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Infinitely many solutions for a gauged nonlinear Schrödinger equation with a perturbation*

Jiafa Xu[a], **Jie Liu**[b,1], **Donal O'Regan**[c]

[a]School of Mathematical Sciences,
Chongqing Normal University,
Chongqing 401331, China
xujiafa292@sina.com

[b]School of Mathematics and Information Science,
Henan Polytechnic University,
Jiaozuo 454000, China
jzyzliujie@hpu.edu.cn

[c]School of Mathematics, Statistics and Applied Mathematics,
National University of Ireland,
Galway, Ireland
donal.oregan@nuigalway.ie

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Abstract. In this paper, we use the Fountain theorem under the Cerami condition to study the gauged nonlinear Schrödinger equation with a perturbation in $\mathbb{R}^2$. Under some appropriate conditions, we obtain the existence of infinitely many high energy solutions for the equation.

Keywords: gauged Schrödinger equation, infinitely many solutions, Fountain theorem.

1 Introduction

In this paper, we study the existence of infinitely many high energy solutions for the following gauged nonlinear Schrödinger equation with a perturbation in $\mathbb{R}^2$:

$$-\Delta u + \lambda V(x)u + \left(\frac{h^2(|x|)}{|x|^2} + \int_{|x|}^{\infty} \frac{h(s)}{s} u^2(s)\,\mathrm{d}s\right)u = f(u) - \mu g(x)|u|^{q-2}u. \tag{1}$$

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[1]Corresponding author.

We first list our assumptions for our problem (1):

(V1) $V \in C(\mathbb{R}^2, \mathbb{R})$, and $\inf_{x\in\mathbb{R}^2} V(x) \geqslant V_0 > 0$, where V_0 is a positive constant.
(V2) There exists $b > 0$ such that $\operatorname{meas}\{x \in \mathbb{R}^2\colon V(x) \leqslant b\}$ is finite; here meas denotes the Lebesgue measure.

(H1) $f \in C(\mathbb{R}, \mathbb{R})$, and $f(u) = o(|u|)$ as $|u| \to 0$.
(H2) There exists $R_0 \geqslant 0$ such that $F(u) = \int_0^u f(t)\,\mathrm{d}t \geqslant 0$ and $\mathbf{F}(u) = f(u)u/6 - F(u) \geqslant 0$ for $|u| \geqslant R_0$.
(H3) $f(u)u/|u|^6 \to +\infty$ as $|u| \to \infty$.
(H4) There exist $\alpha_0, R_1 > 0$, and $\tau \in (1, +\infty)$ such that $|f(u)|^\tau \leqslant \alpha_0 \mathbf{F}(u)|u|^\tau$ for $|u| \geqslant R_1$.
(H5) $f(-u) = -f(u)$ for $u \in \mathbb{R}$.

(g) $g \in L^{q'}(\mathbb{R}^2)$, and $g(x) \geqslant 0$ ($\not\equiv 0$) for $x \in \mathbb{R}^2$, where $q' \in (1, 2/(2-q))$, $q \in (1, 2)$.

Problem (1) arises in the study of standing wave solutions for the gauged nonlinear Schrödinger equation

$$\begin{aligned} &\mathrm{i}D_0\phi + (D_1D_1 + D_2D_2)\phi + g(\phi) = 0,\\ &\partial_0 A_1 - \partial_1 A_0 = -\operatorname{Im}(\overline{\phi} D_2\phi),\\ &\partial_0 A_2 - \partial_2 A_0 = \operatorname{Im}(\overline{\phi} D_1\phi),\\ &\partial_1 A_2 - \partial_2 A_1 = -\frac{1}{2}|\phi|^2, \end{aligned}$$

where i denotes the imaginary unit, $\partial_0 = \partial/\partial_t$, $\partial_1 = \partial/\partial x_1$, $\partial_2 = \partial/\partial x_2$ for $(t, x_1, x_2) \in \mathbb{R}^{1+2}$, $\phi : \mathbb{R}^{1+2} \to \mathbb{C}$ is the complex scalar field, $A_\kappa : \mathbb{R}^{1+2} \to \mathbb{R}$ is the gauge field, and $D_\kappa = \partial_\kappa + \mathrm{i}A_\kappa$ is the covariant derivative for $\kappa = 0, 1, 2$. From the initial study in [8, 9] many papers on this system appeared in the literature; we refer the reader to [1, 2, 4–8, 10, 11, 13, 14, 18–21, 25, 26, 28, 29] and the references therein.

When $\lambda = 1$, the authors [12] obtained the existence and multiplicity of solutions for (1) with concave-convex nonlinearities $\mu g(x, u) + \nu f(x, u)$, where g has sublinear growth, and f has asymptotically linear or superlinear growth. In [20], the authors studied the existence, nonexistence, and multiplicity of standing waves for (1) ($\lambda = 1$, $\mu = 0$) with asymptotically linear nonlinearities and external potential, and in [1, 2, 4–7, 11, 13, 14, 18, 19, 21, 25, 26, 28, 29], the authors studied the existence and multiplicity of solutions (including sign-changing solutions and ground state solutions) for gauged nonlinear Schrödinger equation

$$-\Delta u + \omega u + \left(\frac{h^2(|x|)}{|x|^2} + \int_{|x|}^{+\infty} \frac{h(s)}{s} u^2(s)\,\mathrm{d}s\right) u = f(u), \quad x \in \mathbb{R}^2.$$

Moreover, in [26], the authors also discussed the energy doubling property, i.e., the energy of sign-changing solutions is strictly larger than two times the least energy. In [10], the authors studied the existence and multiplicity of the positive standing wave with $f(u)+\epsilon k(x)$, where the nonlinearity f behaves like $\exp(\alpha|u|^2)$ as $|u| \to \infty$. Moreover, they obtained a mountain-pass type solution when $\epsilon = 0$.

There also are some papers in the literature, which consider perturbation terms; see [15, 17, 22, 23, 27] and the references therein. In [15, 17], the authors used the famous Ambrosetti–Rabinowitz conditions to study the existence of solutions for the following fractional equations:

$$M\left(\iint\limits_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,\mathrm{d}x\,\mathrm{d}y\right)(-\Delta)_p^s u + V(x)|u|^{p-2}u \\ = f(x,u) + g(x) \quad \text{in } \mathbb{R}^N,$$

and

$$(I-\Delta)^s u + \lambda V(x)u = f(x,u) + \mu\xi(x)|u|^{p-2}u \quad \text{in } \mathbb{R}^N,$$

where $(-\Delta)_p^s$ is the fractional p-Laplacian operator, and $(I-\Delta)^s$ is the fractional Bessel operator. Moreover, [17] also considered the effect of the parameter λ, μ on the existence of solutions for their problem.

