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A fractional q-difference equation eigenvalue problem with $p(t)$-Laplacian operator*

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Abstract. This article is devoted to studying a nonhomogeneous boundary value problem involving Stieltjes integral for a more general form of the fractional q-difference equation with $p(t)$-Laplacian operator. Here $p(t)$-Laplacian operator is nonstandard growth, which has been used more widely than the constant growth operator. By using fixed point theorems of $\varphi - (h,e)$-concave operators some conditions, which guarantee the existence of a unique positive solution, are derived. Moreover, we can construct an iterative scheme to approximate the unique solution. At last, two examples are given to illustrate the validity of our theoretical results.

Keywords: unique solution, fractional q-difference equation, $p(t)$-Laplacian operator, $\varphi - (h,e)$-concave operators.

1 Introduction

Fractional calculus, appeared at the beginning of twentieth century, has provided many hot topics of research in many disciplines such as biological sciences, engineering, aerodynamics and communications (see [3–5, 13] for example). Originally, the study on fractional q-difference calculus can be traced back to Agarwal [1] and Al-Salam [4], then it inspired much interest in theoretical research, many remarkable results have been arisen, which can be found in [2, 6, 7, 14].

Naturally, the widespread applications of fractional q-calculus have lead to a new development direction of fractional q-difference equations, which has exhibited adamantine incorporation to application in fluid mechanics and quantum calculus. After that, kinds of fixed point theorems have been used to deal with various fractional q-difference equation boundary value problems; see [1, 3, 6, 7, 10–12, 15, 16, 18, 19] for instance. As we know, the study of existence, uniqueness and multiplicity of solutions are abundant. In 2011,

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Ferreira [7] studied a fractional q-difference equation

$$D_q^\alpha y(x) = -f\bigl(x, y(x)\bigr), \quad x \in (0,1), \tag{1}$$

with boundary conditions

$$y(0) = D_q y(0) = 0, \qquad D_q y(1) = \beta \geqslant 0, \tag{2}$$

where $0 < q < 1$, $2 < \alpha \leqslant 3$, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a nonnegative continuous function. By employing Krasnosel'skii fixed point theorem the existence of positive solutions was enunciated.

In 2017, Wang [17] studied twin iterative positive solutions for a fractional q-difference Schrödinger equation

$$\begin{aligned} &\bigl(D_q^\alpha\bigr)u(x) + \lambda h(x) f\bigl(u(x)\bigr) = 0, \quad 0 < x < 1, \\ &u(0) = D_q u(0) = D_q u(1) = 0, \end{aligned} \tag{3}$$

where $0 < q < 1$, $2 < \alpha < 3$, $f \in C([0,\infty),(0,\infty))$, $h \in C((0,1),(0,\infty))$. The author obtained the existence of twin iterative positive solutions by using a fixed point theorem in cones associated with monotone iterative method. In 2020, Mao et al. [12] generalized the results in [17], the general research problem is

$$\begin{aligned} &\bigl(D_q^\alpha\bigr)u(t) + f\bigl(t, u(t), v(t)\bigr) = 0, \quad 0 < t < 1, \\ &u(0) = 0, D_q u(0) = D_q u(1) = 0, \end{aligned} \tag{4}$$

where $0 < q < 1$, $2 < \alpha \leqslant 3$, f may be singular at $v = 0$, $t = 0, 1$. By the iterative algorithm the author obtained a unique positive solution, where the nonlinear term has two space variables. In 2017, we have studied this boundary value problem in [19] by using the monotone iterative technique and lower-upper solution method, the existence of positive or negative solutions are obtained under the nonlinear term is local continuity and local monotonicity.

Since Leibenson [9] presented the p-Laplacian operator $\phi_p(x(t))$ in the turbulent flow model, recently, the fractional differential equations with p-Laplacian operator attracted much attention of scholars; see [8, 10, 18]. In 2016, Li et al. [10] investigated a fractional q-difference equation nonhomogeneous boundary value problem

$$D_q^\gamma\bigl(\phi\bigl(D_q^\alpha u(t)\bigr)\bigr) = \lambda f\bigl(u(t)\bigr), \quad 0 < t < 1, \tag{5}$$

restricted to

$$u(0) = (D_q u)(0) = 0, \qquad (D_q u)(1) = \beta, \qquad D_q^\alpha u(0) = 0, \tag{6}$$

where $0 < \gamma < 1$, $2 < \alpha < 3$, $\phi : \mathbb{R} \to \mathbb{R}$ is a generalized p-Laplacian operator, which includes two cases: $\phi(u) = u$ and $\phi(u) = |u|^{p-2}u$, $p \geqslant 1$. They gave the existence of positive solutions by some fixed point theorems in cones. As a generalized form of p-Laplacian operator, $p(t)$-Laplacian operator arises from image restoration, elastic mechanics, nonlinear electro-rheological fluids, which has been widely used in different fields such as physics, image processing, bioengineering, etc, with respect to some valuable results that we can see [6, 22].

Different from the above-mentioned works, in this article, we discuss the following nonhomogeneous two-point boundary value problem of a fractional q-difference equation containing $p(t)$-Laplacian operator:

$$\begin{aligned} &D_q^{\beta}\big(\varphi_{p(t)}\big(D_q^{\alpha}u(t)-g(t)\big)\big)+\mu f\big(t,u(t)\big)=0, \quad 0<t<1,\\ &u(0)=0, \quad (D_qu)(0)=0, \qquad (D_qu)(1)-\lambda[u]=\gamma, \qquad D_q^{\alpha}u(t)|_{t=0}=0, \end{aligned} \tag{7}$$

where $0<q<1$, $2<\alpha\leqslant 3$, $0<\beta<1$, $\gamma\geqslant 0$, $f,g\in C(R_0)$, $R_0=\{t\colon 0\leqslant t\leqslant 1\}\in T_q$, where T_q denotes the time scale defined by $T_q=\{q^n\colon n\in \mathbf{N}\}\cup\{0\}$. $D_q^{\alpha}, D_q^{\beta}$ denote the standard Riemann–Liouville fractional q-derivatives, $\mu>0$ is a parameter, $\lambda[u]$ denotes a linear functional given by

$$\lambda[u]=\int\limits_0^1 u(t)\,\mathrm{d}\Lambda(t)$$

involving Stieltjes integral with respect to a suitable function $\Lambda:[0,1]\to\mathbb{R}$ of bounded variation. The measure $\mathrm{d}\Lambda$ can be a signed measure. $\varphi_{p(t)}(z)=|z|^{p(t)-1}\operatorname{sgn} z$ is $p(t)$-Laplacian operator, $p(t)\in C^1[0,1]$ with $p(t)>1$, and it has the following characteristics:

(a) $\varphi:\mathbb{R}\to\mathbb{R}$ is an odd and strictly monotone increasing homeomorphism;
(b) the inverse continuous operator $\varphi_{p(t)}^{-1}(z)=|z|^{(2-p(t))/(p(t)-1)}z$, $z\in\mathbb{R}\setminus\{0\}$, and $\varphi_{p(t)}^{-1}(0)=0$.

Indeed, if $\lambda[u]=g(t)=0$, $\gamma>0$, $\varphi_{p(t)}$ degenerates to constant p, then our research problem turns into (5)–(6); if $\lambda[u]=g(t)=0$, $\varphi_{p(t)}(z)=z$, $\alpha=0$, $\mu=1$, then it turns into (1)–(2); on this basis, if $\gamma=0$, then it becomes homogeneous boundary value problem (3); if f has two space variables, then it changed into (4), so the boundary condition we studied in this paper is more extensive. The form $\varphi_{p(t)}(D_q^{\alpha}u(t)-g(t))$ has not been seen in existing works.

