V1) There exists a positive definite function $V \in C^{2,1}(S_k \times [0, +\infty); \mathbb{R}^+)$ such that for all $(X(t), t) \in (S_k \times [0, +\infty))$, $\alpha \in (1/2, 1)$,

$$\begin{aligned}
L^\alpha V\big(X(t), t\big) &:= V_t\big(X(t), t\big) \\
&\quad + \frac{\alpha-1}{\Gamma(\alpha)} V_X\big(X(t), t\big) \int_0^t f\big(X(\tau), \tau\big)(t-\tau)^{\alpha-2}\,\mathrm{d}\tau \\
&\leqslant 0. \qquad (5)
\end{aligned}$$

From (V1), $V(0, t) \equiv 0$, and there is a continuous nondecreasing function μ such that $V(X(t), t) \geqslant \mu(|X(t)|)$ for all $(X(t), t) \in (S_k \times [0, +\infty))$.

The following result is motivated by [15, p. 111, Thm. 2.2].

Theorem 2. *Assume* (H1)*,* (H2) *and* (V1) *hold,* $\alpha \in (1/2, 1)$*. Then the trivial solution of* (1) *is stochastically stable.*

Proof. Since the proof is similar to [15, p. 111, Thm. 2.2], we only give the sketch of proofs.

Let $\varepsilon \in (0,1)$ and $r > 0$ be arbitrary and assume that $r < k$. By the continuity of V and $V(\mathbf{0}, 0) = 0$ one can find $\delta = \delta(\varepsilon, r) > 0$ such that

$$\frac{1}{\varepsilon} \sup_{X \in S_\delta} V\big(X(t), t\big) \leqslant \mu(r). \tag{6}$$

Obviously, $\delta < r$. Fix $X_0 \in S_\delta$ and let η be the first exit time of $X(t)$ from S_r, i.e., $\eta = \inf\{t > 0\colon X(t) \notin S_r\}$.

By Theorem 1, for any $t > 0$, we can get

$$\begin{aligned}
&V\big(X(\eta \wedge t), \eta \wedge t\big) \\
&\quad = V(X_0, 0) + \int_0^{\tau \wedge t} V_s\big(X(s), s\big)\,\mathrm{d}s \\
&\qquad + \frac{\alpha - 1}{\Gamma(\alpha)} \int_0^{\eta \wedge t} V_X\big(X(s), s\big) \int_0^s (s-\tau)^{\alpha-2} f\big(X(\tau), \tau\big)\,\mathrm{d}\tau\,\mathrm{d}s \\
&\qquad + \frac{\alpha - 1}{\Gamma(\alpha)} \int_0^{\eta \wedge t} V_X\big(X(s), s\big) \int_0^s (s-\tau)^{\alpha-2} g\big(X(\tau), \tau\big)\,\mathrm{d}W(\tau)\,\mathrm{d}s \\
&\quad = V(X_0, 0) + \int_0^{\eta \wedge t} L^\alpha V\big(X(\tau), \tau\big)\,\mathrm{d}\tau\,\mathrm{d}t \\
&\qquad + \frac{\alpha - 1}{\Gamma(\alpha)} \int_0^{\eta \wedge t} V_X\big(X(s), s\big) \int_0^s (s-\tau)^{\alpha-2} g\big(X(\tau), \tau\big)\,\mathrm{d}W(\tau)\,\mathrm{d}s.
\end{aligned} \tag{7}$$

Take the expectation on (7) and note $L^\alpha V \leqslant 0$. By modulus inequality, for any $t > 0$, $1/2 < \alpha < 1$, we can get

$$\begin{aligned}
&\frac{\alpha - 1}{\Gamma(\alpha)} \left| \mathbf{E}\left(\int_0^{\eta \wedge t} V_X\big(X(s), s\big) \int_0^s (s-\tau)^{\alpha-2} g\big(X(\tau), \tau\big)\,\mathrm{d}W(\tau)\,\mathrm{d}s \right) \right| \\
&\quad \leqslant \frac{\alpha - 1}{\Gamma(\alpha)} \mathbf{E} \left| \int_0^{\eta \wedge t} V_X\big(X(s), s\big) \left(\int_0^s (s-\tau)^{\alpha-2} g\big(X(\tau), \tau\big)\,\mathrm{d}W(\tau) \right) \mathrm{d}s \right| \\
&\quad \leqslant 0.
\end{aligned}$$

