Nonlinear Analysis: Modelling and Control, Vol. 26, No. 2, 271–292
https://doi.org/10.15388/namc.2021.26.22429

Vilnius University Press

Tykhonov triples and convergence results for hemivariational inequalities*

Rong Hu[a], **Mircea Sofonea**[b], **Yi-Bin Xiao**[a,1]

[a]School of Mathematical Sciences,
University of Electronic Science and Technology of China,
Chengdu, Sichuan, 611731, China
xiaoyb9999@hotmail.com

[b]Laboratoire de Mathématiques et Physique,
University of Perpignan Via Domitia,
52 Avenue Paul Alduy, 66860 Perpignan, France

Received: January 18, 2020 / **Revised:** August 26, 2020 / **Published online:** March 1, 2021

Abstract. Consider an abstract Problem $\mathcal{P}$ in a metric space (X, d) assumed to have a unique solution u. The aim of this paper is to compare two convergence results $u_n' \to u$ and $u_n'' \to u$, both in X, and to construct a relevant example of convergence result $u_n \to u$ such that the two convergences above represent particular cases of this third convergence. To this end, we use the concept of Tykhonov triple. We illustrate the use of this new and nonstandard mathematical tool in the particular case of hemivariational inequalities in reflexive Banach space. This allows us to obtain and to compare various convergence results for such inequalities. We also specify these convergences in the study of a mathematical model, which describes the contact of an elastic body with a foundation and provide the corresponding mechanical interpretations.

Keywords: Tykhonov triple, well-posedness, hemivariational inequality, contact problem, unilateral constraint.

1 Introduction

In this paper, we study various convergence results at three different levels. The first one concerns generic abstract problems in metric spaces. The second one concerns hemivariational inequalities in abstract reflexive Banach space. Finally, the third level concerns boundary value problems, which model the contact of an elastic body with an obstacle, the so-called foundation. For this reason, we use notation and arguments arising from the

*This research was supported by the European Union's Horizon 2020 Research and Innovation Programme under the Marie Sklodowska-Curie grant agreement No. 823731 CONMECH, the National Natural Science Foundation of China (11771067), the Applied Basic Project of Sichuan Province (2019YJ0204) and the Fundamental Research Funds for the Central Universities (ZYGX2019J095).

[1]Corresponding author.

abstract well-posedness theory in the sense of Tykhonov, the theory of hemivariational inequalities and the mathematical theory of contact mechanics. A brief description of these three topics is the following.

The concept of well-posedness in the sense of Tykhonov was introduced in [29] for a minimization problem. It is based on two main ingredients: the existence and uniqueness of the solution and the convergence to the unique solution of any approximating sequence. This concept was generalized to variational inequalities in [15, 16] and to hemivariational inequalities in [8]. References in the field include [1, 13, 27, 28, 31]. An extension of this concept in the study of generic problems in metric spaces was considered in our recent paper [26]. There some abstract results have been proved and then used in the study of nonlinear equations, history-dependent equations, variational and hemivariational inequalities. In this way, a number of convergence results have been obtained.

Hemivariational inequalities represent a special class of inequalities, which arise in the study of nonsmooth boundary value problems. In contrast with the variational inequalities (which are governed by convex functions), hemivariational inequalities are governed by locally Lipschitz functions, which could be nonconvex. For this reason, their study requires prerequisites on nonsmooth analysis. Hemivariational inequalities have been introduced by Panagiotopoulos in early eighties in the context of applications in engineering problems. Later, they have been studied in a large number of papers, including the books [17,19,22]. The mathematical literature on hemivariational inequalities concerns existence, uniqueness, regularity and convergence results, among others. It grew up rapidly in the last decade, motivated by important applications in physics, mechanics and engineering sciences. A recent reference is the book [25], which provides the state of the art in the field together with relevant applications in contact mechanics.

The mathematical theory of contact mechanics deals with the study of systems of partial differential equations, which describe processes of contact with different constitutive laws, different geometries and different interface laws. Such kind of processes arise in industry and daily life, and therefore, a large effort has been put into their modeling, analysis and numerical simulations. The literature on this field is extensive. It deals with the analysis of various models of contact, which are expressed in terms of strongly elliptic, time-dependent or evolutionary nonlinear boundary value problems. References in the field include [4, 5, 10, 14, 20, 22] and, more recently, [2, 3, 17, 23–25]. There various existence and uniqueness results have been proved by using arguments of variational and hemivariational inequalities. Once existence and uniqueness of solutions have been established, related important questions arise, such as convergence results, which provide the link between the solutions of different models.

The aim of this paper is to introduce a functional framework in which various convergence results can be studied. More precisely, if u represents the solution of a given problem, we provide a framework in which two convergence results $u'_n \to u$ and $u''_n \to u$ can be compared.

To this end, we use arguments and ingredients on Tykhonov well-posedness, including a new result in the study of Tykhonov triples, Theorem 2. The theorem is nonstandard and represents the first trait of novelty of this current paper. Then we illustrate the use of

Theorem 2 in the study of hemivariational inequalities of the form

$$u \in K, \quad \langle Au, v-u\rangle + j^0(u; v-u) \geqslant \langle f, v-u\rangle \quad \forall v \in K. \tag{1}$$

Here X is a real reflexive Banach space, $\langle\cdot,\cdot\rangle$ denotes the duality pairing between X and its dual X^*, $K \subset X$, $A : X \to X^*$, $j : X \to \mathbb{R}$ and $f \in X^*$. We assume that j is a locally Lipschitz function, and we denote by $j^0(u; v)$ the Clarke directional derivative of j at the point u in the direction v. Using Theorem 2 in the study of inequality problems of the form (1) allows us to recover, to compare and to complete our convergence results in [18, 25–27] obtained by using different functional methods. Presenting these results in a unified way consists the second trait of novelty of this paper. Finally, we provide a new and nonstandard model of contact for which our results hold, which consists the third novelty of this work.

The rest of the paper is structured as follows. In Section 2, we provide several elementary convergence results for inequalities of the form (1) in the particular case when j vanishes. Moreover, we establish the link between these convergence results. In Section 3, we introduce the concept of Tykhonov triple, and based on the examples in Section 2, we prove our main abstract result, Theorem 2. It provides sufficient conditions which allow us to construct a majorant for a finite family of Tykhonov triples which can be used in the proof of convergence results. In Sections 4 and 5 we apply Theorem 2 in the study of hemivariational inequalities of the form (1) and establish a general convergence result, respectively. Finally, in Section 6 we consider a mathematical model which describes the equilibrium of an elastic body in contact with a foundation. In a variational formulation, the model leads to a hemivariational inequality for the displacement field. We illustrate the use of the abstract results in Section 4 in the study of this inequality and provide the corresponding mechanical interpretations.

We end this section with the remark that in Sections 4 and 5 of this paper, X is a real reflexive Banach space, unless stated otherwise. We use $\|\cdot\|_X$ and $\|\cdot\|_{X^*}$ for the norm on X and its dual X^* and 0_X, 0_{X^*} for the zero element of X and X^*, respectively. We denote by $K_n \xrightarrow{\mathrm{M}} K$ the convergence in the sense of Mosco recalled in (M1), (M2), and ∂j will represent the Clarke subdifferential of the function j assumed to be locally Lipschitz. All the limits, upper and lower limits will be considered as $n \to \infty$ even if we do not mention it explicitly. Moreover, the symbols "$\rightharpoonup$" and "$\to$" denote the weak and the strong convergence in various spaces, which will be specified. In addition, for a nonempty set F, we use the notation $\mathcal{S}(F)$ for the set of sequences whose elements belongs to F and 2^F for the set of nonempty parts of F.

2 Problem statement

We start with an existence and uniqueness result for a class of hemivariational inequalities, which extends (1) and which is needed in the rest of the paper. Thus, besides the data K, A, j and f, we consider a function $\varphi : X \times X \to \mathbb{R}$. Then we formulate the following inequality problem: find an element $u \in K$ such that

$$\langle Au, v-u\rangle + \varphi(u, v) - \varphi(u, u) + j^0(u; v-u) \geqslant \langle f, v-u\rangle \quad \forall v \in K. \tag{2}$$

Note that the function φ is supposed to be convex with respect to the second argument, and for this reason, following the terminology in [21], we refer to inequality (2) as a variational-hemivariational inequality. In the case when j vanishes, it represents a pure variational inequality, and in the case when φ vanishes, it is a pure hemivariational inequality.

In the study of (2), we consider the following assumptions:

(A1) K is nonempty, closed and convex subset of X;

(A2) $A : X \to X^*$ is pseudomonotone and strongly monotone, i.e.,

- (i) if A is bounded and $u_n \rightharpoonup u$ in X with $\limsup\langle Au_n, u_n - u\rangle \leqslant 0$, then $\liminf\langle Au_n, u_n - v\rangle \geqslant \langle Au, u - v\rangle$ for all $v \in X$,
- (ii) there exists $m_A > 0$ such that for all $u, v \in X$,

$$\langle Au - Av, u - v\rangle \geqslant m_A\|u - v\|_X^2;$$

(A3) $j : X \to \mathbb{R}$ is such that

- (i) j is locally Lipschitz,
- (ii) $\|\xi\|_{X^*} \leqslant c_0 + c_1\|v\|_X$ for all $v \in X$, $\xi \in \partial j(v)$ with $c_0, c_1 \geqslant 0$,
- (iii) there exists $\alpha_j \geqslant 0$ such that for all $v_1, v_2 \in X$,

$$j^0(v_1; v_2 - v_1) + j^0(v_2; v_1 - v_2) \leqslant \alpha_j\|v_1 - v_2\|_X^2;$$

(A4) $f \in X^*$;

(A5) $\varphi : X \times X \to \mathbb{R}$ is such that

- (i) $\varphi(\eta, \cdot) : X \to \mathbb{R}$ is convex lower semicontinuous for all $\eta \in X$,
- (ii) for all $\eta_1, \eta_2, v_1, v_2 \in X$, there exists $\alpha_\varphi \geqslant 0$ such that

$$\begin{aligned}\varphi(\eta_1, v_2) - \varphi(\eta_1, v_1) + \varphi(\eta_2, v_1) - \varphi(\eta_2, v_2)\\ \leqslant \alpha_\varphi\|\eta_1 - \eta_2\|_X\|v_1 - v_2\|_X;\end{aligned}$$

(A6) $\alpha_\varphi + \alpha_j < m_A$.