We study problem (7) by using some fixed point theorems of increasing $\varphi-(h,e)$-concave operators. Several new existence-uniqueness criteria of nontrivial solutions for problem (7) are obtained. In addition, we can construct a convergent monotone iterative scheme for approximating the unique solution, and the existence of lower-upper solutions is not required, thus our result weakened the restrictions in [19]. It should be pointed out that the compactness condition is not required, when $g(t)\equiv 0$, our unique results are also new.

Throughout this paper, let $\bar{p}=\max_{t\in[0,1]}p(t)$, $\underline{p}=\min_{t\in[0,1]}p(t)$, we assume that

(H1) $f:R_0\times[-\hat{e},+\infty)\to[0,+\infty)$ is increasing with respect to the second variable, where $\hat{e}=\max\{e(t)\colon t\in[0,1]\}$, $f(t,0)\not\equiv 0$ (there exists at least one point t_0 such that $f(t_0,0)\neq 0$);
(H2) for any $\lambda\in(0,1)$, there exists $\psi(\lambda)\in(0,1)$ with $\ln\psi(\lambda)>(\underline{p}-1)\ln\lambda$ such that $f(t,\lambda x+(\lambda-1)y)\geqslant\psi(\lambda)f(t,x)$ for all $t\in R_0$, $x\in\mathbb{R}$, $y\in[0,\hat{e}]$.

Remark 1. Let $y = \theta$, $\psi(\lambda) = \lambda^\sigma$, then assumption (H2) become condition (H1) in [17]. It can be regarded as a particular case of (H2) due to the characteristic of $p(t)$-Laplacian operator, here $\sigma + 1 - \underline{p} > 0$ is demanded. Besides, this condition covers the superlinear, sublinear and mixed types of superlinear and sublinear functions.

Remark 2. Condition (H2) implies that, for all $\lambda \geqslant 1$, we have

$$f\bigl(t,\, \lambda x + (\lambda - 1)y\bigr) \leqslant \psi(\lambda) f(t, x).$$

The paper is organized as follows. Section 2 contains some definitions and lemmas that will be used later. In Section 3, the local unique positive solution of problem (7) is obtained by using fixed point theorems in cones. Two examples are added to illustrate the main results in Section 4.

2 Preliminaries and previous results

We present some necessary definitions and lemmas about fractional q-calculus; for details, we can see [1, 4].

For fixed point $q \in \mathbb{R}$, V is a sunset of complex set $\mathcal{C}$, V is called q-geometric if $qt \in V$ whenever $t \in V$, that is to say, if V is q-geometric, then it includes all geometric sequences $\{tq^n\}_{n=0}^{\infty}$. The definition of q-analogue for $\alpha \in \mathbb{R}$ is

$$[\alpha]_q = \frac{1 - q^\alpha}{1 - q}.$$

The q-analogue of the Pochhammer symbol is defined by

$$(a; q)_0 = 1, \qquad (a; q)_n = \prod_{i=0}^{n-1} \bigl(1 - aq^i\bigr), \quad n \in \mathbf{N}.$$

Let f be a real-valued continuous function defined on a q-geometric set V, $|q| \neq 1$, the q-derivative of f is defined by

$$D_q f(t) = \frac{d_q}{d_q t} f(t) = \frac{f(qt) - f(t)}{(q - 1)t}, \quad t \neq 0,$$

and

$$D_q f(0) = \begin{cases} \frac{d_q}{d_q t} f(t)|_{t=0} = \lim_{n\to\infty} \frac{f(q^n t) - f(0)}{q^n t}, & |q| \neq 1, \\ D_{q^{-1}} f(0), & |q| > 1. \end{cases}$$

Furthermore, the nth q-derivative D_q^n can be represented by

$$D_q^n f(t) = (1 - q)^{-n} t^{-n} \sum_{r=0}^{n} q^r \frac{(q^{-n}; q)_r}{(q; q)_r} f\bigl(tq^r\bigr), \quad t \in V \setminus \{0\}.$$

The q-integral of a function f in the interval $[0,b]$ is defined by

$$(I_q f)(t) = \int_0^t f(s)\,\mathrm{d}_q s = (1-q)\sum_{n=0}^{\infty} f\bigl(tq^n\bigr)tq^n, \quad t \in [0,b].$$

Definition 1. (See [5].) Let $\alpha \geqslant 0$ and f be a function defined on $[0,1]$. The fractional q-integral of Riemann–Liouville type is $(I_q^0 f)(t) = f(t)$, and

$$\bigl(I_q^\alpha f\bigr)(t) = \frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)} f(s)\,\mathrm{d}_q s, \quad \alpha > 0.$$

Further, $(I_q^\alpha f)(t) = (I_q f)(t)$ when $\alpha = 1$.

Definition 2. [5] The fractional q-derivative of Riemann–Liouville type of order $\alpha \geqslant 0$ is defined by

$$\bigl(D_q^\alpha f\bigr)(t) = \bigl(D_q^{\lceil\alpha\rceil} I_q^{\lceil\alpha\rceil-\alpha} f\bigr)(t), \quad t \in [0,1],$$

where $\lceil\alpha\rceil$ is the smallest integer greater than or equal to α.

Moreover,

$$\bigl(I_q^\alpha D_q^p f\bigr)(t) = \bigl(D_q^p I_q^\alpha f\bigr)(t) - \sum_{n=0}^{p-1} \frac{t^{\alpha-p+n}}{\Gamma_q(\alpha-p+n+1)}\bigl(D_q^n f\bigr)(0), \quad p \in \mathbf{N}. \tag{8}$$

Remark 3. (See [19].) Assume that $f(t)$ is a continuous function on $[0,1]$ and there exists $t_0 \in (0,1)$ such that $f(t_0) \neq 0$. If $f(t) \geqslant 0$, then we have $\int_0^1 f(t)\,\mathrm{d}_q t > 0$, $t \in [0,1]$.

First, we consider the following boundary value problem:

$$\begin{aligned} &D_q^\alpha u(t) + y(t) = 0, \quad t \in (0,1),\\ &u(0) = 0, \quad (D_q u)(0) = 0, \qquad (D_q u)(1) - \lambda[u] = \gamma. \end{aligned} \tag{9}$$

We require the following assumption:

(H0) $\Lambda : [0,1] \to \mathbb{R}$ is a function of bounded variation and $[\alpha-1]_q - A \neq 0$, $A = \int_0^1 t^{\alpha-1}\,\mathrm{d}\Lambda(t)$, $\zeta(s) = \int_0^1 G_1(t,qs)\,\mathrm{d}\Lambda(t)$ for

$$G_1(t,qs) = \frac{1}{\Gamma_q(\alpha)}\begin{cases} (1-qs)^{(\alpha-2)}t^{\alpha-1} - (t-qs)^{(\alpha-1)}, & 0 \leqslant qs \leqslant t \leqslant 1,\\ (1-qs)^{(\alpha-2)}t^{\alpha-1}, & 0 \leqslant t \leqslant qs \leqslant 1. \end{cases}$$

Lemma 1. *Assume* (H0) *holds and* $y \in C[0,1]$, *then problem* (9) *has a unique solution*

$$u(t) = \frac{\gamma}{[\alpha-1]_q - A}t^{\alpha-1} + \int_0^1 G(t,qs)y(s)\,\mathrm{d}_q s,$$

where

$$G(t,qs) = G_1(t,qs) + \frac{t^{\alpha-1}}{[\alpha-1]_q - A}\zeta(s). \tag{10}$$

Proof. By Definitions 1, 2 and (8) we can reduce above problem to

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3} - I_q^{\alpha} y(t), \quad c_i \in \mathbb{R},\ i = 1, 2, 3.$$

It follows from the condition $u(0) = D_q u(0) = 0$ that $c_2 = c_3 = 0$, then

$$u(t) = c_1 t^{\alpha-1} - I_q^{\alpha} y(t), \quad c_1 \in \mathbb{R}. \tag{11}$$