Following the same procedure in the proof of [15, p. 111, Thm. 2.2] via (6), one has $\mathbf{P}\{\eta \leqslant t\} \leqslant \varepsilon$. Let $t \to +\infty$, i.e., $\mathbf{P}\{\eta < +\infty\} \leqslant \varepsilon$.

Then we have $\mathbf{P}\{|X(t)| \leqslant r\} \geqslant 1 - \varepsilon$ for all $t \geqslant 0$. By Definition 4(i) the trivial solution of (1) is stochastically stable. □

(V2) The positive definite decreasing function $V \in C^{2,1}(S_k \times [0, +\infty); \mathbb{R}^+)$ such that $L^\alpha V < 0$, $\alpha \in (1/2, 1)$, where $L^\alpha V$ is defined in (5).

From (V2), $V(0, t) \equiv 0$, and there exist continuous nondecreasing functions μ_1, μ_2, μ_3 such that

$$\mu_1\bigl(|X(t)|\bigr) \leqslant V\bigl(X(t), t\bigr) \leqslant \mu_2\bigl(|X(t)|\bigr), \qquad L^\alpha V\bigl(X(t), t\bigr) \leqslant -\mu_3\bigl(|X(t)|\bigr)$$

for all $(X(t), t) \in (S_k \times [0, +\infty))$.

The following result is motivated by [15, p. 112, Thm. 2.3].

Theorem 3. *Assume* (H1), (H2) *and* (V2) *hold. Then the trivial solution of* (1) *is stochastically asymptotically stable.*

Proof. From Theorem 2 the trivial solution of (1) is stochastically stable. Following the same procedure in the proof of [15, p. 112, Thm. 2.3] and using Theorem 1, one can show that there exists $\delta_0 = \delta_0(\varepsilon) > 0$ such that $\mathbf{P}\{\lim_{t\to+\infty} X(t) = 0\} \geqslant 1-\varepsilon$ for $|X_0| < \delta_0$, $\varepsilon \in (0, 1)$.

Based on Definition 4(ii), the trivial solution of (1) is stochastically asymptotically stable. □

(V3) $V \in C^{2,1}(\mathbb{R} \times [0, +\infty); \mathbb{R}^+)$, and there exist constants $c_1 > 1$, $c_2 \in \mathbb{R}$, $c_3 \geqslant 0$ such that
 (i) $c_1|X(t)| \leqslant V(X(t), t)$,
 (ii) $L^\alpha V(X(t), t) \leqslant c_2 V(X(t), t)$,
 (iii) $|V_X(X(t), t)|^2 \int_0^t |g(X(t), t)(s - \tau)^{\alpha-2}|^2 \, \mathrm{d}\tau \geqslant c_3 V^2(X(t), t)$ for all $X(t) \neq 0$, $\alpha \in (1/2, 1)$ and $t \geqslant 0$.

The following result is motivated by [15, p. 121, Thm. 3.3].

Theorem 4. *Assume* (H1), (H2) *and* $(V3)$ *hold. Then*

$$\lim_{t\to+\infty} \sup \frac{1}{t} \ln|X(t)| \leqslant -\frac{1}{\ln c_1} \frac{1-\alpha}{\Gamma(\alpha)}(c_2 + c_3) \quad \textit{a.s.} \tag{8}$$

In particular, if $c_2 + c_3 > 0$*, the trivial solution of* (1) *is almost surely exponentially stable.*