Concerning condition (A2), we follow [25] and recall that an operator $A : X \to X^*$ is pseudomonotone, i.e., it satisfies condition (A2)(i) if and only if it is bounded and $u_n \to u$ weakly in X together with $\limsup\langle Au_n, u_n - u\rangle \leqslant 0$ implies $\lim\langle Au_n, u_n - u\rangle = 0$ and $Au_n \to Au$ weakly in X^*.

The unique solvability of inequality (2) is given by the following result [18, 25].

Theorem 1. *Assume* (A1)–(A6). *Then inequality* (2) *has a unique solution* $u \in K$.

We now consider the following versions of inequality (2):

$$u \in K, \quad \langle Au, v - u\rangle \geqslant \langle f, v - u\rangle \quad \forall v \in K, \tag{3}$$

$$u_n \in K, \quad \langle Au_n, v - u_n\rangle + \theta_n\|v - u_n\|_X \geqslant \langle f, v - u_n\rangle \quad \forall v \in K, \tag{4}$$

$$u_n \in K, \quad \langle Au_n, v - u_n\rangle + \varphi_n(v) - \varphi_n(u_n) \geqslant \langle f, v - u_n\rangle \quad \forall v \in K, \tag{5}$$

$$u_n \in K, \quad \langle Au_n, v - u_n\rangle \geqslant \langle f_n, v - u_n\rangle \quad \forall v \in K, \tag{6}$$

$$u_n \in K_n, \quad \langle Au_n, v - u_n\rangle \geqslant \langle f, v - u_n\rangle \quad \forall v \in K_n, \tag{7}$$

where

(A7) $K_n \subset X$, $f_n \in X^*$, $\theta_n \geqslant 0$, φ_n is a seminorm on X for each $n \in \mathbb{N}$.

Then, assuming (A1), (A2) and (A4), we deduce the unique solvability of inequality (3). Denote in what follows by u the unique solution to this inequality and consider the following assumptions:

(B1) u_n is a solution to inequality (4), and $\theta_n \to 0$;
(B2) u_n is a solution to inequality (5), for each $n \in \mathbb{N}$, there exists $\varepsilon_n > 0$ such that $\varphi_n(v) \leqslant \varepsilon_n \|v\|_X$ for all $v \in X$ and, moreover, $\varepsilon_n \to 0$;
(B3) u_n is a solution to inequality (6) and $f_n \to f$ in X^*;
(B4) u_n is a solution to inequality (7) and $K_n \overset{\mathrm{M}}{\to} K$.

Recall that notation $K_n \overset{\mathrm{M}}{\to} K$ represents a short-hand notation for the convergence in the sense of Mosco, i.e., by definition it means that the following properties hold:

(M1) For every $v \in K$, there exists a sequence $\{v_n\} \subset X$ such that $v_n \in K_n$ for each $n \in \mathbb{N}$ and $v_n \to v$ in X;
(M2) For each sequence $\{v_n\}$ such that $v_n \in K_n$ for each $n \in \mathbb{N}$ and $v_n \rightharpoonup v$ in X, we have $v \in K$.

Then we have the following result.

Proposition 1. *Under conditions* (A1)*,* (A2)*,* (A4) *and* (A7)*, each of assumptions* (B1)–(B4) *above implies the convergence*

$$u_n \to u \quad \textit{in } X. \tag{8}$$

The proof of the convergence (8) under assumptions (B1)–(B3) is standard, and therefore, we skip it. We restrict ourselves to mention that it follows by using estimates based on the strong monotonicity of the operator A, (A2)(ii). The proof of the convergence (8) under assumptions (B4) is more delicate since it is based on arguments of compactness and pseudomonotonicity. Nevertheless, it follows from the proof of a more general convergence result in the study of variational-hemivariational inequality (2) obtained in [30, 32], and therefore, we skip it, too.

We now complete Proposition 1 with the following remark.

Remark 1. Assume (A1), (A2), (A4) and (A7) and let $n \in \mathbb{N}$. Then:

(a) If φ_n is a seminorm on X, then assumption (B2) implies (B1) with $\theta_n = \varepsilon_n$;
(b) If $f_n \in X^*$, then assumption (B3) implies (B1) with $\theta_n = \|f_n - f\|_{X^*}$;
(c) If $f_n \in X^*$, then assumption (B3) implies (B2) with $\varphi_n(v) = \langle f - f_n, v\rangle$ for all $v \in V$;
(d) In general, if (B4) holds, we cannot find $\theta_n \geqslant 0$ such that (B1) holds. Similarly, we cannot find a seminorm φ_n such that (B2) is satisfied nor an element $f_n \in X^*$ such that (B3) holds.

The proof of statements (a)–(c) is elementary. For instance, assume that φ_n is a seminorm and (B2) holds. We have $\varphi_n(v) - \varphi_n(u_n) \leqslant \varphi_n(v - u_n) \leqslant \varepsilon_n\|v - u_n\|_X$ for all $v \in K$, and therefore, inequality (5) implies (4) with $\theta_n = \varepsilon_n$, which proves (a). In addition, the proof of (b) and (c) can be easily obtained, so we omit it. The proof of (d) is a consequence of the following counter-example. Let $X = \mathbb{R}$, $K = [-1, 1]$,

$K_n = [-1 - 1/n, 1 + 1/n]$, $Au = u$ and $f > 2$. Then it is easy to see that $K_n \xrightarrow{\mathrm{M}} K$. Moreover, the solution of inequality (7) is the projection of f in the set K_n, i.e., $u_n = 1 + 1/n$. This shows that u_n cannot satisfy inequalities (4)–(6) since $u_n \notin K$.

It follows from Remark 1 that the convergence (8) under assumptions (B2) or (B3) is a consequence of the convergence (8) under assumption (B1). Also, the convergence (8) under assumption (B3) can be viewed as a consequence of the convergence (8) under assumption (B2). Moreover, the convergence (8) under assumption (B4) cannot be deduced from fact that it holds under assumptions (B1), (B2) or (B3). In other words, it follows from above that some of the convergences imply other convergences and some do not.

Assume now that (X, d) is a metric space and consider Problem $\mathcal{P}$, which has a unique solution denoted by u. An example of Problem $\mathcal{P}$ is given by the problem of finding a solution to the variational inequality (3) in a reflexive space X. It follows from the results above in this section that, in general, there exist several sequences $\{u_n\}$, which converge to u in X. Moreover, for each $n \in \mathbb{N}$, u_n is a solution of Problem $\mathcal{P}_n$, which represents a perturbation of Problem $\mathcal{P}$. Examples are provided by the variational inequalities (4)–(7). Nevertheless, these convergences do not have the same status since, as shown above, some of the convergences could imply others and some of the convergences cannot be implied by others. Therefore, there is a need to compare two convergences, i.e., to establish an order relation between them. And on this matter, we formulate the following question:

($\mathcal{Q}$) Is it possible to compare two convergence results $u_n' \to u$ and $u_n'' \to u$, both in X, in the study of Problem $\mathcal{P}$?

A possible answer to this question will be provided in the next section. There we also construct a relevant example of convergence $u_n \to u$ such that the two convergences in the statement of the question $\mathcal{Q}$ represent particular cases of this third convergence.

3 Tykhonov triples for abstract problems

Everywhere in this section, we assume that (X, d) is a metric space and $\mathcal{P}$ is an abstract mathematical object called generic "problem". We also assume that Problem $\mathcal{P}$ has a unique "solution" $u \in X$, and we use the notation $\mathcal{S}_{\mathcal{P}} = \{u\}$. Next, following our previous paper [31] and using the notation $\mathcal{S}(F)$ and 2^F in introduction, we recall the following definition.

Definition 1.

(a) A Tykhonov triple is a mathematical object of the form $\mathcal{T} = (I, \Omega, \mathcal{C})$, where I is a given nonempty set, $\Omega : I \to 2^X$ and $\mathcal{C}$ is a nonempty subset of the set $\mathcal{S}(I)$.

(b) Given a Tykhonov triple $\mathcal{T} = (I, \Omega, \mathcal{C})$, a sequence $\{u_n\} \in \mathcal{S}(X)$ is called a $\mathcal{T}$-approximating sequence if there exists a sequence $\{\theta_n\} \in \mathcal{C}$ such that $u_n \in \Omega(\theta_n)$ for each $n \in \mathbb{N}$.

(c) Given a Tykhonov triple $\mathcal{T} = (I, \Omega, \mathcal{C})$, Problem $\mathcal{P}$ is said to be well-posed with $\mathcal{T}$ or, equivalently, $\mathcal{T}$-well-posed if it has a unique solution and every $\mathcal{T}$-approximating sequence converges in X to this solution.

Below in this paper, we refer to I as the set of parameters. A typical element of I will be denoted by θ. We refer to the family of sets $\{\Omega(\theta)\}_{\theta\in I}$ as the family of approximating sets, and moreover, we say that $\mathcal{C}$ define the criterion of convergence. Note that $\mathcal{T}$-approximating sequences always exist since, by assumption, $\mathcal{C}\neq\emptyset$ and, moreover, for any sequence $\{\theta_n\}\in\mathcal{C}$ and any $n\in\mathbb{N}$, the set $\Omega(\theta_n)$ is not empty. We also remark that in most of the references in introduction, including [26], the well-posed of a problem was studied with respect to a canonical triple, and for this reason, in that papers, the terminology "approximating sequence" and "well-posedness" was used without any reference to that Tykhonov triple. In contrast, in the current paper, we consider several Tykhonov triples in the study of the same problem, which implicitly requires the use of terminology "$\mathcal{T}$-approximating sequence" and "$\mathcal{T}$-well-posedness".