Further, one has

$$D_q u(t) = [\alpha-1]_q c_1 t^{\alpha-2} - I_q^{\alpha-1} y(t), \quad t \in [0,1].$$

The condition $D_q u(1) - \lambda(u) = \gamma$ implies that

$$[\alpha-1]_q c_1 - I_q^{\alpha-1} y(1) - \lambda[u] = \gamma.$$

By simple calculation we get

$$\begin{aligned}
c_1 &= \frac{\gamma}{[\alpha-1]_q - A} + \frac{1}{([\alpha-1]_q - A)\Gamma_q(\alpha-1)} \int_0^1 (1-qs)^{(\alpha-2)} y(s)\,\mathrm{d}_q s \\
&\quad - \frac{1}{([\alpha-1]_q - A)\Gamma_q(\alpha)} \int_0^1 \int_0^t (t-qs)^{(\alpha-1)} y(s)\,\mathrm{d}_q s\,\mathrm{d}\Lambda(t),
\end{aligned}$$

and so, substituting it into (11), we deduce that

$$\begin{aligned}
u(t) &= \frac{t^{\alpha-1}(1 - \frac{1}{[\alpha-1]_q}\int_0^1 t^{\alpha-1}\,\mathrm{d}\Lambda(t) + \frac{1}{[\alpha-1]_q}\int_0^1 t^{\alpha-1}\,\mathrm{d}\Lambda(t))}{(1 - \frac{1}{[\alpha-1]_q}\int_0^1 t^{\alpha-1}\,\mathrm{d}\Lambda(t))\Gamma_q(\alpha)} \\
&\quad \times \int_0^1 (1-qs)^{(\alpha-2)} y(s)\,\mathrm{d}_q s + \frac{\gamma t^{\alpha-1}}{[\alpha-1]_q - A} - \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} y(s)\,\mathrm{d}_q s \\
&\quad - \frac{t^{\alpha-1}}{([\alpha-1]_q - A)\Gamma_q(\alpha)} \int_0^1 \int_0^t (t-qs)^{(\alpha-1)} y(s)\,\mathrm{d}_q s\,\mathrm{d}\Lambda(t) \\
&= \frac{\gamma t^{\alpha-1}}{[\alpha-1]_q - A} + \frac{1}{\Gamma_q(\alpha)} \int_0^1 t^{\alpha-1}(1-qs)^{(\alpha-2)} y(s)\,\mathrm{d}_q s \\
&\quad + \frac{t^{\alpha-1}}{([\alpha-1]_q - A)\Gamma_q(\alpha)} \Biggl\{ \int_0^1 \int_0^1 t^{\alpha-1}(1-qs)^{(\alpha-2)} y(s)\,\mathrm{d}_q s\,\mathrm{d}\Lambda(t) \\
&\quad - \int_0^1 \int_0^t (t-qs)^{(\alpha-1)} y(s)\,\mathrm{d}_q s\,\mathrm{d}\Lambda(t) \Biggr\} - \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} y(s)\,\mathrm{d}_q s
\end{aligned}$$

$$= \frac{\gamma t^{\alpha-1}}{[\alpha-1]_q - A} + \int_0^1 G_1(t,qs)y(s)\,\mathrm{d}_q s + \frac{t^{\alpha-1}}{[\alpha-1]_q - A}\int_0^1\int_0^1 G_1(t,qs)y(s)\,\mathrm{d}_q s\,\mathrm{d}\Lambda(t)$$
$$= \frac{\gamma t^{\alpha-1}}{[\alpha-1]_q - A} + \int_0^1 G(t,qs)y(s)\,\mathrm{d}_q s.$$

The proof is complete. □

Lemma 2. *The function $G_1(t,qs)$ has the following properties*:

(i) $G_1(t,qs) \geqslant 0$, $G_1(t,qs) \leqslant G_1(1,qs)$, $0 \leqslant t,s \leqslant 1$;
(ii) $G_1(t,qs) \geqslant t^{\alpha-1}G_1(1,qs)$, $0 \leqslant t,s \leqslant 1$;
(iii) $G_1(t,qs) \leqslant (1-qs)^{(\alpha-2)}t^{\alpha-1}/\Gamma_q(\alpha) \leqslant 1/\Gamma_q(\alpha)$, $0 \leqslant t,s \leqslant 1$.

Proof. The proof is similar to Lemma 3.0.7. of [7], we omit it. □

Remark 4. From Lemma 1 and (10) we have

$$t^{\alpha-1}\left[G(1,qs) + \frac{\zeta(s)}{[\alpha-1]_q - A}\right] \leqslant G(t,qs) \leqslant G(1,qs) + \frac{\zeta(s)}{[\alpha-1]_q - A},$$

and

$$G(t,qs) \leqslant t^{\alpha-1}\left[\frac{1}{\Gamma_q(\alpha)}(1-qs)^{(\alpha-2)} + \frac{\zeta(s)}{[\alpha-1]_q - A}\right].$$

Next, we consider the following boundary value problem:

$$\begin{aligned} &D_q^\beta\bigl(\varphi_{p(t)}\bigl(D_q^\alpha u(t) - g(t)\bigr)\bigr) + y(t) = 0, \quad 0 < t < 1,\\ &u(0) = 0, \quad (D_q u)(0) = 0, \qquad (D_q u)(1) - \lambda[u] = \gamma \geqslant 0,\\ &D_q^\alpha u(t)|_{t=0} = 0. \end{aligned} \tag{12}$$

Lemma 3. *Let $g \in C[0,1]$ be a given function with $g(0) = 0$. Then problem* (12) *has a unique solution*

$$u(t) = \int_0^1 G(t,qs)\varphi_{p(s)}^{-1}\left(\frac{1}{\Gamma_q(\beta)}\int_0^s (s-q\tau)^{(\beta-1)}y(\tau)\,\mathrm{d}_q\tau\right)\mathrm{d}_q s$$
$$+ \frac{\gamma}{[\alpha-1]_q - A}t^{\alpha-1} - \int_0^1 G(t,qs)g(s)\,\mathrm{d}_q s,$$

where $G(t,qs)$ is defined as in (10).

Proof. First, we deduce $D_q^\beta \sigma(t) = -y(t)$, here $\sigma = \varphi_{p(t)}(\varsigma - g)$, $\varsigma = D_q^\alpha u$. Then

$$\begin{aligned}&(D_q^\beta)\sigma(t) + y(t) = 0, \quad t \in (0,1),\\ &\sigma(0) = \varphi_{p(t)}\bigl(\varsigma(0) - g(0)\bigr) = 0\end{aligned}$$

has the solution $\sigma(t) = -I_q^\beta y(t) + c_0 t^{\beta-1}$. From the condition $D_q^\alpha u(t)|_{t=0} = 0$ and $g(0) = 0$, which implies $c_0 = 0$, we know $\sigma(t) = -I_q^\beta y(t)$. Noticing that $D_q^\alpha u(t) = \varphi_{p(t)}^{-1}(\sigma(t)) + g(t)$, we translate into considering

$$\begin{aligned}&D_q^\alpha u(t) = \varphi_{p(t)}^{-1}\bigl(-I_q^\beta y(t)\bigr) + g(t), \quad t \in (0,1),\\ &u(0) = (D_q u)(0) = 0, \qquad (D_q u)(1) - \lambda[u] = \gamma \geqslant 0,\\ &D_q^\alpha u(t)|_{t=0} = 0.\end{aligned}$$

Lemma 1 implies that problem (12) has a unique solution

$$\begin{aligned}u(t) &= \frac{\gamma}{[\alpha-1]_q - A} t^{\alpha-1} - \int_0^1 G(t,qs)\bigl(\varphi_{p(s)}^{-1}\bigl(-I_q^\beta y(s)\bigr) + g(s)\bigr)\,\mathrm{d}_q s\\ &= \int_0^1 G(t,qs)\varphi_{p(s)}^{-1}\left(\frac{1}{\Gamma_q(\beta)}\int_0^s (s-q\tau)^{(\beta-1)} y(\tau)\,\mathrm{d}_q\tau\right)\mathrm{d}_q s\\ &\quad + \frac{\gamma}{[\alpha-1]_q - A} t^{\alpha-1} - \int_0^1 G(t,qs) g(s)\,\mathrm{d}_q s.\end{aligned}$$

The proof is complete. □

Moreover, we collect some notations that are already known in literatures [20, 21].