Proof. Fix any $X_0 \neq 0$. From Theorem 1 and (V3)(ii), (iii), for $\alpha \in (1/2, 1)$, we have

$$\ln V\bigl(X(t), t\bigr) = \ln V(X_0, 0) + \int_0^t \frac{(V_s(X(s), s)}{V(X(s), s)} \, \mathrm{d}s + \frac{\alpha - 1}{\Gamma(\alpha)} \int_0^t \frac{V_X(X(s), s) \int_0^s f(X(\tau), \tau)(s - \tau)^{\alpha-2} \, \mathrm{d}\tau}{V(X(s), s)} \, \mathrm{d}s$$

$$
\begin{aligned}
&+ \frac{\alpha-1}{\Gamma(\alpha)} \int_0^t \frac{V_X(X(s),s) \int_0^s g\big(X(\tau),\tau\big)(s-\tau)^{\alpha-2}\big) \,\mathrm{d}W(\tau)}{V(X(s),s)} \,\mathrm{d}s \\
&\leqslant \ln V(X_0,0) + \int_0^t \frac{L^\alpha V(X(s),s)}{V(X(s),s)} \,\mathrm{d}s \\
&+ \frac{\alpha-1}{\Gamma(\alpha)} \int_0^t \frac{V_X(X(s),s) \int_0^s g(X(\tau),\tau)(s-\tau)^{\alpha-2} \,\mathrm{d}W(\tau)}{V(X(s),s)} \,\mathrm{d}s.
\end{aligned}
$$

Set $M(t) = \int_0^t (V_X(X(s),s) \int_0^s g(X(\tau),\tau)(s-\tau)^{\alpha-2} \,\mathrm{d}W(\tau)/V(X(s),s)) \,\mathrm{d}s$. Then let $n = 1, 2, \dots$. For arbitrary $\varepsilon \in (0,1)$, by (V3)(iii) we can get

$$
\mathbf{P}\left\{ \sup_{0\leqslant t\leqslant n} \left| M(t) + \varepsilon \int_0^t \frac{V_X^2(X(s),s) \int_0^s |g(X(\tau),\tau)(s-\tau)^{\alpha-2}|^2 \,\mathrm{d}\tau}{V^2(X(s),s)} \,\mathrm{d}s \right| \geqslant c_3 t \right\} \leqslant \varepsilon.
$$

Using Lemma 2, we get that almost surely

$$
\begin{aligned}
M(t) &\leqslant c_3 t - \varepsilon \int_0^t \frac{V_X^2(X(s),s) \int_0^s |g(X(\tau),\tau)(s-\tau)^{\alpha-2}|^2 \,\mathrm{d}\tau}{V^2(X(s),s)} \,\mathrm{d}s \\
&= (1-\varepsilon) c_3 t.
\end{aligned} \tag{9}
$$

Thus, by (V3)(iii) and (9) we have

$$
\ln V\big(X(t),t\big) \leqslant \ln V(X_0,0) - \frac{1-\alpha}{\Gamma(\alpha)} \big[c_2 + (1-\varepsilon)c_3\big] t.
$$

Then we can get

$$
\frac{1}{t} \ln V\big(X(t),t\big) \leqslant -\frac{1-\alpha}{\Gamma(\alpha)} \big[c_2 + (1-\varepsilon)c_3\big] + \frac{\ln V(X_0,0)}{t}.
$$

Thus

$$
\lim_{t\to+\infty} \sup \frac{1}{t} \ln V\big(X(t),t\big) \leqslant -\frac{1-\alpha}{\Gamma(\alpha)} \big[c_2 + (1-\varepsilon)c_3\big].
$$

Using (V3)(i),

$$
\lim_{t\to+\infty} \sup \frac{1}{t} \ln c_1 \big|X(t)\big| \leqslant \lim_{t\to\infty} \frac{1}{t} \ln V\big(X(t),t\big) \leqslant -\frac{1-\alpha}{\Gamma(\alpha)} \big[c_2 + (1-\varepsilon)c_3\big].
$$

Finally, we get

$$
\lim_{t\to+\infty} \sup \frac{1}{t} \ln\big|X(t)\big| \leqslant -\frac{1}{\ln c_1} \frac{1-\alpha}{\Gamma(\alpha)} \big[c_2 + (1-\varepsilon)c_3\big].
$$

Since ε is arbitrarily, we have (8).

