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Hopf-zero bifurcation of the ring unidirectionally coupled Toda oscillators with delay*

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Abstract. In this paper, the Hopf-zero bifurcation of the ring unidirectionally coupled Toda oscillators with delay was explored. First, the conditions of the occurrence of Hopf-zero bifurcation were obtained by analyzing the distribution of eigenvalues in correspondence to linearization. Second, the stability of Hopf-zero bifurcation periodic solutions was determined based on the discussion of the normal form of the system, and some numerical simulations were employed to illustrate the results of this study. Lastly, the normal form of the system on the center manifold was derived by using the center manifold theorem and normal form method.

Keywords: Toda oscillators, normal form, Hopf-zero.

1 Introduction

In more recent decades, the bifurcation theory in dynamical system has become a research hotspot, which is widely used in the fields of physics, chemistry, medicine, finance, biology, engineering and so on [3,12,13,15,19,20,22,24,30–35]. In particular, the bifurcation phenomenon of the system is analyzed, and the dynamic characteristics of the nonlinear

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model are characterized. In real life, the mathematical models to solve practical problems have been described based on nonlinear dynamical systems, in which, how to study the dynamic characteristics of the high-dimensional nonlinear system is very important. In engineering, Toda oscillators (named after the Morikazu–Toda) refer to a dynamical system employed to model a chain of particles with exponential potential interaction between neighbors. The Toda oscillators model has been extensively applied in engineering [21, 23, 25]. It is noteworthy that it acts as a simple model to clarify the phenomenon of self-pulsation, namely, a quasiperiodic pulsation of the output intensity of a solid-state laser in the transient regime [16].

For the different control parameters of the chains of coupled Toda oscillators with external force, most scholars have investigated the effect of external force (F, ω) on the system from the numerical continuation and experimental simulation, they suggest that the system eventually produces the coexistence stable limit cycle in synchronous region [10, 11, 17]. Under the action of external force (F, ω), the coexistence stability limit cycle is generated in the synchronous region. This shows that the final trajectories of the Toda oscillators are asymptotically stable in the synchronous region, that is to say, there is a regular stable transmission between the oscillators, and the industrial productivity reaches the maximum. Thus far, the bifurcation phenomenon caused by coupled delay in nonlinear differential equations has been extensively studied, especially, the Hopf-zero bifurcation [1, 2, 4, 5, 18, 26–29], but no studies have been conducted on the Hopf-zero bifurcation of the ring unidirectionally coupled Toda oscillators with delay as we know today. Therefore, it is of great significance to probe into the Hopf-zero bifurcation of system (1). Under the guidance of the bifurcation theory, our study is majorally under the premise that the external force does not exist (considering only the coupled effect between oscillators).

In this study, we investigate the ring system of three unidirectionally coupled Toda oscillators [6]

$$\begin{aligned} \dot{x}_1 &= y_1, \\ \dot{y}_1 &= 1 - \exp(x_1) - \alpha y_1 + \gamma H(x_3) + F\sin(\omega t), \\ \dot{x}_2 &= y_2, \\ \dot{y}_2 &= 1 - \exp(x_2) - \alpha y_2 + \gamma H(x_1), \\ \dot{x}_3 &= y_3, \\ \dot{y}_3 &= 1 - \exp(x_3) - \alpha y_3 + \gamma H(x_2), \end{aligned} \tag{1}$$

where x_1, y_1, x_2, y_2, x_3 and y_3 are the dynamical variables of system (1), α is the damping coefficient, γ is the coupled coefficient $(\gamma > 0)$, $H(x) = \exp(x) - 1$ is the coupling function, F and ω represent the amplitude and the frequency of the external force, respectively. The symbol (F, ω) denotes the effect of external force on the coupled system (1). In [6], with XPPAUT software package, the author has summarized the dynamic characteristics of system (1) with numerical integration of differential equations and Runge–Kutta method, such as exponential spectrum, projections of phase portraits, bifurcation diagram and periodic multiplier on the parameter plane. The author mainly investigated the peculiar phenomena of the ring structure of the coupled oscillators with the change of coupled

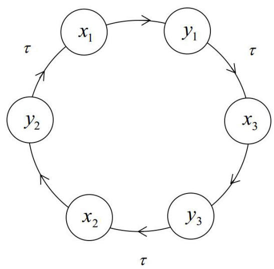


Figure 1. The ring of unidirectional coupled Toda oscillator model with time delay. Arrows denote coupled direction.

coefficient γ under the effect of external force (F, ω). In other words, with the threshold of self-oscillation birth, the further increase of coupled coefficient γ will be followed by the new ring resonance phenomenon, besides when the threshold of self-oscillation birth is exceeded, the interaction between external oscillations and the inner oscillatory mode of system (1) will result in the synchronization region in the parameter plane [6, 7].

In the small neighborhood of the equilibrium point of system (1), the coupled time delay should be considered since the propagations between oscillators x_1, x_2 and x_3 are no longer instantaneous. Assuming $(F, \omega) = 0$, system (1) generated the rich dynamic characteristics with the joint change of the coupled coefficient γ, the coupled time delay τ and control parameters. Furthermore, the appropriate coupled time delay is introduced to facilitate the understanding of high-dimensional bifurcation analysis followed by the generation of a new Toda oscillators model with delay, and the coupled direction in the model is shown in Fig. 1.

In this paper, we mainly study the dynamic behaviors of the following simplified system:

$$\begin{aligned}
\dot{x}_1 &= y_1, \\
\dot{y}_1 &= 1 - \exp(x_1) - \alpha y_1 + \gamma\big[\exp\big(x_3(t-\tau)\big) - 1\big], \\
\dot{x}_2 &= y_2, \\
\dot{y}_2 &= 1 - \exp(x_2) - \alpha y_2 + \gamma\big[\exp\big(x_1(t-\tau)\big) - 1\big], \\
\dot{x}_3 &= y_3, \\
\dot{y}_3 &= 1 - \exp(x_3) - \alpha y_3 + \gamma\big[\exp\big(x_2(t-\tau)\big) - 1\big].
\end{aligned} \tag{2}$$

In engineering fields, such a time-delay feedback indicates an unidirectional interaction between oscillators x_1, x_2 and x_3. That is to say, the dynamic characteristics of system (2) are determined uniquely by the effect of connection when the oscillators have coupled delay in a ring structure.

The rest of the paper is organized as follows. In Section 2, the existence of Hopf-zero bifurcation in the ring of unidirectionally coupled Toda oscillators with delay is considered. In Section 3, according to the normal form of system (2), the dynamic bifurcation analysis and numerical simulation are conducted. Lastly, in Section 4, the conclusion is drawn.

2 Existence of Hopf-zero bifurcation

In this section, we will give the existence conditions of Hopf-zero bifurcation of system (2). To convenience for expression, we make the following assumptions:

$$\begin{aligned} x_1(t) &= u_1(t), & y_1(t) &= u_2(t), & x_2(t) &= u_3(t), \\ y_3(t) &= u_4(t), & x_3(t) &= u_5(t), & y_3(t) &= u_6(t). \end{aligned}$$

Then system (2) can be written as

$$\begin{aligned} \dot{u}_1 &= u_2, \\ \dot{u}_2 &= 1 - \exp(u_1) - \alpha u_2 + \gamma\bigl[\exp\bigl(u_5(t-\tau)\bigr) - 1\bigr], \\ \dot{u}_3 &= u_4, \\ \dot{u}_4 &= 1 - \exp(u_3) - \alpha u_4 + \gamma\bigl[\exp\bigl(u_1(t-\tau)\bigr) - 1\bigr], \\ \dot{u}_5 &= u_6, \\ \dot{u}_6 &= 1 - \exp(u_5) - \alpha u_6 + \gamma\bigl[\exp\bigl(u_3(t-\tau)\bigr) - 1\bigr]. \end{aligned} \tag{3}$$

Clearly, system (3) is equivalent to system (2), where $(u_1, \dots, u_6) = (0, \dots, 0)$ is an equilibrium point of system (3). The characteristic equation according to the linear part of system (3) at the equilibrium point is given by

$$\bigl(\alpha\lambda + \lambda^2 + 1\bigr)^3 - \gamma^3 \mathrm{e}^{-3\lambda\tau} = \Lambda_1(\lambda)\Lambda_2(\lambda)\Lambda_3(\lambda) = 0, \tag{4}$$

where

$$\begin{aligned} \Lambda_1(\lambda) &= \alpha\lambda + \lambda^2 + 1 - \gamma\mathrm{e}^{-\lambda\tau}, \\ \Lambda_2(\lambda) &= \alpha\lambda + \lambda^2 + 1 - \frac{1}{2}\gamma\mathrm{e}^{-\lambda\tau} + \mathrm{i}\frac{\sqrt{3}}{2}\gamma\mathrm{e}^{-\lambda\tau}, \\ \Lambda_3(\lambda) &= \alpha\lambda + \lambda^2 + 1 - \frac{1}{2}\gamma\mathrm{e}^{-\lambda\tau} - \mathrm{i}\frac{\sqrt{3}}{2}\gamma\mathrm{e}^{-\lambda\tau}. \end{aligned}$$

There exists three types of bifurcations as follows:

Case 1: Fixed-point bifurcation. Substituting $\lambda = 0$ into equation (4), we obtain $\gamma = 1$ under which the three roots of equation (4) with $\tau = 0$ are $\lambda_1 = 0$, $\lambda_{2,3} = -1/2 \pm \sqrt{3}\mathrm{i}/2$, then system (3) undergoes a fixed-point bifurcation.