Let $\mathcal{T}=(I,\Omega,\mathcal{C})$ be a Tykhonov triple. We denote $\widetilde{\mathcal{S}}_{\mathcal{P}}$ the set of sequences of X, which converge to u and by $\widetilde{\mathcal{S}}_{\mathcal{T}}$ the set of $\mathcal{T}$-approximating sequences, that is,

$$\widetilde{\mathcal{S}}_{\mathcal{P}}=\bigl\{\{u_n\}\in\mathcal{S}(X)\colon\ u_n\to u \text{ in } X\bigr\},\tag{9}$$

$$\widetilde{\mathcal{S}}_{\mathcal{T}}=\bigl\{\{u_n\}\in\mathcal{S}(X)\colon\ \{u_n\} \text{ is a } \mathcal{T}\text{-approximating sequence}\bigr\}.\tag{10}$$

Next, we use Definition 1(c) and equalities (9), (10) to see that

(C) Problem $\mathcal{P}$ is $\mathcal{T}$-well-posed if and only if $\widetilde{\mathcal{S}}_{\mathcal{T}}\subset\widetilde{\mathcal{S}}_{\mathcal{P}}$.

Moreover, the set $\widetilde{\mathcal{S}}_{\mathcal{T}}$ of $\mathcal{T}$-approximating sequences suggests us to introduce the following definition.

Definition 2. Given two Tykhonov triples $\mathcal{T}_1=(I_1,\Omega_1,\mathcal{C}_1)$ and $\mathcal{T}_2=(I_2,\Omega_2,\mathcal{C}_2)$, we say that:

(a) $\mathcal{T}_1$ and $\mathcal{T}_2$ are equivalent if their sets of approximating sequences are the same, i.e., $\widetilde{\mathcal{S}}_{\mathcal{T}_1}=\widetilde{\mathcal{S}}_{\mathcal{T}_2}$. In this case, we use the notation $\mathcal{T}_1\approx\mathcal{T}_2$.
(b) $\mathcal{T}_1$ is smaller than $\mathcal{T}_2$ if $\widetilde{\mathcal{S}}_{\mathcal{T}_1}\subset\widetilde{\mathcal{S}}_{\mathcal{T}_2}$. In this case, we use the notation $\mathcal{T}_1\leqslant\mathcal{T}_2$.

It is easy to see that "$\approx$" represents an equivalence relation on the set of Tykhonov triples, while "$\leqslant$" defines a relation of order on the same set. Moreover, using (C), we deduce that the following statements hold:

(C1) If $\mathcal{T}_1\approx\mathcal{T}_2$, then Problem $\mathcal{P}$ is $\mathcal{T}_1$-well-posed if and only if it is $\mathcal{T}_2$-well posed.
(C2) If $\mathcal{T}_1\leqslant\mathcal{T}_2$ and Problem $\mathcal{P}$ is $\mathcal{T}_2$-well-posed, then Problem $\mathcal{P}$ is $\mathcal{T}_1$-well-posed, too.

Next, we remark that any Tykhonov triple such that Problem $\mathcal{P}$ is $\mathcal{T}$-well-posed gives rise to a convergence result to the solution of Problem $\mathcal{P}$ since, following Definition 1(c), we have

(C3) If $\{u_n\}\in\widetilde{\mathcal{S}}_{\mathcal{T}}$ and Problem $\mathcal{P}$ is $\mathcal{T}$-well-posed, then $u_n\to u$ in X.

Assume in what follows that the converse of this implication is also true, i.e.,

(C4) If $u_n\to u$ in X, then there exists a Tykhonov triple $\mathcal{T}$ such that Problem $\mathcal{P}$ is $\mathcal{T}$-well-posed and $\{u_n\}\in\widetilde{\mathcal{S}}_{\mathcal{T}}$.

Note that condition (C4) is valid for the examples presented below in this paper as well as for most of the convergence results in the literatures, see, for instance [26, 27] and the references therein. Moreover, in all these examples, $\mathcal{T}$ can be chosen in a canonical way. Then, using implications (C3) and (C4), we can associate to each convergence result $u_n \to u$ in X a (canonical) Tykhonov triple $\mathcal{T}$ such that Problem $\mathcal{P}$ is $\mathcal{T}$-well-posed and vice versa. This allows us to replace the study of the convergence results by the study of the Tykhonov triples. Moreover, since the implications above are governed by the set $\widetilde{\mathcal{S}}_{\mathcal{T}}$, using Definition 2(a), we deduce that a convergence result to the solution of Problem $\mathcal{P}$ is characterized by an equivalence class of Tykhonov triples with whom Problem $\mathcal{P}$ is well-posed. In addition, since Definition 2(b) allows us to compare two (equivalence classes of) Tykhonov triples, we deduce that it allows us to compare two convergence results, too. This provides a possible answer to the question $\mathcal{Q}$ formulated at the end of Section 2.

We turn now to the construction of a relevant example of Tykhonov triple, which will be used in the next sections. Let $p \in \mathbb{N}$, $\mathcal{T}_i = (I_i, \Omega_i, \mathcal{C}_i)$ be Tykhonov triples with $i = 1, \dots, p$ and consider the following assumptions:

(D1) For each $i = 1, \dots, p$, there exists $c_i \in I_i$ such that the sequence $\boldsymbol{\theta}_i = \{\theta_n^i\}$ defined by $\theta_n^i = c_i$ for each $n \in \mathbb{N}$ belongs to $\mathcal{C}_i$;

(D2) There exists a multifunction $\Omega : I_1 \times I_2 \times \cdots \times I_p \to 2^X$ such that

$$\begin{aligned}
&\Omega_1(\theta_1) \subset \Omega(\theta_1, c_2, c_3, \dots, c_{p-1}, c_p) \quad \forall \theta_1 \in I_1,\\
&\Omega_2(\theta_2) \subset \Omega(c_1, \theta_2, c_3, \dots, c_{p-1}, c_p) \quad \forall \theta_2 \in I_2,\\
&\qquad\dots\\
&\Omega_p(\theta_p) \subset \Omega(c_1, c_2, c_3, \dots, c_{p-1}, \theta_p) \quad \forall \theta_p \in I_p.
\end{aligned}$$

Our main result in this section is the following.

Theorem 2. *Assume* (D1) *and* (D2). *Then there exists a Tykhonov triple* $\mathcal{T} = (I, \Omega, C)$ *such that* $\mathcal{T}_i \leqslant \mathcal{T}$ *for all* $i = 1, \dots, p$.

Proof. Define $I = I_1 \times I_2 \times \cdots \times I_p$, $\mathcal{C} = \mathcal{C}_1 \times \mathcal{C}_2 \times \cdots \times \mathcal{C}_p$, and let $\mathcal{T} = (I, \Omega, \mathcal{C})$, where $\Omega : I \to 2^X$ is the multifunction provided by assumption (D2). We shall prove that $\mathcal{T}_i \leqslant \mathcal{T}$ for all $i = 1, \dots, p$.

Let $\{u_n^1\}$ be a $\mathcal{T}_1$-approximating sequence for Problem $\mathcal{P}$. Then it follows from Definition 1(b) that there exists a sequence $\{\theta_n^1\} \in \mathcal{C}_1$ such that $u_n^1 \in \Omega_1(\theta_n^1)$ for each $n \in \mathbb{N}$. We now use the first inclusion in assumption (D2) to see that

$$u_n^1 \in \Omega(\theta_n^1, c_2, c_3, \dots, c_p) \quad \forall n \in \mathbb{N}. \tag{11}$$

On the other hand, assumption (D1) guarantees that the sequence $\boldsymbol{\theta} = \{\theta_n\}$ with $\theta_n = (\theta_n^1, c_2, c_3, \dots, c_p)$ belongs to $\mathcal{C}$. Combining this result with (11), we deduce that

$$\boldsymbol{\theta} = \{\theta_n\} \in \mathcal{C} \quad \text{and} \quad u_n^1 \in \Omega(\theta_n) \quad \forall n \in \mathbb{N}. \tag{12}$$

We now use (12) and Definition 1(b) to see that the sequence $\{u_n^1\}$ is a $\mathcal{T}$-approximating sequence for Problem $\mathcal{P}$. It follows from here that $\widetilde{\mathcal{S}}_{\mathcal{T}_1} \subset \widetilde{\mathcal{S}}_{\mathcal{T}}$, and using Definition 2(b), we find that $\mathcal{T}_1 \leqslant \mathcal{T}$. A similar argument leads to the inequalities $\mathcal{T}_2 \leqslant \mathcal{T}, \dots, \mathcal{T}_p \leqslant \mathcal{T}$, which concludes the proof. □

We now use implication (C2) and end this section with the following remark: under the assumption of Theorem 2, if Problem $\mathcal{P}$ is $\mathcal{T}$-well-posed, then it is $\mathcal{T}_i$-well-posed for each $i = 1, 2, \dots, p$. In terms of convergence, this remark can be stated as follows: the convergence generated by $\mathcal{T}$ implies each of the convergences generated by $\mathcal{T}_i$ with $i = 1, 2, \dots, p$. The importance of this abstract result arises in the fact that it can be used in order to obtain various convergence results in an unified way as we exemplify in Sections 4–6 below.

4 The case of hemivariational inequalities

In this section, we assume that X is a reflexive Banach space, and we apply Theorem 2 in the study of the hemivariational inequality (1). Therefore, Problem $\mathcal{P}$ under consideration is as follows.

Problem $\mathcal{P}$. Find $u \in X$ such that (1) holds.

Everywhere in this section, we assume that (A1)–(A4) hold, and moreover,

(A6′) $\alpha_j < m_A$.

Then, using Theorem 1 with $\varphi \equiv 0$, we deduce that Problem $\mathcal{P}$ has a unique solution denoted in what follows by u. Next, we consider a function $h : [0, +\infty) \times X \to \mathbb{R}$ such that

(E1) $h(\varepsilon, u) \geqslant 0$ and $h(0, u) = 0$ for all $u \in X$, $\varepsilon \geqslant 0$,
(E2) $h(\varepsilon_n, u_n) \to 0$ whenever $0 \leqslant \varepsilon_n \to 0$, and $\{u_n\} \subset X$ is bounded,
(E3) there exists $L_h : [0, +\infty) \to \mathbb{R}$ such that
  (i) $|h(\varepsilon, u) - h(\varepsilon, v)| \leqslant L_h(\varepsilon)\|u - v\|_X$ for all $u, v \in X$, $\varepsilon > 0$,
  (ii) $L_h(\varepsilon_n) \to 0$ whenever $0 \leqslant \varepsilon_n \to 0$.

Moreover, we assume the following additional condition on the function j:

(E4) For all sequences $\{u_n\}, \{v_n\} \subset X$ such that $u_n \rightharpoonup u$ in X, $v_n \to v$ in X, we have $\limsup j^0(u_n; v_n - u_n) \leqslant j^0(u; v - u)$.