Let $(E, \|\cdot\|)$ be a real Banach space and it is partially ordered by a cone $K \subset E$. For any $x, y \in E$, the notation $x \sim y$ means that there exist $\mu > 0$ and $\nu > 0$ such that $\mu x \leqslant y \leqslant \nu x$. For fixed $h > \theta (i.e., h \geqslant \theta$ and $h \neq \theta)$, θ denotes the zero element of E. Define a set $K_h = \{x \in E\colon x \sim h\}$. Clearly, $K_h \subset K$. Take $e \in K$ with $\theta \leqslant e \leqslant h$, we define $K_{h,e} = \{x \in E\colon x + e \in K_h\}$, that is, $K_{h,e} = \{x \in E\colon$ there exist $\mu = \mu(h,e,x) > 0$, $\nu = \nu(h,e,x) > 0$ such that $\mu h \leqslant x + e \leqslant \nu h\}$.

Definition 3. (See [21].) Let $T : K_{h,e} \to E$ be a given operator. For any $x \in K_{h,e}$, $\lambda \in (0,1)$, there exists $\varphi(\lambda) > \lambda$ such that

$$T\bigl(\lambda x + (\lambda - 1)e\bigr) \geqslant \varphi(\lambda)Tx + \bigl(\varphi(\lambda) - 1\bigr)e.$$

Then T is called a $\varphi - (h,e)$-concave operator.

Now we consider problem (7) in Banach space $E = C[0,1]$ endowed with the norm $\|u\| = \sup\{|u(t)|\colon t \in [0,1]\}$. Set the standard cone $K = \{x \in E\colon x(t) \geqslant 0$, $\min_{t\in[\tau,1]} x(t) \geqslant \tau^{\alpha-1}\|x\|$, $t \in [0,1]\}$. Obviously, $K \subset E$ is normal. Define the operator $T : K \to E$ by

$$Tu(t) = \int_0^1 G(t, qs)\varphi_{p(s)}^{-1}\left(\frac{\mu}{\Gamma_q(\beta)}\int_0^s (s-q\tau)^{(\beta-1)} f\big(\tau, u(\tau)\big)\,\mathrm{d}_q\tau\right)\mathrm{d}_q s + \frac{\gamma}{[\alpha-1]_q - A}t^{\alpha-1} - \int_0^1 G(t, qs)g(s)\,\mathrm{d}_q s.$$

Further, let

$$e(t) = \int_0^1 G(t, qs)g(s)\,\mathrm{d}_q s, \quad t \in [0,1],\ h(t) = Ht^{\alpha-1},$$

where

$$H \geqslant \int_0^1 \left[\frac{1}{\Gamma_q(\alpha)}(1-qs)^{(\alpha-2)} + \frac{\zeta(s)}{[\alpha-1]_q - A}\right] g(s)\,\mathrm{d}_q s.$$

Lemma 4. *Let assumptions* (H0)–(H2) *hold. In addition, we assume that*

(H3) $0 \leqslant A < [\alpha-1]_q$, $\zeta(s) \geqslant 0$, *where* A, ζ *are defined as in assumption* (H0).

If $g(t) \geqslant 0$ *with* $g(t) \not\equiv 0$, $g(0) = 0$ *for* $t \in [0,1]$, *then* $T : K_{h,e} \to E$ *is a* $\varphi - (h,e)$-*concave operator.*

Proof. For $t \in [0,1]$, we have

$$h(t) = Ht^{\alpha-1} \geqslant t^{\alpha-1}\int_0^1 \left[\frac{1}{\Gamma_q(\alpha)}(1-qs)^{(\alpha-2)} + \frac{\zeta(s)}{[\alpha-1]_q - A}\right] g(s)\,\mathrm{d}_q s \geqslant 0.$$

Since $g(t) \geqslant 0$, $g(t) \not\equiv 0$, thus $h(t) \not\equiv 0$. Then we show that $0 \leqslant e(t) \leqslant h(t)$. By Lemma 2 and (H3) we have

$$e(t) = \int_0^1 G(t, qs)g(s)\,\mathrm{d}_q s \geqslant 0, \quad t \in [0,1],$$

and

$$\min_{t\in[\tau,1]} e(t) \geqslant \tau^{\alpha-1}\int_0^1 G(1, qs)g(s)\,\mathrm{d}_q s = \tau^{\alpha-1}\|e\|,$$

that is, $e \in K$. Further, from Remark 4 one has

$$e(t) = \int_0^1 G(t, qs)g(s)\,\mathrm{d}_q s \leqslant \int_0^1 \left[\frac{1}{\Gamma_q(\alpha)}(1-qs)^{(\alpha-2)} + \frac{\zeta(s)}{[\alpha-1]_q - A}\right] g(s)\,\mathrm{d}_q s \cdot t^{\alpha-1} \leqslant Ht^{\alpha-1} = h(t),$$

hence, $0 \leqslant e(t) \leqslant h(t)$. Moreover, $K_{h,e} = \{u \in C[0,1]\colon\ u + e \in K_h\}$. In view of Lemma 3, the solution $u(t)$ of problem (7) can be expressed as

$$\begin{aligned} u(t) &= \int_0^1 G(t,qs)\varphi_{p(s)}^{-1}\left(\frac{\mu}{\Gamma_q(\beta)}\int_0^s (s-q\tau)^{(\beta-1)} f\bigl(\tau,u(\tau)\bigr)\,\mathrm{d}_q\tau\right)\mathrm{d}_q s \\ &\quad + \frac{\gamma}{[\alpha-1]_q - A}t^{\alpha-1} - \int_0^1 G(t,qs)g(s)\,\mathrm{d}_q s \\ &= \int_0^1 G(t,qs)\varphi_{p(s)}^{-1}\left(\frac{\mu}{\Gamma_q(\beta)}\int_0^s (s-q\tau)^{(\beta-1)} f\bigl(\tau,u(\tau)\bigr)\,\mathrm{d}_q\tau\right)\mathrm{d}_q s \\ &\quad + \frac{\gamma}{[\alpha-1]_q - A}t^{\alpha-1} - e(t). \end{aligned}$$

For any $u \in K_{h,e}$, we consider the operator T, which can also be written as

$$\begin{aligned} Tu(t) &= \int_0^1 G(t,qs)\varphi_{p(s)}^{-1}\left(\frac{\mu}{\Gamma_q(\beta)}\int_0^s (s-q\tau)^{(\beta-1)} f\bigl(\tau,u(\tau)\bigr)\,\mathrm{d}_q\tau\right)\mathrm{d}_q s \\ &\quad + \frac{\gamma}{[\alpha-1]_q - A}t^{\alpha-1} - e(t). \end{aligned}$$

Evidently, $u(t)$ is the solution of problem (7) if and only if u is the fixed point of T.