Note that $c_1 > 1$, then, if $c_2 + c_3 > 0$, we have $-(1/\ln c_1)((1-\alpha)/\Gamma(\alpha)) \times (c_2 + c_3) < 0$. By Definition 5 the trivial solution of (1) is almost surely exponentially stable. □

(V4) $V \in C^{2,1}(\mathbb{R} \times [0,+\infty); \mathbb{R}^+)$, and there exist constants $c_1 > 1$, $c_2 \in \mathbb{R}$, $c_3 \geqslant 0$ such that
- (i) $c_1|X(t)| \geqslant V(X(t),t) > 0$,
- (ii) $L^\alpha V(X(t),t) \geqslant c_2 V(X(t),t)$,
- (iii) $|V_X(X(t),t)|^2 \int_0^t |g(X(t),t)(s-\tau)^{\rho-1}|^2 \,\mathrm{d}\tau \leqslant c_3 V^2(X(t),t)$

for all $X(t) \neq 0$ and $t \geqslant 0$, $1/2 < \alpha < 1$.

The following result is motivated by [15, p. 123, Thm. 3.5].

Remark 1. Assume (H1), (H2) and (V4) hold. Then

$$\lim_{t\to+\infty} \inf \frac{1}{t} \ln\bigl|X(t)\bigr| \geqslant -\frac{1}{\ln c_1} \frac{1-\alpha}{\Gamma(\alpha)} (c_2 + c_3) \quad \text{a.s.} \tag{10}$$

If $c_2 + c_3 < 0$, then $-(1/\ln c_1)((1-\alpha)/\Gamma(\alpha))(c_2 + c_3) > 0$. Thus, almost all the sample paths of $X(\cdot)$ will tend to infinity (see (10)), i.e., the trivial solution of (1) is almost surely exponentially unstable.

(H3) The following inequality holds for $K > 0$:

$$\int_0^t f\bigl(X(\tau),\tau\bigr)(t-\tau)^{\alpha-2} \,\mathrm{d}\tau \leqslant K\bigl|X(t)\bigr|, \quad (X,t) \in \mathbb{R} \times [0,+\infty). \tag{11}$$

The following result is motivated by [15, p. 128, Thm. 4.2].

Theorem 5. *Assume* (H1), (H2) *and* (H3) *hold. Then the travail solution of* (1) *is pth moment exponentially stable* (*also almost surely exponentially stable*).

Proof. Let $n = 1, 2, \dots$. By Theorem 1 and (11), for any $n-1 \leqslant t \leqslant n$,

$$\begin{aligned}
\bigl|X(t)\bigr|^p &= |X_0|^p + \frac{\alpha-1}{\Gamma(\alpha)} \int_0^t p\bigl|X(s)\bigr|^{p-1} \int_0^s f\bigl(X(\tau),\tau\bigr)(s-\tau)^{\alpha-2} \,\mathrm{d}\tau \,\mathrm{d}s \\
&\quad + \frac{\alpha-1}{\Gamma(\alpha)} \int_0^t p\bigl|X(s)\bigr|^{p-1} \int_0^s g\bigl(X(\tau),\tau\bigr)(s-\tau)^{\alpha-2} \,\mathrm{d}W(\tau) \,\mathrm{d}s \\
&\leqslant |X_0|^p + pK\frac{\alpha-1}{\Gamma(\alpha)} \int_0^t \bigl|X(s)\bigr|^p \,\mathrm{d}s \\
&\quad + \frac{\alpha-1}{\Gamma(\alpha)} \int_0^t p\bigl|X(s)\bigr|^{p-1} \int_0^s g\bigl(X(\tau),\tau\bigr)(s-\tau)^{\alpha-2} \,\mathrm{d}W(\tau) \,\mathrm{d}s.
\end{aligned}$$

