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Feng–Liu-type fixed point result in orbital b-metric spaces and application to fractal integral equation*

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Abstract. In this manuscript, we establish two Wardowski–Feng–Liu-type fixed point theorems for orbitally lower semicontinuous functions defined in orbitally complete b-metric spaces. The obtained results generalize and improve several existing theorems in the literature. Moreover, the findings are justified by suitable nontrivial examples. Further, we also discuss ordered version of the obtained results. Finally, an application is presented by using the concept of fractal involving a certain kind of fractal integral equations. An illustrative example is presented to substantiate the applicability of the obtained result in reducing the energy of an antenna.

Keywords: b-metric space, F-contraction, fixed point of a multivalued mapping, orbitally lower semicontinuous.

1 Introduction

Through the whole of the last century, mathematicians engrossed themselves with touching up the underlying metric framework of the acclaimed Banach contraction principle. In their research articles, Bakhtin [3] and afterwards Czerwik [6–8] put forward another

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natural and impressive setting as an extension of metric spaces, b-metric spaces, to work with. Since then, plenty of fixed point, common fixed point and related results involving various classes of single-valued and multi-valued operators defined in such kind of spaces or in allied structures are published (see [1, 2, 5, 10, 11, 13–18, 20]). Here we recall some basic definitions, notations and essential results, which will play crucial roles in this manuscript. Throughout this article, $\mathbb{R}^+$ and $\mathbb{R}_0^+$ denote the set of all positive, respectively, nonnegative real numbers, and $\mathbb{N}$ stands for the set of positive integers. First of all, we recollect the definition of b-metric spaces.

Definition 1. A b-metric on a nonempty set X is a function $d_b : X \times X \to \mathbb{R}_0^+$ such that for a constant $b \geqslant 1$ and all $x, y, z \in X$, the following conditions hold:

(M1) $d_b(x, y) = 0$ if and only if $x = y$;
(M2) $d_b(x, y) = d_b(y, x)$;
(M3) $d_b(x, y) \leqslant b(d_b(x, z) + d_b(z, y))$.

Then (X, d_b) is called a b-metric space.

One may note that each metric space is a b-metric space considering $b = 1$. However, the converse does not hold. Furthermore, the topology on a b-metric space, the concept of Cauchy sequences, convergent sequences and the completeness of the setting are analogous to that of standard metric spaces. However, in general, a b-metric is not a continuous mapping in both variables (see, e.g., [13]).

On the other hand, Cosentino et al. in [5] introduced the set of functions $\mathfrak{T}F_b$ in the line of Wardowski's [19] approach to b-metric space as follows:

Definition 2. Let $b \geqslant 1$ be a real number. $\mathfrak{T}F_b$ denotes the family of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ with the following properties:

(F1) F is strictly increasing;
(F2) For each sequence $\{a_n\}$ of positive numbers, $\lim_{n\to\infty} a_n = 0$ if and only if $\lim_{n\to\infty} F(a_n) = -\infty$;
(F3) For each sequence $\{a_n\}$ of positive numbers with $\lim_{n\to\infty} a_n = 0$, there exists $k \in (0, 1)$ such that $\lim_{n\to\infty} a_n^k F(a_n) = 0$;
(F4) There exists $\tau \in \mathbb{R}^+$ such that for each sequence $\{a_n\}$ of positive numbers, if $\tau + F(ba_n) \leqslant F(a_{n-1})$ for all $n \in \mathbb{N}$, then $\tau + F(b^n a_n) \leqslant F(b^{n-1}a_{n-1})$ for all $n \in \mathbb{N}$.

Definition 3. Let $b \geqslant 1$ be a real number. $\mathfrak{T}F_b^*$ denotes the family of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ having the properties (F1)–(F4) and the following property:

(F5) $F(\inf A) = \inf F(A)$ for all $A \subset (0, \infty)$ with $\inf A > 0$.

It is easy to see that the function $F(x) = \ln x$ or $F(x) = x + \ln x$ satisfies properties (F1)–(F5) for $x > 0$.

We recall the following result from [5]. In this result, $\mathcal{H}_b$ denotes the b-Hausdorff–Pompeiu metric.

Theorem 1. (*See* [5, *Thm.* 3.4].) *Let (X, d_b) be a complete b-metric space, and let $T : X \to \mathcal{P}_{cb}(X)$. Assume that there exists a continuous from the right function $F \in \mathfrak{T}F_b$ and $\tau \in \mathbb{R}^+$ such that*

$$2\tau + F\big(b\mathcal{H}_b(Tx, Ty)\big) \leqslant F\big(d_b(x, y)\big)$$

for all $x, y \in X$, $Tx \neq Ty$. Then T has a fixed point.