Now we show that $T : K_{h,e} \to E$ is a $\varphi - (h,e)$-concave operator. For $\lambda \in (0,1)$, $u \in K_{h,e}$, by condition (H2) we can obtain

$$\begin{aligned} &T\bigl(\lambda u + (\lambda-1)e\bigr)(t) \\ &\quad= \int_0^1 G(t,qs)\varphi_{p(s)}^{-1}\left(\frac{\mu}{\Gamma_q(\beta)}\int_0^s (s-q\tau)^{(\beta-1)} f\bigl(\tau,\lambda u(\tau) + (\lambda-1)e(\tau)\bigr)\,\mathrm{d}_q\tau\right)\mathrm{d}_q s \\ &\qquad + \frac{\gamma}{[\alpha-1]_q - A}t^{\alpha-1} - e(t) \\ &\quad\geqslant \bigl(\psi(\lambda)\bigr)^{1/(\underline{p}-1)}\int_0^1 G(t,qs)\varphi_{p(s)}^{-1}\left(\frac{\mu}{\Gamma_q(\beta)}\int_0^s (s-q\tau)^{(\beta-1)} f\bigl(\tau,u(\tau)\bigr)\,\mathrm{d}_q\tau\right)\mathrm{d}_q s \\ &\qquad + \frac{(\psi(\lambda))^{1/(\underline{p}-1)}\gamma}{[\alpha-1]_q - A}t^{\alpha-1} - e(t) \\ &\quad= \bigl(\psi(\lambda)\bigr)^{1/(\underline{p}-1)}\left[\int_0^1 G(t,qs)\varphi_{p(s)}^{-1}\left(\frac{\mu}{\Gamma_q(\beta)}\int_0^s (s-q\tau)^{(\beta-1)} f\bigl(\tau,u(\tau)\bigr)\,\mathrm{d}_q\tau\right)\mathrm{d}_q s \right.\\ &\qquad \left. + \frac{\gamma}{[\alpha-1]_q - A}t^{\alpha-1} - e(t)\right] + \bigl(\psi(\lambda)\bigr)^{1/(\underline{p}-1)}e(t) - e(t) \\ &\quad= \bigl(\psi(\lambda)\bigr)^{1/(\underline{p}-1)}Tu(t) + \bigl[\bigl(\psi(\lambda)\bigr)^{1/(\underline{p}-1)} - 1\bigr]e(t). \end{aligned}$$

Let $\varphi(\lambda) := (\psi(\lambda))^{1/(\underline{p}-1)}$. For $\lambda \in (0,1)$, because $\ln\psi(\lambda) > (\underline{p}-1)\ln\lambda$ in (H2), we have $\ln\psi(\lambda)/\ln\lambda < \underline{p}-1$, and thus

$$\begin{aligned}\ln\varphi(\lambda) &= \frac{1}{\underline{p}-1}\ln\psi(\lambda) = \frac{1}{\underline{p}-1}\cdot\frac{\ln\psi(\lambda)}{\ln\lambda}\cdot\ln\lambda \\ &> \frac{1}{\underline{p}-1}\cdot\underline{p}-1\cdot\ln\lambda = \ln\lambda.\end{aligned}$$

So we have $\varphi(\lambda) > \lambda$, $\lambda \in (0,1)$. For $u \in K_{h,e}$, we get

$$T(\lambda u + (\lambda-1)e) \geqslant \varphi(\lambda)Tu + [\varphi(\lambda)-1]e, \quad \lambda \in (0,1).$$

According to Definition 3, we know that T is a $\varphi-(h,e)$-concave operator. The proof is complete. □

Remark 5. If $g(t) \leqslant 0$ with $g(0) = 0$, (H0)–(H3) hold, it is clear that T is a $\varphi-(h,\theta)$-concave operator.

Remark 6. Note that the inequalities $A \geqslant 0$, $\zeta(s) \geqslant 0$ of condition (H3) are general satisfied provided that $\mathrm{d}\Lambda$ is positive. Consider the case when the measure $\mathrm{d}\Lambda$ changes the sign, particularly, take $\mathrm{d}\Lambda(t) = (at-b)\,\mathrm{d}_q t$, $a, b > 0$. It changes sign and one can see

$$A = \int_0^1 t^{\alpha-1}(at-b)\,\mathrm{d}_q t = \frac{(b-a)(a-bq)(1-q)(q^\alpha - q^{\alpha+1})}{(1-q^\alpha)(1-q^{\alpha+1})}.$$

Let $0 \leqslant A < 1$, then it requires that

$$0 \leqslant (b-a)(a-bq) < \frac{(1-q^\alpha)(1-q^{\alpha+1})}{(1-q)(q^\alpha - q^{\alpha+1})}.$$

If $\alpha = 5/2$, $q = 1/2$, then $0 \leqslant (b-a)(a-b/2) < (65\sqrt{2}-24)/4$, while if $\alpha = 3$, $q = 1/2$, then $0 \leqslant (b-a)(a-b/2) < 105/4$. Further, we know that if $q \to 1$, $\mathrm{d}\Lambda(t) = (at-b)\,\mathrm{d}t$, $a, b > 0$. Then

$$A = \int_0^1 t^{\alpha-1}(at-b)\,\mathrm{d}_q t = \frac{(a-b)\alpha - b}{\alpha(\alpha+1)}.$$

Similarly, let $0 \leqslant A < 1$, it requires that $b/\alpha \leqslant a-b < \alpha+1$. If $\alpha = 5/2$, then $2b/5 \leqslant a-b < 7/2$, while if $\alpha = 3$, then $b/4 \leqslant a-b < 4$.

Lemma 5. (*See* [21].) *Let K be normal and T be an increasing $\varphi-(h,e)$-concave operator with $Th \in K_{h,e}$. Then T has a unique fixed point x^* in $K_{h,e}$. Moreover, for any $w_0 \in K_{h,e}$, constructing the sequence $w_n = Tw_{n-1}$, $n = 1, 2, \dots$, then $\|w_n - x^*\| \to 0$ as $n \to \infty$.*

Remark 7. If $e = \theta$, i.e., T is an increasing $\varphi-(h,\theta)$-concave operator, the above result is still holds.

3 Local unique solutions

In this section, we can formulate some results giving sufficient conditions for the existence and uniqueness of solution to problem (7).

Theorem 1. *Assume that* (H0)–(H3) *hold,* $g(t) \geqslant 0$ *with* $g(t) \not\equiv 0$, $g(0) = 0$. *Then problem* (7) *has a unique solution* u^* *in* $K_{h,e}$. *Further, for any given* $v_0 \in K_{h,e}$, *constructing a sequence*

$$v_n(t) = \int_0^1 G(t, qs)\varphi_{p(s)}^{-1}\left(\frac{\mu}{\Gamma_q(\beta)}\int_0^s (s - q\tau)^{(\beta-1)} f\bigl(\tau, v_{n-1}(\tau)\bigr)\,\mathrm{d}_q\tau\right)\mathrm{d}_q s + \frac{\gamma}{[\alpha-1]_q - A} t^{\alpha-1} - \int_0^1 G(t, qs)g(s)\,\mathrm{d}_q s,$$

one has $v_n(t) \to u^*(t)$ *as* $n \to \infty$.

Proof. By means of Lemma 4 we know that $T : K_{h,e} \to E$ is a $\varphi - (h, e)$-concave operator. Now we prove that $T : K_{h,e} \to E$ is increasing. For $u \in K_{h,e}$, we have $u + e \in K_h$, and then there exists $m > 0$ such that $u(t) + e(t) \geqslant mh(t)$. We obtain

$$u(t) \geqslant mh(t) - e(t) \geqslant -e(t) \geqslant -\hat{e}, \quad t \in [0, 1].$$

By using the condition (H1) we know $T : K_{h,e} \to E$ is increasing.