Note that this condition can be avoided in the proof of Theorem 3 below. Nevertheless, we keep it for two reasons: first, it allows us to simplify the proof of this theorem; second, it is satisfied in the example we present in Section 6 below. Moreover, we mention that examples of functions j, which satisfy this conditions, are given in [25].

Next, we consider the Tykhonov triples $\mathcal{T}_1 = (I_1, \Omega_1, \mathcal{C}_1)$, $\mathcal{T}_2 = (I_2, \Omega_2, \mathcal{C}_2)$, $\mathcal{T}_3 = (I_3, \Omega_3, \mathcal{C}_3)$ defined as follows: $I_1 = [0, +\infty)$, $I_2 = \{g\colon g \in X^*\}$, $I_3 = \{\widetilde{K}\colon \widetilde{K}$ is a nonempty closed convex subset of $X\}$,

$$\begin{aligned} \Omega_1(\varepsilon) = \bigl\{u \in K\colon \langle Au, v - u\rangle + j^0(u; v - u) + h(\varepsilon, u)\|v - u\|_X \\ \geqslant \langle f, v - u\rangle\ \forall v \in K\bigr\} \quad \forall \varepsilon \in I_1, \end{aligned} \tag{13}$$
$$\mathcal{C}_1 = \bigl\{\{\varepsilon_n\}\colon \varepsilon_n \in I_1\ \forall n \in \mathbb{N},\ \varepsilon_n \to 0 \text{ as } n \to \infty\bigr\};$$

$$\begin{aligned}\Omega_2(g) = \big\{u \in K\colon\ &\langle Au, v-u\rangle + j^0(u; v-u)\\ &\geqslant \langle g, v-u\rangle\ \forall v \in K\big\} \quad \forall g \in I_2,\\ \mathcal{C}_2 = \big\{\{g_n\}\colon\ &g_n \in I_2\ \forall n \in \mathbb{N},\ g_n \to f \text{ in } X^* \text{ as } n \to \infty\big\};\end{aligned} \tag{14}$$

$$\begin{aligned}\Omega_3(\widetilde{K}) = \big\{u \in \widetilde{K}\colon\ &\langle Au, v-u\rangle + j^0(u; v-u)\\ &\geqslant \langle f, v-u\rangle\ \forall v \in \widetilde{K}\big\} \quad \forall \widetilde{K} \in I_3,\\ \mathcal{C}_3 = \big\{\{K_n\}\colon\ &K_n \in I_3\ \forall n \in \mathbb{N},\ K_n \xrightarrow{\mathrm{M}} K \text{ as } n \to \infty\big\}.\end{aligned} \tag{15}$$

Our main result in this section is the following.

Theorem 3. *Assume* (A1)–(A4), (A6′), (E1)–(E4)*, and let* $\mathcal{T}_1$*,* $\mathcal{T}_2$*,* $\mathcal{T}_3$ *be defined by* (13)–(15)*, respectively. Then there exists a Tykhonov triple* $\mathcal{T} = (I, \Omega, \mathcal{C})$ *such that*:

(a) $\mathcal{T}_1 \leqslant \mathcal{T}$, $\mathcal{T}_2 \leqslant \mathcal{T}$ *and* $\mathcal{T}_3 \leqslant \mathcal{T}$;
(b) *Problem* $\mathcal{P}$ *is* $\mathcal{T}$*-well-posed.*

Proof. (a) Consider the sequences $\boldsymbol{\theta}_1 = \{\theta_n^1\}$, $\boldsymbol{\theta}_2 = \{\theta_n^2\}$, $\boldsymbol{\theta}_3 = \{\theta_n^3\}$ defined by $\theta_n^1 = 0$, $\theta_n^2 = f$, $\theta_n^3 = K$ for all $n \in \mathbb{N}$. Then, using definitions (13)–(15) and assumptions (A1), (A4), we see that $0 \in I_1$, $f \in I_2$, $K \in I_3$ and, moreover, $\boldsymbol{\theta}_1 \in \mathcal{C}_1$, $\boldsymbol{\theta}_2 \in \mathcal{C}_2$, $\boldsymbol{\theta}_3 \in \mathcal{C}_3$. We conclude from here that condition (D1) is satisfied with $p = 3$, $c_1 = 0$, $c_2 = f$ and $c_3 = K$.

On the other hand, define the multifunction $\Omega : I_1 \times I_2 \times I_3 \to 2^X$ by equality

$$\begin{aligned}\Omega(\theta) = \big\{u \in \widetilde{K}\colon\ &\langle Au, v-u\rangle + j^0(u; v-u) + h(\varepsilon, u)\|v-u\|_X\\ &\geqslant \langle g, v-u\rangle\ \forall v \in \widetilde{K}\big\} \quad \forall \theta = (\varepsilon, g, \widetilde{K}) \in I_1 \times I_2 \times I_3.\end{aligned} \tag{16}$$

Then, using assumptions (E1)–(E3) and (13)–(16), it is easy to see that

$$\begin{gathered}\Omega_1(\varepsilon) = \Omega(\varepsilon, f, K) \quad \forall \varepsilon \in I_1, \qquad \Omega_2(g) = \Omega(0, g, K) \quad \forall g \in I_2,\\ \Omega_3(\widetilde{K}) = \Omega(0, f, \widetilde{K}) \quad \forall \widetilde{K} \in I_3,\end{gathered}$$

which shows that condition (D2) is satisfied, too.

It follows from above that we are in a position to apply Theorem 2. Therefore, we conclude that the Tykhonov triple $\mathcal{T} = (I, \Omega, \mathcal{C})$, defined by

$$\begin{aligned}I &= I_1 \times I_2 \times I_3 = \big\{\theta = (\varepsilon, g, \widetilde{K})\colon\ \varepsilon \in I_1, g \in I_2, \widetilde{K} \in I_3\big\},\\ \Omega(\theta) &= \big\{u \in \widetilde{K}\colon\ \langle Au, v-u\rangle + j^0(u; v-u) + h(\varepsilon, u)\|v-u\|_X\\ &\qquad \geqslant \langle g, v-u\rangle\ \forall v \in \widetilde{K}\big\} \quad \forall \theta \in I,\\ \mathcal{C} &= \mathcal{C}_1 \times \mathcal{C}_2 \times \mathcal{C}_3\\ &= \big\{\{\theta_n\}\colon\ \theta_n = (\varepsilon_n, g_n, K_n),\ \{\varepsilon_n\} \in \mathcal{C}_1, \{g_n\} \in \mathcal{C}_2,\ \{K_n\} \in \mathcal{C}_3\big\},\end{aligned} \tag{17}$$

satisfies the inequalities $\mathcal{T}_1 \leqslant \mathcal{T}$, $\mathcal{T}_2 \leqslant \mathcal{T}$ and $\mathcal{T}_3 \leqslant \mathcal{T}$, which concludes the proof of the first part of the theorem.

(b) For the second part of the theorem, we consider a $\mathcal{T}$-approximating sequence of Problem $\mathcal{P}$ denoted by $\{u_n\}$. Then Definition 1(a) and (17) show that there exists a sequence $\boldsymbol{\theta} = \{\theta_n\} \in \mathcal{C}$ with $\theta_n = (\varepsilon_n, g_n, K_n)$ such that

$$\begin{aligned} u_n \in K_n, \quad & \langle Au_n, v - u_n\rangle + j^0(u_n; v - u_n) + h(\varepsilon_n, u_n)\|v - u_n\|_X \\ & \geqslant \langle g_n, v - u_n\rangle \quad \forall v \in K_n. \end{aligned} \tag{18}$$

Recall also that inclusion $\boldsymbol{\theta} = \{\theta_n\} \in \mathcal{C}$ implies the following convergences:

$$\varepsilon_n \to 0, \tag{19}$$

$$g_n \to f \quad \text{in } X^*, \tag{20}$$

$$K_n \xrightarrow{\mathrm{M}} K. \tag{21}$$

We shall prove that $u_n \to u$ in X, and to this end, we divide the proof in three steps described below.

Step 1. The sequence $\{u_n\}$ is bounded in X.

Let $v \in K$ be a given element. Then the convergence $K_n \xrightarrow{\mathrm{M}} K$ as $n \to \infty$, guaranteed by (21), implies that there exists a sequence $\{v_n\} \subset X$ such that $v_n \in K_n$ for all $n \in \mathbb{N}$ and $v_n \to v$ in K. Letting $v = v_n$ in inequality (18), we have

$$\begin{aligned} & \langle Au_n, v_n - u_n\rangle + j^0(u_n; v_n - u_n) + h(\varepsilon_n, u_n)\|v_n - u_n\|_X \\ & \quad \geqslant \langle g_n, v_n - u_n\rangle. \end{aligned} \tag{22}$$

We now use assumption (A2)(ii) to see that

$$\begin{aligned} m_A\|v_n - u_n\|_X^2 & \leqslant \langle Av_n - Au_n, v_n - u_n\rangle \\ & = \langle Av_n, v_n - u_n\rangle - \langle Au_n, v_n - u_n\rangle, \end{aligned}$$

and therefore, inequality (22) yields

$$\begin{aligned} m_A\|v_n - u_n\|_X^2 & \leqslant \big(\|Av_n - g_n\|_{X^*} + h(\varepsilon_n, u_n)\big)\|v_n - u_n\|_X \\ & \quad + j^0(u_n; v_n - u_n). \end{aligned} \tag{23}$$

Moreover, note that condition (E3)(i) implies that

$$h(\varepsilon_n, u_n) \leqslant L_h(\varepsilon_n)\|v_n - u_n\|_X + h(\varepsilon_n, v_n). \tag{24}$$

On the other hand, using assumption (A3) and the properties of the Clarke directional derivative, we have

$$\begin{aligned} j^0(u_n; v_n - u_n) & = j^0(u_n; v_n - u_n) + j^0(v_n; u_n - v_n) - j^0(v_n; u_n - v_n) \\ & \leqslant j^0(u_n; v_n - u_n) + j^0(v_n; u_n - v_n) + \big|j^0(v_n; u_n - v_n)\big| \\ & \leqslant \alpha_j\|u_n - v_n\|_X^2 + \big|\max\big\{\langle \xi, u_n - v_n\rangle \colon \xi \in \partial j(v_n)\big\}\big| \\ & \leqslant \alpha_j\|u_n - v_n\|_X^2 + \big(c_0 + c_1\|v_n\|_X\big)\|u_n - v_n\|_X. \end{aligned} \tag{25}$$

We now combine inequalities (23)–(25) to see that

$$\begin{aligned}&\big(m_A-\alpha_j-L_h(\varepsilon_n)\big)\|u_n-v_n\|_X\\&\quad\leqslant\|Av_n-g_n\|_{X^*}+h(\varepsilon_n,v_n)+c_0+c_1\|v_n\|_X,\end{aligned}$$

and using assumptions (E3)(ii) and (A6′), we find that there exists a positive constant C_0, which does not depend on n such that

$$\|u_n-v_n\|_X\leqslant C_0\big(\|Av_n-g_n\|_{X^*}+h(\varepsilon_n,v_n)+c_0+c_1\|v_n\|_X\big) \tag{26}$$

for n large enough. Therefore, since the sequences $\{v_n\}$ and $\{g_n\}$ are bounded in X and X^*, respectively, and A is a bounded operator, using the convergence (19) and assumption (E2), we deduce from inequality (26) that the sequence $\{u_n-v_n\}$ is bounded in X. This implies that $\{u_n\}$ is a bounded sequence in X, which concludes the proof of this step.