Moreover, Feng and Liu obtained the subsequent result (recall that a function $f : X \to \mathbb{R}$ is said to be lower semicontinuous if for all sequences $\{x_n\}$ in X with $\lim_{n\to\infty} x_n = x \in X$, it satisfies $f(x) \leqslant \liminf_{n\to\infty} f(x_n)$).

Theorem 2. *Let (X, d) be a metric space, $T : X \to CL(X)$, and let the function $f : X \to \mathbb{R}$, $f(x) = d(x, Tx)$ be lower semicontinuous. If there exist $b, c \in (0, 1)$ with $b < c$ such that for any $x \in X$, there is $y \in Tx$ satisfying*

$$c\, d(x, y) \leqslant f(x) \quad \text{and} \quad f(y) \leqslant b\, d(x, y),$$

then T has a fixed point.

Against this background, we obtain fixed point results for multivalued mappings satisfying Wardowski–Feng–Liu-type conditions for orbitally lower semicontinuous functions in orbitally complete b-metric spaces. These results generalize, complement and unify the findings proposed in [2, 12]. Besides, we illustrate a couple of examples to validate our obtained results, and also, we consider the ordered version of the attained results. Finally, we dish out an interesting application concerning our derived theorem and employing the notion of fractals to a certain type of fractal integral equations. An illustrative example is presented to show the applicability of the obtained result in reducing the energy of an antenna.

2 Main results

Let $CL(X)$ denotes the family of nonempty closed subsets of X. Let $T : X \to CL(X)$ be a multivalued map, $F \in \mathfrak{T}F_b$ and $\eta : (0, \infty) \to (0, \infty)$. For $x \in X$ with $d_b(x, Tx) > 0$, define a set $F^x_\eta \subseteq X$ as

$$F^x_\eta = \big\{y \in Tx\colon\ F\big(d_b(x, y)\big) \leqslant F\big(d_b(x, Tx)\big) + \eta\big(d_b(x, y)\big)\big\}.$$

Let $T : X \to X$, and for some $x_0 \in X$, $\mathcal{O}(x_0) = \{x_0, Tx_0, T^2x_0, \dots\}$ be the orbit of x_0. A function $f : X \to \mathbb{R}$ is called T-orbitally lower semicontinuous if $f(x) \leqslant \liminf_{n\to\infty} f(x_n)$ for all sequences $\{x_n\} \subset \mathcal{O}(x_0)$ with $\lim_{n\to\infty} x_n = x \in X$.

Ćirić defined the orbit and orbital completeness in case of multivalued mappings in his paper [4]. In a similar fashion, we define these concepts in case of b-metric spaces. For a set X, we use the notation 2^X to denote the set of all subsets of X.

Definition 4. Let $T : X \to 2^X$ be a multivalued mapping on a b-metric space (X, d_b). An orbit for T at a point $x_0 \in X$ is denoted by $\mathcal{O}(x_0)$ and is defined as a sequence $\{x_n\colon\ x_n \in Tx_{n-1}\}$.

Definition 5. Let $T : X \to 2^X$ be a multivalued mapping on a b-metric space (X, d_b). A b-metric space X is said to be T-orbitally complete if every Cauchy sequence of the form $\{x_n\colon x_n \in Tx_{n-1}\}$ converges in X.

It is obvious that an orbitally complete b-metric space may not be complete. Now we present one of our main result.

Theorem 3. *Let (X, d_b) be an orbitally complete b-metric space with $b \geqslant 1$, $T\colon X \to CL(X)$ and $F \in \mathfrak{T}F_b^*$. Assume that the following conditions hold*:

(i) *The mapping $x \mapsto d_b(x, Tx)$ is orbitally lower semicontinuous*;
(ii) *There exist functions $\tau, \eta : (0, \infty) \to (0, \infty)$ such that for all $t \geqslant 0$,*

$$\tau(t) > \eta(t), \quad \liminf_{s \to t^+} \tau(s) > \liminf_{s \to t^+} \eta(s);$$

(iii) *For any $x \in X$ with $d_b(x, Tx) > 0$, there exists $y \in F_\eta^x$ satisfying*

$$\tau\bigl(d_b(x, y)\bigr) + F\bigl(bd_b(y, Ty)\bigr) \leqslant F\bigl(d_b(x, y)\bigr).$$

Then T has a fixed point.