Motivated by the aforementioned works, in this paper, we study the existence of infinitely many high energy solutions under some appropriate conditions, which are weaker than the Ambrosetti–Rabinowitz conditions, and also consider the effect of the parameters and the perturbation terms on the existence of solutions.

Now, we state our main result:

Theorem 1. *Suppose that* (V1)*,* (V2)*,* (H1)–(H5)*, and* (g) *hold. Then for any* $\mu > 0$*, there exists* $\Lambda > 0$ *such that system* (1) *possesses infinitely many high energy solutions when* $\lambda \geqslant \Lambda$*.*

Remark 1. By virtue of (H1), (H2), and (H4) we can obtain a growth condition for f. Using (H2) and (H4), for $|u| \geqslant R_2 := \max\{R_0, R_1\}$, we have

$$\left|f(u)\right|^\tau \leqslant \alpha_0 \mathbf{F}(u)|u|^\tau = \alpha_0\left(\frac{1}{6}f(u)u - F(u)\right)|u|^\tau \leqslant \frac{\alpha_0}{6}\left|f(u)\right||u|^{\tau+1},$$

and

$$\left|f(u)\right| \leqslant \sqrt[\tau-1]{\frac{\alpha_0}{6}}|u|^{\frac{\tau+1}{\tau-1}}.$$

Let $p = (\tau+1)/(\tau-1)+1 = 2\tau/(\tau-1)$. Then from (H4) we have $p \in (2, +\infty)$, and

$$\left|f(u)\right| \leqslant \sqrt[\tau-1]{\frac{\alpha_0}{6}}|u|^{p-1} \quad \text{for } |u| \geqslant R_2.$$

On the other hand, using (H1), for all $\varepsilon > 0$, we have

$$\left|f(u)\right| \leqslant \varepsilon|u| \quad \text{for } |u| \leqslant R_2.$$

Therefore, by the above two inequalities we have the growth condition for f:

$$\bigl|f(u)\bigr| \leqslant \varepsilon|u| + c_\varepsilon|u|^{p-1}, \quad u \in \mathbb{R},\ c_\varepsilon := \sqrt[\tau-1]{\frac{\alpha_0}{6}}. \tag{2}$$

Note the relation F and f, and we obtain

$$\bigl|F(u)\bigr| \leqslant \frac{\varepsilon}{2}|u|^2 + \frac{c_\varepsilon}{p}|u|^p, \quad u \in \mathbb{R}. \tag{3}$$

Remark 2. Let $f(t) = t^5(6\log|t| + 1)$, $t \in \mathbb{R}$, and $t \neq 0$. Then $F(t) = t^6 \ln|t|$, and we can check that f, F satisfy (H1)–(H5). For example, if we take $\tau \in (1, 3/2)$, we have

$$\lim_{|t|\to+\infty} \frac{6\ln|t| + 1}{|t|^{\frac{6-4\tau}{\tau}}} = \lim_{|t|\to+\infty} \frac{6\tau}{6-4\tau}\frac{1}{|t|^{\frac{6-4\tau}{\tau}}} = 0.$$

Consequently, for $|t|$ large, we obtain

$$\frac{(6\ln|t| + 1)^\tau}{|t|^{6-4\tau}} \leqslant \frac{\alpha_0}{6}$$

and

$$\bigl|f(t)\bigr|^\tau = \bigl|t^5\bigl(6\ln|t| + 1\bigr)\bigr|^\tau \leqslant \frac{\alpha_0}{6}|t|^{6+\tau} = \alpha_0\mathbf{F}(t)|t|^\tau.$$

This implies that (H4) holds. Moreover, this function also satisfies (H1)–(H3) and (H5). However, this function does not satisfy the Ambrosetti–Rabinowitz condition, namely:

(AR) There exists $\mu > 6$ such that $0 < \mu F(u) \leqslant f(u)u$ for $u \in \mathbb{R} \setminus \{0\}$.

2 Preliminaries

Note the parameter λ, and we can consider the work space

$$E := \left\{u \in H^1\bigl(\mathbb{R}^2\bigr)\colon \int_{\mathbb{R}^2} \bigl(|\nabla u|^2 + \lambda V(x)u^2\bigr)\,\mathrm{d}x < +\infty\right\}.$$

Then E is a Hilbert space with the inner product and norm

$$(u, v) = \int_{\mathbb{R}^2} \bigl(\nabla u \cdot \nabla v + \lambda V(x)uv\bigr)\,\mathrm{d}x, \quad \|u\| = \sqrt{(u, u)}.$$

Moreover, by [24] we have that the embedding $E \hookrightarrow L^r(\mathbb{R}^2)$ is continuous for $r \in [2, +\infty)$ and $E \hookrightarrow L^r(\mathbb{R}^2)$ is compact for $r \in (2, +\infty)$, i.e., there are constants $\gamma_r > 0$ such that $\|u\|_r \leqslant \gamma_r\|u\|$ for $2 \leqslant r < \infty$, where $\|\cdot\|_r$ is the norm in the usual Lebesgue space $L^r(\mathbb{R}^2)$.