Step 2. The sequence $\{u_n\}$ converges weakly to the solution u of Problem $\mathcal{P}$.

Using the step 1 and the reflexivity of the space X, we deduce that passing to a subsequence if necessary, we have that

$$u_n\rightharpoonup\widetilde{u}\quad\text{as } n\to\infty \tag{27}$$

with some $\widetilde{u}\in X$. Our aim in what follows is to prove that $\widetilde{u}$ is a solution to Problem $\mathcal{P}$. To this end, we remark that the convergences (21) and (27) imply that

$$\widetilde{u}\in K. \tag{28}$$

Consider now an arbitrary element $v\in K$ and a sequence $\{v_n\}\subset X$ such that $v_n\in K_n$ for all $n\in\mathbb{N}$ and $v_n\to v$ in K. Then we use inequality (22) to see that

$$\langle Au_n,u_n-v_n\rangle\leqslant j^0(u_n;v_n-u_n)+h(\varepsilon_n,u_n)\|v_n-u_n\|_X+\langle g_n,u_n-v_n\rangle.$$

Passing to the upper limit in this inequality, we find that

$$\begin{aligned}&\limsup\langle Au_n,u_n-v_n\rangle\\&\quad\leqslant\limsup j^0(u_n;v_n-u_n)+\limsup h(\varepsilon_n,u_n)\|v_n-u_n\|_X\\&\qquad+\limsup\langle g_n,u_n-v_n\rangle.\end{aligned} \tag{29}$$

We now use the convergences (19), (20), (27), $v_n\to v$ in X and assumptions (E2), (E4) to deduce that

$$\limsup\, j^0(u_n;v_n-u_n)\leqslant j^0(\widetilde{u};v-\widetilde{u}), \tag{30}$$

$$h(\varepsilon_n,u_n)\|v_n-u_n\|_X\to 0, \tag{31}$$

$$\langle g_n,u_n-v_n\rangle\to\langle f,\widetilde{u}-v\rangle. \tag{32}$$

We now combine relations (29)–(32) to find that

$$\limsup\langle Au_n,u_n-v_n\rangle\leqslant j^0(\widetilde{u};v-\widetilde{u})+\langle f,\widetilde{u}-v\rangle. \tag{33}$$

Next, we write

$$\langle Au_n, u_n - v_n\rangle = \langle Au_n, u_n - v\rangle + \langle Au_n, v - v_n\rangle,$$

then use assumptions (A2)(i) and the convergence $v_n \to v$ in X to see $\langle Au_n, v - v_n\rangle \to 0$, and therefore,

$$\limsup\langle Au_n, u_n - v_n\rangle = \limsup\langle Au_n, u_n - v\rangle. \tag{34}$$

We now combine inequalities (33) and (34) to obtain that

$$\limsup\langle Au_n, u_n - v\rangle \leqslant j^0(\widetilde{u}; v - \widetilde{u}) + \langle f, \widetilde{u} - v\rangle. \tag{35}$$

Next, we take $v = \widetilde{u}$ in (35) and use the property $j^0(\widetilde{u}; 0_X) = 0$ of the Clarke directional derivative to deduce that

$$\limsup\langle Au_n, u_n - \widetilde{u}\rangle \leqslant 0. \tag{36}$$

Exploiting now the pseudomonotonicity of the operator A, from (27) and (36) we have

$$\langle A\widetilde{u}, \widetilde{u} - v\rangle \leqslant \liminf\langle Au_n, u_n - v\rangle \quad \forall v \in X. \tag{37}$$

Next, from (28), (35) and (37) we obtain that $\widetilde{u}$ is a solution to Problem $\mathcal{P}$ as claimed. Thus, by the uniqueness of the solution we find that $\widetilde{u} = u$.

A careful analysis of the results presented above indicates that every subsequence of $\{u_n\}$, which converges weakly in X, has the same weak limit u. On the other hand, $\{u_n\}$ is bounded in X. Therefore, we deduce that the whole sequence $\{u_n\}$ converges weakly to u in X as $n \to \infty$, which concludes the proof of this step.

Step 3. The sequence $\{u_n\}$ converges strongly to the solution u of Problem $\mathcal{P}$.

We take $v = \widetilde{u} \in K$ in both (35) and (37), then we use equality $\widetilde{u} = u$ to obtain

$$0 \leqslant \liminf\langle Au_n, u_n - u\rangle \leqslant \limsup\langle Au_n, u_n - u\rangle \leqslant 0,$$

which shows that $\lim\langle Au_n, u_n - u\rangle = 0$. Therefore, using the strong monotonicity of the operator A and the convergence $u_n \rightharpoonup u$ in X, we have

$$\begin{aligned} m_A\|u_n - u\|_X^2 &\leqslant \langle Au_n - Au, u_n - u\rangle \\ &= \langle Au_n, u_n - u\rangle - \langle Au, u_n - u\rangle. \end{aligned}$$

Taking limit at both sides of the above inequality yields $u_n \to u$ in X, which concludes the proof of this step.

So, we proved that any $\mathcal{T}$-approximating sequence converges to the solution of Problem $\mathcal{P}$. Therefore, using Definition 1(c) it follows that Problem $\mathcal{P}$ is $\mathcal{T}$-well-posed, which concludes the proof of the theorem. □

We end this section with the following corollary, which represents a direct consequence of Theorem 3.

Corollary 1. *Assume* (A1)–(A4)*,* (A6′)*,* (E1)–(E4)*, and let* $\mathcal{T}_1$*,* $\mathcal{T}_2$ *and* $\mathcal{T}_3$ *be the Tykhonov triples defined by* (13)*,* (14) *and* (15)*, respectively. Then Problem* $\mathcal{P}$ *is* $\mathcal{T}_i$*-well-posed for each* $i = 1, 2, 3$.

5 A convergence result

In this section, we provide a consequence of Theorem 3. To this end, besides the data K, A, j and f in Problem $\mathcal{P}$, for each $n \in \mathbb{N}$, we consider a set K_n, an operator A_n, a function φ_n and an element f_n such that the following hold:

(F1) K_n is nonempty, closed and convex subset of X, and $K_n \overset{\mathrm{M}}{\to} K$ as $n \to \infty$;
(F2) $A_n : X \to X^*$, and there exist $T : X \to X^*$ and $\omega_n \geqslant 0$ such that

(i) $A_n v = Av + \omega_n Tv$ for all $v \in X$,
(ii) $\|Tu - Tv\|_{X^*} \leqslant L_T \|u - v\|_X$ for all $u, v \in X$ with $L_T > 0$,
(iii) $\langle Tu - Tv, u - v\rangle \geqslant 0$ for all $u, v \in X$,
(iv) $\omega_n \to 0$ as $n \to \infty$;

(F3) $\varphi_n : X \times X \to \mathbb{R}$ satisfies condition (A5) with $\alpha_n = \alpha_{\varphi_n} \geqslant 0$, and there exists $\delta_n \geqslant 0$ such that

(i) $\varphi_n(\eta, v_1) - \varphi_n(\eta, v_2) \leqslant \delta_n \|\eta\|_X \|v_1 - v_2\|_X$ for all $\eta, v_1, v_2 \in X$,
(ii) $\delta_n \to 0$ as $n \to \infty$;

(F4) $\alpha_j + \alpha_n < m_A$;
(F5) $f_n \in X^*$ and $f_n \to f$ in X^* as $n \to \infty$;
(F6) $\alpha_n \to 0$ as $n \to \infty$.

With this data, we consider a perturbation of the variational-hemivariational inequality (2),

$$u_n \in K_n, \quad \langle A_n u_n, v - u_n\rangle + \varphi_n(u_n, v) - \varphi_n(u_n, u_n) + j^0(u_n; v - u_n) \geqslant \langle f_n, v - u_n\rangle \quad \forall v \in K_n \tag{38}$$

together with the following problem.

Problem $\mathcal{P}_n$. Find $u_n \in X$ such that (38) holds.

Our main result in this section is the following.

Theorem 4. *Assume* (A1)–(A4), (E4) *and* (F1)–(F6). *Then the following statements hold*:

(a) *There exists a unique solution u to Problem $\mathcal{P}$, and for each $n \in \mathbb{N}$, there exists a unique solution u_n to Problem $\mathcal{P}_n$;*
(b) *The sequence $\{u_n\}$ converges to the unique solution u in X.*

Proof. (a) Note that since $\alpha_n \geqslant 0$, assumption (F4) implies (A6′). Therefore, the existence of the unique solution to Problem $\mathcal{P}$ is a direct consequence of Theorem 1 with $\varphi \equiv 0$.

