



Bipartite multi-tracking in MASs with intermittent communication*

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Abstract. In this paper, a novel cluster consensus problem related with the bipartition of the graph of multi-agent systems (MASs) is studied. To track the virtual leaders and reach the expected consensus, a new type of pinning consensus protocol with aperiodic intermittent effects is designed according to the graph structure, and a new kind of aperiodic intermittent communication is defined. Moreover, the protocol is applied to construct networked systems with intermittent communications. Lyapunov functional is applied to get sufficient conditions for solving the multi-tracking problem under a dual subsystem framework. Finally, some numerical simulations are given to illustrate the effectiveness of the theoretical results.

Keywords: multi-agent system, layered intermittence, bipartite cluster consensus, intermittent communication.

1 Introduction

Consensus problem is a class of distributed coordinative control problem of multi-agent systems (MASs). In consensus models, the agents are required to exchange information under the network structure so that they can effectively cooperate and achieve the goal.

As one of important interdisciplinary topics in coordination problems, consensus problem has attracted many researchers due to the fact that it has been widely applied in the cooperative control of unmanned aerial vehicles, mobile sensor networks, satellite clusters, etc. The consensus of multi-agent systems have been studied in various aspects,

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such as consensus problems with time delays [14], consensus with second order [13,21], consensus via pinning strategy [1].

As one of significant research branches of the consensus problem, cluster consensus means that all agents in the same cluster achieve an identical state, while agents in different clusters have different goals. In fact, many cooperative tasks need to be done by partitioning all the agents into different subgroups, and one can see that cluster consensus is a common phenomenon in real-world situations, such as the emergence of subgroups opinions and cluster formation in a flock of aero crafts. The cluster consensus problems with multi-leaders is named multi-tracking for which the aim is to design suitable control protocols in order that the states of agents of each cluster can keep consensus with the status of their leader node.

In recent years, the coordination problem of clustered networked system has been paid more and more attention, and lots of valuable works have been done [2,5,10–13,16,18,19,23,25,28,29]. In [13], the paper addresses the group consensus problem of second-order multi-agent systems through leader-following approach and pinning control. In [25], the cluster mixed synchronization of these networks is studied by using some linear pinning control schemes, only the nodes in one community, which have direct connections to the nodes in other communities, are controlled. In reality, due to the fact that there may exist various time delays in the clustered network model, some research on the cluster coordinative problems with different types of delays has been done [6,26,27]. The delays include the inherent delay of the dynamics, the delay between different leaders, the delays between one cluster and its leader, etc. Besides, we know that the consensus problem with delayed tracking pattern can achieve some missions, which need to prevent the traffic or aircraft block. Therefore, a new delayed relationship of goal states between corresponding clusters is designed in this work naturally.

In real scenarios, many networks of coordination problems have the bipartite graph structure. For instances, outer synchronization [24], lag synchronization [17], bipartite consensus [7], etc. in which the nodes of the networks are partitioned into two sets. In [22], the distributed node-to-node consensus problem is addressed, and it is assumed that the network structure of multi-agent systems consist of two layers. Inspired by [22], it seems that a cluster to cluster consensus tracking problem can comprise the factors of both cluster and bipartition. It is interesting to noticed that if the clustered structures are appropriately involved in the bipartite subnetworks, and then both the traditional cluster consensus and the final state relations of different clusters might be solved by constructing a suitable multi-tracking model, and some meaningful results can be acquired.

In real circumstances, the agents may only communicate with a portion of the neighbours at some disconnected time intervals because of the limitation of sensing ranges or the encounter of obstacles. Therefore, the coordination problems with discontinuous communication are worth noting, and many valuable research works on this topic have been done [3,8,21]. In practical systems, the graph of a discontinues network may be more complicated and may have the clustered structure due to the achievement of multi-missions.

Based on the discussions above, the idea of a novel intermittent communication based on bipartite clustered tracking model is naturally proposed. In this work, a sort of clus-

ter consensus problem named bipartite cluster consensus (BCC) is studied via pinning approach. Specifically, the main contributions of this paper are presented as follows.

1. This article defines the notion of bipartite cluster consensus (BCC). The control strategy is designed by setting pinning intermittent effects to both the two subnetworks instead of controlling only one of them.
2. A new sort of layered intermittence is proposed based on the bipartite multi-tracking model. Furthermore, some novel systems with aperiodic intermittent communication are established, and some sufficient criteria is derived to ensure the achievement of BCC.

In this work, the term “cluster” refers to a subset of nodes, which have the common goal of achieving a desired state. The derivation for solving the cluster consensus problem is based on the graph theory, matrix theory and Lyapunov stability method.

The rest of the paper is organized as follows. Some preliminaries and the model description are given in Section 2. The main results are discussed in Section 3. Numerical simulations are given to verify the theoretical results in Section 4. Finally, the conclusions are made in Section 5.