Since $1/2 < \alpha < 1$, by modulus inequality we obtain

$$\begin{aligned}
&\frac{\alpha-1}{\Gamma(\alpha)}\left|\mathbf{E}\left(\int_0^t p\bigl|X(s)\bigr|^{p-1}\int_0^s g\bigl(X(\tau),\tau\bigr)(s-\tau)^{\alpha-2}\,\mathrm{d}W(\tau)\,\mathrm{d}s\right)\right| \\
&\quad\leqslant \frac{p(\alpha-1)}{\Gamma(\alpha)}\mathbf{E}\left|\int_0^t \bigl|X(s)\bigr|^{p-1}\left(\int_0^s g\bigl(X(\tau),\tau\bigr)(s-\tau)^{\alpha-2}\,\mathrm{d}W(\tau)\right)\mathrm{d}s\right| \\
&\quad\leqslant 0.
\end{aligned}$$

Hence,

$$\mathbf{E}\Bigl(\sup_{0\leqslant t\leqslant n}\bigl|X(t)\bigr|^p\Bigr) \leqslant |X_0|^p + pK\frac{\alpha-1}{\Gamma(\alpha)}\int_0^t \mathbf{E}\Bigl(\sup_{0\leqslant t\leqslant n}\bigl|X(s)\bigr|^p\Bigr)\,\mathrm{d}s.$$

Using Gronwall's inequality,

$$\mathbf{E}\Bigl(\sup_{0\leqslant t\leqslant n}\bigl|X(t)\bigr|^p\Bigr) \leqslant |X_0|^p\mathrm{e}^{-\lambda t}, \quad \lambda = \frac{1}{\Gamma(\alpha)}pK(1-\alpha) > 0.$$

By Definition 6 the trivial solution of (1) is pth moment exponentially stable. Now, let $\varepsilon \in (0,1)$ be arbitrary, then

$$\begin{aligned}
\mathbf{P}\Bigl\{\sup_{0\leqslant t\leqslant n}\bigl|X(t)\bigr|^p > \mathrm{e}^{-(\lambda-\varepsilon)t}\Bigr\} &\leqslant \mathrm{e}^{(\lambda-\varepsilon)t}\cdot\mathbf{E}\Bigl(\sup_{0\leqslant t\leqslant n}\bigl|X(t)\bigr|^p\Bigr) \\
&\leqslant |X_0|^p\mathrm{e}^{-\varepsilon t}.
\end{aligned}$$

By Lemma 2 we get

$$\sup_{n-1\leqslant t\leqslant n}\bigl|X(t)\bigr|^p \leqslant \mathrm{e}^{-(\lambda-\varepsilon)t} \quad \text{a.s.}$$

Consequently, for almost all $\omega \in \Omega$, $n-1 \leqslant t \leqslant n$,

$$\frac{1}{t}\ln\bigl|X(t)\bigr| \leqslant \frac{1}{pt}\ln\bigl|X(t)\bigr|^p \leqslant -\frac{1}{pt}(\lambda-\varepsilon)t \leqslant -\frac{(\lambda-\varepsilon)t}{pn}.$$

Hence,

$$\lim_{t\to+\infty}\sup\frac{1}{t}\ln\bigl|X(t)\bigr| \leqslant -\frac{\lambda-\varepsilon}{p} \quad \text{a.s.}$$

Since $\varepsilon > 0$ is arbitrary, we obtain

$$\lim_{t\to+\infty}\sup\frac{1}{t}\ln\bigl|X(t)\bigr| \leqslant -\frac{\lambda}{p} \quad \text{a.s.}$$

By Definition 5 the trivial solution of (1) is almost surly exponential stability. □

5 Examples

Example 1. Consider the stochastic differential equation on

$$\begin{aligned} {}^{C}D^{\alpha}_{0+}X(t) &= f\big(X(t),t\big) + g\big(X(t),t\big)\frac{\mathrm{d}W(t)}{\mathrm{d}t}, \quad t \geqslant 0,\ \alpha \in \left(\frac{1}{2},1\right), \\ X(0) &= X_0, \end{aligned} \tag{12}$$

where $X_0 \in \mathbb{R}$ and $X_0 \neq \infty$, $f : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$, $g : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$.