In what follows, we present the energy functional $\mathcal{I}: E \to \mathbb{R}$ for problem (1) defined as

$$\mathcal{I}(u) = \frac{1}{2}\int_{\mathbb{R}^2} \left(|\nabla u|^2 + \lambda V(x)u^2\right) \mathrm{d}x + B(u) - \int_{\mathbb{R}^2} F(u)\,\mathrm{d}x + \frac{\mu}{q}\int_{\mathbb{R}^2} g(x)|u|^q\,\mathrm{d}x,$$

where

$$B(u) := \frac{1}{2}\int_{\mathbb{R}^2} \frac{u^2}{|x|^2}\left(\int_0^{|x|} \frac{r}{2}u^2(r)\,\mathrm{d}r\right)^2 \mathrm{d}x = \frac{1}{2}\int_{\mathbb{R}^2} \frac{u^2}{|x|^2}\left(\int_{B_{|x|}} \frac{u^2}{4\pi}\right)^2 \mathrm{d}x.$$

Note (3) and (g). We obtain that $\mathcal{I}$ is of class C^1 and its derivative is

$$\begin{aligned}\langle \mathcal{I}'(u), \varphi\rangle &= \int_{\mathbb{R}^2} \left(\nabla u \nabla\varphi + \lambda V(x)u\varphi\right) \mathrm{d}x + \langle B'(u), \varphi\rangle - \int_{\mathbb{R}^2} f(u)\varphi\,\mathrm{d}x \\ &\quad + \mu \int_{\mathbb{R}^2} g(x)|u|^{q-2}u\varphi\,\mathrm{d}x \quad \forall \varphi \in E,\end{aligned}$$

where

$$\langle B'(u), \varphi\rangle = \int_{\mathbb{R}^2} \left(\frac{h^2(|x|)}{|x|^2} + \int_{|x|}^{+\infty} \frac{h(s)}{s}u^2(s)\,\mathrm{d}s\right) u(x)\varphi(x)\,\mathrm{d}x \quad \forall \varphi \in E.$$

Lemma 1. (*See* [1,13,14,29].) *Suppose that* $\{u_n\}$ *converges weakly to a function* u *in* E *as* $n \to \infty$. *Then*

(i) $\lim_{n\to+\infty} B(u_n) = B(u)$,
(ii) $\lim_{n\to+\infty}\langle B'(u_n), u_n\rangle = \langle B'(u), u\rangle$,
(iii) $\lim_{n\to+\infty}\langle B'(u_n), \varphi\rangle = \langle B'(u), \varphi\rangle$,
(iv) $\langle B'(u), u\rangle = 6B(u)$,
(v) $B(u) \leqslant C_0\|u\|_4^4\|u\|_2^2 \leqslant C_0\gamma_2^2\gamma_4^4\|u\|^6 := C_1\|u\|^6$ *for some* $C_0, C_1 > 0$.

In order to obtain our main result, we need to introduce the Fountain theorem under the Cerami condition (C).

Definition 1. (See [16].) Assume that X is a Banach space. We say that J satisfies the Cerami condition if

(C) $J \in C^1(X, \mathbb{R})$, and for all $c \in \mathbb{R}$,
  (i) any bounded sequence $\{u_n\} \subset X$ satisfying $J(u_n) \to c$, $J'(u_n) \to 0$ possesses a convergent subsequence;
  (ii) there exist $\sigma, R, \beta > 0$ such that for any $u \in J^{-1}([c-\sigma, c+\sigma])$ with $\|u\| \geqslant R$, $\|J'(u)\|\|u\| \geqslant \beta$.

Lemma 2. (*See* [16].) *Assume that* $X = \overline{\bigoplus_{j=1}^{\infty} X_j}$, *where* X_j *are finite dimensional subspaces of* X. *For each* $k \in \mathbb{N}$, *let* $Y_k = \bigoplus_{j=1}^{k} X_j$, $Z_k = \overline{\bigoplus_{j=k}^{\infty} X_j}$. *Suppose that* $J \in C^1(X, \mathbb{R})$ *satisfies the Cerami condition* (C) *and* $J(-u) = J(u)$. *Assume for each* $k \in \mathbb{N}$, *there exist* $\rho_k > r_k > 0$ *such that*

(i) $b_k = \inf_{u \in Z_k \cap S_{r_k}} J(u) \to +\infty$, $k \to \infty$,
(ii) $a_k = \max_{u \in Y_k \cap S_{\rho_k}} J(u) \leqslant 0$, *where* $S_\rho = \{u \in X\colon \|u\| = \rho\}$.

Then J *has a sequence of critical points* u_n *such that* $J(u_n) \to +\infty$ *as* $n \to \infty$.

3 Proof of Theorem 1

Lemma 3. *Let sequence* $\{u_n\}$ *converge weakly to a function* u *in* E, $u_n(x) \to u(x)$ *a.e. in* $\mathbb{R}^2$ *as* $n \to \infty$. *Then*

$$\mathcal{I}(u_n) = \mathcal{I}(u_n - u) + \mathcal{I}(u) + o(1) \quad \text{as } n \to \infty, \tag{4}$$

$$\langle \mathcal{I}'(u_n), \varphi \rangle = \langle \mathcal{I}'(u_n - u), \varphi \rangle + \langle \mathcal{I}'(u), \varphi \rangle + o(1) \quad \forall \varphi \in E \text{ as } n \to \infty. \tag{5}$$

In particular, if

$$\mathcal{I}(u_n) \to c, \quad \mathcal{I}'(u_n) \to 0 \quad \text{as } n \to \infty,$$

then

$$\mathcal{I}(u_n - u) = c - \mathcal{I}(u) + o(1) \quad \text{as } n \to \infty, \tag{6}$$

and

$$\langle \mathcal{I}'(u_n - u),\, \varphi \rangle = o(1) \quad \forall \varphi \in E \text{ as } n \to \infty. \tag{7}$$

Proof. From the compactness of $E \hookrightarrow L^r(\mathbb{R}^2)$, for $r \in (2, +\infty)$, we have

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{weakly in } E, \\ u_n &\to u \quad \text{strongly in } L^p(\mathbb{R}^2) \text{ for } p \in (2, +\infty), \\ u_n &\to u \quad \text{for a.e. } x \in \mathbb{R}^2. \end{aligned}$$

Let $w_n = u_n - u$. Then we have

$$\begin{aligned} w_n &\rightharpoonup 0 \quad \text{weakly in } E, \\ w_n &\to 0 \quad \text{strongly in } L^p(\mathbb{R}^2) \text{ for } p \in (2, +\infty), \\ w_n &\to 0 \quad \text{for a.e. } x \in \mathbb{R}^2. \end{aligned}$$

Since $u_n \rightharpoonup u$ in E, we have $(u_n - u, u) \to 0$ as $n \to \infty$, which implies

$$\|u_n\|^2 = (w_n + u,\, w_n + u) = \|w_n\|^2 + \|u\|^2 + o(1) \quad \text{as } n \to \infty.$$