Case 2: Hopf-bifurcation. Substituting $\lambda = \mathrm{i}\omega_1$ ($\omega_1 > 0$, $\mathrm{i}^2 = -1$) into $\Lambda_1(\lambda) = 0$ and separating the real and imaginary part, we obtain

$$\cos\omega_1\tau = 1 - \omega_1^2, \qquad \sin\omega_1\tau = -\alpha\omega_1. \tag{5}$$

Then the time delay τ can be solved from system (5) as

$$\tau_j^1 = \frac{1}{\omega_1}\bigl[2\pi(j+1) - \arccos\bigl(1-\omega_1^2\bigr)\bigr], \tag{6}$$

where $j = 0, 1, 2 \dots$. Under the assumption $0 < \alpha < \sqrt{2}$, we have $\omega_1 = \sqrt{2 - \alpha^2}$. After a simple calculation, the transversality conditions are shown by

$$\begin{aligned} \left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^{-1} &= -\frac{\alpha + 2\lambda}{\lambda \mathrm{e}^{-\lambda\tau}} - \frac{\tau}{\lambda}, \\ \mathrm{Re}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^{-1}_{\tau=\tau_j^1} &= 2 - \alpha^2 > 0, \end{aligned} \tag{7}$$

where "Re" represents the real part.

Next, let $\lambda = \mathrm{i}\omega_2$, $(\omega_2 > 0)$ be the root of $\Lambda_2(\lambda) = 0$. It is not difficult to verify that $\lambda = -\mathrm{i}\omega_2$ is the root of $\Lambda_3(\lambda) = 0$. In order to study the distribution of the roots of $\Lambda_3(\lambda) = 0$, we only need to investigate the distribution of the roots of $\Lambda_2(\lambda) = 0$. Substituting the root $\lambda = \mathrm{i}\omega_2$ into $\Lambda_2(\lambda) = 0$ and separating the real and imaginary parts, we obtain

$$\begin{aligned} \cos\omega_2\tau &= \frac{1}{2}\left(-\omega_2^2 - \sqrt{3}\alpha\omega_2 + 1\right), \\ \sin\omega_2\tau &= \frac{1}{2}\left(\sqrt{3}\left(\omega_2^2 - 1\right) - \alpha\omega_2\right), \end{aligned} \tag{8}$$

then the time delay τ can be solved from system (5) as

$$\tau_j^2 = \begin{cases} \frac{1}{\omega_2}[\arccos(\frac{-\omega_2^2 - \sqrt{3}\alpha\omega_2 + 1}{2}) + 2j\pi], & \sqrt{3}(\omega_2^2 - 1) - \alpha\omega_2 > 0, \\ \frac{1}{\omega_2}[(2j+1)\pi - \arccos(\frac{-\omega_2^2 - \sqrt{3}\alpha\omega_2 + 1}{2})], & \sqrt{3}(\omega_2^2 - 1) - \alpha\omega_2 < 0. \end{cases} \tag{9}$$

Under the assumption $0 < \alpha < \sqrt{2}$, we have $\omega_2 = \sqrt{2 - \alpha^2}$. After a simple calculation, the transversality conditions are shown as follows:

$$\begin{aligned} \left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^{-1} &= \frac{\alpha + 2\lambda}{\lambda \mathrm{e}^{-\lambda\tau}(-\frac{1}{2} + \mathrm{i}\frac{\sqrt{3}}{2}) - \frac{\tau}{\lambda}}, \\ \mathrm{Re}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^{-1}_{\tau=\tau_j^2} &= 2 - \alpha^2 > 0. \end{aligned} \tag{10}$$

Case 3: Hopf-zero bifurcation. Combining cases 1 and 2, we have the following theorem.

Theorem 1. *Assume $\gamma = 1$ and $0 < \alpha < \sqrt{2}$, we have $\omega_0 = \omega_{1,2} = \sqrt{2 - \alpha^2}$. The critical time delays and the transversality conditions are given by systems* (6), (9), (7) *and* (10), *respectively. Moreover, when $\tau \in [0, \tau_0)$, where $\tau_0 = \min\{\tau_0^1, \tau_0^2\}$, the roots of characteristic equation* (4) *are* 0 *and* $-\alpha$. *Then the Hopf-zero bifurcation occurs associated with eigenvalues* 0 *and* $\pm\mathrm{i}\omega$ *of system* (3). *When $\tau > \tau_0$, the trivial equilibrium of system* (3) *is asymptotically unstable.*

3 Hopf-zero Bifurcation analysis and Numerical simulations

In the present section, a clearer analysis of the dynamic bifurcation of system (3) is to be conducted. The normal form on the center manifold of system (3) is reduced to the following form:

$$\begin{aligned}
\dot{x}_1 &= \mathrm{i}\omega x_1 + (a_{11}\mu_1 + a_{12}\mu_2)x_1 + a_{13}x_1x_3 + (b_{11} + c_{11})x_1^2x_2 \\
&\quad + (b_{12} + c_{12})x_1x_3^2 + \text{h.o.t.}, \\
\dot{x}_2 &= -\mathrm{i}\omega x_2 + (\bar{a}_{11}\mu_1 + \bar{a}_{12}\mu_2)x_2 + \bar{a}_{13}x_2x_3 + (\bar{b}_{11} + \bar{c}_{11})x_1^2x_2 \\
&\quad + (\bar{b}_{12} + \bar{c}_{12})x_1x_3^2 + \text{h.o.t.}, \\
\dot{x}_3 &= a_{21}\mu_2x_3 + c_{31}x_1x_2x_3 + \text{h.o.t.}
\end{aligned} \tag{11}$$

The detailed calculation and analysis of system (11) are made in the Appendix. Since $x_1 = \bar{x}_2$, through the change of variables $x_1 = \alpha_1 - \mathrm{i}\alpha_2$, $x_2 = \alpha_1 + \mathrm{i}\alpha_2$, $x_3 = \alpha_3$ and $x_1x_2 = \rho^2$, if we use double polar coordinates $\alpha_1 = \rho\cos\chi$, $\alpha_2 = \rho\sin\chi$, $\alpha_3 = \zeta$, then we get $x_1 = \rho\mathrm{e}^{-\mathrm{i}\chi} = \rho\cos\chi - \mathrm{i}\rho\sin\chi$, $x_2 = \rho\mathrm{e}^{\mathrm{i}\chi} = \rho\cos\chi + \mathrm{i}\rho\sin\chi$, $x_1x_2 = \rho^2$, and system (11) becomes

$$\begin{aligned}
\dot{\rho} &= \big(\mathrm{Re}[a_{11}]\mu_1 + \mathrm{Re}[a_{12}]\mu_2\big)\rho + \big(\mathrm{Re}[b_{11}] + \mathrm{Re}[c_{11}]\big)\rho^3 \\
&\quad + \big(\mathrm{Re}[b_{12}] + \mathrm{Re}[c_{12}]\big)\rho\zeta^2 + \mathrm{Re}[a_{13}]\rho\zeta + \text{h.o.t.}, \\
\dot{\zeta} &= a_{21}\mu_2\zeta + c_{31}\rho^2\zeta + \text{h.o.t.}, \\
\dot{\chi} &= -\big[\big(\mathrm{Im}[a_{11}]\mu_1 + \mathrm{Im}[a_{12}]\mu_2\big) + \omega + \mathrm{Im}[a_{13}]\zeta \\
&\quad + \big(\mathrm{Im}[b_{11}] + \mathrm{Im}[c_{11}]\big)\rho^2 + \big(\mathrm{Im}[b_{12}] + \mathrm{Im}[c_{12}]\big)\zeta^2\big] + \text{h.o.t.}
\end{aligned}$$

Since the third equation describes a rotation around the χ-axis, it is irrelevant to our discussion, and we shall omit it. Hence we get the system in the ρ,ζ-plane up to the third order

$$\begin{aligned}
\dot{\rho} &= \kappa_1(\mu)\rho + \beta_{11}\rho\zeta + \beta_{30}\rho^3 + \beta_{12}\rho\zeta^2 + \text{h.o.t.}, \\
\dot{\zeta} &= \kappa_2(\mu)\zeta + \eta_{21}\rho^2\zeta + \text{h.o.t.},
\end{aligned} \tag{12}$$

where

$$\begin{gathered}
\kappa_1(\mu) = \big(\mathrm{Re}[a_{11}]\mu_1 + \mathrm{Re}[a_{12}]\mu_2\big), \qquad \kappa_2(\mu) = a_{21}\mu_2, \qquad \eta_{21} = c_{31}, \\
\beta_{11} = \mathrm{Re}[a_{13}], \qquad \beta_{12} = \big(\mathrm{Re}[b_{12}] + \mathrm{Re}[c_{12}]\big), \qquad \beta_{30} = \big(\mathrm{Re}[b_{11}] + \mathrm{Re}[c_{11}]\big).
\end{gathered}$$

From Section 2 we have $\mathrm{e}^{\pm\mathrm{i}\tau\omega} = \mp\mathrm{i}\alpha\omega - \omega^2 + 1$, then the coefficient β_{11} is sufficiently small. For simplicity, we only discuss the case of $\beta_{11} = 0$. So system (12) becomes

$$\begin{aligned}
\dot{\rho} &= \kappa_1(\mu)\rho + \beta_{30}\rho^3 + \beta_{12}\rho\zeta^2 + \text{h.o.t.}, \\
\dot{\zeta} &= \kappa_2(\mu)\zeta + \eta_{21}\rho^2\zeta + \text{h.o.t.}
\end{aligned} \tag{13}$$

System (13) is equivalent to system (3), we can analyze the dynamic behaviors of system (13) in the neighborhood of the bifurcation critical point, which are obtained by

$\dot{\rho} = \dot{\zeta} = 0$. Note that: $M_0 = (\rho, \zeta) = (0, 0)$ stands for the coexistence equilibrium point; $M_1 = (\rho, \zeta) = (\sqrt{-\kappa_1(\mu)/\beta_{30}}, 0)$ is a Homogeneous periodic solution for $\kappa_1(\mu)/\beta_{30} < 0$; $M_2^{\pm} = (\sqrt{-\kappa_2(\mu)/\eta_{21}}, \pm\sqrt{(\beta_{30}\kappa_2(\mu) - \eta_{21}\kappa_1(\mu))/(\beta_{12}\eta_{21})})$ is a pair of inhomogeneous periodic solutions for $\kappa_2(\mu)/\eta_{21} < 0$ and $(\beta_{30}\kappa_2(\mu) - \eta_{21}\kappa_1(\mu))/(\beta_{12}\eta_{21}) > 0$.