Let $n \in \mathbb{N}$. It is well known that a monotone and Lipschitz continuous operator is pseudomonotone and the sum of two pseudomonotone operators is pseudomonotone. Therefore, assumptions (A2) and (F2) show that the operator A_n is pseudomonotone. Moreover, it is strongly monotone with the constant m_A. It follows from here that operator A_n satisfies condition (A2), too. On the other hand, assumption (F3) implies that (A5) holds and (F4) implies (A6), both with $\alpha_\varphi = \alpha_n$. Recall also that K_n is a nonempty, closed and convex subset of X, $f_n \in X^*$ and j satisfies (A3). All these ingredients show that we are in a position to use Theorem 1 with K_n, A_n, φ_n and f_n instead of K, A, φ and f, respectively. In this way, we deduce the unique solvability of Problem $\mathcal{P}_n$, which concludes the proof of the first part of the theorem.

(b) For this part of the proof, we use the Tykhonov triple $\mathcal{T} = (I, \Omega, \mathcal{C})$ defined by (17). Let $n \in \mathbb{N}$ and $v \in X$. We use assumption (F2)(i) and (38) to see that

$$\begin{aligned} u_n \in K_n, \quad & \langle Au_n, v - u_n\rangle + \omega_n\langle Tu_n, v - u_n\rangle + \varphi_n(u_n, v) - \varphi_n(u_n, u_n) \\ & + j^0(u_n; v - u_n) \geqslant \langle f_n, v - u_n\rangle \quad \forall v \in K_n. \end{aligned} \tag{39}$$

Then we write

$$\begin{aligned} \omega_n\langle Tu_n, v - u_n\rangle &\leqslant \omega_n\|Tu_n\|_{X^*}\|v - u_n\|_X \\ &\leqslant \omega_n\big(\|Tu_n - T0_X\|_{X^*} + \|T0_X\|_{X^*}\big)\|v - u_n\|_X, \end{aligned}$$

and using assumption (F2)(ii), we find that

$$\omega_n\langle Tu_n, v - u_n\rangle \leqslant \omega_n\big(L_T\|u_n\|_X + \|T0_X\|_{X^*}\big)\|v - u_n\|_X. \tag{40}$$

Next, assumption (F3)(i) implies that

$$\varphi_n(u_n, v) - \varphi_n(u_n, u_n) \leqslant \delta_n\|u_n\|_X\|v - u_n\|_X. \tag{41}$$

We now denote

$$\varepsilon_n = \max\{\omega_n, \delta_n\}, \tag{42}$$

and then we combine (39)–(42) to see that u_n satisfies inequality (18) with $g_n = f_n$ and

$$h(\varepsilon, u) = \varepsilon\big((L_T + 1)\|u\|_X + \|T0_X\|_{X^*}\big).$$

Note that the function h satisfies assumptions (E1)–(E3) with $L_h(\varepsilon) = \varepsilon(L_T + 1)$. On the other hand, (F2)(iv), (F3)(iii) and (42) show that $\varepsilon_n \to 0$, and therefore, (19) holds. Moreover, (F5) implies (20) with $g_n = f_n$, and (F1) implies (21). It follows from here that the sequence $\{\theta_n\}$ with $\theta_n = (\varepsilon_n, f_n, K_n)$ belongs to the set $\mathcal{C}$ defined in (17), and therefore, $\{u_n\}$ is a $\mathcal{T}$-approximating sequence for Problem $\mathcal{P}$. The convergence $u_n \to u$ in X follows from the $\mathcal{T}$-well-posedness of Problem $\mathcal{P}$ guaranteed by Theorem 3. □

We end this section with the following comments. First, the convergence of the solution of inequality (38) to the solution of inequality (2) was obtained in [30, 32] under different assumptions on the data. The aim of these papers was to obtain results on the variational-hemivariational inequality (2). In contrast, in the current paper, we focus on the hemivariational inequality (1), and we consider its perturbation (38) in order to show that its solution approaches the solution of (1) as $n \to \infty$. Such kind of results are important in various applications since they could establish the link between the solutions of different mathematical models. An example arising in contact mechanics will be provided in Section 6 below.

Next, consider the following particular versions of inequalities (18) and (38):

$$u_n \in K_n, \quad \langle Au_n, v - u_n\rangle + j^0(u_n; v - u_n) \geqslant \langle f, v - u_n\rangle \quad \forall v \in K_n, \tag{43}$$

$$u_n \in K, \quad \langle Au_n, v - u_n\rangle + j^0(u_n; v - u_n) \geqslant \langle f_n, v - u_n\rangle \quad \forall v \in K, \tag{44}$$

$$u_n \in K, \quad \langle A_n u_n, v - u_n\rangle + j^0(u_n; v - u_n) \geqslant \langle f, v - u_n\rangle \quad \forall v \in K, \tag{45}$$

$$\begin{aligned} u_n \in K, \quad & \langle A u_n, v - u_n\rangle + j^0(u_n; v - u_n) + \varepsilon_n \|v - u_n\|_X \\ & \geqslant \langle f, v - u_n\rangle \quad \forall v \in K. \end{aligned} \tag{46}$$

Then, using Theorem 3 and Theorem 4(b), we can obtain the convergence of the solution of each of inequalities (43)–(46) to the solution of inequality (1) under appropriate assumptions. The convergence of the solution of (44) and (45) to the solution of inequality (1) stands for a continuous dependence result of the solution of Problem $\mathcal{P}$ with respect to the element f and the operator A, respectively. Moreover, the convergence of the solution of (46) to the solution of inequality (1) extends a result proved in [27] in the particular case when $h(\varepsilon, u) = \varepsilon$.

The convergence of the solution of inequality (43) to the solution of (1) shows the continuous dependence of the solution of Problem $\mathcal{P}$ with respect to the convex set K. This result is important in the numerical analysis of the hemivariational inequality (1). There assumption $K_n \xrightarrow{\mathrm{M}} K$ shows that K_n represents an approximation of the set K in the sense used in [6, 7]. The approximation is external if $K_n \not\subset K$ and is internal if $K_n \subset K$. The internal approximation of hemivariational inequalities with the choice $K_n = X_n \cap K$, where X_n is a finite dimensional subspace of X, was used in [12], for instance. More details on abstract approximation of hemivariational inequalities can be found in [9, 11, 12]. There, besides the convergence of the solution of the discrete scheme (38) to the solution of Problem $\mathcal{P}$, various error estimates have been obtained.

6 A contact problem

Hemivariational inequalities of the forms (1), (2) arise in the study of a number of contact problems with elastic materials. Details on the construction and variational analysis of these problems, including some technical results, can be found in our books [17, 23, 25], and therefore, we skip them. We restrict in this section to illustrate the convergence result provided by Theorem 4 in the study of two contact models.

Let $d \in \{2, 3\}$. We denote by $\mathbb{S}^d$ the space of second-order symmetric tensors on $\mathbb{R}^d$ and use the notation "$\cdot$", $\|\cdot\|$ for the inner product and norm on $\mathbb{R}^d$ and $\mathbb{S}^d$. The zero element of the spaces $\mathbb{R}^d$ and $\mathbb{S}^d$ will be denoted by $\mathbf{0}$. Also, consider a domain $\Omega \subset \mathbb{R}^d$ with smooth boundary Γ divided into three measurable disjoint parts Γ_1, Γ_2 and Γ_3 such that $\mathrm{meas}(\Gamma_1) > 0$ and denote by $\boldsymbol{\nu}$ the unit outward normal vector to Γ.

We use the standard notation for Sobolev and Lebesgue spaces associated to Ω and Γ, and for an element $\boldsymbol{v} \in H^1(\Omega)^d$, we still write $\boldsymbol{v}$ for the trace of $\boldsymbol{v}$ to Γ. In addition, we consider the space $V = \{\boldsymbol{v} \in H^1(\Omega)^d\colon \boldsymbol{v} = \mathbf{0} \text{ on } \Gamma_1\}$, which is real Hilbert space endowed with the canonical inner product $(\boldsymbol{u}, \boldsymbol{v})_V = \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\,\mathrm{d}x$ and the associated norm $\|\cdot\|_V$. Here and below $\boldsymbol{\varepsilon}$ represents the symmetric part of the gradient of $\boldsymbol{v}$. We denote by $\mathbf{0}_V$ the zero element of V and, for an element $\boldsymbol{v} \in V$, v_ν and $\boldsymbol{v}_\tau$ will represent its normal and tangential components on Γ given by $v_\nu = \boldsymbol{v} \cdot \boldsymbol{\nu}$ and $\boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu \boldsymbol{\nu}$, respectively. Finally, V^* represents the dual of V, $\langle \cdot, \cdot \rangle$ denotes the duality pairing between V^* and V, and $\|\gamma\|$ is norm of the trace operator $\gamma : V \to L^2(\Gamma)^d$. Recall that the following inequality holds:

$$\|\boldsymbol{v}\|_{L^2(\Gamma)^d} \leqslant \|\gamma\|\|\boldsymbol{v}\|_V \quad \forall \boldsymbol{v} \in V. \tag{47}$$

The first contact model we consider in this section is constructed by using the data $\mathcal{F}$, $\boldsymbol{f}_0$, $\boldsymbol{f}_2$, k and p assumed to satisfy the following conditions:

(G1) $\mathcal{F} : \mathbb{S}^d \to \mathbb{S}^d$ and
- (i) there exists $L_{\mathcal{F}} > 0$ such that $\|\mathcal{F}(\boldsymbol{\varepsilon}_1) - \mathcal{F}(\boldsymbol{\varepsilon}_2)\| \leqslant L_{\mathcal{F}}\|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|$ for all $\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d$,
- (ii) there exists $m_{\mathcal{F}} > 0$ such that $(\mathcal{F}(\boldsymbol{\varepsilon}_1) - \mathcal{F}(\boldsymbol{\varepsilon}_2)) \cdot (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \geqslant m_{\mathcal{F}}\|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|^2$ for all $\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d$,
- (iii) $\mathcal{F}(\boldsymbol{0}) = \boldsymbol{0}$;

(G2) $\boldsymbol{f}_0 \in L^2(\Omega)^d$, $\boldsymbol{f}_2 \in L^2(\Gamma_2)^d$;

(G3) $k \geqslant 0$;

(G4) $p : \mathbb{R} \to \mathbb{R}$ is a continuous function such that
- (i) $|p(r)| \leqslant \bar{c}_0 + \bar{c}_1|r|$ for all $r \in \mathbb{R}$ with $\bar{c}_0, \bar{c}_1 \geqslant 0$,
- (ii) $(p(r_1) - p(r_2))(r_2 - r_1) \leqslant \alpha_p|r_1 - r_2|^2$ for all $r_1, r_2 \in \mathbb{R}$ with $\alpha_p \geqslant 0$.