Note Lemma 1(v), and we have $B(u_n-u)\leqslant C_0\|u_n-u\|_4^4\|u_n-u\|_2^2\to 0$ as $n\to\infty$. Consequently, to obtain (4), by Lemma 1(i) we only need to check that

$$\int\limits_{\mathbb{R}^2}\big(F(u_n)-F(w_n)-F(u)\big)\,\mathrm{d}x=o(1)\quad\text{as } n\to\infty \tag{8}$$

and

$$\int\limits_{\mathbb{R}^2} g(x)\big(|u_n|^q-|w_n|^q-|u|^q\big)\,\mathrm{d}x=o(1)\quad\text{as } n\to\infty. \tag{9}$$

Note the definition of $(\cdot,\cdot)$, for all $n\in\mathbb{N}$, we have $(u_n,\varphi)=(u_n-u,\varphi)+(u,\varphi)$. Moreover, since $w_n\rightharpoonup 0$ in E and by Lemma 1(iii), to prove (5), it suffices to show that

$$\sup_{\|\varphi\|=1}\int\limits_{\mathbb{R}^2}\big(f(u_n)-f(w_n)-f(u)\big)\varphi\,\mathrm{d}x=o(1)\quad\text{as } n\to\infty \tag{10}$$

and

$$\begin{aligned}&\sup_{\|\varphi\|=1}\int\limits_{\mathbb{R}^2} g(x)\big(|u_n|^{q-2}u_n-|u_n-u|^{q-2}(u_n-u)-|u|^{q-2}u\big)\varphi\,\mathrm{d}x\\&\quad=o(1)\quad\text{as } n\to\infty.\end{aligned} \tag{11}$$

We first prove that (9) and (11). Using the inequality from page 13 in [17] and the Hölder inequality, for $qq'/(q'-1)>2$, we have

$$\begin{aligned}\bigg|\int\limits_{\mathbb{R}^2} g(x)\big(|u_n|^q-|u|^q\big)\,\mathrm{d}x\bigg|&\leqslant\int\limits_{\mathbb{R}^2} g(x)|w_n|^q\,\mathrm{d}x\leqslant\|g\|_{q'}\|w_n\|^q_{\frac{qq'}{q'-1}}\\&\to 0\quad\text{as } n\to\infty.\end{aligned}$$

Hence, (9) holds. From Lemma 1 in [3] there exists $C_q>0$ such that $||u_n|^{q-2}u_n-|u|^{q-2}u|\leqslant C_q|u_n-u|^{q-1}$. Therefore, from (g) and the Hölder inequality we only need to prove:

$$\begin{aligned}&\sup_{\|\varphi\|=1}\bigg|\int\limits_{\mathbb{R}^2} g(x)|w_n|^{q-2}w_n\varphi\,\mathrm{d}x\bigg|\\&\quad\leqslant\sup_{\|\varphi\|=1}\int\limits_{\mathbb{R}^2} g(x)|w_n|^{q-1}|\varphi|\,\mathrm{d}x\\&\quad\leqslant\|g\|_{q'}\bigg(\int\limits_{\mathbb{R}^2}|\varphi|^{\frac{qq'}{q'-1}}\,\mathrm{d}x\bigg)^{\frac{q'-1}{qq'}}\bigg(\int\limits_{\mathbb{R}^2}|w_n|^{\frac{qq'}{q'-1}}\,\mathrm{d}x\bigg)^{\frac{q-1}{q'-1}qq'}\\&\quad\leqslant\|g\|_{q'}\gamma_{\frac{qq'}{q'-1}}\|w_n\|^{q-1}_{\frac{qq'}{q'-1}}\to 0\quad\text{as } n\to\infty.\end{aligned}$$

Consequently, (11) is true. Note that we can use similar methods in Lemma 4.7 of [30] to prove (10). In what follows, we prove (8). Using the ideas in [17, 22, 23], we have

$$F(u_n) - F(u_n - u) = -\int_0^1 \left(\frac{\mathrm{d}}{\mathrm{d}t} F(u_n - tu)\right) \mathrm{d}t = \int_0^1 f(u_n - tu)u \,\mathrm{d}t.$$

Hence, from (2) we obtain

$$\left|F(u_n) - F(u_n - u)\right| \leqslant \varepsilon_1 |u_n||u| + \varepsilon_1 |u|^2 + C_{\varepsilon_1} |u_n|^{p-1}|u| + C_{\varepsilon_1}|u|^p$$

for some $\varepsilon_1, C_{\varepsilon_1} > 0$, where $p > 2$. Therefore, together with (3), using the Young inequality with ε (for all $\varepsilon > 0$), we obtain

$$\begin{aligned}&\left|F(u_n) - F(w_n) - F(u)\right| \\ &\quad\leqslant C_{\varepsilon_1, C_{\varepsilon_1}}\left[\varepsilon|u_n|^2 + C_{\varepsilon,\varepsilon_1}|u|^2 + \varepsilon|u_n|^p + C_{\varepsilon, C_{\varepsilon_1}, c_\varepsilon}|u|^p\right].\end{aligned}$$

Consequently, we consider the function $\widetilde{f}_n$ defined as

$$\widetilde{f}_n(x) := \max\left\{\left|F(u_n) - F(w_n) - F(u)\right| - C_{\varepsilon_1, C_{\varepsilon_1}}\varepsilon\left(|u_n|^2 + |u_n|^p\right),\ 0\right\}.$$

Then

$$0 \leqslant \widetilde{f}_n(x) \leqslant C_{\varepsilon_1, C_{\varepsilon_1}} C_{\varepsilon,\varepsilon_1}|u|^2 + C_{\varepsilon_1,\varepsilon_1} C_{\varepsilon, C_{\varepsilon_1}, c_\varepsilon}|u|^p \in L^1(\mathbb{R}^2),$$

and by the Lebesgue dominated convergence theorem we have

$$\int_{\mathbb{R}^2} \widetilde{f}_n(x)\,\mathrm{d}x \to 0 \quad \text{as } n \to \infty. \tag{12}$$

Note that

$$\left|F(u_n) - F(w_n) - F(u)\right| \leqslant \widetilde{f}_n(x) + C_{\varepsilon_1, C_{\varepsilon_1}}\varepsilon\left(|u_n|^2 + |u_n|^p\right).$$

Using (12) shows that (8) holds.