Proof. Suppose that T has no fixed points. Then for all $x \in X$, we have $d_b(x, Tx) > 0$. Since $Tx \in CL(X)$ for every $x \in X$ and $F \in \mathfrak{T}F_b^*$, then it is easy to prove that the set F_η^x is nonempty for every $x \in X$ (proof will follow in the line of [12]). If $x_0 \in X$ is any initial point, then there exists $x_1 \in F_\eta^{x_0}$ such that

$$\tau\bigl(d_b(x_0, x_1)\bigr) + F\bigl(bd_b(x_1, Tx_1)\bigr) \leqslant F\bigl(d_b(x_0, x_1)\bigr),$$

and for $x_1 \in X$, there exists $x_2 \in F_\eta^{x_1}$ satisfying

$$\tau\bigl(d_b(x_1, x_2)\bigr) + F\bigl(bd_b(x_2, Tx_2)\bigr) \leqslant F\bigl(d_b(x_1, x_2)\bigr).$$

Continuing this process, we get an iterative sequence $\{x_n\}$, where $x_{n+1} \in F_\eta^{x_n}$ and

$$\tau\bigl(d_b(x_n, x_{n+1})\bigr) + F\bigl(bd_b(x_{n+1}, Tx_{n+1})\bigr) \leqslant F\bigl(d_b(x_n, x_{n+1})\bigr) \tag{1}$$

for all $n \in \mathbb{N} \cup \{0\}$. It follows by (1) and property (F4) that

$$\tau\bigl(d_b(x_n, x_{n+1})\bigr) + F\bigl(b^{n+1}d_b(x_{n+1}, Tx_{n+1})\bigr) \leqslant F\bigl(b^n d_b(x_n, x_{n+1})\bigr) \tag{2}$$

for all $n \in \mathbb{N}$. We verify that $\{x_n\}$ is a Cauchy sequence. Since $x_{n+1} \in F_\eta^{x_n}$, then by the definition of $F_\eta^{x_n}$ we have

$$F\bigl(d_b(x_n, x_{n+1})\bigr) \leqslant F\bigl(d_b(x_n, Tx_n)\bigr) + \eta\bigl(d_b(x_n, x_{n+1})\bigr),$$

which implies that

$$F\bigl(b^n d_b(x_n, x_{n+1})\bigr) \leqslant F\bigl(b^n d_b(x_n, Tx_n)\bigr) + \eta\bigl(d_b(x_n, x_{n+1})\bigr). \tag{3}$$

From (2) and (3) we have

$$F\big(b^{n+1}d_b(x_{n+1},Tx_{n+1})\big) \leqslant F\big(b^n d_b(x_n,Tx_n)\big) + \eta\big(d_b(x_n,x_{n+1})\big) - \tau\big(d_b(x_n,x_{n+1})\big), \tag{4}$$

i.e.,

$$F\big(b^{n+1}d_b(x_{n+1},x_{n+2})\big) \leqslant F\big(b^n d_b(x_n,x_{n+1})\big) + \eta\big(d_b(x_n,x_{n+1})\big) - \tau\big(d_b(x_n,x_{n+1})\big). \tag{5}$$

Let $\varrho_n = d_b(x_n,x_{n+1})$ for $n \in \mathbb{N}$; then $\varrho_n > 0$, and from (5) $\{\varrho_n\}$ is decreasing. Therefore, there exists $\delta > 0$ such that $\lim_{n\to\infty} \varrho_n = \delta$. Now let $\delta > 0$. Let $\beta(t) = \liminf_{t\to s^+} \tau(t) - \liminf_{t\to s^+} \eta(t) > 0$. Then using (5), the following holds:

$$\begin{aligned} F\big(b^{n+1}\varrho_{n+1}\big) &\leqslant F\big(b^n\varrho_n\big) - \beta(\varrho_n) \\ &\leqslant F\big(b^{n-1}\varrho_{n-1}\big) - \beta(\varrho_n) - \beta(\varrho_{n-1}) \\ &\dots \\ &\leqslant F(\varrho_0) - \beta(\varrho_n) - \beta(\varrho_{n-1}) - \dots - \beta(\varrho_0). \end{aligned} \tag{6}$$

Let p_n be the greatest number in $\{0,1,\dots,n-1\}$ such that

$$\beta(\varrho_{p_n}) = \min\big\{\beta(\varrho_0),\beta(\varrho_1),\dots,\beta(\varrho_n)\big\}$$

for all $n \in \mathbb{N}$. In this case, $\{\varrho_n\}$ is a nondecreasing sequence. From (6) we get

$$F(b^n\varrho_n) \leqslant F(\varrho_0) - n\beta(\varrho_{p_n}). \tag{7}$$

In a similar way, from (4) we can obtain

$$F\big(b^{n+1}d_b(x_{n+1},Tx_{n+1})\big) \leqslant F\big(d_b(x_0,x_1)\big) - n\beta(\varrho_{p_n}). \tag{8}$$

Now consider the sequence $\{\beta(\varrho_{p_n})\}$. We distinguish two cases.