As follows, we prove that $Th \in K_{h,e}$, so we have to prove $Th + e \in K_h$. From Lemma 2 and (H1) we get

$$\begin{aligned}
&Th(t) + e(t) \\
&= \int_0^1 G(t, qs)\varphi_{p(s)}^{-1}\left(\frac{\mu}{\Gamma_q(\beta)}\int_0^s (s - q\tau)^{(\beta-1)} f\bigl(\tau, H\tau^{\alpha-1}\bigr)\,\mathrm{d}_q\tau\right)\mathrm{d}_q s \\
&\quad + \frac{\gamma}{[\alpha-1]_q - A} t^{\alpha-1} \\
&\leqslant \frac{\gamma}{[\alpha-1]_q - A} t^{\alpha-1} + \int_0^1 \left[\frac{1}{\Gamma_q(\alpha)}(1 - qs)^{(\alpha-2)} + \frac{\zeta(s)}{[\alpha-1]_q - A}\right] \\
&\quad \times \varphi_{p(s)}^{-1}\left(\frac{\mu}{\Gamma_q(\beta)}\int_0^s (s - q\tau)^{(\beta-1)} f(\tau, H)\,\mathrm{d}_q\tau\right)\mathrm{d}_q s \cdot t^{\alpha-1} \\
&\leqslant \frac{\gamma}{H(1 - q^{\alpha-1} - A)} h(t) + \frac{1}{H}\int_0^1 \left[\frac{1}{\Gamma_q(\alpha)}(1 - qs)^{(\alpha-2)} + \frac{\zeta(s)}{[\alpha-1]_q - A}\right] \\
&\quad \times \varphi_{p(s)}^{-1}\left(\frac{\mu}{\Gamma_q(\beta)}\int_0^s (s - q\tau)^{(\beta-1)} f(\tau, H)\,\mathrm{d}_q\tau\right)\mathrm{d}_q s \cdot h(t)
\end{aligned}$$

and

$$
\begin{aligned}
Th(t)+e(t) \geqslant{}& \frac{\gamma}{[\alpha-1]_q-A}t^{\alpha-1}+\int_0^1 \varphi_{p(s)}^{-1}\left(\frac{\mu}{\Gamma_q(\beta)}\int_0^s (s-q\tau)^{(\beta-1)} f(\tau,0)\,\mathrm{d}_q\tau\right) \\
&\times\left[\frac{(1-qs)^{(\alpha-2)}-(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}+\frac{\zeta(s)}{[\alpha-1]_q-A}\right]\mathrm{d}_q s\cdot t^{\alpha-1} \\
\geqslant{}& \frac{\gamma}{H(\frac{1}{1-q}-A)}h(t)+\frac{1}{H}\int_0^1 \varphi_{p(s)}^{-1}\left(\frac{\mu}{\Gamma_q(\beta)}\int_0^s (s-q\tau)^{(\beta-1)} f(\tau,0)\,\mathrm{d}_q\tau\right) \\
&\times\left[\frac{(1-qs)^{(\alpha-2)}-(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}+\frac{\zeta(s)}{[\alpha-1]_q-A}\right]\mathrm{d}_q s\cdot h(t).
\end{aligned}
$$

Let

$$
\begin{aligned}
r_1 ={}& \frac{1}{H}\int_0^1 \left[\frac{1}{\Gamma_q(\alpha)}(1-qs)^{(\alpha-2)}+\frac{\zeta(s)}{[\alpha-1]_q-A}\right] \\
&\times \varphi_{p(s)}^{-1}\left(\frac{\mu}{\Gamma_q(\beta)}\int_0^s (s-q\tau)^{(\beta-1)} f(\tau,H)\,\mathrm{d}_q\tau\right)\mathrm{d}_q s+\frac{\gamma}{H(1-q^{\alpha-1}-A)},
\end{aligned}
$$

$$
\begin{aligned}
r_2 ={}& \frac{1}{H}\int_0^1 \left[\frac{(1-qs)^{(\alpha-2)}-(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}+\frac{\zeta(s)}{[\alpha-1]_q-A}\right] \\
&\times \varphi_{p(s)}^{-1}\left(\frac{\mu}{\Gamma_q(\beta)}\int_0^s (s-q\tau)^{(\beta-1)} f(\tau,0)\,\mathrm{d}_q\tau\right)\mathrm{d}_q s+\frac{\gamma}{H(\frac{1}{1-q}-A)}.
\end{aligned}
$$

Since $\beta\geqslant 0$, $1/(1-q^\alpha)>1-q$, from (H1) we can easily get

$$
\begin{aligned}
r_1 ={}& \frac{1}{H}\int_0^1 \left[\frac{1}{\Gamma_q(\alpha)}(1-qs)^{(\alpha-2)}+\frac{\zeta(s)}{[\alpha-1]_q-A}\right] \\
&\times \varphi_{p(s)}^{-1}\left(\frac{\mu}{\Gamma_q(\beta)}\int_0^s (s-q\tau)^{(\beta-1)} f(\tau,H)\,\mathrm{d}_q\tau\right)\mathrm{d}_q s+\frac{\gamma}{H(1-q^{\alpha-1}-A)} \\
\geqslant{}& \frac{1}{H}\int_0^1 \left[\frac{1}{\Gamma_q(\alpha)}(1-qs)^{(\alpha-2)}+\frac{\zeta(s)}{[\alpha-1]_q-A}\right] \\
&\times \varphi_{p(s)}^{-1}\left(\frac{\mu}{\Gamma_q(\beta)}\int_0^s (s-q\tau)^{(\beta-1)} f(\tau,0)\,\mathrm{d}_q\tau\right)\mathrm{d}_q s+\frac{(1-q)\gamma}{H(1-q^{\alpha}-A)}
\end{aligned}
$$

$$\begin{aligned}
&\geqslant \frac{1}{H}\int_0^1 \left[\frac{(1-qs)^{(\alpha-2)}-(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}+\frac{\zeta(s)}{[\alpha-1]_q-A}\right] \\
&\quad\times \varphi_{p(s)}^{-1}\left(\frac{\mu}{\Gamma_q(\beta)}\int_0^s (s-q\tau)^{(\beta-1)} f(\tau,0)\,\mathrm{d}_q\tau\right)\mathrm{d}_q s+\frac{\gamma}{H(\frac{1}{1-q}-A)} \\
&= r_2>0,
\end{aligned}$$

hence we have $r_1 \geqslant r_2 > 0$ and $r_2 h \leqslant Th + e \leqslant r_1 h$, which implies that $Th + e \in K_h$. In view of Lemma 5, the operator T has a unique fixed point u^* in $K_{h,e}$, and

$$\begin{aligned}
u^*(t) &= \int_0^1 G(t,qs)\varphi_{p(s)}^{-1}\left(\frac{\mu}{\Gamma_q(\beta)}\int_0^s (s-q\tau)^{(\beta-1)} f\big(\tau,u^*(\tau)\big)\,\mathrm{d}_q\tau\right)\mathrm{d}_q s \\
&\quad+\frac{\gamma}{[\alpha-1]_q-A}t^{\alpha-1}-e(t).
\end{aligned}$$

Namely, $u^*(t)$ is the solution of problem (7). In addition, for any $v_0 \in K_{h,e}$, the sequence $v_n = Tv_{n-1}$, $n = 1, 2, \dots$, satisfies $v_n \to u^*$ as $n \to \infty$, that is,

$$\begin{aligned}
v_n(t) &= \int_0^1 G(t,qs)\varphi_{p(s)}^{-1}\left(\frac{\mu}{\Gamma_q(\beta)}\int_0^s (s-q\tau)^{(\beta-1)} f(\tau,v_{n-1}(\tau))\,\mathrm{d}_q\tau\right)\mathrm{d}_q s \\
&\quad+\frac{\gamma}{[\alpha-1]_q-A}t^{\alpha-1}-\int_0^1 G(t,qs)g(s)\,\mathrm{d}_q s,\quad n=1,2,\dots,
\end{aligned}$$

and $v_n(t) \to u^*(t)$ as $n \to \infty$. □

Corollary 1. *Let* $\rho(s) := (\mu/\Gamma_q(\beta)) \int_0^s (s-q\tau)^{(\beta-1)} f(\tau,0)\,\mathrm{d}_q\tau)^{1/(p(s)-1)}$. *If the conditions of Theorem* 1 *hold and*

$$\rho(s)-g(s)\geqslant 0 \quad \textit{with } \rho(s)-g(s)\not\equiv 0,\quad s\in[0,1],$$

then problem (7) *has a unique nontrivial positive solution in* $K_{h,e}$. *In addition, we can also construct an iterative scheme*

$$\begin{aligned}
v_n(t) &= \int_0^1 G(t,qs)\varphi_{p(s)}^{-1}\left(\frac{\mu}{\Gamma_q(\beta)}\int_0^s (s-q\tau)^{(\beta-1)} f\big(\tau,v_{n-1}(\tau)\big)\,\mathrm{d}_q\tau\right)\mathrm{d}_q s \\
&\quad+\frac{\gamma}{[\alpha-1]_q-A}t^{\alpha-1}-\int_0^1 G(t,qs)g(s)\,\mathrm{d}_q s,\quad n=1,2,\dots,
\end{aligned}$$

approximating the unique nontrivial positive solution $u^*(t)$.