When $\rho = 0$, system (13) has no solution, then the spatially inhomogeneous steady states does not exist.

In order to give a more clear bifurcation picture, we choose $\alpha = 1.35$, which satisfy the assumption $0 < \alpha < \sqrt{2}$ of Theorem 1. Then we have

$$\begin{aligned} \omega = \sqrt{2 - \alpha^2} = 0.421307, \qquad -\sqrt{3}\alpha\omega - \omega^2 + 1 = -0.16263 < 0, \\ \tau_0^1 = \frac{1}{\omega}\left(2\pi - \arccos\left(1 - \omega^2\right)\right) = 13.4775, \\ \tau_0^2 = \frac{1}{\omega}\left(2\pi - \arccos\left(\frac{-\sqrt{3}\alpha\omega - \omega^2 + 1}{2}\right)\right) = 10.9919. \end{aligned}$$

We take $\tau_0 = \tau_0^2 = 10.9919$, the characteristic equation (4) has a zero root and a pair of purely imaginary eigenvalues ± 0.421307i, and all the other eigenvalues have negative real part. Assume that system (3) undergoes a Hopf-zero bifurcation from the equilibrium point $(0, 0, 0, 0, 0, 0)$. After a simple calculation, we get $\beta_{30} = \mathrm{Re}[b_{11} + c_{11}] = 0.534691$, $\beta_{12} = \mathrm{Re}[b_{11} + c_{31}] = -4.55396$, $\eta_{21} = \mathrm{Re}[c_{31}] = -4.07092$, $\kappa_1(\mu) = 0.0359584\mu_1 - 0.0873093\mu_2$, $\kappa_2(\mu) = 0.890617\mu_2$.

By analyzing the above contents we can obtain the bifurcation critical lines

$$\begin{aligned} &L_1\colon \left\{\left(\kappa_1(\mu), \kappa_2(\mu) \mid \kappa_1(\mu) = 0,\ \kappa_2(\mu) < 0\right)\right\}, \\ &L_2\colon \left\{\left(\kappa_1(\mu), \kappa_2(\mu) \mid \kappa_1(\mu) < 0,\ \kappa_2(\mu) = -7.61359\kappa_1(\mu)\right)\right\}, \\ &L_3\colon \left\{\left(\kappa_1(\mu), \kappa_2(\mu) \mid \kappa_1(\mu) = 0,\ \kappa_2(\mu) > 0\right)\right\}, \\ &L_4\colon \left\{\left(\kappa_1(\mu), \kappa_2(\mu) \mid \kappa_1(\mu) > 0,\ \kappa_2(\mu) = 0\right)\right\} \end{aligned}$$

and phase portraits as shown in Fig. 2.

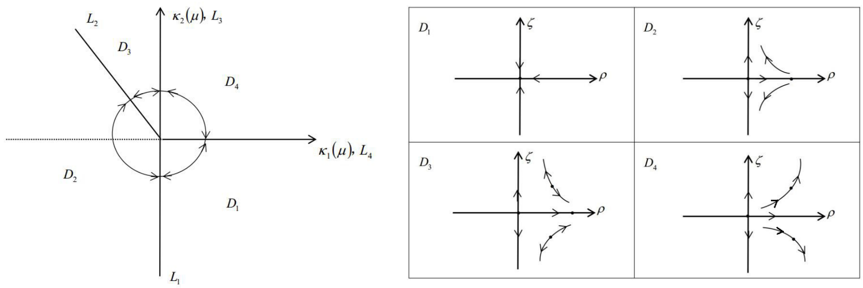
From [14] the dynamic characteristic of system (13) in D_1–D_4 near the critical parameters (α, τ_0) are as follows:

In D_1, system (13) has only one trivial equilibrium M_0, which is a sink.

In D_2, the trivial equilibrium (corresponding to M_0) becomes a saddle from a sink, and an unstable periodic orbit (corresponding to M_1) appears.

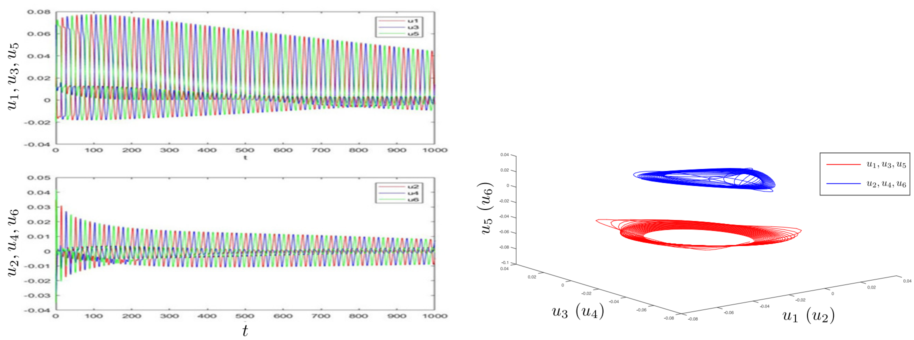
In D_3, the trivial equilibrium (corresponding to M_0) becomes a source from a saddle, the periodic orbit (corresponding to M_1) becomes stable, and a pair of unstable periodic solution equilibria (corresponding to $M_2^{\pm}$) appears.

In D_4, the trivial equilibrium (corresponding to M_0) becomes a saddle from a source, the periodic orbit (corresponding to M_1) remains unstable, and the unstable periodic orbit (corresponding to $M_2^{\pm}$) disappear.



(a) (b)

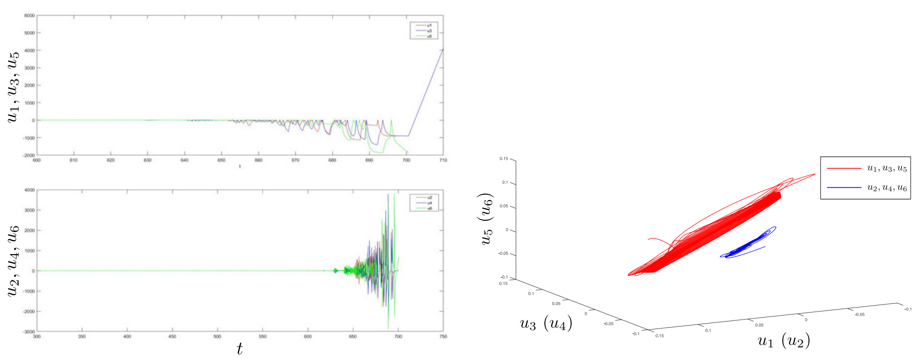
Figure 2. (a) Bifurcation diagrams of system (13) with parameter $(\kappa_1(\mu), \kappa_2(\mu))$ around $(0, 0)$; (b) phase portraits of region $D_1 - D_4$.



(a) Solution curves for the variables u_1, u_2, u_3 and u_4, u_5, u_6, respectively.

(b) Phase portrait of the variables u_1, u_2, u_3 and u_4, u_5, u_6.

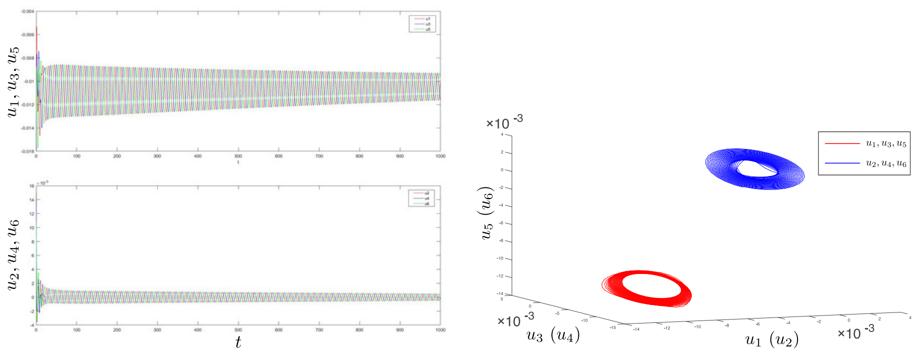
Figure 3. Region D_1: the stable trivial equilibrium point of system (3) with initial values $(-0.012, -0.05, -0.08, -0.015, -0.0015, -0.0135)$.



(a) Solution curves for the variables u_1, u_2, u_3 and u_4, u_5, u_6, respectively.

(b) Phase portrait of the variables u_1, u_2, u_3 and u_4, u_5, u_6.

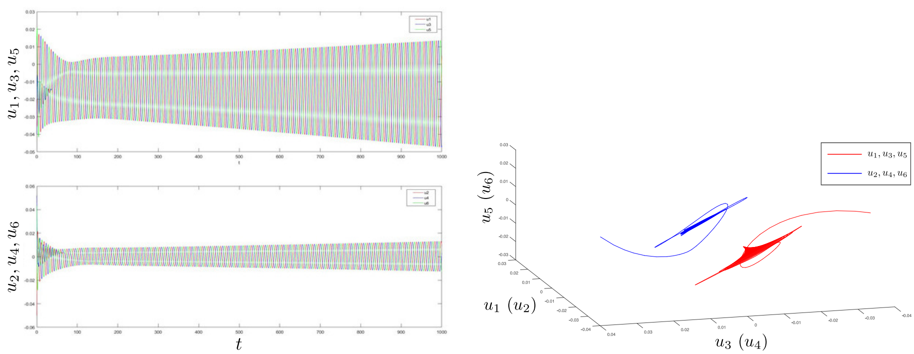
Figure 4. Region D_2: the solution curves of system (3) with initial values $(-0.0492, -0.05, 0.051, 0.055, 0.0415, -0.035)$.