Let $q : \mathbb{R} \to \mathbb{R}$ is the function defined by

$$q(r) = \int_0^r p(s)\,\mathrm{d}s \quad \forall r \in \mathbb{R}, \tag{48}$$

and introduce the following notations:

$$K = \{\boldsymbol{v} \in V\colon\ v_\nu \leqslant k \text{ a.e. on } \Gamma_3\}, \tag{49}$$

$$A : V \to V^*, \quad \langle A\boldsymbol{u}, \boldsymbol{v}\rangle = \int_\Omega \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\,\mathrm{d}x, \tag{50}$$

$$j : V \to \mathbb{R}, \quad j(\boldsymbol{v}) = \int_{\Gamma_3} q(v_\nu)\,\mathrm{d}a, \tag{51}$$

$$\boldsymbol{f} \in V^*, \quad \langle \boldsymbol{f}, \boldsymbol{v}\rangle = \int_\Omega \boldsymbol{f}_0 \cdot \boldsymbol{v}\,\mathrm{d}x + \int_{\Gamma_2} \boldsymbol{f}_2 \cdot \gamma\boldsymbol{v}\,\mathrm{d}a$$

for all $\boldsymbol{u}, \boldsymbol{v} \in V$. Then we consider the following problem.

Problem $\mathcal{S}$. Find a displacement field $\boldsymbol{u} \in K$ such that

$$\langle A\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}\rangle + j^0(\boldsymbol{u}; \boldsymbol{v} - \boldsymbol{u}) \geqslant \langle \boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u}\rangle \quad \forall \boldsymbol{v} \in K.$$

Following the arguments in [25], it can be shown that Problem $\mathcal{S}$ represents the variational formulation of a mathematical model, which describes the equilibrium of an elastic body in frictionless contact with a foundation under the action of external forces. It is assumed that the foundation is made of a rigid material covered by a layer of deformable material. The function $\mathcal{F}$ is a constitutive function, while $\boldsymbol{f}_0$ and $\boldsymbol{f}_2$ denote the density of body forces and applied tractions, which act on the body and the surface Γ_2, respectively. Finally, k represents the thickness of the deformable layer, and p is a normal compliance function, which describes its reaction towards the elastic body.

For the second contact model, we consider a set B such that

(H1) B is a closed convex subset of $\mathbb{S}^d$ and $\mathbf{0} \in B$.

Moreover, for each $n \in \mathbb{N}$, we assume that ω_n, μ_n, $\boldsymbol{f}_{0n}$, $\boldsymbol{f}_{2n}$, k_n are given and satisfy

(H2) $\omega_n > 0$, $\omega_n \to 0$;
(H3) $\mu_n > 0$, $\mu_n \to 0$;
(H4) $\boldsymbol{f}_{0n} \in L^2(\Omega)^d$, $\boldsymbol{f}_{2n} \in L^2(\Gamma_2)^d$;
(H5) $\boldsymbol{f}_{0n} \to \boldsymbol{f}_0$ in $L^2(\Omega)^d$, $\boldsymbol{f}_{2n} \to \boldsymbol{f}_2$ in $L^2(\Gamma_2)^d$;
(H6) $k_n \geqslant 0$, $k_n \to k$;
(H7) $(\alpha_p + \mu_n)\|\gamma\|^2 < m_{\mathcal{F}}$.

Note that here we consider only the homogeneous case, for simplicity. Nevertheless, we remark that the results below can be easily extended to the case when the functions $\mathcal{F}$, p, as well as ω_n, μ_n, depend on the spatial variable $\boldsymbol{x} \in \Omega \cup \Gamma$.

We now introduce the following notations:

$$K_n = \{\boldsymbol{v} \in V\colon\ v_\nu \leqslant k_n \text{ a.e. on } \Gamma_3\}, \tag{52}$$

$$A_n : V \to V^*, \quad \langle A_n\boldsymbol{u}, \boldsymbol{v}\rangle = \langle A\boldsymbol{u}, \boldsymbol{v}\rangle + \omega_n \int\limits_\Omega \big(\boldsymbol{\varepsilon}(\boldsymbol{u}) - P_B\boldsymbol{\varepsilon}(\boldsymbol{u})\big) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\,\mathrm{d}x, \tag{53}$$

$$\varphi_n : V \times V \to \mathbb{R}, \quad \varphi_n(\boldsymbol{u}, \boldsymbol{v}) = \mu_n \int\limits_{\Gamma_3} u_\nu^+ \|\boldsymbol{v}_\tau\|\,\mathrm{d}a, \tag{54}$$

$$\boldsymbol{f}_n \in V^*, \quad \langle \boldsymbol{f}_n, \boldsymbol{v}\rangle = \int\limits_\Omega \boldsymbol{f}_{0n} \cdot \boldsymbol{v}\,\mathrm{d}x + \int\limits_{\Gamma_2} \boldsymbol{f}_{2n} \cdot \gamma\boldsymbol{v}\,\mathrm{d}a$$

for all $\boldsymbol{u}, \boldsymbol{v} \in V$. Here and below $P_B : \mathbb{S}^d \to B$ denotes the projection operator on the set B, and r^+ represents the positive part of r, i.e., $r^+ = max\{r, 0\}$. Then, for each $n \in \mathbb{N}$, we consider the following problem.

Problem $\mathcal{S}_n$. Find a displacement field $\boldsymbol{u}_n$ such that

$$\begin{aligned} \boldsymbol{u}_n \in K_n, \quad & \langle A_n\boldsymbol{u}_n, \boldsymbol{v} - \boldsymbol{u}_n\rangle + \varphi_n(\boldsymbol{u}_n, \boldsymbol{v}) - \varphi_n(\boldsymbol{u}_n, \boldsymbol{u}_n) + j^0(\boldsymbol{u}_n; \boldsymbol{v} - \boldsymbol{u}_n) \\ & \geqslant \langle \boldsymbol{f}_n, \boldsymbol{v} - \boldsymbol{u}_n\rangle \quad \forall \boldsymbol{v} \in K_n. \end{aligned}$$

Note that, in contrast to Problem $\mathcal{S}$, Problem $\mathcal{S}_n$ represents the variational formulation of a mathematical model, which describes the equilibrium of an elastic body in frictional contact. The friction is described with the function φ_n in which μ_n represents the coefficient of friction. Moreover, in the statement of Problem $\mathcal{S}_n$, the constitutive law is perturbed by using the elasticity coefficient ω_n and the projection on the convex set B, the densities of body forces and surface tractions are replaced by their perturbation

$\boldsymbol{f}_{0n}$ and $\boldsymbol{f}_{2n}$, respectively, and the thickness of the deformable material is replaced by its perturbation k_n.

Our result in the study of Problem $\mathcal{S}$ and Problem $\mathcal{S}_n$ is the following.

Theorem 5. *Assume* (G1)–(G4), (H1)–(H6). *Then the following statements hold*:

(a) *There exists a unique solution $\boldsymbol{u}$ to Problem $\mathcal{S}$, and for each $n \in \mathbb{N}$, there exists a unique solution $\boldsymbol{u}_n$ to Problem $\mathcal{S}_n$.*

(a) *The sequence $\{\boldsymbol{u}_n\}$ converges to the unique solution $\boldsymbol{u}$ in V.*

Proof. We use Theorem 4 with $X = V$, and to this end, we have check the validity of conditions (A1)–(A4), (E4) and (F1)–(F6). First, we remark that condition (A1) is obviously satisfied. Moreover, the operator A defined by (50) satisfies condition (A2). Indeed, using assumption (G1)(i), we find that

$$\langle A\boldsymbol{u} - A\boldsymbol{v}, \boldsymbol{w}\rangle \leqslant L_{\mathcal{F}}\|\boldsymbol{u}-\boldsymbol{v}\|_V\|\boldsymbol{w}\|_V \quad \forall \boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\in V.$$

This implies that

$$\|A\boldsymbol{u} - A\boldsymbol{v}\|_{V^*} \leqslant L_{\mathcal{F}}\|\boldsymbol{u}-\boldsymbol{v}\|_V \quad \forall \boldsymbol{u},\boldsymbol{v}\in V$$

and shows that A is Lipschitz continuous. On the other hand, using assumption (G1)(ii) yields

$$\langle A\boldsymbol{u} - A\boldsymbol{v}, \boldsymbol{u}-\boldsymbol{v}\rangle_{V^*\times V} \geqslant m_{\mathcal{F}}\|\boldsymbol{u}-\boldsymbol{v}\|_V^2 \quad \forall \boldsymbol{u},\boldsymbol{v}\in V.$$

This shows that condition (A2)(ii) is satisfied with $m_A = m_{\mathcal{F}}$. Since A is Lipschitz continuous and monotone, it follows that A is pseudomonotone, and therefore, (A2)(i) holds.

On the other hand, it is obvious to see that the function q defined by (48) is a locally Lipschitz function. Moreover, using the properties of p and equality $q^0(r;s) = p(r)s$, valid for all $r, s \in \mathbb{R}$, it follows that the function q satisfies condition (A3) on $X = \mathbb{R}$. Therefore, using the arguments in [25, p. 219], we deduce that the function j given by (51) satisfies (A3) with $\alpha_j = \alpha_p\|\gamma\|^2$. In addition, condition (A4) is guaranteed by assumption (G2). Next, we remark condition (F1) is a direct consequence of definitions (49), (52) and assumption (H5).

Consider now the operator $T : V \to V^*$ given by

$$\langle T\boldsymbol{u}, \boldsymbol{v}\rangle = \int\limits_{\Omega} \big(\boldsymbol{\varepsilon}(\boldsymbol{u}) - P_B\boldsymbol{\varepsilon}(\boldsymbol{u})\big)\cdot\boldsymbol{\varepsilon}(\boldsymbol{v})\,\mathrm{d}x \quad \forall \boldsymbol{u},\boldsymbol{v}\in V.$$

Then using the nonexpansivity of the projection yields

$$\|T\boldsymbol{u} - T\boldsymbol{v}\|_{V^*} \leqslant 2\|\boldsymbol{u}-\boldsymbol{v}\|_V, \quad \langle T\boldsymbol{u} - T\boldsymbol{v}, \boldsymbol{u}-\boldsymbol{v}\rangle \geqslant 0 \quad \forall \boldsymbol{u},\boldsymbol{v}\in V.$$

Therefore, assumption 6 and notation (53) imply that condition (F2) is satisfied.