Let the functions f, g satisfy assumptions (H1) and (H2), and for any $t \geqslant 0$,

$$\int_0^t f\big(X(\tau),\tau\big)(t-\tau)^{\alpha-2}\,\mathrm{d}\tau < +\infty. \tag{13}$$

Then for any $t \geqslant 0$, there is a pair of positive constants θ and K such that

$$-K \leqslant \frac{\alpha-1}{\Gamma(\alpha)}\int_0^t f\big(X(\tau),\tau\big)(t-\tau)^{\alpha-2}\,\mathrm{d}\tau + \theta \leqslant K. \tag{14}$$

Next, let $t \geqslant 0$, we define

$$V\big(X(t),t\big) = \mathrm{e}^{|X(t)|}\exp\left[-\frac{\alpha-1}{\Gamma(\alpha)}\int_0^t f\big(X(\tau),\tau\big)(t-\tau)^{\alpha-2}\,\mathrm{d}\tau + \theta\right].$$

By (14), for $t \geqslant 0$, we can get

$$\mathrm{e}^{|X(t)|}\mathrm{e}^{-\varepsilon K} \leqslant V\big(X(t),t\big) \leqslant \mathrm{e}^{|X(t)|}\mathrm{e}^{\varepsilon K}.$$

Hence, V is positive definite. Next, by (13), for $t \geqslant 0$, we have

$$\begin{aligned} L^{\alpha}V\big(X(t),t\big) &= \mathrm{e}^{|X(t)|}\exp\left[-\frac{\alpha-1}{\Gamma(\alpha)}\int_0^t f\big(X(\tau),\tau\big)(t-\tau)^{\alpha-2}\,\mathrm{d}\tau + \theta\right] \\ &\quad\times\left(\frac{\alpha-1}{\Gamma(\alpha)}\int_0^t f\big(X(\tau),\tau\big)(t-\tau)^{\alpha-2}\,\mathrm{d}\tau + \theta\right) \\ &\quad+ \mathrm{e}^{|X(t)|}\exp\left[-\frac{\alpha-1}{\Gamma(\alpha)}\int_0^t f\big(X(\tau),\tau\big)(t-\tau)^{\alpha-2}\,\mathrm{d}\tau + \theta\right] \\ &\quad\times\frac{\alpha-1}{\Gamma(\alpha)}\int_0^t f\big(X(\tau),\tau\big)(t-\tau)^{\alpha-2}\,\mathrm{d}\tau \\ &\leqslant -\theta\mathrm{e}^{-K}\mathrm{e}^{|X(t)|} < 0. \end{aligned}$$

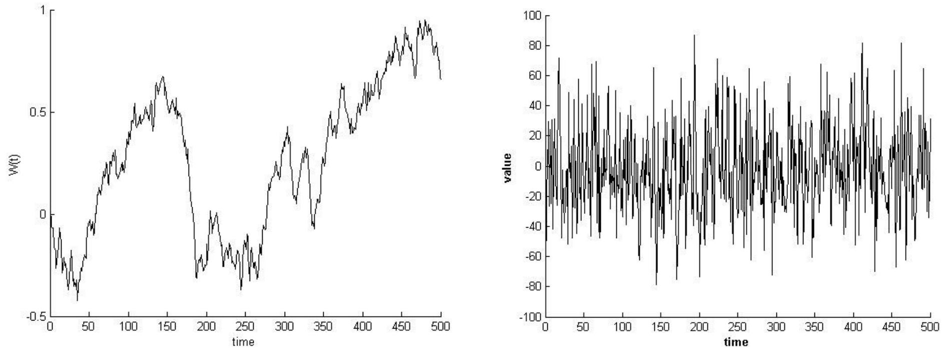

Figure 1. Brownian motion $W(t)$ and white noise $\mathrm{d}W(t)/\mathrm{d}t$, $t \in [0, 500]$.