(a) Solution curves for the variables u_1, u_2, u_3 and u_4, u_5, u_6, respectively.

(b) Phase portrait of the variables u_1, u_2, u_3 and u_4, u_5, u_6.

Figure 5. Region D_3: a stable periodic solution of system (3) with initial values $(-0.012, 0.015, -0.016, 0.014, -0.013, 0.011)$.



(a) Solution curves for the variables u_1, u_2, u_3 and u_4, u_5, u_6, respectively.

(b) Phase portrait of the variables u_1, u_2, u_3 and u_4, u_5, u_6.

Figure 6. Region D_4: the solution curves of system (3) with initial values $(-0.02, -0.05, -0.04, 0.052, 0.011, 0.043)$.

The first, we choose a group of perturbation parameter value: $(\kappa_1(\mu), \kappa_1(\mu)) = (0.012, -0.1)$, which determined by $(\mu_1, \mu_2) = (0.012, -0.1)$ and belong to region D_1 (see Fig. 3).

The second, we choose a group of perturbation parameter value: $(\kappa_1(\mu), \kappa_1(\mu)) = (-0.006, -0.0116)$, which determined by $(\mu_1, \mu_2) = (-0.2, -0.013)$ and belong to the region D_2 (see Fig. 4).

The third, we choose a group of perturbation parameter value: $(\kappa_1(\mu), \kappa_1(\mu)) = (-0.00141, 0.089)$, which determined by $(\mu_1, \mu_2) = (-0.015, 0.01)$ and belong to the region D_3 (see Fig. 5).

The last, we choose a group of perturbation parameter value: $(\kappa_1(\mu), \kappa_1(\mu)) = (0.0045, 0.0089)$, which determined by $(\mu_1, \mu_2) = (0.15, 0.01)$ and belong to the region D_4 (see Fig. 6).

4 Conclusions

In this paper, we have investigated the Hopf-zero bifurcation in the ring of unidirectionally coupled Toda oscillators with delay. We mainly conclude the singularity in a small enough neighborhood of system (3) near equilibrium point, such as the phenomenon of stable coexistence periodic solution and unstable quasiperiodic solution caused by the self-pulsation without external force (F, ω). Moreover, the Hopf zero bifurcation of system (3) on the parameter plane is determined according to the change of control parameters (μ_1, μ_2) and the time delay τ. It shows that there may exhibit a stable trivial equilibrium point, an unstable periodic equilibrium point orbit and a pair of unstable periodic solutions (see Fig. 4(b)). Firstly, the stability of equilibrium and existence of Hopf-zero bifurcation induced by delay were discussed through characteristic equation. It is then followed by the normal form on the center manifold so that the bifurcation direction and stability of Hopf-zero bifurcation could be determined. Finally, in two-dimensional space, we presented numerical simulations to illustrate a homogeneous periodic solution bifurcating from the positive equilibrium when $\alpha = 1.35$ and $\tau = \tau_0$.

According to the physical meaning of the Toda oscillator model, the trajectory of the oscillator in the synchronous region is asymptotically stable because the more stable the system is, the more regular the transmission between the oscillators is, and the higher the industrial productivity is. In our conclusion, Hopf-zero bifurcation of the model with $(F, \omega) = 0$, $\gamma = 1$ occurs when the control parameters α and time delay τ vary in a small range. Then we find that the dynamic characteristics of system (3) will change from stable to unstable, that is, an unstable region will be generated near the equilibrium point. However, the neighborhood is small enough to be ignored in industrial production and can almost be regarded as a stable region. We further confirm that the effect of time delay and the mutual coupling between oscillators will eventually lead to the synchronization region in the parameter plane indicating that the external oscillations and internal oscillations of the system are equally important. Therefore, it is meaningful to analyze the Hopf-zero bifurcation of the Toda oscillators model without external intervention (F, ω).

Appendix

In this section, we will derive the normal form of system (3) by using the Faria and Magalhães method [8, 9] regrading γ and τ as bifurcation parameters. Suppose $0 < \alpha < \sqrt{2}$, then system (3) undergoes a Hopf-zero bifurcation from the trivial equilibrium at the critical point $(\gamma, \tau) = (1, \tau_0)$. After scaling $t \to t/\tau$, system (3) can be written as

$$\begin{aligned}
\dot{u}_1 &= \tau u_2, \\
\dot{u}_2 &= \tau - \tau \exp(u_1) - \tau\alpha u_2 + \tau\gamma\bigl[\exp\bigl(u_5(t-1)\bigr) - 1\bigr], \\
\dot{u}_3 &= \tau u_4, \\
\dot{u}_4 &= \tau - \tau \exp(u_3) - \tau\alpha u_4 + \tau\gamma\bigl[\exp\bigl(u_1(t-1)\bigr) - 1\bigr], \\
\dot{u}_5 &= \tau u_6, \\
\dot{u}_6 &= \tau - \tau \exp(u_5) - \tau\alpha u_6 + \tau\gamma\bigl[\exp\bigl(u_3(t-1)\bigr) - 1\bigr].
\end{aligned} \tag{A.1}$$

Consider the linearization of system (A.1)

$$\dot{X}(t) = L_0 X_t.$$

Choose the space $C = C([-1,0], \mathbb{R}^6)$ and, for any $\phi \in C$, define that $\|\phi\| = \sup_{-1\leqslant\theta\leqslant 0} |\phi(\theta)|$, then the phase space C can be chosen as the Banach space of continuous function from $[-1,0]$ to $\mathbb{R}^3$ with the supremum norm. Since L_0 is a bounded linear operator, then we define

$$L_0\varphi = \int_{-1}^{0} \mathrm{d}\eta(\theta)\varphi(\theta), \quad \varphi \in C,$$

where $\eta : [-1,0] \to \mathbb{R}^{3\times 3}$ is a matrix-valued function whose components are bounded variation, and its definition is given by

$$\eta(\theta) = \tau_0 \mathbf{N}_1 \delta(\theta) + \tau_0 \mathbf{N}_2 \delta(\theta+1), \quad \delta(\theta) = \begin{cases} 0, & \theta \neq 0, \\ 1, & \theta = 0, \end{cases}$$

and

$$\mathbf{N}_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & -\alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -\alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & -\alpha \end{pmatrix}, \qquad \mathbf{N}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Take the Taylor expansion of system (A.1) at the equilibrium point, and let $\gamma = \mu_1 + 1$, $\tau = \mu_2 + \tau_0$. Set $\mu = (\mu_1, \mu_2)$, then μ is the bifurcation parameter, and system (A.1) becomes

$$\begin{aligned}
\dot{u}_1 &= (\mu_2+\tau_0)u_2, \\
\dot{u}_2 &= (\mu_2+\tau_0)(-u_1-\alpha u_2) + (\mu_2+\tau_0)(\mu_1+1)u_5(t-1) \\
&\quad + (\mu_2+\tau_0)\left(-\frac{u_1^3}{6} - \frac{u_1^2}{2}\right) + (\mu_2+\tau_0)(\mu_1+1)\left(\frac{u_5^2(t-1)}{2} + \frac{u_5^3(t-1)}{6}\right), \\
\dot{u}_3 &= (\mu_2+\tau_0)u_4, \\
\dot{u}_4 &= (\mu_2+\tau_0)(-u_3-\alpha u_4) + (\mu_2+\tau_0)(\mu_1+1)u_1(t-1) \\
&\quad + (\mu_2+\tau_0)\left(-\frac{u_3^3}{6} - \frac{u_3^2}{2}\right) + (\mu_2+\tau_0)(\mu_1+1)\left(\frac{u_1(t-1)^2}{2} + \frac{u_1^3(t-1)}{6}\right), \\
\dot{u}_5 &= (\mu_2+\tau_0)u_6, \\
\dot{u}_6 &= (\mu_2+\tau_0)(-u_5-\alpha u_6) + (\mu_2+\tau_0)(\mu_1+1)u_3(t-1) \\
&\quad + (\mu_2+\tau_0)\left(-\frac{u_5^3}{6} - \frac{u_5^2}{2}\right) + (\mu_2+\tau_0)(\mu_1+1)\left(\frac{u_3(t-1)^2}{2} + \frac{u_3^3(t-1)}{6}\right).
\end{aligned} \tag{A.2}$$