Next, we apply the trace inequality (47) to see that the function φ_n defined by (54) satisfies condition (F3) with $\alpha_n = \delta_n = \mu_n\|\gamma\|^2$, and therefore, assumption (H6) shows that condition (F4) holds. Moreover, we note that assumptions (H3) and (H4) imply condition (F5) and, since $\alpha_n = \mu_n\|\gamma\|^2$, (H2) shows that (F6) holds, too.

Finally, condition (E4) follows from the compactness of the trace operator γ. Indeed, since the function q is regular, Lemma 8 in [25] guarantees that

$$j^0(\boldsymbol{u};\boldsymbol{v}) = \int_{\Gamma_3} q^0(u_\nu;v_\nu)\,\mathrm{d}a = \int_{\Gamma_3} p(u_\nu)v_\nu\,\mathrm{d}a \quad \forall \boldsymbol{u},\boldsymbol{v}\in V. \tag{55}$$

Therefore, if $\boldsymbol{u}_n \rightharpoonup \boldsymbol{u}$ and $\boldsymbol{v}_n \to \boldsymbol{v}$ in V, using (55), we deduce that

$$\begin{aligned}
&\limsup j^0(\boldsymbol{u}_n;\boldsymbol{v}_n-\boldsymbol{u}_n)\\
&\quad= \limsup \int_{\Gamma_3} p(u_{n\nu})(v_{n\nu}-u_{n\nu})\,\mathrm{d}a \geqslant \int_{\Gamma_3} p(u_\nu)(v_\nu-u_\nu)\,\mathrm{d}a\\
&\quad= j^0(\boldsymbol{u};\boldsymbol{v}-\boldsymbol{u}),
\end{aligned}$$

which shows that condition (E4) holds, as claimed.

It follows from above that we are in a position to use Theorem 4 in order to conclude the proof of Theorem 5. □

We end this section with the following mechanical interpretations. First, Theorem 5 provides the unique weak solvability of two contact models: a frictionless model in which the stress tensor $\boldsymbol{\sigma}$ satisfies the elastic constitutive law $\boldsymbol{\sigma} = \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u})$ and a frictional model in which the stress tensor $\boldsymbol{\sigma}$ satisfies the elastic constitutive law $\boldsymbol{\sigma} = \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}) + \omega(\boldsymbol{\varepsilon}(\boldsymbol{u}) - P_B\boldsymbol{\varepsilon}(\boldsymbol{u}))$. Second, it establishes the link between the weak solutions of these contact models constructed by using different mechanical assumptions. Third, it provides the continuous dependence of the weak solution of the first model with respect to the density of body forces, the surface tractions and the thickness of the deformable layer. All these ingredients show that, in addition of the mathematical interest in Theorem 5, it is important from mechanical point of view.

References

1. D.L. Cai, M. Sofonea, Y.-B. Xiao, Tykhonov well-posedness of a mixed variational problem, *Optimization*, 2020, https://doi.org/10.1080/02331934.2020.1808646.
2. D.L. Cai, M. Sofonea, Y.-B. Xiao, Convergence results for elliptic variational-hemivariational inequalities, *Adv. Nonlinear Anal.*, **10**:2–23, 2021, https://doi.org/10.1515/anona-2020-0107.
3. A. Capatina, *Variational Inequalities and Frictional Contact Problems*, Springer, New York, 2014, https://doi.org/10.1007/978-3-319-10163-7.
4. G. Duvaut, J.L. Lions, *Inequalities in Mechanics and Physics*, Springer, Berlin, 1976, https://doi.org/10.1115/1.3424078.
5. C. Eck, J. Jarušek, M. Krbeč, *Unilateral Contact Problems: Variational Methods and Existence Theorems*, Pure Appl. Math., Vol. 270, Chapman/CRC Press, Boca Raton, FL, 2005, https://doi.org/10.1201/9781420027365.

6. R. Glowinski, *Numerical Methods for Nonlinear Variational Problems*, Springer, New York, 1984, `https://doi.org/10.1115/1.3169136`.
7. R. Glowinski, J.L. Lions, R. Trémolières, *Numerical Analysis of Variational Inequalities*, North-Holland, Amsterdam, 1981, `https://doi.org/10.1016/S0168-2024(08)70199-1`.
8. D. Goeleven, D. Mentagui, Well-posed hemivariational inequalities, *Numer. Funct. Anal. Optim.*, **16**(7):909–921, 1995, `https://doi.org/10.1080/01630569508816652`.
9. W. Han, Numerical analysis of stationary variational-hemivariational inequalities with applications in contact mechanics, *Math. Mech. Solids*, **23**(3):279–293, 2018, `https://doi.org/10.1177/1081286517713342`.
10. W. Han, M. Sofonea, *Quasistatic Contact Problems in Viscoelasticity and Viscoplasticity*, AMS/IP Studies in Advanced Mathematics, Vol. 30, AMS/International Press, Providence, RI/Somerville, MA, 2002, `https://doi.org/10.1090/amsip/030`.
11. W. Han, M. Sofonea, Numerical analysis of hemivariational inequalities in contact mechanics, *Acta Numerica*, **28**:175–286, 2019, `https://doi.org/10.1017/S0962492919000023`.
12. W. Han, M. Sofonea, D. Danan, Numerical analysis of stationary variational-hemivariational inequalities, *Numer. Math.*, **139**(3):563–592, 2018, `https://doi.org/10.1007/s00211-018-0951-9`.
13. R. Hu, M. Sofonea, Y.-B. Xiao, A Tykhonov-type well-posedness concept for elliptic hemivariational inequalities, *Z. Angew. Math. Phys.*, **71**(4):120, 2020, `https://doi.org/10.1007/s00033-020-01337-1`.
14. N. Kikuchi, J.T. Oden, *Contact Problems in Elasticity: A Study of Variational Inequalities and Finite Element Methods*, SIAM, Philadelphia, 1988, `https://doi.org/10.1137/1.9781611970845`.
15. R. Lucchetti, F. Patrone, A characterization of Tykhonov well-posedness for minimum problems with applications to variational inequalities, *Numer. Funct. Anal. Optim.*, **3**(4):461–476, 1981, `https://doi.org/10.1080/01630568108816100`.
16. R. Lucchetti, F. Patrone, Some properties of "well–posed" variational inequalities governed by linear operators, *Numer. Funct. Anal. Optim.*, **5**(3):349–361, 1983, `https://doi.org/10.1080/01630568308816145`.
17. S. Migórski, A. Ochal, M. Sofonea, *Nonlinear Inclusions and Hemivariational Inequalities: Models and Analysis of Contact Problems*, Adv. Mech. Math., Vol. 26, Springer, New York, 2013, `https://doi.org/10.1007/978-1-4614-4232-5`.
18. S. Migórski, A. Ochal, M. Sofonea, A class of variational-hemivariational inequalities in reflexive Banach spaces, *J. Elasticity*, **127**(2):151–178, 2017, `https://doi.org/10.1007/s10659-016-9600-7`.
19. Z. Naniewicz, P. D. Panagiotopoulos, *Mathematical Theory of Hemivariational Inequalities and Applications*, CRC Press, Boca Raton, FL, 1994.
20. P.D. Panagiotopoulos, *Inequality Problems in Mechanics and Applications*, Birkhäuser, Boston, MA, 1985, `https://doi.org/10.1007/978-1-4612-5152-1`.
21. P.D. Panagiotopoulos, Nonconvex problems of semipermeable media and related topics, *Z. Angew. Math. Mech.*, **65**(1):29–36, 1985, `https://doi.org/10.1002/zamm.19850650116`.

22. P.D. Panagiotopoulos, *Hemivariational Inequalities: Applications in Mechanics and Engineering*, Springer, Berlin, 1993, https://doi.org/10.1007/978-3-642-51677-1.

23. M. Sofonea, A. Matei, *Mathematical Models in Contact Mechanics*, Lond. Math. Soc. Lect. Note Ser., Vol. 398, Cambridge Univ. Press, Cambridge, 2012, https://doi.org/10.1017/CBO9781139104166.

24. M. Sofonea, A. Matei, Y.-B. Xiao, Optimal control for a class of mixed variational problems, *Z. Angew. Math. Phys.*, **70**(4):127, 2019, https://doi.org/10.1007/s00033-019-1173-4.

25. M. Sofonea, S. Migórski, *Variational-Hemivariational Inequalities with Applications*, Pure Appl. Math., Chapman & Hall/CRC Press, Boca Raton, FL, 2018, https://doi.org/10.1201/9781315153261.

26. M. Sofonea, Y.-B. Xiao, On the well-posedness concept in the sense of Tykhonov, *J. Optim. Theory Appl.*, **183**(1):139–157, 2019, https://doi.org/10.1007/s10957-019-01549-0.

27. M. Sofonea, Y.-B. Xiao, Tykhonov well-posedness of elliptic variational-hemivariational inequalities, *Electron. J. Differ. Equ.*, **2019**:64, 2019, https://doi.org/1912.11340.

28. M. Sofonea, Y.-B. Xiao, Tykhonov well-posedness of a viscoplastic contact problem, *Evol. Equ. Control Theory*, **9**(4):1167–1185, 2020, https://doi.org/10.3934/eect.2020048.

29. A.N. Tykhonov, On the stability of functional optimization problems, *U.S.S.R Comput. Math. Math. Phys.*, **6**(4):28–33, 1966, https://doi.org/10.1016/0041-5553(66)90003-6.

30. Y.-B. Xiao, M. Sofonea, On the optimal control of variational-hemivariational inequalities, *J. Math. Anal. Appl.*, **475**(1):364–384, 2019, https://doi.org/10.1016/j.jmaa.2019.02.046.

31. Y.-B. Xiao, M. Sofonea, Tykhonov triples, well-posedness and convergence results, *Carpathian J. Math.*, to appear.

32. B. Zeng, Z. Liu, S. Migorski, On convergence of solutions to variational-hemivariational inequalities, *Z. Angew. Math. Phys.*, **69**(3):87, 2018, https://doi.org/10.1007/s00033-018-0980-3.