Further, similar to the proof of Theorem 1, by using Remark 7 we have the following result:

Theorem 2. *Assume that* (H0), (H3) *hold,* $g(t) \leqslant 0$*,* $g(0) = 0$ *and*

(H4) $f : R_0 \times \mathbb{R}^+ \to \mathbb{R}^+$ *is increasing with respect to the second variable with* $f(t,0) \not\equiv 0$;

(H5) *for any* $\lambda \in (0,1)$*, there exists* $\psi(\lambda) \in (0,1)$ *with* $\ln \psi(\lambda) > (\underline{p} - 1)\ln \lambda$ *such that*

$$f(t,\lambda x) \geqslant \psi(\lambda) f(t,x), \quad t \in [0,1],\ x \in \mathbb{R}^+.$$

Then problem (7) *has a unique positive solution* u^* *in* K_h*, where* $h(t) = t^{\alpha-1}$*,* $t \in [0,1]$*. Further, making a monotone iterative sequence*

$$v_n(t) = \int_0^1 G(t,qs)\left(\frac{\mu}{\Gamma_q(\beta)}\int_0^s (s-q\tau)^{(\beta-1)} f(\tau, v_{n-1}(\tau))\,\mathrm{d}_q\tau\right)^{1/(p(s)-1)} \mathrm{d}_q s + \frac{\gamma}{[\alpha-1]_q - A} t^{\alpha-1} - \int_0^1 G(t,qs) g(s)\,\mathrm{d}_q s, \quad n = 1,2,\dots,$$

for any $v_0 \in K_h$*, we have* $v_n(t) \to u^*(t)$ *as* $n \to \infty$*.*

Next, we consider a special case of problem (7) with homogeneous boundary condition

$$\begin{gathered} D_q^\beta\bigl(\varphi_{p(t)}\bigl(D_q^\alpha u(t) - g(t)\bigr)\bigr) + f\bigl(t,u(t)\bigr) = 0, \quad 0<t<1, \\ u(0) = (D_q u)(0) = 0, \qquad (D_q u)(1) - \lambda[u] = 0, \qquad D_q^\alpha u(t)|_{t=0} = 0. \end{gathered} \tag{13}$$

Corollary 2. *Assume* (H0)–(H3) *hold and* $g(t) \geqslant 0$*,* $g(t) \not\equiv 0$*,* $g(0) = 0$*. Then problem* (13) *has a unique solution* u^* *in* $K_{h,e}$*. Further, for any* $v_0 \in K_{h,e}$*, making a monotone iterative sequence*

$$v_n(t) = \int_0^1 G(t,qs)\left(\frac{\mu}{\Gamma_q(\beta)}\int_0^s (s-q\tau)^{(\beta-1)} f\bigl(\tau, v_{n-1}(\tau)\bigr)\,\mathrm{d}_q\tau\right)^{1/(p(s)-1)} \mathrm{d}_q s - \int_0^1 G(t,qs) g(s)\,\mathrm{d}_q s, \quad n = 1,2,\dots, \tag{14}$$

one has $v_n(t) \to u^*(t)$ *as* $n \to \infty$*.*

Corollary 3. *In Corollary* 1*, if only requires* $\rho(s) - g(s) \not\equiv 0$*,* $s \in [0,1]$*, then problem* (13) *has a unique nontrivial solution in* $K_{h,e}$*. In addition, we can also construct an iterative scheme shown as* (14) *approximating the unique nontrivial solution* $u^*(t)$*.*

Corollary 4. *If* $g(t) \leqslant 0$, $g(0) = 0$ *and assumptions* (H3)–(H5) *hold, then problem* (13) *has a unique nontrivial positive solution in* K_h. *In addition, we can make an iterative scheme shown as* (14) *approximating the unique nontrivial solution* $u^*(t)$.

Remark 8.

(i) From Theorem 1 and Lemma 5 we can see that the unique solution u^* of problem (1) is in a special set $K_{h,e}$. That is, there exist $\mu, \nu > 0$ such that $u^* \in [\mu h - e, \nu h + e]$. So we say u^* is a local solution.

(ii) From Theorem 2 and Remark 7 the unique solution u^* of problem (1) is in a special set K_h. That is, there exist $\mu, \nu > 0$ such that $u^* \in [\mu h, \nu h]$, and thus u^* is a positive solution.

(iii) For fractional q-difference equations, our main results has not been seen in previous works. The method used here is relatively new, which cannot only guarantee the existence of unique solution, but also can approximate to the unique solution by making an iterative scheme.

4 Examples

Example 1. Consider the following boundary value problem:

$$\begin{aligned} &D_q^{1/2}\big(\varphi_2\big(D_q^{5/2}u(t) - t^2\big)\big) + f\big(t, u(t)\big) = 0, \quad t \in (0,1], \\ &u(0) = (D_q u)(0) = 0, \qquad (D_q u)(1) - \lambda[u] = 1, \qquad D_q^{5/2}u(t)|_{t=0} = 0, \end{aligned} \tag{15}$$

where, for $t \in (0,1]$,

$$f(t,u) = \left[\left(\frac{128}{512}u + \frac{64\sqrt{2}}{(538\sqrt{2} - 353)\Gamma_q(\frac{7}{2})}\right)\left(\frac{32 - \sqrt{2}}{8}t^{3/2} - (4 - \sqrt{2})t^{9/2}\right)\right]^{2/5}.$$

Let $\lambda[u] = 0$, $\alpha = 5/2$, $q = 1/2$, $p(t) = 2$, $\gamma = 1$, $g(t) = t^2$, and

$$e(t) = \frac{64\sqrt{2}t^{3/2}}{(538\sqrt{2} - 353)\Gamma_q(\frac{7}{2})}\left(\frac{16\sqrt{2} - 1}{4\sqrt{2}} - \frac{4\sqrt{2} - 2}{\sqrt{2}}t^3\right), \quad h(t) = Ht^{3/2},$$

for $t \in (0,1]$ with $H \geqslant ((16\sqrt{2} - 1)/(48\sqrt{2} - 24))^{1/3}$. Then we obtain that $A = 0$, $\zeta(s) = 0$,

$$\hat{e} = \max\big\{e(t)\colon t \in [0,1]\big\} = \left(\frac{127 + 12\sqrt{2}}{336}\right)^{1/3},$$

$$e(t) \geqslant \frac{448\sqrt{2}t^{3/2}}{(4304 - 1412\sqrt{2})\Gamma_q(\frac{7}{2})} \geqslant 0$$

and

$$e(t) \leqslant \frac{512}{(1076 - 353\sqrt{2})\Gamma_q(\frac{7}{2})}t^{3/2} \leqslant Ht^{3/2} = h(t).$$