Compare (4), (5) with (6), (7). We only need to prove that $\langle \mathcal{I}'(u), \varphi\rangle = 0$ for all $\varphi \in E$. Note Lemma 1(iii), (10), (11), and $(u_n - u, \varphi) \to 0$ as $n \to \infty$. It suffices to check that $\int_{\mathbb{R}^2} f(w_n)\varphi\,\mathrm{d}x = o(1)$ as $n \to \infty$. Note the arbitrariness of ε in (2), and $w_n \to 0$ in $L^p(\mathbb{R}^2)$, $p > 2$. Therefore, from (2) we have

$$\begin{aligned}\left|\int_{\mathbb{R}^2} f(w_n)\varphi\,\mathrm{d}x\right| &\leqslant \int_{\mathbb{R}^2} \left(\varepsilon|w_n| + c_\varepsilon|w_n|^{p-1}\right)|\varphi|\,\mathrm{d}x \\ &\leqslant \varepsilon\gamma_2^2\|w_n\|\|\varphi\| + c_\varepsilon\gamma_p\|w_n\|_p^{p-1}\|\varphi\| \\ &\to 0 \quad \text{as } n \to \infty.\end{aligned}$$

This completes the proof. □

Lemma 4. *Suppose that all the assumptions in Theorem* 1 *hold. Then $\mathcal{I}$ satisfies the Cerami condition* (C).

Proof. For all $c \in \mathbb{R}$, suppose that there exists $\{u_n\}_{n\in\mathbb{N}} \subset E$ is bounded and

$$\mathcal{I}(u_n) \to c, \quad \mathcal{I}'(u_n) \to 0 \quad \text{as } n \to \infty.$$

Using $\langle \mathcal{I}'(u), \varphi\rangle = 0$ for all $\varphi \in E$ in Lemma 3 and noting Lemma 1(iv), we have

$$\begin{aligned}\mathcal{I}(u) &= \mathcal{I}(u) - \frac{1}{6}\big\langle \mathcal{I}'(u), u\big\rangle \\ &= \frac{1}{3}\|u\|^2 + \int\limits_{\mathbb{R}^2} \mathbf{F}(u)\,\mathrm{d}x + \mu\left(\frac{1}{q} - \frac{1}{6}\right)\int\limits_{\mathbb{R}^2} g(x)|u|^q\,\mathrm{d}x.\end{aligned}$$

This implies that

$$\begin{aligned}\frac{1}{3}\|u\|^2 + \int\limits_{\mathbb{R}^2} \mathbf{F}(u)\,\mathrm{d}x &= \mathcal{I}(u) - \frac{1}{6}\big\langle \mathcal{I}'(u), u\big\rangle - \mu\left(\frac{1}{q} - \frac{1}{6}\right)\int\limits_{\mathbb{R}^2} g(x)|u|^q\,\mathrm{d}x \\ &\leqslant \mathcal{I}(u) - \frac{1}{6}\big\langle \mathcal{I}'(u), u\big\rangle.\end{aligned}$$

Recall $w_n = u_n - u$. From (6) and (7) we have

$$\begin{aligned}&\frac{1}{3}\|w_n\|^2 + \int\limits_{\mathbb{R}^2} \mathbf{F}(w_n)\,\mathrm{d}x \\ &\quad\leqslant \mathcal{I}(w_n) - \frac{1}{6}\big\langle \mathcal{I}'(w_n), w_n\big\rangle \leqslant c - \mathcal{I}(u) + o(1) \\ &\quad= c - \left[\frac{1}{3}\|u\|^2 + \int\limits_{\mathbb{R}^2} \mathbf{F}(u)\,\mathrm{d}x + \mu\left(\frac{1}{q} - \frac{1}{6}\right)\int\limits_{\mathbb{R}^2} g(x)|u|^q\,\mathrm{d}x\right] + o(1) \\ &\quad\leqslant \widetilde{M} \quad \text{for some } \widetilde{M} > 0.\end{aligned}$$

As $V(x) < b$ on a set of finite measure and $w_n \rightharpoonup 0$ in E, we have

$$\begin{aligned}\|w_n\|_2^2 &= \int\limits_{\mathbb{R}^2} |w_n|^2\,\mathrm{d}x \leqslant \frac{1}{\lambda b}\int\limits_{V\geqslant b} \lambda V(x)|w_n|^2\,\mathrm{d}x + \int\limits_{V<b} |w_n|^2\,\mathrm{d}x \\ &\leqslant \frac{1}{\lambda b}\|w_n\|^2 + o(1).\end{aligned}$$

Combining this and the Hölder inequality, recall $p = 2\tau/(\tau - 1) \in (2, +\infty)$, fixed $\nu \in (p, +\infty)$, we have

$$\begin{aligned}\|w_n\|_p^p &= \int\limits_{\mathbb{R}^2} |w_n|^p\,\mathrm{d}x = \int\limits_{\mathbb{R}^2} |w_n|^{\frac{2(\nu-p)}{\nu-2}}|w_n|^{p-\frac{2(\nu-p)}{\nu-2}}\,\mathrm{d}x \\ &\leqslant \left(\int\limits_{\mathbb{R}^2} |w_n|^{\frac{2(\nu-p)}{\nu-2}\frac{\nu-2}{\nu-p}}\,\mathrm{d}x\right)^{\frac{\nu-p}{\nu-2}} \left(\int\limits_{\mathbb{R}^2} |w_n|^{(p-\frac{2(\nu-p)}{\nu-2})\frac{\nu-2}{p-2}}\,\mathrm{d}x\right)^{\frac{p-2}{\nu-2}}\end{aligned}$$

$$
\begin{aligned}
&= \left(\int_{\mathbb{R}^2} |w_n|^2 \,\mathrm{d}x\right)^{\frac{\nu-p}{\nu-2}} \left(\int_{\mathbb{R}^2} |w_n|^\nu \,\mathrm{d}x\right)^{\frac{p-2}{\nu-2}} \\
&\leqslant \left(\frac{1}{\lambda b}\right)^{\frac{\nu-p}{\nu-2}} \gamma_\nu^{\frac{\nu(p-2)}{\nu-2}} \|w_n\|^{\frac{2(\nu-p)}{\nu-2}} \|w_n\|^{\frac{\nu(p-2)}{\nu-2}} \\
&= \left(\frac{1}{\lambda b}\right)^{\frac{\nu-p}{\nu-2}} \gamma_\nu^{\frac{\nu(p-2)}{\nu-2}} \|w_n\|^p \quad \text{for } \gamma_\nu > 0.
\end{aligned}
$$