Case 1. For each $n \in \mathbb{N}$, there is $m > n$ such that $\beta(\varrho_{p_n}) > \beta(\varrho_{p_m})$. Then we obtain a subsequence $\{\varrho_{p_{n_k}}\}$ of $\{\varrho_{p_n}\}$ with $\beta(\varrho_{p_{n_k}}) > \beta(\varrho_{p_{n_{k+1}}})$ for all k. Since $\varrho_{p_{n_k}} \to \delta^+$, we deduce that

$$\liminf_{t\to s^+} \beta(\varrho_{p_{n_k}}) > 0.$$

Hence,

$$F\big(b^{n_k}\varrho_{n_k}\big) \leqslant F(\varrho_0) - n^k\beta(\varrho_{p_{n_k}})$$

for all k. Consequently, $\lim_{k\to\infty} F(b^{n_k}\varrho_{n_k}) = -\infty$, and by (F2) $\lim_{k\to\infty} b^{n_k}\varrho_{n_k} = 0$, which contradicts the fact that $\lim_{k\to\infty}\varrho_{n_k} > 0$ as $b > 1$.

Case 2. There is $n_0 \in \mathbb{N}$ such that $\beta(\varrho_{p_0}) > \beta(\varrho_{p_m})$ for all $m > n_0$. Then $F(\varrho_m) \leqslant F(\varrho_0) - m\beta(\varrho_{p_{n_0}})$ for all $m > n_0$. Hence, $\lim_{m\to\infty} F(\varrho_m) = -\infty$, and by (F2) $\lim_{m\to\infty}\varrho_m = 0$, which contradicts the fact that $\lim_{m\to\infty}\varrho_m > 0$. Thus, $\lim_{m\to\infty}\varrho_m = 0$. From (F3) there exists $k \in (0,1)$ such that

$$\lim_{n\to\infty} (b^n\varrho_n)^k F\big(b^n\varrho_n\big) = -\infty.$$

By (7) the following holds for all $n \in \mathbb{N}$:

$$\begin{aligned}
&\left(b^n \varrho_n\right)^k F\left(b^n \varrho_n\right) - \left(b^n \varrho_n\right)^k F(\varrho_0) \\
&\quad \leqslant \left(b^n \varrho_n\right)^k \left(F(\varrho_0) - n\beta(\varrho_{p_n})\right) - (\varrho_n)^k F(\varrho_0) \\
&\quad = -n\left(b^n \varrho_n\right)^k \beta(\varrho_{p_n}) \leqslant 0. \qquad (9)
\end{aligned}$$

Passing to limit as $n \to \infty$ in (9), we obtain

$$\lim_{n\to\infty} n(b^n \varrho_n)^k \beta(\varrho_{p_n}) = 0.$$

Since $\zeta := \liminf_{n\to\infty} \beta(\varrho_{p_n}) > 0$, then there exists $n_0 \in \mathbb{N}$ such that $\beta(\varrho_{p_n}) > \zeta/2$ for all $n \neq n_0$. Thus,

$$n\left(b^n \varrho_n\right)^k \frac{\zeta}{2} < n\left(b^n \varrho_n\right)^k \beta(\varrho_{p_n}) \qquad (10)$$

for all $n \geqslant n_0$. Letting $n \to \infty$ in (10), we have $0 \leqslant \lim_{n\to\infty} n(b^n \varrho_n)^k \zeta/2 < \lim_{n\to\infty} n(b^n \varrho_n)^k \beta(\varrho_{p_n}) = 0$, that is,

$$\lim_{n\to\infty} n\left(b^n \varrho_n\right)^k = 0. \qquad (11)$$

From (11) there exits $n_1 \in \mathbb{N}$ such that $n(\varrho_n)^k \leqslant 1$ for all $n \geqslant n_1$. So, we have $\varrho_n \leqslant 1/(b^n n^{1/k})$ for all $n \geqslant n_1$. Now, the last limit implies that the series $\sum_{n=1}^{+\infty} b^n \varrho_n$ is convergent, and hence, $\{x_n\}$ is a Cauchy sequence in X. Since X is a orbitally complete b-metric space, there exists $z \in \mathcal{O}(x_0)$ such that $x_n \to z$ as $n \to \infty$. On the other hand, from (8) and (F2) we have $\lim_{n\to\infty} d_b(x_n, Tx_n) = 0$. Since $x \mapsto d_b(x, Tx)$ is orbitally lower semicontinuous, we have

$$\begin{aligned}
0 &\leqslant d_b(z, Tz) \leqslant \liminf_{n\to\infty} d_b(x_n, Tx_n) \leqslant \lim_{n\to\infty} d_b(x_n, x_{n+1}) \\
&\leqslant \lim_{n\to\infty} b\left[d_b(x_n, x) + d_b(x, x_{n+1})\right] = 0.
\end{aligned}$$

Therefore, $z \in Tz$. Hence, T has a fixed point. □

Next, we present an example in order to substantiate the above result.