It is clear that $f:[0,1]\times[-(127+12\sqrt{2})/336)^{1/3},+\infty)\to[0,+\infty)$ is continuous and increasing with respect to the second variable,

$$f(t,0)=\left[\frac{64\sqrt{2}}{(538\sqrt{2}-353)\Gamma_q(\frac{7}{2})}\left(\frac{32-\sqrt{2}}{8}t^{3/2}-(4-\sqrt{2})t^{9/2}\right)\right]^{2/5}\not\equiv 0,$$

$$\begin{aligned}f(t,u)&=\left[\left(\frac{128}{512}u+\frac{64\sqrt{2}}{(538\sqrt{2}-353)\Gamma_q(\frac{7}{2})}\right)\left(\frac{32-\sqrt{2}}{8}t^{3/2}-(4-\sqrt{2})t^{9/2}\right)\right]^{2/5}\\&=\left[\frac{64\sqrt{2}t^{3/2}}{(538\sqrt{2}-353)\Gamma_q(\frac{7}{2})}\left(\frac{32-\sqrt{2}}{8}-(4-\sqrt{2})t^{3}\right)\right.\\&\quad\left.+\frac{128}{512}u\left(\frac{32-\sqrt{2}}{8}t^{3/2}-(4-\sqrt{2})t^{9/2}\right)\right]^{2/5}\\&=\left(\frac{(1076-353\sqrt{2})\Gamma_q(\frac{7}{2})}{512}u+1\right)^{2/5}\left[e(t)\right]^{2/5}\\&=\left(\frac{(1076-353\sqrt{2})\Gamma_q(\frac{7}{2})}{512}ue(t)+e(t)\right)^{2/5}.\end{aligned}$$

Further, for $\lambda\in(0,1)$, $x\in\mathbb{R}$, $y\in[0,\hat{e}]$, one has

$$\begin{aligned}&f\big(t,\ \lambda x+(\lambda-1)y\big)\\&\quad=\left(\frac{(1076-353\sqrt{2})\Gamma_q(\frac{7}{2})}{512}e(t)\big[\lambda x+(\lambda-1)y\big]+e(t)\right)^{2/5}\\&\quad=\lambda^{2/5}\left(\frac{(1076-353\sqrt{2})\Gamma_q(\frac{7}{2})}{512}e(t)\left[x+\left(1-\frac{1}{\lambda}\right)y\right]+\frac{e(t)}{\lambda}\right)^{2/5}\\&\quad\geqslant\lambda^{2/5}\left(\frac{(1076-353\sqrt{2})\Gamma_q(\frac{7}{2})}{512}e(t)x+\left(1-\frac{1}{\lambda}\right)e(t)+\frac{e(t)}{\lambda}\right)^{2/5}\\&\quad=\lambda^{2/5}\left(\frac{(1076-353\sqrt{2})\Gamma_q(\frac{7}{2})}{512}e(t)x+e(t)\right)^{2/5}\\&\quad=\lambda^{2/5}f(t,x)=\psi(\lambda)f(t,x),\end{aligned}$$

here $\psi(\lambda):=\lambda^{2/5}$, $\lambda\in(0,1)$. Hence, for $\lambda\in(0,1)$, we have

$$\ln\psi(\lambda)=\frac{2}{5}\ln\lambda>(\underline{p}-1)\ln\lambda,$$

we claim that condition (H2) holds. Therefore, Theorem 1 implies that problem (15) has a unique solution $u^*\in K_{h,e}$. For $v_0\in K_{h,e}$, we construct a sequence

$$v_n(t)=\int_0^1 G(t,qs)\varphi_2^{-1}\left(\int_0^s(s-q\tau)^{(-1/2)}\left(\frac{128}{512}v_{n-1}(\tau)+\frac{64\sqrt{2}}{(538\sqrt{2}-353)\Gamma_q(\frac{7}{2})}\right)^{2/5}\right.$$

$$\times \frac{1}{\Gamma_q(\frac{1}{2})}\left(\frac{32-\sqrt{2}}{8}\tau^{3/2}-(4-\sqrt{2})\tau^{9/2}\right)^{2/5} \mathrm{d}_q\tau\Bigg) \mathrm{d}_q s + \frac{4-\sqrt{2}}{2}t^{3/2}$$
$$-\frac{64\sqrt{2}t^{3/2}}{(538\sqrt{2}-353)\Gamma_q(\frac{7}{2})}\left(\frac{16\sqrt{2}-1}{4\sqrt{2}}-\frac{4\sqrt{2}-2}{\sqrt{2}}t^3\right), \quad n=1,2,\dots,$$

and we have $\lim_{n\to\infty} v_n(t) = u^*(t)$, $t \in [0,1]$.

Example 2. Consider the following boundary value problem:

$$\begin{aligned} &D_q^{1/2}\big(\varphi_{7/2}(D_q^{5/2}u(t)+\sqrt{t})\big) \\ &\quad + \left[\frac{\ln(2+t)}{t}\big(u^{3/2}+\sin^2 t+1\big)\right]^{2/5} = 0, \quad t\in(0,1], \\ &u(0)=(D_qu)(0)=0, \qquad (D_qu)(1)=2, \qquad D_q^{5/2}u(t)|_{t=0}=0, \end{aligned} \tag{16}$$

where

$$f(t,u)=\left[\frac{\ln(2+t)}{t}\big(u^{3/2}+\sin^2 t+1\big)\right]^{2/5}, \quad t\in(0,1],$$

and $q=\beta=1/2$, $\alpha=5/2$, $p(t)=7/2$, $\gamma=2$, $\lambda[u]=0$, $g(t)=-\sqrt{t}\leqslant 0$ with $g(0)=0$, $t\in(0,1]$. Take $h(t)=t^{3/2}$, it can be seen that $f: R_0\times\mathbb{R}^+\to\mathbb{R}^+$ is continuous and increasing with respect to u, and $f(t,0)=[(\ln(2+t)/t)(\sin^2 t+1)]^{2/5}\geqslant 0$ with $f(t,0)\not\equiv 0$, then condition (H4) holds. For $\lambda\in(0,1)$, we get

$$\begin{aligned} f(t,\lambda u) &= \left[\frac{\ln(2+t)}{t}\big((\lambda u)^{3/2}+\sin^2 t+1\big)\right]^{2/5} \\ &\geqslant \left[\frac{\ln(2+t)}{t}\big((\lambda u)^{3/2}+\lambda^{3/2}\sin^2 t+\lambda^{3/2}\big)\right]^{2/5} \\ &= \lambda^{3/5}\left[\frac{\ln(2+t)}{t}\big(u^{3/2}+\sin^2 t+1\big)\right]^{2/5} = \psi(\lambda)f(t,u) \end{aligned}$$

for all $t\in[0,1]$, $u\in\mathbb{R}^+$, where $\psi(\lambda):=\lambda^{3/5}$, so $\ln\psi(\lambda)=3/5\ln\lambda>(\underline{p}-1)\ln\lambda$. Hence condition (H5) is satisfied. Considering Theorem 2, problem (16) has a unique positive solution $u^*\in K_h$. For $v_0\in K_h$, making a sequence

$$v_n(t)=\frac{8+2\sqrt{2}}{7}t^{3/2}+\frac{t^2(8-\sqrt{2}-(8-2\sqrt{2})t)}{(17-6\sqrt{2})\Gamma_q(\frac{5}{2})}+\frac{1}{\Gamma_q^{2/5}(\frac{1}{2})}\int_0^1 G(t,qs)$$
$$\times\left(\int_0^s (s-q\tau)^{(-1/2)}\left[\frac{\ln(2+\tau)}{\tau}\big(v_{n-1}^{3/2}(\tau)+\sin^2\tau+1\big)\right]^{2/5}\mathrm{d}_q\tau\right)^{2/5}\mathrm{d}_q s$$

for $n=1,2,\dots$, we have $\lim_{n\to\infty} v_n(t)=u^*(t)$, $t\in[0,1]$.

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