From (H1), for all $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that $|f(u)| \leqslant \varepsilon|u|$ for $x \in \mathbb{R}^2$ and $|u| \leqslant \delta$. Without loss of generality, we can choose this $\delta > R_1$, where R_1 is defined in (H4). Therefore, we have

$$
\int_{|w_n|\leqslant R_1} f(w_n)w_n \,\mathrm{d}x \leqslant \varepsilon \int_{|w_n|\leqslant R_1} |w_n|^2 \,\mathrm{d}x \leqslant \frac{\varepsilon}{\lambda b}\|w_n\|^2 + o(1).
$$

On the other hand, when $|w_n| \geqslant R_1$, from (H4) we have

$$
\begin{aligned}
\int_{|w_n|\geqslant R_1} f(w_n)w_n \,\mathrm{d}x &= \int_{|w_n|\geqslant R_1} \frac{f(w_n)}{w_n} w_n^2 \,\mathrm{d}x \\
&\leqslant \left(\int_{|w_n|\geqslant R_1} \left|\frac{f(w_n)}{w_n}\right|^\tau \mathrm{d}x\right)^{1/\tau} \left(\int_{|w_n|\geqslant R_1} |w_n|^{\frac{2\tau}{\tau-1}} \,\mathrm{d}x\right)^{(\tau-1)/\tau} \\
&\leqslant \left(\int_{|w_n|\geqslant R_1} \alpha_0 \mathbf{F}(w_n) \,\mathrm{d}x\right)^{1/\tau} \|w_n\|_p^2 \\
&\leqslant (\alpha_0 \widetilde{M})^{1/\tau} \left(\frac{1}{\lambda b}\right)^{\frac{2(\nu-p)}{p(\nu-2)}} \gamma_\nu^{\frac{2\nu(p-2)}{p(\nu-2)}} \|w_n\|^2 + o(1).
\end{aligned}
$$

Consequently, from (7) we obtain

$$
\begin{aligned}
o(1) &= \langle J'(w_n), w_n \rangle \\
&= \|w_n\|^2 + \langle B'(w_n), w_n \rangle - \int_{\mathbb{R}^2} f(w_n)w_n \,\mathrm{d}x + \mu \int_{\mathbb{R}^2} g(x)|w_n|^q \,\mathrm{d}x \\
&\geqslant \left[1 - \frac{\varepsilon}{\lambda b} - (\alpha_0 \widetilde{M})^{1/\tau} \left(\frac{1}{\lambda b}\right)^{\frac{2(\nu-p)}{p(\nu-2)}} \gamma_\nu^{\frac{2\nu(p-2)}{p(\nu-2)}}\right] \|w_n\|^2 + o(1).
\end{aligned}
$$

Thus, given the arbitrariness of ε, there exists $\varLambda > 0$ such that $w_n \to 0$ in E when $\lambda > \varLambda$. This implies that $u_n \to u$ in E, and Definition 1(i) holds.

Finally, we prove that Definition 1(ii) holds. We argue indirectly, i.e., suppose that there exist $c \in \mathbb{R}$ and $\{u_n\}_{n\in\mathbb{N}} \subset E$ such that

$$
\mathcal{I}(u_n) \to c, \quad \|u_n\| \to \infty, \quad \|\mathcal{I}'(u_n)\|\|u_n\| \to 0 \quad \text{as } n \to \infty. \tag{13}
$$

Then we have

$$c+o(1)=\mathcal{I}(u_n)-\frac{1}{6}\langle\mathcal{I}'(u_n),u_n\rangle$$
$$=\frac{1}{3}\|u_n\|^2+\int_{\mathbb{R}^2}\mathbf{F}(u_n)\,\mathrm{d}x+\mu\left(\frac{1}{q}-\frac{1}{6}\right)\int_{\mathbb{R}^2}g(x)|u_n|^q\,\mathrm{d}x$$
$$\geqslant\int_{\mathbb{R}^2}\mathbf{F}(u_n)\,\mathrm{d}x. \tag{14}$$

Using Lemma 1(iv), (13), and (g), we obtain

$$1=\frac{\|u_n\|^2}{\|u_n\|^2}$$
$$=\frac{\langle\mathcal{I}'(u_n),u_n\rangle}{\|u_n\|^2}-\frac{\langle B'(u_n),u_n\rangle}{\|u_n\|^2}+\frac{\int_{\mathbb{R}^2}f(u_n)u_n\,\mathrm{d}x}{\|u_n\|^2}-\frac{\mu\int_{\mathbb{R}^2}g(x)|u_n|^q\,\mathrm{d}x}{\|u_n\|^2}$$
$$\leqslant\limsup_{n\to\infty}\left[\frac{\langle\mathcal{I}'(u_n),u_n\rangle}{\|u_n\|^2}+\frac{\int_{\mathbb{R}^2}f(u_n)u_n\,\mathrm{d}x}{\|u_n\|^2}+\frac{\mu\|g\|_{q'}\gamma^q_{\frac{qq'}{q'-1}}\|u_n\|^q}{\|u_n\|^2}\right]$$
$$\leqslant\limsup_{n\to\infty}\frac{\int_{\mathbb{R}^2}f(u_n)u_n\,\mathrm{d}x}{\|u_n\|^2}. \tag{15}$$

Let $v_n=u_n/\|u_n\|$. Then $\|v_n\|=1$, and there exists a function $v\in E$ such that $v_n\rightharpoonup v$ weakly in E, $v_n\to v$ strongly in $L^r(\mathbb{R}^2)$ with $r\in(2,+\infty)$, $v_n(x)\to v(x)$ for a.e. $x\in\mathbb{R}^2$. Define a set $\Omega_n(a,b)=\{x\in\mathbb{R}^2\colon a\leqslant|u_n(x)|<b\}$ with $0\leqslant a<b$, and consider the following two possible cases.