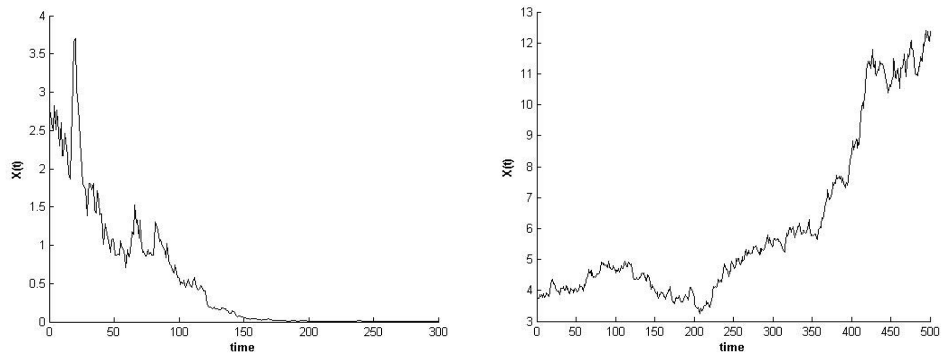

Figure 2. Stochastic stability and unstability when $\alpha = 1$.

From above (V2) holds. By Theorem 2 we can conclude that under (13) and (14), the trivial solution of (12) is stochastically stable.

Let r and β be constants when $\alpha = 1$, $f(X,t) = rX$ and $g(X,t) = \beta X$, the solution of (12) can be see in [15, Ex. 5.5]. Our chosen Brownian motion and white noise are shown in Fig. 1. Using MATLAB software, trajectories of stochastically stable on $r < \beta^2/2$ and $r > \beta^2/2$ and corresponding unstable results are shown in Fig. 2.

Example 2. We consider the one-dimensional fractional Langevin equation on $t \geqslant 0$

$$\begin{aligned} {}^{C}D^{\alpha}_{0+}X(t) &= -rX(t) + \sigma\frac{\mathrm{d}W(t)}{\mathrm{d}t}, \quad \alpha \in \left(\frac{1}{2}, 1\right),\ r > 0, \\ X(0) &= X_0. \end{aligned} \tag{15}$$

The solution of (15) can be expressed as

$$X(t) = X_0 t^{\alpha-1} E_{\alpha,\alpha}\bigl(-rt^{\alpha}\bigr) + \sigma\int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\bigl(-r(t-\tau)^{\alpha}\bigr)\,\mathrm{d}W(\tau), \quad t \geqslant 0.$$

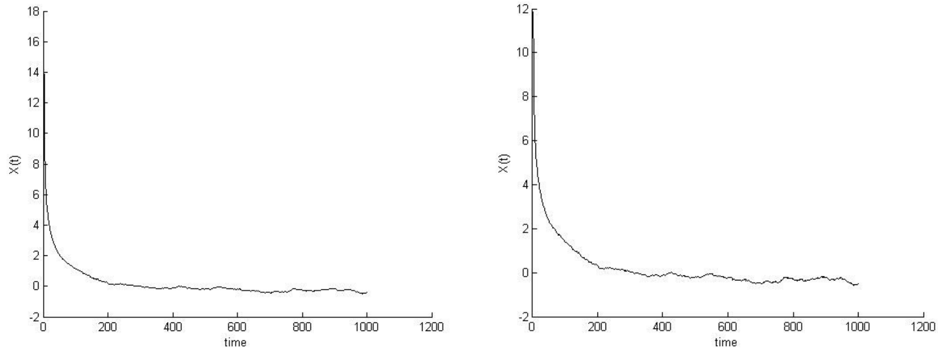

Figure 3. Almost surly exponential stability of fractional Langevin equation when $\alpha = 0.6$ and 0.7.