Let

$$X = (u_1, u_2, u_3, u_4, u_5, u_6)^{\mathrm{T}},$$

$$F(X_t, \mu) = \left(F_1^2 + F_1^3,\ F_2^2 + F_2^3,\ F_3^2 + F_3^3,\ F_4^2 + F_4^3,\ F_5^2 + F_5^3,\ F_6^2 + F_6^3\right)^{\mathrm{T}},$$

where

$$\begin{aligned}
&F_1^2 = \mu_2 u_2, \qquad F_1^3 = 0,\\
&F_2^2 = \mu_2(-u_1 - \alpha u_2) + (\mu_1\tau_0 + \mu_2)u_5(t-1) - \frac{\tau_0}{2}u_1^2 + \frac{\tau_0}{2}u_5^2(t-1),\\
&F_2^3 = \mu_1\mu_2 u_5(t-1) - \frac{\mu_2}{2}u_1^2 - \frac{\tau_0}{6}u_1^3 + \frac{\mu_1\tau_0 + \mu_2}{2}u_5^2(t-1) + \frac{\tau_0}{6}u_5^3(t-1),\\
&F_3^2 = \mu_2 u_4, \qquad F_3^3 = 0,\\
&F_4^2 = \mu_2(-u_3 - \alpha u_4) + (\mu_1\tau_0 + \mu_2)u_1(t-1) - \frac{\tau_0}{2}u_3^2 + \frac{\tau_0}{2}u_1^2(t-1),\\
&F_4^3 = \mu_1\mu_2 u_1(t-1) - \frac{\mu_2}{2}u_3^2 - \frac{\tau_0}{6}u_3^3 + \frac{\mu_1\tau_0 + \mu_2}{2}u_1^2(t-1) + \frac{\tau_0}{6}u_1^3(t-1),\\
&F_5^2 = \mu_2 u_6, \qquad F_5^3 = 0,\\
&F_6^2 = \mu_2(-u_5 - \alpha u_6) + (\mu_1\tau_0 + \mu_2)u_3(t-1) - \frac{\tau_0}{2}u_5^2 + \frac{\tau_0}{2}u_3^2(t-1),\\
&F_6^3 = \mu_1\mu_2 u_3(t-1) - \frac{\mu_2}{2}u_5^2 - \frac{\tau_0}{6}u_5^3 + \frac{\mu_1\tau_0 + \mu_2}{2}u_3^2(t-1) + \frac{\tau_0}{6}u_3^3(t-1).
\end{aligned}$$

Then system (A.2) can be transformed into

$$\dot{X}(t) = L_\mu X_t + F(X_t, \mu), \quad X_t = X(t+\theta), \quad -1 \leqslant \theta \leqslant 0,$$

and the bilinear form on C and C^* ($*$ stand for adjoin) is

$$\big(\psi(s), \varphi(\theta)\big) = \psi(0)\varphi(0) - \int_{-1}^{0}\int_{0}^{\theta} \psi(\xi - \theta)\,\mathrm{d}\eta(\theta, 0)\varphi(\xi)\,\mathrm{d}\xi, \quad \psi \in C^*,\ \varphi \in C,$$

where

$$\begin{aligned}
\varphi(\theta) &= \big(\varphi_1(\theta), \varphi_2(\theta), \varphi_3(\theta), \varphi_4(\theta), \varphi_5(\theta), \varphi_6(\theta)\big) \in C,\\
\psi(s) &= \big(\psi_1(s), \psi_2(s), \psi_3(s), \psi_4(s), \psi_5(s), \psi_6(s)\big)^{\mathrm{T}} \in C^*.
\end{aligned}$$

Because L_0 has a simple 0 and a pair of purely imaginary eigenvalues $\pm\mathrm{i}\omega_0$, all other eigenvalues have negative real part. Let $\Lambda = \{\mathrm{i}\omega_0\tau_0, -\mathrm{i}\omega_0\tau_0, 0\}$, and let P_Λ can be the generalized eigenspace associated with Λ and P_Λ^* the space adjoint with P_Λ. Then C can be decomposed as $C = P_\Lambda \bigoplus Q_\Lambda$, where $Q_\Lambda = \{\varphi \in C\colon \langle\psi, \varphi\rangle = 0 \text{ for all } \psi \in P_\Lambda^*\}$. Choose the bases Φ and Ψ for P_Λ and P_Λ^* such that $(\Psi(s), \Phi(\theta)) = I$, $\dot{\Phi} = \Phi\mathbf{J}$ and $-\Psi = \mathbf{J}\Psi$, where $\mathbf{J} = \mathrm{diag}(\mathrm{i}\omega_0\tau_0, -\mathrm{i}\omega_0\tau_0, 0)$. Suppose that the bases for P_Λ and P_Λ^* are given by

$$\Phi = \big(\phi_1(\theta), \bar{\phi}_1(\theta), \phi_2(\theta)\big), \qquad \Psi(s) = \big(\bar{\psi}_1(\theta), \psi_1(\theta), \psi_2(\theta)\big)^{\mathrm{T}},$$

where

$$\begin{aligned}
\phi_1(\theta) &= \mathrm{e}^{\mathrm{i}\tau_0\omega_0\theta}(1, \mathrm{i}\omega_0, 1, \mathrm{i}\omega_0, 1, \mathrm{i}\omega_0)^{\mathrm{T}}, \qquad \phi_2(\theta) = (1, 0, 1, 0, 1, 0)^{\mathrm{T}}, \\
\psi_1(\theta) &= D\mathrm{e}^{-\mathrm{i}s\tau_0\omega_0} \\
&\quad \times \bigl((\alpha+\mathrm{i}\omega_0)\mathrm{e}^{-2\mathrm{i}\tau_0\omega_0}, \mathrm{e}^{-2\mathrm{i}\tau_0\omega_0}, (\alpha+\mathrm{i}\omega_0)\mathrm{e}^{-4\mathrm{i}\tau_0\omega_0}, \mathrm{e}^{-4\mathrm{i}\tau_0\omega_0}, (\alpha+\mathrm{i}\omega_0), 1\bigr), \\
\psi_2(\theta) &= D_1(\alpha, 1, \alpha, 1, \alpha, 1), \\
D &= \bigl[\bigl(1 + \mathrm{e}^{-2\mathrm{i}\tau_0\omega_0} + \mathrm{e}^{-4\mathrm{i}\tau_0\omega_0}\bigr)\bigl(\tau_0\mathrm{e}^{-\mathrm{i}\tau_0\omega_0} + 2\mathrm{i}\omega_0 + \alpha\bigr)\bigr]^{-1}, \\
D_1 &= \bigl[3(\alpha + \tau_0)\bigr]^{-1}.
\end{aligned}$$

Thus system (A.2) becomes an abstract ODE in the space BC, which can be identified as $C \times \mathbb{R}^3$

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \mathcal{A}u + X_0\widetilde{F}(u_t, \mu). \tag{A.3}$$

The elements of BC can be written as $\psi = \varphi + \eta_0 c$ with $\varphi \in C$, $c \in \mathbb{R}^3$, and

$$\eta_0(\theta) = \begin{cases} 0, & \theta \in [-1, 0), \\ I, & \theta = 0, \end{cases}$$

where $u \in C$ and $\mathcal{A}$ is defined by

$$\mathcal{A}\colon C \to BC, \quad \mathcal{A}u = \dot{u} + \eta_0\bigl[L_0 u - \dot{u}(0)\bigr]$$

and

$$\tilde{F}(u, \mu) = \bigl[L(\mu) - L_0\bigr]u + F(u, \mu).$$

Then the enlarged phase space BC can be decomposed as $BC = P_\Lambda \oplus \operatorname{Ker}\pi$. Let $u_t = \Phi x(t) + y_t$ with $x(t) = (x_1, x_2, x_3)^{\mathrm{T}}$, $y_t \in Q^1 := Q \cap C^1 \subset \operatorname{Ker}\pi$, and $\mathcal{A}_{Q^1}$ is the restriction of $\mathcal{A}$ as an operator from Q^1 to the Banach space $\operatorname{Ker}\pi$, namely,

$$\begin{pmatrix} u_1(\theta) \\ u_2(\theta) \\ u_3(\theta) \\ u_4(\theta) \\ u_5(\theta) \\ u_6(\theta) \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\theta\tau_0\omega_0}x_1 + \mathrm{e}^{-\mathrm{i}\theta\tau_0\omega_0}x_2 + x_3 + y_1(\theta) \\ \mathrm{i}\omega_0\mathrm{e}^{\mathrm{i}\theta\tau_0\omega_0}x_1 - \mathrm{i}\omega_0\mathrm{e}^{-\mathrm{i}\theta\tau_0\omega_0}x_2 + y_2(\theta) \\ \mathrm{e}^{\mathrm{i}\theta\tau_0\omega_0}x_1 + \mathrm{e}^{-\mathrm{i}\theta\tau_0\omega_0}x_2 + x_3 + y_3(\theta) \\ \mathrm{i}\omega_0\mathrm{e}^{\mathrm{i}\theta\tau_0\omega_0}x_1 - \mathrm{i}\omega_0\mathrm{e}^{-\mathrm{i}\theta\tau_0\omega_0}x_2 + y_4(\theta) \\ \mathrm{e}^{\mathrm{i}\theta\tau_0\omega_0}x_1 + \mathrm{e}^{-\mathrm{i}\theta\tau_0\omega_0}x_2 + x_3 + y_5(\theta) \\ \mathrm{i}\omega_0\mathrm{e}^{\mathrm{i}\theta\tau_0\omega_0}x_1 - \mathrm{i}\omega_0\mathrm{e}^{-\mathrm{i}\theta\tau_0\omega_0}x_2 + y_6(\theta) \end{pmatrix}.$$

Equation (A.3) is decomposed to the form

$$\dot{x} = \mathbf{J} + \Psi(0)\tilde{F}(\Phi x + y_t,\, \mu), \qquad \dot{y}_t = \mathcal{A}_{Q^1}\tilde{\varphi} + (I - \pi)X_0\tilde{F}(\Phi x + y_t,\, \mu).$$