Case 1. The function v is a zero function in E, i.e., $v=0$, and $v_n\rightharpoonup 0$ weakly in E, $v_n(x)\to 0$ for a.e. $x\in\mathbb{R}^2$. From (2) we have

$$\int_{\Omega_n(0,R_1)}\frac{f(u_n)u_n}{\|u_n\|^2}\,\mathrm{d}x=\int_{\Omega_n(0,R_1)}\frac{f(u_n)u_n}{|u_n|^2}|v_n|^2\mathrm{d}x\leqslant\left(\varepsilon+c_\varepsilon R_1^{p-2}\right)\int_{\Omega_n(0,R_1)}|v_n|^2\,\mathrm{d}x$$
$$\leqslant\left(\varepsilon+c_\varepsilon R_1^{p-2}\right)\int_{\mathbb{R}^2}|v_n|^2\,\mathrm{d}x\to 0. \tag{16}$$

On the other hand, by the Hölder inequality, (14), and (H4) we obtain

$$\int_{\Omega_n(R_1,\infty)}\frac{f(u_n)u_n}{\|u_n\|^2}\,\mathrm{d}x=\int_{\Omega_n(R_1,\infty)}\frac{f(u_n)u_n}{|u_n|^2}|v_n|^2\,\mathrm{d}x$$
$$\leqslant\left(\int_{\Omega_n(R_1,\infty)}\left(\frac{f(u_n)u_n}{|u_n|^2}\right)^\tau\mathrm{d}x\right)^{\frac{1}{\tau}}\left(\int_{\Omega_n(R_1,\infty)}|v_n|^{\frac{2\tau}{\tau-1}}\,\mathrm{d}x\right)^{\frac{\tau-1}{\tau}}$$

$$\leqslant \left(\int_{\Omega_n(R_1,\infty)} \left| \frac{f(u_n)}{u_n} \right|^\tau \mathrm{d}x \right)^{\frac{1}{\tau}} \left(\int_{\Omega_n(R_1,\infty)} |v_n|^p \, \mathrm{d}x \right)^{\frac{2}{p}}$$
$$\leqslant \left(\int_{\Omega_n(R_1,\infty)} \alpha_0 \mathbf{F}(u_n) \, \mathrm{d}x \right)^{\frac{1}{\tau}} \left(\int_{\Omega_n(R_1,\infty)} |v_n|^p \, \mathrm{d}x \right)^{\frac{2}{p}}$$
$$\leqslant \left[\alpha_0(c+1) \right]^{\frac{1}{\tau}} \|v_n\|_p^2 \to 0. \tag{17}$$

Combining (16) and (17), we have

$$\int_{\mathbb{R}^2} \frac{f(u_n)u_n}{\|u_n\|^2} \, \mathrm{d}x = \int_{\Omega_n(0,R_1)} \frac{f(u_n)u_n}{\|u_n\|^2} \, \mathrm{d}x + \int_{\Omega_n(R_1,\infty)} \frac{f(u_n)u_n}{\|u_n\|^2} \, \mathrm{d}x \to 0,$$

which contradicts (15).

Case 2. The function v is not a zero function in E, i.e., $v(x) \not\equiv 0$, $x \in \mathbb{R}^2$. Hence, if we set $A = \{x \in \mathbb{R}^2\colon v(x) \neq 0\}$, then $\operatorname{meas} A > 0$. For $x \in A$, we have $\lim_{n\to\infty} |u_n(x)| = \infty$, and hence $A \subset \Omega_n(R_1, \infty)$ for large n. By (H3) and Lemma 1(iv), (v), noting the nonnegativity of $f(u)u$, Fatou's Lemma enables us to obtain

$$\begin{aligned}
0 &= \lim_{n\to\infty} \frac{\langle \mathcal{I}'(u_n), u_n \rangle}{\|u_n\|^6} \\
&= \lim_{n\to\infty} \left[\frac{\|u_n\|^2}{\|u_n\|^6} + \frac{\langle B'(u_n), u_n \rangle}{\|u_n\|^6} - \frac{\int_{\mathbb{R}^2} f(u_n)u_n \, \mathrm{d}x}{\|u_n\|^6} + \frac{\mu \int_{\mathbb{R}^2} g(x)|u_n|^q \, \mathrm{d}x}{\|u_n\|^6} \right] \\
&\leqslant \lim_{n\to\infty} \left[\frac{\|u_n\|^q}{\|u_n\|^6} \mu \|g\|_{q'} \gamma^q_{\frac{qq'}{q'-1}} + 6C_1 \frac{\|u_n\|^6}{\|u_n\|^6} \right. \\
&\quad \left. - \int_{\Omega_n(0,R_1)} \frac{f(u_n)u_n}{\|u_n\|^6} \, \mathrm{d}x - \int_{\Omega_n(R_1,\infty)} \frac{f(u_n)u_n}{|u_n|^6} |v_n|^6 \, \mathrm{d}x \right] \\
&\leqslant 6C_1 + \limsup_{n\to\infty} \int_{\Omega_n(0,R_1)} \frac{f(u_n)u_n}{\|u_n\|^6} \, \mathrm{d}x - \liminf_{n\to\infty} \int_{\Omega_n(R_1,\infty)} \frac{f(u_n)u_n}{|u_n|^6} |v_n|^6 \, \mathrm{d}x \\
&\leqslant 6C_1 + \limsup_{n\to\infty} \frac{\varepsilon R_1^2 + c_\varepsilon R_1^p}{\|u_n\|^6} \operatorname{meas}\bigl(\Omega_n(0, R_1)\bigr) \\
&\quad - \liminf_{n\to\infty} \int_{\Omega_n(R_1,\infty)} \frac{f(u_n)u_n}{|u_n|^6} \bigl[\chi_{\Omega_n(R_1,\infty)}(x)\bigr] |v_n|^6 \, \mathrm{d}x \\
&\leqslant 6C_1 - \int_{\Omega_n(R_1,\infty)} \liminf_{n\to\infty} \frac{f(u_n)u_n}{|u_n|^6} \bigl[\chi_{\Omega_n(R_1,\infty)}(x)\bigr] |v_n|^6 \, \mathrm{d}x \\
&\to -\infty.
\end{aligned}$$

This is also a contradiction.