Example 1. Let us consider the set $X = [1/9, \infty)$ and define a mapping $d_b : X \times X \to \mathbb{R}$ by $d_b(x, y) = |x - y|^2$. Then d_b is a b-metric on X with $b = 2$. Next, we define a mapping $T : X \to CL(X)$ by

$$Tx = \left\{\frac{x+1}{4}, x + 2^x\right\}$$

for all $x \in X$. Let us take $F(\alpha) = \ln \alpha$ for all $\alpha \in \mathbb{R}^+$ and $\tau(t) = 1/4$, $\eta(t) = 1/6$ for all $t \in (0, \infty)$. Then, clearly, $F \in \mathfrak{T}F_b$ and $\tau(t) > \eta(t)$, $\liminf_{s\to t^+} \tau(s) > \liminf_{s\to t^+} \eta(s)$ for all $t \geqslant 0$.

For any $x \in X$, we have $d_b(x,Tx) = ((3x-1)/4)^2$. So the mapping $x \mapsto d_b(x,Tx)$ is orbitally lower semicontinuous. Again, for any $x \in X$, we have $y = (x+1)/4 \in F_\eta^x$, and for this y, we have

$$\begin{aligned}
&\tau\bigl(d_b(x,y)\bigr) + F\bigl(bd_b(y,Ty)\bigr)\\
&\quad= \frac{1}{4} + \ln\biggl(2\biggl(\frac{3x-1}{16}\biggr)^2\biggr) \leqslant \ln\biggl(3\biggl(\frac{3x-1}{16}\biggr)^2\biggr) \leqslant \ln\biggl(\biggl(\frac{3x-1}{4}\biggr)^2\biggr)\\
&\quad= F\bigl(d_b(x,y)\bigr).
\end{aligned}$$

Thus, we see that all the conditions of Theorem 3 hold true. So by that theorem T has a fixed point, and note that $z = 1/3$ is a fixed point of T.

Our second result is related to multivalued mappings T on the b-metric space X, where Tx is compact for all $x \in X$. By taking into account Case 1, we can take $\eta \geqslant 0$. Therefore, the proof of the following theorem is obvious.

Let us denote $K(X)$ as the set of all nonempty compact subsets of X.

Theorem 4. *Let (X,d_b) be an orbitally complete b-metric space, $b \geqslant 1$, $T: X \to K(X)$ and $F \in \mathfrak{T}F_b$. Assume that the following conditions hold*:

(i) *The mapping $x \mapsto d_b(x,Tx)$ is orbitally lower semicontinuous*;

(ii) *There exist functions $\tau : (0,\infty) \to (0,\infty)$ and $\eta : (0,\infty) \to [0,\infty)$ such that, for all $t \geqslant 0$,*

$$\tau(t) > \eta(t), \quad \liminf_{s\to t^+} \tau(s) > \liminf_{s\to t^+} \eta(s);$$

(iii) *For any $x \in X$ with $d_b(x,Tx) > 0$, there exists $y \in F_\eta^x$ satisfying*

$$\tau\bigl(d_b(x,y)\bigr) + F\bigl(bd_b(y,Ty)\bigr) \leqslant F\bigl(d_b(x,y)\bigr).$$

Then T has a fixed point.

Now we have the ordered version of the above results. To do this, we recall the definition of ordered b-metric space. $(X,d_b,\preccurlyeq)$ is called an ordered b-metric space if d_b is a b-metric space on X and $(X,\preccurlyeq)$ is a partially ordered set. Further, if $(X,\preccurlyeq)$ is a partially ordered set, then $x,y \in X$ are called comparable if $x \preccurlyeq y$ or $y \preccurlyeq x$ holds.