Using the method of Faria and Magalhães [8, 9], the above system can be written as

$$\dot{x} = \mathbf{J}x + \sum_{j \geqslant 2}\frac{1}{j!}f_j^1(x, y, \mu), \qquad \dot{y} = \mathcal{A}_{Q^1}y_t + \sum_{j \geqslant 2}\frac{1}{j!}f_j^2(x, y, \mu), \tag{A.4}$$

where $f_i(x, y, \mu)$ is homogeneous polynomials of degree j about (x, y, μ) with coefficients in C^3. Here, define M_j to be the operator in $V_j^5(C^3 \times \operatorname{Ker}\pi)$ with the range in the same space by

$$M_j^1\colon V_j^5(C^3) \to V_j^5(C^3), \quad (M_j^1 p)(x, \mu) = D_x p(x, \mu)\mathbf{J}x - \mathbf{J}p(x, \mu),$$

where $V_j^5(C^3)$ represents the linear space of the second-order homogeneous polynomials in seven variables $(x_1, x_2, x_3, \mu_1, \mu_2)$ with coefficients in C^3, and it is easy to verify that one can choose such a decomposition

$$V_2^5(C^3) = \operatorname{Im}(M_2^1) \oplus \operatorname{Ker}(M_2^1), \qquad V_3^5(C^3) = \operatorname{Im}(M_3^1) \oplus \operatorname{Ker}(M_3^1),$$

where $\operatorname{Ker}(M_2^1)$ and $\operatorname{Ker}(M_3^1)$ represent the complimentary space, respectively, and the span elements of $\operatorname{Ker}(M_2^1)$ are

$$\{x_3^2 e_3,\ x_1x_2e_3,\ x_1\mu_i e_3,\ \mu_1\mu_2 e_3,\ x_2x_3e_2,\ x_2\mu_i e_2,\ x_1x_3e_1,\ x_1\mu_i e_1\}, \quad i = 1, 2.$$

Similarly, we can get that $\operatorname{Ker}(M_3^1)$ is spanned by

$$\begin{aligned}&\{x_3^3e_3,\ x_1x_2x_3e_3,\ \mu_1\mu_2x_3e_3,\ x_3^2\mu_ie_3,\ \mu_ix_1x_2e_3,\ \mu_i^2x_3e_3,\\ &\ x_3^2x_2e_2,\ x_1x_2^2e_2,\ x_2\mu_i^2e_2,\ \mu_1\mu_2x_2e_2,\ \mu_i^2x_2e_2,\ x_3^2x_1e_1,\\ &\ x_1^2x_2e_1,\ \mu_ix_1x_3e_1,\ \mu_i^2x_1e_1,\ \mu_1\mu_2x_1e_1\}, \quad i = 1, 2.\end{aligned}$$

Let

$$\Psi(0) = \begin{pmatrix} \psi_{11} & \psi_{12} & \psi_{13} & \psi_{14} & \psi_{15} & \psi_{16} \\ \psi_{21} & \psi_{22} & \psi_{23} & \psi_{24} & \psi_{25} & \psi_{26} \\ \psi_{31} & \psi_{32} & \psi_{33} & \psi_{34} & \psi_{35} & \psi_{36} \end{pmatrix},$$

where

$$\begin{aligned}
&\psi_{11} = \bar{D}e^{2i\tau_0\omega_0}(\alpha - i\omega_0), && \psi_{12} = \bar{D}e^{2i\tau_0\omega_0}, && \psi_{13} = \bar{D}e^{4i\tau_0\omega_0}(\alpha - i\omega_0),\\
&\psi_{14} = \bar{D}e^{4i\tau_0\omega_0}, && \psi_{15} = \bar{D}(\alpha - i\omega_0), && \psi_{16} = \bar{D}\\
&\psi_{21} = De^{-2i\tau_0\omega_0}(\alpha + i\omega_0), && \psi_{22} = De^{-2i\tau_0\omega_0}, && \psi_{23} = De^{-4i\tau_0\omega_0}(\alpha + i\omega_0),\\
&\psi_{24} = De^{-4i\tau_0\omega_0}, && \psi_{25} = D(\alpha + i\omega_0), && \psi_{26} = D,
\end{aligned}$$

$$\psi_{31} = \psi_{33} = \psi_{35} = D_1\alpha, \qquad \psi_{32} = \psi_{34} = \psi_{36} = D_1.$$

Next, we will do a little more detailed calculation:

$$\begin{aligned}
\Psi(0)F(x_t, \mu) &= \frac{1}{2!}f_2^1(x, y, \mu) + \frac{1}{3!}f_3^1(x, y, \mu) + \text{h.o.t.}\\
&= \begin{pmatrix} \psi_{11}F_1^2 + \psi_{12}F_2^2 + \psi_{13}F_3^2 + \psi_{14}F_4^2 + \psi_{15}F_5^2 + \psi_{16}F_6^2 \\ \psi_{21}F_1^2 + \psi_{22}F_2^2 + \psi_{23}F_3^2 + \psi_{24}F_4^2 + \psi_{25}F_5^2 + \psi_{26}F_6^2 \\ \psi_{31}F_1^2 + \psi_{32}F_2^2 + \psi_{33}F_3^2 + \psi_{34}F_4^2 + \psi_{35}F_5^2 + \psi_{36}F_6^2 \end{pmatrix}\\
&\quad + \begin{pmatrix} \psi_{12}F_2^3 + \psi_{14}F_4^3 + \psi_{16}F_6^3 \\ \psi_{22}F_2^3 + \psi_{24}F_4^3 + \psi_{26}F_6^3 \\ \psi_{32}F_2^3 + \psi_{34}F_4^3 + \psi_{36}F_6^3 \end{pmatrix} + \text{h.o.t.},
\end{aligned}$$

then we have

$$\frac{1}{2!}f_2^1(x,y,\mu)=\begin{pmatrix}\psi_{11}F_1^2+\psi_{12}F_2^2+\psi_{13}F_3^2+\psi_{14}F_4^2+\psi_{15}F_5^2+\psi_{16}F_6^2\\ \psi_{21}F_1^2+\psi_{22}F_2^2+\psi_{23}F_3^2+\psi_{24}F_4^2+\psi_{25}F_5^2+\psi_{26}F_6^2\\ \psi_{31}F_1^2+\psi_{32}F_2^2+\psi_{33}F_3^2+\psi_{34}F_4^2+\psi_{35}F_5^2+\psi_{36}F_6^2\end{pmatrix}$$

and

$$\frac{1}{3!}f_3^1(x,y,\mu)=\begin{pmatrix}\psi_{12}F_2^3+\psi_{14}F_4^3+\psi_{16}F_6^3\\ \psi_{22}F_2^3+\psi_{24}F_4^3+\psi_{26}F_6^3\\ \psi_{32}F_2^3+\psi_{34}F_4^3+\psi_{36}F_6^3\end{pmatrix}.$$

On the center manifold, the first equation of system (A.4) can be transform as the following normal form:

$$\dot{x}=\mathbf{J}z+\frac{1}{2!}g_2^1(x,0,\mu)+\frac{1}{3!}g_3^1(x,0,0)+\text{h.o.t.}$$

with $g_2^1(x,0,\mu)=\operatorname{Proj}_{\operatorname{Ker}(M_2^1)}f_2^1(x,0,\mu)$ and $g_3^1(x,0,\mu)=\operatorname{Proj}_{\operatorname{Ker}(M_3^1)}f_3^1(x,0,\mu)$.

Step 1. Calculate $g_2^1(x,0,\mu)$:

$$\begin{aligned}\frac{1}{2!}g_2^1(x,0,\mu)&=\frac{1}{2!}\operatorname{Proj}_{\operatorname{Ker}(M_2^1)}f_2^1(x,0,\mu)+O(|\mu^2|)\\ &=\begin{pmatrix}(a_{11}\mu_1+a_{12}\mu_2)x_1+a_{13}x_1x_3\\ (\bar{a}_{11}\mu_1+\bar{a}_{12}\mu_2)x_2+\bar{a}_{13}x_2x_3\\ (a_{21}\mu_1+a_{22}\mu_2)x_3+a_{23}x_1x_2+a_{24}x_3^2\end{pmatrix},\end{aligned}$$

where

$$\begin{aligned}a_{11}&=\bar{D}e^{-i\tau_0\omega_0}\left(e^{2i\tau_0\omega_0}+e^{4i\tau_0\omega_0}+1\right)\left(1+\left(\omega_0^2-1\right)e^{i\tau_0\omega_0}\right),\\ a_{13}&=-\bar{D}\tau_0e^{-i\tau_0\omega_0}\left(e^{i\tau_0\omega_0}-1\right)\left(e^{2i\tau_0\omega_0}+e^{4i\tau_0\omega_0}+1\right),\\ a_{21}&=3D_1\tau_0\qquad a_{22}=0,\qquad a_{23}=0,\qquad a_{24}=0.\end{aligned}$$

Step 2. Calculate $g_3^1(x,0,\mu)$ according to the expression from [27] as follows:

$$\begin{aligned}&\frac{1}{3!}g_3^1(x,0,\mu)\\ &\quad=\operatorname{Proj}_{\operatorname{Ker}(M_3^1)}\frac{1}{3!}\tilde{f}_3^1(x,0,\mu)\\ &\quad=\operatorname{Proj}_{\operatorname{Ker}(M_3^1)}\frac{1}{3!}\tilde{f}_3^1(x,0,0)+O\left(|\mu|^2|x|\right)+O\left(|\mu||x|^2\right)\\ &\quad=\operatorname{Proj}_{\operatorname{Ker}(M_3^1)}\frac{1}{3!}f_3^1(x,0,0)\\ &\qquad+\operatorname{Proj}_{\operatorname{Ker}(M_3^1)}\left[\frac{1}{4}\left(D_xf_2^1\right)(x,0,0)U_2^1(x,0)+\frac{1}{4}\left(D_yf_2^1\right)(x,0,0)U_2^2(x,0)\right].\end{aligned}$$