Combining the above two cases, we have that Definition 1(ii) holds. Thus, $\mathcal{I}$ satisfies the Cerami condition (C). This completes the proof. □

Proof of Theorem 1. Note that E is a Hilbert space, and let e_j be an orthonomormal basis of E. Then we have

$$Y_k = \bigoplus_{j=1}^{k} X_j, \quad Z_k = \overline{\bigoplus_{j=k}^{\infty} X_j}, \quad k \in \mathbb{N},\ X_j = \mathbb{R}_{e_j}.$$

In what follows, we show that for each $k \in \mathbb{N}$, there exist $\rho_k > r_k > 0$ such that

$$b_k = \inf_{u \in Z_k,\ \|u\| = r_k} \mathcal{I}(u) \to +\infty \quad \text{as } k \to \infty \tag{18}$$

and

$$a_k = \max_{u \in Y_k,\ \|u\| = \rho_k} \mathcal{I}(u) \leqslant 0. \tag{19}$$

Note that the compact embedding $E \hookrightarrow L^r(\mathbb{R}^2)$ with $r \in (2, +\infty)$, and by Lemma 3.8 in [24] we have $\beta_k(r) = \sup_{u \in Z_k,\ \|u\|=1} \|u\|_r \to 0$, $k \to \infty$. This, together with (3), implies that

$$\begin{aligned}
\mathcal{I}(u) &= \frac{1}{2}\|u\|^2 + B(u) - \int_{\mathbb{R}^2} F(u)\,\mathrm{d}x + \frac{\mu}{q}\int_{\mathbb{R}^2} g(x)|u|^q\,\mathrm{d}x \\
&\geqslant \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^2} F(u)\,\mathrm{d}x \geqslant \frac{1}{2}\|u\|^2 - \frac{\varepsilon}{2}\|u\|_2^2 - \frac{c_\varepsilon}{p}\|u\|_p^p \\
&\geqslant \frac{1}{2}\|u\|^2 - \frac{\varepsilon}{2}\gamma_2^2\|u\|^2 - \frac{c_\varepsilon}{p}\|u\|_p^p.
\end{aligned} \tag{20}$$

Note that $p = 2\tau/(\tau-1)$, and if we take $\varepsilon \leqslant 1/(\gamma_2^2(2\tau-1))$ and $r_k = (c_\varepsilon \beta_k^p)^{1/(2-p)}$, by (20), for $u \in Z_k$ and $\|u\| = r_k$, we find

$$\begin{aligned}
\mathcal{I}(u) &\geqslant \frac{\tau-1}{2\tau-1}\|u\|^2 - \frac{c_\varepsilon}{p}\beta_k^p\|u\|^p \geqslant \left(\frac{\tau-1}{2\tau-1} - \frac{\tau-1}{2\tau}\right)\left(c_\varepsilon\beta_k^p\right)^{\frac{2}{2-p}} \\
&\to +\infty \quad \text{as } k \to +\infty
\end{aligned}$$

with $\tau > 1$, $p > 2$. Therefore, (18) holds.

On the other hand, for any finite dimensional subspace $\widetilde{E} \subset E$, we show that

$$\mathcal{I}(u) \to -\infty, \quad \|u\| \to \infty, \quad u \in \widetilde{E}. \tag{21}$$

Arguing indirectly, assume that for some sequence $\{u_n\} \subset \widetilde{E}$ with $\|u_n\| \to \infty$, there exists $M > 0$ such that $\mathcal{I}(u_n) \geqslant -M$ for all $n \in \mathbb{N}$. Let $v_n = u_n/\|u_n\|$. Then $\|v_n\| = 1$, and there is a function $v \in \widetilde{E}$ such that $v_n \rightharpoonup v$ in $\widetilde{E}$. Since $\dim \widetilde{E} < \infty$, we have $v_n \to v$ in $\widetilde{E}$, $v_n(x) \to v(x)$ for a.e. $x \in \mathbb{R}^2$, and $\|v\| = 1$. Let $\Omega = \{x \in \mathbb{R}^2\colon v(x) \neq 0\}$. Then meas $\Omega > 0$, and $\lim_{n\to\infty} |u_n(x)| \to \infty$ for a.e. $x \in \Omega$. From Lemma 1(v) and (g) we have

$$\begin{aligned}\lim_{n\to\infty} \int\limits_{\mathbb{R}^2} \frac{F(u_n)}{\|u_n\|^6}\,\mathrm{d}x &= \lim_{n\to\infty} \frac{\frac{1}{2}\|u_n\|^2 + B(u_n) - \mathcal{I}(u_n) + \frac{\mu}{q}\int_{\mathbb{R}^2} g(x)|u_n|^q\,\mathrm{d}x}{\|u_n\|^6} \\ &\leqslant \lim_{n\to\infty} \frac{\frac{1}{2}\|u_n\|^2 + C_1\|u_n\|^6 - \mathcal{I}(u_n) + \frac{\mu}{q}\|g\|_{q'}\gamma^q_{\frac{qq'}{q'-1}}\|u_n\|^q}{\|u_n\|^6} \\ &= C_1. \end{aligned} \tag{22}$$

From the L'Hôspital's rule and (H3) we have

$$\lim_{|u|\to\infty} \frac{F(u)}{|u|^6} = +\infty \quad \text{uniformly in } x \in \mathbb{R}^2.$$

Fatou's lemma implies that

$$\begin{aligned}\lim_{n\to\infty} \int\limits_{\mathbb{R}^2} \frac{F(u_n)}{\|u_n\|^6}\,\mathrm{d}x &\geqslant \lim_{n\to\infty} \int\limits_{\Omega} \frac{F(u_n)}{\|u_n\|^6}\,\mathrm{d}x \geqslant \liminf_{n\to\infty} \int\limits_{\Omega} \frac{F(u_n)}{|u_n|^6}|v_n|^6\,\mathrm{d}x \\ &\geqslant \int\limits_{\Omega} \liminf_{n\to\infty} \frac{F(u_n)}{|u_n|^6}|v_n|^6\,\mathrm{d}x \geqslant \int\limits_{\Omega} \liminf_{n\to\infty} \frac{F(u_n)}{|u_n|^6}\big[\chi_\Omega(x)\big]|v_n|^6\,\mathrm{d}x \\ &= +\infty. \end{aligned}$$

This contradicts (22), and thus (21) holds. As a result, we can take $u \in Y_k$ and large ρ_k ($\rho_k > r_k$) such that

$$J(u) \leqslant 0 \quad \text{for } u \in Y_k,\ \|u\| = \rho_k.$$

Thus, (19) holds.

Finally, (H5) implies that $\mathcal{I}$ is an even functional on E, and by Lemma 4 $\mathcal{I}$ satisfies all the conditions of Lemma 2. Then $\mathcal{I}$ has a sequence of critical points $\{u_n\}$ such that $\mathcal{I}(u_n) \to +\infty$ as $n \to \infty$. This means that (1) has infinitely many high energy solutions. This completes the proof. □

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