For $x \in X$ with $d_b(x,Tx) > 0$, define a set $F_{\eta,\preccurlyeq}^x \subseteq X$ as

$$\begin{aligned}
F_{\eta,\preccurlyeq}^x = \bigl\{y \in Tx\colon\ &F\bigl(d_b(x,y)\bigr) \leqslant F\bigl(d_b(x,Tx)\bigr) + \eta\bigl(d_b(x,y)\bigr),\\
&F\bigl(kd_b(x,y)\bigr) \leqslant F\bigl(kd_b(x,Tx)\bigr) + \eta\bigl(d_b(x,y)\bigr),\ x \preccurlyeq y,\ k > 1\bigr\}.
\end{aligned}$$

Theorem 5. *Let $(X,d_b,\preccurlyeq)$ be an ordered orbitally complete b-metric space with $b \geqslant 1$, $T: X \to CL(X)$ and $F \in \mathfrak{T}F_b$. Assume that the following conditions hold*:

(i) *The mapping $x \mapsto d_b(x,Tx)$ is orbitally lower semicontinuous;*

(ii) *There exist functions $\tau,\eta : (0,\infty) \to (0,\infty)$ such that, for all $t \geqslant 0$,*

$$\tau(t) > \eta(t), \quad \liminf_{s\to t^+} \tau(s) > \liminf_{s\to t^+} \eta(s);$$

(iii) *For any* $x \in X$ *with* $d_b(x,Tx) > 0$, *there exists* $y \in F^x_{\eta,\preccurlyeq}$ *satisfying*

$$\tau\big(d_b(x,y)\big) + F\big(bd_b(y,Ty)\big) \leqslant F\big(d_b(x,y)\big).$$

If the condition

(C) *If* $\{x_n\} \subset X$ *is a nondecreasing sequence with* $x_n \to z$ *in* X, *then* $x_n \preccurlyeq z$ *for all* n

holds, then T *has a fixed point.*

Proof. Following the line of proof of Theorem 3 and definition of $F^x_{\eta,\preccurlyeq} \subseteq X$, we can show that $\{x_n\}$ is a Cauchy sequence in $(X, d_b, \preccurlyeq)$ with $x_{n-1} \preccurlyeq x_n$ for $n \in \mathbb{N}$. From the orbital completeness of X there exists a $z \in \mathcal{O}(x_0)$ such that $x_n \to z$ as $n \to \infty$. By assumption (C), $x_n \preccurlyeq z$ for all n. Rest is followed from the proof of Theorem 3. $\square$

Theorem 6. *Let* $(X, d, \preccurlyeq)$ *be an ordered orbitally complete b-metric space with* $b \geqslant 1$, $T : X \to K(X)$ *and* $F \in \mathfrak{T}F_b$. *Assume that the following conditions hold*:

(i) *The mapping* $x \mapsto d_b(x,Tx)$ *is orbitally lower semicontinuous*;

(ii) *There exist functions* $\tau : (0,\infty) \to (0,\infty)$ *and* $\eta : (0,\infty) \to [0,\infty)$ *such that, for all* $t \geqslant 0$,

$$\tau(t) > \eta(t), \quad \liminf_{s \to t^+} \tau(s) > \liminf_{s \to t^+} \eta(s);$$

(iii) *For any* $x \in X$ *with* $d_b(x,Tx) > 0$, *there exists* $y \in F^x_{\eta,\preccurlyeq}$ *satisfying*

$$\tau\big(d_b(x,y)\big) + F\big(bd_b(y,Ty)\big) \leqslant F\big(d_b(x,y)\big).$$

Then T *has a fixed point provided* (C) *holds.*

Now we present an example to authenticate Theorem 5.

Example 2. Let us take $X = \mathbb{N} \cup \{0\}$ and consider a relation $\preccurlyeq$ on X by defining $x \preccurlyeq y$ if and only if x divides y. Then it is easy to verify that $(X, \preccurlyeq)$ is a partially ordered set. Now we define $d_b : X \times X \to \mathbb{R}$ by

$$d_b(x,y) = \begin{cases} 0 & \text{if } x = y; \\ 3(\frac{1}{n} + \frac{1}{m}) & \text{if } x = n,\ y = m \text{ and } n \neq m; \\ \frac{1}{n} & \text{if } x = n, y = 0 \text{ or } x = 0,\ y = n. \end{cases}$$

Then (X, d_b) is a partially ordered b-metric space with $b = 3$.