We need to calculate step by step. First, calculate $\mathrm{Proj}_{\mathrm{Ker}(M_3^1)}(1/3!)f_3^1(x,0,0)$:

$$\frac{1}{3!}f_3^1(x,0,0)=\frac{\tau_0}{6}\begin{pmatrix}\bar{D}\Theta_1(-\Theta_2^3+\Theta_3^3)\\ D\bar{\Theta}_1(-\Theta_2^3+\Theta_3^3)\\ 3D_1(-\Theta_2^3+\Theta_3^3)\end{pmatrix},$$

where

$$\Theta_1=\bigl(1+\mathrm{e}^{2\mathrm{i}\tau_0\omega_0}+\mathrm{e}^{4\mathrm{i}\tau_0\omega_0}\bigr),\qquad \Theta_2=x_1+x_2+x_3,$$
$$\Theta_3=\mathrm{e}^{-\mathrm{i}\tau_0\omega_0}x_1+\mathrm{e}^{\mathrm{i}\tau_0\omega_0}x_2+x_3,$$

then we have

$$\frac{1}{3!}\,\mathrm{Proj}_{\mathrm{Ker}(M_3^1)}\,f_3^1(x,0,0)=\begin{pmatrix}b_{11}x_1^2x_2+b_{12}x_1x_3^2\\ \bar{b}_{11}x_1x_2^2+\bar{b}_{12}x_2x_3^2\\ b_{21}x_1x_2x_3+b_{22}x_3^3\end{pmatrix},$$

where

$$b_{11}=b_{12}=-\frac{1}{2}\bar{D}\tau_0\Theta_1(1-\mathrm{e}^{-\mathrm{i}\tau_0\omega_0}),\qquad b_{21}=b_{22}=0.$$

Second, calculate $(1/4)\,\mathrm{Proj}_{\mathrm{Ker}(M_3^1)}[(D_xf_2^1(x,0,0))U_2^1(x,0)]$ since

$$U_2^1(x,0)=U_2^1(x,\mu)|_{\mu=0}=(M_2^1)^{-1}\,\mathrm{Proj}_{\mathrm{Im}(M_2^1)}\,f_2^1(x,0,0)=(\Upsilon_1,\Upsilon_2,\Upsilon_3)^{\mathrm{T}},$$

where

$$\begin{aligned}\Upsilon_1={}&\frac{\bar{D}\tau_0}{\mathrm{i}\omega_0}\bigl\{x_1^2\bigl(\mathrm{e}^{-2\mathrm{i}\tau_0\omega_0}-\mathrm{e}^{4\mathrm{i}\tau_0\omega_0}\bigr)-3^{-1}x_2^2\bigl(\mathrm{e}^{6\mathrm{i}\tau_0\omega_0}-1\bigr)\\&-x_2x_3\bigl(\mathrm{e}^{\mathrm{i}\tau_0\omega_0}-\mathrm{e}^{2\mathrm{i}\tau_0\omega_0}+\mathrm{e}^{3\mathrm{i}\tau_0\omega_0}-\mathrm{e}^{4\mathrm{i}\tau_0\omega_0}+\mathrm{e}^{5\mathrm{i}\tau_0\omega_0}-1\bigr)\bigr\},\\ \Upsilon_2={}&\frac{D\tau_0}{\mathrm{i}\omega_0}\bigl\{3^{-1}x_1^2\bigl(\mathrm{e}^{-6\mathrm{i}\tau_0\omega_0}-1\bigr)-x_2^2\bigl(\mathrm{e}^{2\mathrm{i}\tau_0\omega_0}-\mathrm{e}^{-4\mathrm{i}\tau_0\omega_0}\bigr)\\&+x_1x_3\bigl(\mathrm{e}^{-\mathrm{i}\tau_0\omega_0}+\mathrm{e}^{-5\mathrm{i}\tau_0\omega_0}-\mathrm{e}^{-4\mathrm{i}\tau_0\omega_0}+\mathrm{e}^{-3\mathrm{i}\tau_0\omega_0}-\mathrm{e}^{-2\mathrm{i}\tau_0\omega_0}-1\bigr)\bigr\},\\ \Upsilon_3={}&\frac{3D_1\tau_0}{\mathrm{i}\omega_0}\bigl\{2^{-1}x_1^2\bigl(\mathrm{e}^{-2\mathrm{i}\tau_0\omega_0}-1\bigr)-2^{-1}x_2^2\bigl(\mathrm{e}^{2\mathrm{i}\tau_0\omega_0}-1\bigr)\\&-2x_2x_3\bigl(\mathrm{e}^{\mathrm{i}\tau_0\omega_0}-1\bigr)+2x_1x_3\bigl(\mathrm{e}^{-\mathrm{i}\tau_0\omega_0}-1\bigr)\bigr\}\end{aligned}$$

with the elements of $\mathrm{Im}(M_2^1)$ $(\mu=0)$

$$\bigl\{x_1x_2e_1,\ x_2x_3e_1,\ x_1^2e_1,\ x_2^2e_1,\ x_3^2e_1,\ x_1x_2e_2,\ x_1x_3e_2,$$
$$x_1^2e_2,\ x_2^2e_2,\ x_3^2e_2,\ x_1x_3e_3,\ x_2x_3e_3,\ x_1^2e_3,\ x_2^2e_3\bigr\}.$$

Then we have

$$D_x(f_2^1(x,0,0))=\begin{pmatrix}\mathbf{g}_{11}&\mathbf{g}_{12}&\mathbf{g}_{13}\\ \mathbf{g}_{21}&\mathbf{g}_{22}&\mathbf{g}_{23}\\ \mathbf{g}_{31}&\mathbf{g}_{32}&\mathbf{g}_{33}\end{pmatrix},$$

where

$$\begin{aligned}
\mathbf{g}_{11} &= 2\bar{D}\tau_0\Theta_1\big(-(x_1+x_3)+\big(x_1\mathrm{e}^{-2\mathrm{i}\tau_0\omega_0}+\mathrm{e}^{-\mathrm{i}\tau_0\omega_0}x_3\big)\big),\\
\mathbf{g}_{12} &= 2\bar{D}\tau_0\Theta_1\big(-(x_2+x_3)+\big(\mathrm{e}^{2\mathrm{i}\tau_0\omega_0}x_2+\mathrm{e}^{\mathrm{i}\tau_0\omega_0}x_3\big)\big),\\
\mathbf{g}_{13} &= 2\bar{D}\tau_0\Theta_1\big(-(x_1+x_2)+\big(x_1\mathrm{e}^{-\mathrm{i}\tau_0\omega_0}+x_2\mathrm{e}^{\mathrm{i}\tau_0\omega_0}\big)\big),\\
\mathbf{g}_{21} &= 2D\tau_0\bar{\Theta}_1\big(-(x_1+x_3)+\big(x_1\mathrm{e}^{-2\mathrm{i}\tau_0\omega_0}+\mathrm{e}^{-\mathrm{i}\tau_0\omega_0}x_3\big)\big),\\
\mathbf{g}_{22} &= 2D\tau_0\bar{\Theta}_1\big(-(x_2+x_3)+\big(\mathrm{e}^{2\mathrm{i}\tau_0\omega_0}x_2+\mathrm{e}^{\mathrm{i}\tau_0\omega_0}x_3\big)\big),\\
\mathbf{g}_{23} &= 2D\tau_0\bar{\Theta}_1\big(-(x_1+x_2)+\big(x_1\mathrm{e}^{-\mathrm{i}\tau_0\omega_0}+x_2\mathrm{e}^{\mathrm{i}\tau_0\omega_0}\big)\big),\\
\mathbf{g}_{31} &= 6D_1\tau_0\big(-\Theta_2+\big(x_1\mathrm{e}^{-2\mathrm{i}\tau_0\omega_0}+x_2+\mathrm{e}^{-\mathrm{i}\tau_0\omega_0}x_3\big)\big),\\
\mathbf{g}_{32} &= 6D_1\tau_0\big(-\Theta_2+\big(x_1+\mathrm{e}^{2\mathrm{i}\tau_0\omega_0}x_2+\mathrm{e}^{\mathrm{i}\tau_0\omega_0}x_3\big)\big),\\
\mathbf{g}_{33} &= 6D_1\tau_0\big(-(x_1+x_2)+\big(x_1\mathrm{e}^{-\mathrm{i}\tau_0\omega_0}+x_2\mathrm{e}^{\mathrm{i}\tau_0\omega_0}\big)\big).
\end{aligned}$$

Combining to the above, we obtain

$$\mathrm{Proj}_{\mathrm{Ker}(M_3^1)}\left[\frac{1}{4}(D_xf_2^1)(x,0,0)U_2^1(x,0)\right] = \begin{pmatrix} c_{11}x_1^2x_2+c_{12}x_1x_3^2\\ c_{21}x_1x_2^2+c_{22}x_2x_3^2\\ c_{31}x_1x_2x_3+c_{32}x_3^3\end{pmatrix},$$

where

$$\begin{aligned}
c_{11} &= \frac{\bar{D}\tau_0^2\mathrm{e}^{-6\mathrm{i}\tau_0\omega_0}}{12\mathrm{i}\omega_0}\big(9D_1\mathrm{e}^{4\mathrm{i}\tau_0\omega_0}\big(\mathrm{e}^{\mathrm{i}\tau_0\omega_0}+\mathrm{e}^{6\mathrm{i}\tau_0\omega_0}-\mathrm{e}^{7\mathrm{i}\tau_0\omega_0}-1\big)\\
&\quad -2D\big(-1+\mathrm{e}^{6\mathrm{i}\tau_0\omega_0}\big)^2\big),\\
c_{12} &= -\frac{\bar{D}D\tau_0^2\mathrm{e}^{-5\mathrm{i}\tau_0\omega_0}}{2\mathrm{i}\omega_0}\big(\mathrm{e}^{\mathrm{i}\tau_0\omega_0}-\mathrm{e}^{2\mathrm{i}\tau_0\omega_0}+\mathrm{e}^{3\mathrm{i}\tau_0\omega_0}-\mathrm{e}^{4\mathrm{i}\tau_0\omega_0}+\mathrm{e}^{5\mathrm{i}\tau_0\omega_0}-1\big)^2,\\
c_{21} &= \bar{c}_{11},\qquad c_{22}=\bar{c}_{12},\\
c_{31} &= \frac{3D_1\tau_0^2\mathrm{e}^{-5\mathrm{i}\tau_0\omega_0}}{2\mathrm{i}\omega_0}\big(-1+\mathrm{e}^{\mathrm{i}\tau_0\omega_0}\big)^2\big(\mathrm{e}^{\mathrm{i}\tau_0\omega_0}+\mathrm{e}^{2\mathrm{i}\tau_0\omega_0}+\mathrm{e}^{3\mathrm{i}\tau_0\omega_0}+\mathrm{e}^{4\mathrm{i}\tau_0\omega_0}\\
&\quad +\mathrm{e}^{5\mathrm{i}\tau_0\omega_0}+1\big)\big(-D+\bar{D}\mathrm{e}^{3\mathrm{i}\tau_0\omega_0}\big),\\
c_{32} &= 0.
\end{aligned}$$