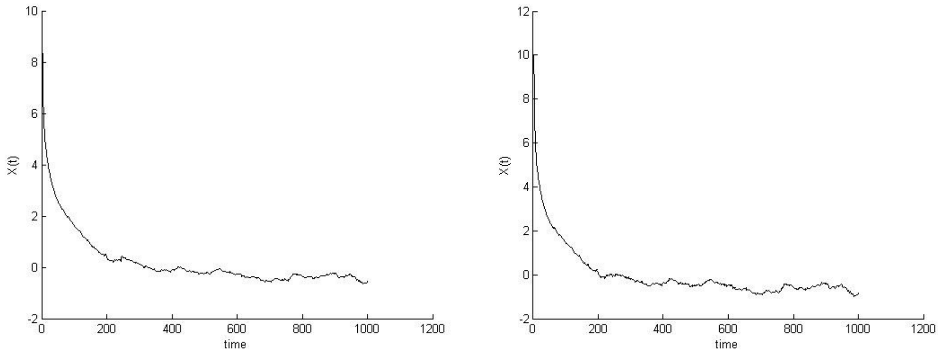

Figure 4. Almost surly exponential stability of fractional Langevin equation when $\alpha = 0.8$ and 0.9.

If $X_0 > 0$ and $X_0 \neq \infty$, according to the results for asymptotic behavior of Mittag-Leffler functions in [7], the solution of (15) is exponentially decaying under the disturbance of white noise.

On the other hand, from Lemma 1 $X(t) \in \mathcal{X}_t := \mathbb{L}^2(\Omega, \mathcal{F}_t, \mathbf{P})$. Since $r > 0$, we can get

$$\int_0^t -rX(\tau)(t-\tau)^{\alpha-2}\,\mathrm{d}\tau < +\infty, \quad t \geqslant 0.$$

Obviously, there are a pair of positive constants $\overline{\theta}$ and $\overline{K}$ such that

$$-\overline{K} \leqslant \frac{\alpha-1}{\Gamma(\alpha)} \int_0^t -rX(\tau)(t-\tau)^{\alpha-2}\,\mathrm{d}\tau + \overline{\theta} \leqslant \overline{K}, \quad t \geqslant 0.$$

From Example 1 the solution of (15) is stochastically stable. Using MATLAB software, the solution trajectories for $\alpha = 0.6, 0.7, 0.8, 0.9$ are shown in Figs. 3 and 4. When

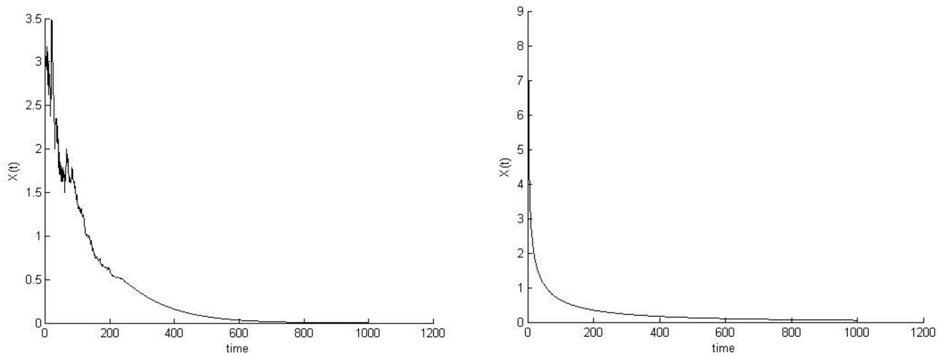

Figure 5. Almost surly exponential stability of fractional Langevin equation when $\alpha = 1$ and white noise vanishes in (15) when $\alpha = 0.6$.

$\alpha = 1$, the solution of (15) can be directly expressed as

$$X(t) = X_0 \mathrm{e}^{-rt} + \sigma \int_0^t \mathrm{e}^{-r(t-\tau)} \,\mathrm{d}W(\tau), \quad t \geqslant 0,$$

and its trajectory is shown in Fig. 5.

6 Conclusion

This paper studies the stability of Caputo-type fractional stochastic differential equations. The Itô's formula of Caputo version is established and used to prove the stochastic stability, stochastic asymptotical stability, almost surly exponential stability and pth moment exponential stability. Numerical examples are given to illustrate the stability results.

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