Now we define a mapping $T : X \to CL(X)$ by

$$Tx = \{2x, 3x, 4x\} \quad \text{for all } x \in X.$$

Let us choose $F(\alpha) = \ln \alpha$ for all $\alpha \in \mathbb{R}^+$ and $\tau(t) = 1/8$, $\eta(t) = 1/10$ for all $t \in (0,\infty)$. Then it is obvious that $F \in \mathfrak{T}F_b$ and $\tau(t) > \eta(t)$, $\liminf_{s \to t^+} \tau(s) > \liminf_{s \to t^+} \eta(s)$ for all $t \geqslant 0$. Further, for $x \in X$, we have

$$d_b(x,Tx) = \begin{cases} 0 & \text{if } x = 0; \\ \frac{15}{4x} & \text{if } x \neq 0. \end{cases}$$

Therefore, $x \mapsto d_b(x, Tx)$ is orbitally lower semicontinuous. For $x \in X$, we have $y = 4x \in F^x_{\eta,\preccurlyeq}$, and for this y, we have

$$\begin{aligned}
&\tau\bigl(d_b(x,y)\bigr) + F\bigl(bd_b(y,Ty)\bigr) \\
&\quad= \frac{1}{8} + \ln\frac{45}{16x} \leqslant \ln\left(\frac{5}{4}\cdot\frac{45}{16x}\right) \leqslant \ln\frac{15}{4x} \\
&\quad= F\bigl(d_b(x,y)\bigr).
\end{aligned}$$

Thus, we see that all conditions of Theorem 5 hold. So by the same theorem T has a fixed point. Indeed $z = 0$ is a fixed point of T.

3 Application

Fredholm equations stand up obviously in the scheme of signal processing, for instance, as the well-known spectral concentration issue. The operators convoluted are the similar as linear filters. They besides normally stand up in linear forward forming and inverse problems. In physics, the result of such integral equations permits for investigation spectra to be correlated to several fundamental disseminations, for example, the mass supply of polymers in a polymeric melts, or the supply of reduction times in the scheme. Moreover, Fredholm integral equations indicate in fluid mechanics issues connecting hydrodynamic connections near finite-sized elastic borders (see [9] for recent work). In antenna manufacturing, side lobes are the lobes (local maxima with local minimum energy) of the distant field radiation design of an antenna or other radioactivity foundation, which are not the chief lobe. The problem statement is that: is there a solution for which the side lobe energy is minimum? The answer of this question is to find a solution for Fredholm integral equation. In our discussion, we use a generalize fractal Fredholm integral equation based on the fractal integral [21] as follows:

$$\chi(t) = \frac{1}{\Gamma(\wp+1)}\int_{\alpha}^{\beta} \Lambda\bigl(t,\varsigma,\chi(\varsigma)\bigr)\,(\mathrm{d}\varsigma)^{\wp} + \sigma(t), \quad t,\varsigma \in J := [\alpha,\beta], \tag{12}$$

where $\Lambda : J \times J \times \mathbb{R} \to \mathbb{R}$ is defined and continuous on a fractal set of fractal dimension $\wp \in (0,1)$, and $\sigma : J \to \mathbb{R}$ is continuous function. Our aim is to show that (12) has a solution by using Theorem 3.

Theorem 7. *Let $\mathbb{X} = C[J,\mathbb{R}]$, and let $\mathcal{Q}$ the operator defined by*

$$(\mathcal{Q}\chi)(t) = \frac{1}{\Gamma(\wp+1)}\int_{\alpha}^{\beta} \Lambda\bigl(t,\varsigma,\chi(\varsigma)\bigr)\,(\mathrm{d}\varsigma)^{\wp} + \sigma(t), \quad t,\varsigma \in J = [\alpha,\beta],\ \wp \in (0,1),$$

where $\Lambda : J \times J \times \mathbb{R} \to \mathbb{R}$ is defined and continuous on a fractal set of fractal dimension $\wp \in (0,1)$, and $\sigma : J \to \mathbb{R}$ is continuous function. Moreover, assume the following conditions:

(i) *There occurs a continuous function* $\gamma : \mathbb{X} \to [0, \infty)$ *satisfying*

$$\left|\Lambda\big(t, \varsigma, \chi(\varsigma)\big) - \Lambda\big(t, \varsigma, (\mathcal{Q}\chi)(\varsigma)\big)\right| \leqslant \gamma(\varsigma)\left|\chi(\varsigma) - (\mathcal{Q}\chi)(\varsigma)\right|;$$

(ii) $\int_\alpha^\beta \gamma(\varsigma)\,(\mathrm{d}\varsigma)^\wp \leqslant \sqrt{\mathrm{e}^{-\mathfrak{T}}}$, $\mathfrak{T} := \tau(d(\chi, \mathcal{Q}\chi)) > 0$.