Step 3. We compute $\mathrm{Proj}_{\mathrm{Ker}(M_3^1)}[(D_yf_2^1)(x,0,0)U_2^2(x.0)]$. Define $h=h(x)(\theta)=U_2^2(x,0)$. Then we have

$$\begin{aligned}
h(\theta) = \begin{pmatrix} h^{(1)}(\theta)\\ h^{(2)}(\theta)\\ h^{(3)}(\theta)\\ h^{(4)}(\theta)\\ h^{(5)}(\theta)\\ h^{(6)}(\theta)\end{pmatrix} &= \begin{pmatrix}
h_{200}^{(1)}x_1^2+h_{020}^{(1)}x_2^2+h_{101}^{(1)}x_1x_3+h_{011}^{(1)}x_2x_3\\
h_{200}^{(2)}x_1^2+h_{020}^{(2)}x_2^2+h_{101}^{(2)}x_1x_3+h_{011}^{(2)}x_2x_3\\
h_{200}^{(3)}x_1^2+h_{020}^{(3)}x_2^2+h_{101}^{(3)}x_1x_3+h_{011}^{(3)}x_2x_3\\
h_{200}^{(4)}x_1^2+h_{020}^{(4)}x_2^2+h_{101}^{(4)}x_1x_3+h_{011}^{(4)}x_2x_3\\
h_{200}^{(5)}x_1^2+h_{020}^{(5)}x_2^2+h_{101}^{(5)}x_1x_3+h_{011}^{(5)}x_2x_3\\
h_{200}^{(6)}x_1^2+h_{020}^{(6)}x_2^2+h_{101}^{(6)}x_1x_3+h_{011}^{(6)}x_2x_3
\end{pmatrix}\\
&= h_{200}x_1^2+h_{020}x_2^2+h_{002}x_3^2+h_{110}x_1x_2+h_{101}x_1x_3+h_{011}x_2x_3
\end{aligned}$$

and

$$F^2(\Phi x, 0) = A_{200}x_1^2 + A_{020}x_2^2 + A_{002}x_3^2 + A_{110}x_1x_2 + A_{101}x_1x_3 + A_{011}x_2x_3,$$

where $h_{200}, h_{020}, h_{002}, h_{110}, h_{101}, h_{011} \in Q^1$, $A_{ijk} \in \mathbf{C}^2$, $0 \leqslant i, j, k \leqslant 2$ and $i + j + k = 2$. Comparing the coefficients of x_1^2, x_2^2, x_3^2, x_1x_2, x_1x_3, x_2x_3 from the equation $A_{002} = A_{110} = 0$ and $h_{200} = \bar{h}_{020}$, $h_{101} = \bar{h}_{011}$, we can get $h_{002} = h_{110} = 0$. According to the definition of $\mathcal{A}_{Q^1}$ and π, we obtain that h_{200}, h_{101} satisfy the following differential equations, respectively:

$$\begin{aligned} &\dot{h}_{200} - 2\mathrm{i}\omega_0\tau_0 h_{200} = \Phi(\theta)\Psi(0)A_{200}, &&\dot{h}_{101} - \mathrm{i}\omega_0\tau_* h_{101} = \Phi(\theta)\Psi(0)A_{101},\\ &\dot{h}_{200}(0) - \mathcal{L}(h_{200}) = A_{200}, &&\dot{h}_{101}(0) - \mathcal{L}(h_{101}) = A_{101}. \end{aligned}$$

Since

$$F^2(u_t, 0) = F^2(\Phi x + y,\, 0) = -\frac{\tau_0}{2}\begin{pmatrix} 0\\ (\Theta_1 + y_1(0))^2 - (\Theta_3 + y_5(-1))^2\\ 0\\ (\Theta_1 + y_3(0))^2 - (\Theta_3 + y_1(-1))^2\\ 0\\ (\Theta_1 + y_5(0))^2 - (\Theta_3 + y_3(-1))^2 \end{pmatrix},$$

we have

$$\begin{aligned} \frac{1}{2}f_2^1(x, y, 0) &= \Psi(0)F^2(\Phi x + y,\, 0)\\ &= \tau_0\begin{pmatrix} \bar{D}\mathrm{e}^{2\mathrm{i}\tau_0\omega_0}[-(\Theta_1 + y_1(0))^2 + (\Theta_3 + y_5(-1))^2]\\ \quad +\bar{D}\mathrm{e}^{4\mathrm{i}\tau_0\omega_0}[-(\Theta_1 + y_3(0))^2 + (\Theta_3 + y_1(-1))^2]\\ \quad +\bar{D}\big[-(\Theta_1 + y_5(0))^2 + (\Theta_3 + y_3(-1))^2]\\ D\mathrm{e}^{-2\mathrm{i}\tau_0\omega_0}[-(\Theta_1 + y_1(0))^2 + (\Theta_3 + y_5(-1))^2]\\ \quad +D\mathrm{e}^{-4\mathrm{i}\tau_0\omega_0}[-(\Theta_1 + y_3(0))^2 + (\Theta_3 + y_1(-1))^2]\\ \quad +D\big[-(\Theta_1 + y_5(0))^2 + (\Theta_3 + y_3(-1))^2]\\ D_1[-(\Theta_1 + y_1(0))^2 + (\Theta_3 + y_5(-1))^2]\\ \quad +D_1[-(\Theta_1 + y_3(0))^2 + (\Theta_3 + y_1(-1))^2]\\ \quad +D_1[-(\Theta_1 + y_5(0))^2 + (\Theta_3 + y_3(-1))^2] \end{pmatrix}. \end{aligned}$$

Continue to find the derivative of y and get the value of $D_y f_2^1(x, 0, 0)$). Then we have

$$\frac{1}{4}D_y f_2^1(x, 0, 0)h(\theta) = \frac{1}{2}\tau_0\begin{pmatrix} \bar{D}\mathrm{e}^{2\mathrm{i}\tau_0\omega_0}[-\Theta_1 h^1(0) + \mathrm{e}^{2\mathrm{i}\tau_0\omega_0}\Theta_3 h^1(-1)]\\ \quad +\bar{D}[-\mathrm{e}^{4\mathrm{i}\tau_0\omega_0}\Theta_1 h^3(0) + \Theta_3 h^3(-1)]\\ \quad +\bar{D}[-\Theta_1 h^5(0) + \mathrm{e}^{2\mathrm{i}\tau_0\omega_0}\Theta_3 h^5(-1)]\\ D\mathrm{e}^{-2\mathrm{i}\tau_0\omega_0}\big[-\Theta_1 h^1(0) + \mathrm{e}^{-2\mathrm{i}\tau_0\omega_0}\Theta_3 h^1(-1)\big]\\ \quad +D[-\mathrm{e}^{-4\mathrm{i}\tau_0\omega_0}\Theta_1 h^3(0) + \Theta_3 h^3(-1)]\\ \quad +D[-\Theta_1 h^5(0) + \mathrm{e}^{-2\mathrm{i}\tau_0\omega_0}\Theta_3 h^5(-1)]\\ D_1[-\Theta_1 h^1(0) + \Theta_3 h^1(-1)]\\ D_1[-\Theta_1 h^3(0) + \Theta_3 h^3(-1)]\\ D_1[-\Theta_1 h^5(0) + \Theta_3 h^5(-1)]) \end{pmatrix}. \qquad \text{(A.5)}$$

By expanding the original equation $F^2(\Phi x + y,\, 0)$ we have

$$A_{200} = \begin{pmatrix} 0 \\ (-1 + \mathrm{e}^{-2\mathrm{i}\tau_0\omega_0}) \\ 0 \\ (-1 + \mathrm{e}^{-2\mathrm{i}\tau_0\omega_0}) \\ 0 \\ (-1 + \mathrm{e}^{-2\mathrm{i}\tau_0\omega_0}) \end{pmatrix}, \qquad A_{101} = \begin{pmatrix} 0 \\ (-2 + 2\mathrm{e}^{-\mathrm{i}\tau_0\omega_0}) \\ 0 \\ (-2 + 2\mathrm{e}^{-\mathrm{i}\tau_0\omega_0}) \\ 0 \\ (-2 + 2\mathrm{e}^{-\mathrm{i}\tau_0\omega_0}) \end{pmatrix},$$

thus $A_{200}^{(1)} = A_{200}^{(3)} = A_{200}^{(5)} = A_{101}^{(1)} = A_{101}^{(3)} = A_{101}^{(5)} = 0$, then $h_{101}^{(1)} = h_{101}^{(3)} = h_{101}^{(5)} = h_{200}^{(1)} = h_{200}^{(3)} = h_{200}^{(5)} = 0$. So we know that $h^1(\theta) = h^3(\theta) = h^5(\theta) = 0$. According to system (A.5), we have $\mathrm{Proj}_{\mathrm{Ker}(M_3^1)}[(D_y f_2^1)(x, 0, 0)U_2^2(x.0)] = 0$.

From Steps 1–3 the normal form (11) has been obtained.

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