Then the integral equation (12) *admits a solution.*

Proof. We have to prove that the operator $\mathcal{Q}$ achieves all the assumptions of Theorem 3 in the single-valued type. Let $\chi \in \mathbb{X}$, then we obtain

$$\begin{aligned}
&\left|(\mathcal{Q}\chi)(t) - \mathcal{Q}\big(\mathcal{Q}\chi(t)\big)\right|^2 \\
&\quad\leqslant \frac{1}{\Gamma^2(\wp+1)}\left(\int_\alpha^\beta \left|\Lambda\big(t, \varsigma, \chi(\varsigma)\big) - \Lambda\big(t, \varsigma, \mathcal{Q}\chi(\varsigma)\big)\right|(\mathrm{d}\varsigma)^\wp\right)^2 \\
&\quad\leqslant \frac{1}{\Gamma^2(\wp+1)}\left(\int_\alpha^\beta \gamma(\varsigma)\left|\chi(\varsigma) - \mathcal{Q}\chi(\varsigma)\right|(\mathrm{d}\varsigma)^\wp\right)^2 \\
&\quad= \frac{1}{\Gamma^2(\wp+1)} d(\chi, \mathcal{Q}\chi)\left(\int_\alpha^\beta \gamma(\varsigma)\,(\mathrm{d}\varsigma)^\wp\right)^2 \\
&\quad\leqslant \frac{1}{\Gamma^2(\wp+1)} \mathrm{e}^{-\mathfrak{T}} d(\chi, \mathcal{Q}\chi).
\end{aligned}$$

Thus, we get the following inequality:

$$\Gamma^2(\wp+1)\left|(\mathcal{Q}\chi)(t) - \mathcal{Q}\big(\mathcal{Q}\chi(t)\big)\right|^2 \leqslant \mathrm{e}^{-\mathfrak{T}} d(\chi, \mathcal{Q}\chi).$$

Since the natural logarithm indicates to be in $\mathfrak{T}F_b$, employing it on above inequality, we conclude that

$$\mathfrak{T} + \ln\left[\Gamma^2(\wp+1)\, d\big(\mathcal{Q}\chi, \mathcal{Q}^2\chi\big)\right] \leqslant \ln\left[d(\chi, \mathcal{Q}\chi)\right].$$

Consequently, we obtain

$$\tau\big(d(\chi, \mathcal{Q}\chi)\big) + \ln\left[\Gamma^2(\wp+1)\, d\big(\mathcal{Q}\chi, \mathcal{Q}^2\chi\big)\right] \leqslant \ln\left[d(\chi, \mathcal{Q}\chi)\right].$$

Now let $b := \Gamma^2(\wp+1)$, thus, we indicate the following inequality:

$$\tau\big(d(\chi, \mathcal{Q}\chi)\big) + \ln\left[b, d\big(\mathcal{Q}\chi, \mathcal{Q}^2\chi\big)\right] \leqslant \ln\left[d(\chi, \mathcal{Q}\chi)\right].$$

Hence, in view of Theorem 3 (single-valued) with $F(\psi) = \ln\psi$, the operator $\mathcal{Q}$ admits at least one fixed point, which is corresponding to the solution of Eq. (12). This completes the proof. $\square$

Example 3. Consider the following data: $J = [0, 1]$, $\Lambda(t, \varsigma, \chi) = \varsigma\chi(\varsigma)/4$, $\varsigma \in J$. From Fig. 1 we can say that the relation between b and $\wp$ certainly minimize the energy of the antenna. Obviously, the value of $\wp = 0.4616$ minimized the energy. It is clear that $\int_0^1 \gamma(\varsigma)\,(d\varsigma)^\wp = 0.25 < \sqrt{\mathrm{e}^{-\mathfrak{T}}} = 0.36$, $\mathfrak{T} = 1 = \tau(1) > 0$. Hence, in view of Theorem 7, Eq. (12) has a solution, which minimize the energy.

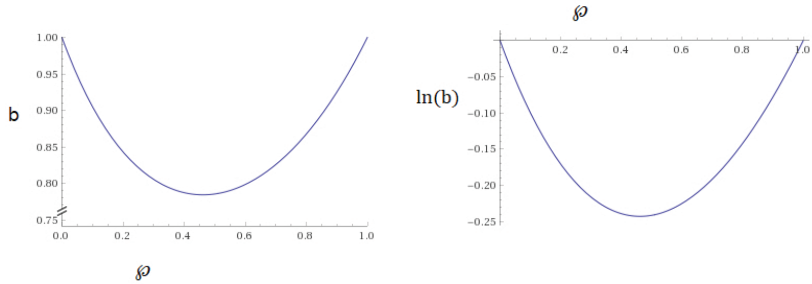

Figure 1. The relation between $b = \Gamma^2(\wp + 1)$ and $\wp$ in the left, while the right graph is the relation between $ln(b) = F$ with respect to $\wp$. The minimum value of b is at $\wp = 0.4616$.

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