Notations. Through out this paper, \mathbb{R} denotes the set of real numbers. Let \mathbb{R}^n denote the n -dimensional Euclidean space and $\mathbb{R}^{M \times N}$ denotes the set of all $M \times N$ real matrices. $O_{N \times N}$ denotes the zero matrix. Let I_n be an n -dimensional identity matrix. For a real matrix $A \in \mathbb{R}^{N \times N}$, let A^T be its transpose and define symmetric matrix $A_s = (A + A^T)/2$, $\lambda_{\max}(A)$ denotes the maximum eigenvalue of A . Vector norm is defined as $\|x\| = (x^T x)^{1/2}$ for $x \in \mathbb{R}^n$. For any real symmetric matrix B , denote $B > 0$ ($B < 0$) if B is positive (negative) definite. For any two nonempty sets \mathcal{P} and \mathcal{Q} , $\mathcal{P} \setminus \mathcal{Q}$ denotes the complementary set of \mathcal{Q} respect to \mathcal{P} . \otimes denotes the Kronecker product. Matrix dimensions, if not explicitly stated, are assumed to be compatible for algebraic operations.

2 Preliminaries and problem formulation

Throughout this paper, the communication structure of a multi-agent system with N nodes is represented by a weakly connected digraph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$, where $\mathcal{V} = \{v_1, v_2, \dots, v_N\}$ is the node set representing agents, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the edge set, and $A = (a_{ij})_{N \times N}$ is the weighted adjacency matrix of \mathcal{G} , which denotes the coupling configuration of the network. A directed edge of \mathcal{G} denoted by (v_i, v_j) means that there is a directed information link point to v_j from v_i . The elements of \mathcal{A} are defined as $a_{ij} > 0$ if $(v_j, v_i) \in \mathcal{E}$; $a_{ij} = 0$ if $(v_j, v_i) \notin \mathcal{E}$. The in-degree of node v_i is defined as $\deg_{\text{in}}(v_i) = \sum_{j=1, j \neq i}^N a_{ij}$.

The Laplacian matrix of \mathcal{G} is denoted by $L = (l_{ij}) \in \mathbb{R}^{N \times N}$ and is defined as $l_{ij} = -a_{ij}$, $i \neq j$. $l_{ii} = \sum_{j=1, j \neq i}^N a_{ij}$, which ensures that $\sum_{j=1}^N l_{ij} = 0$.

Consider a nonlinear first-order MAS consist of two subsystems N_1, N_2 and m leaders. Each follower node of N_1 is modeled as

$$\dot{x}_i(t) = f(x_i(t)) + c \sum_{j \neq i} a_{ij} (x_j(t) - x_i(t)) + u_{i1}(t), \quad i = 1, 2, \dots, N. \quad (1)$$

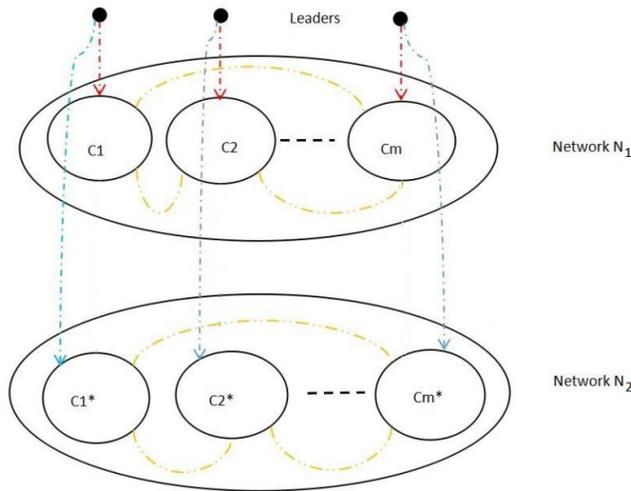


Figure 1. The bipartite multi-tracking model.

The dynamics of the follower in N_2 is modeled by

$$\dot{y}_i(t) = f(y_i(t)) + c \sum_{j \neq i} \bar{a}_{ij} (y_j(t) - y_i(t)) + u_{i2}(t), \quad i = 1, 2, \dots, N, \quad (2)$$

where $x_i(t), y_i(t) \in \mathbb{R}^n$ are the states of the i th agent in subnetwork N_1 and N_2 , respectively, $u_i(t) \in \mathbb{R}^n$ is the control input to be determined later, and $f(x_i(t)) \in \mathbb{R}^n$ is the intrinsic nonlinear dynamics of the i th agent.

Suppose that each of the two subnetworks has m clusters with $2 \leq m < N$, and each cluster has exactly one leader, and the corresponding two clusters, which are designed to track the common leader, will maintain a delayed final state relation as time goes on. Denote V_p as the node set of the p th cluster ($p = 1, 2, \dots, m$) in N_1 , thus we have $V = \{v_1, v_2, \dots, v_N\} = V_1 \cup V_2 \cup \dots \cup V_m$ and $V_p \cap V_q = \emptyset$ ($p \neq q$). Denote the set of leaders by $V^* = \{v_1^*, v_2^*, \dots, v_m^*\}$, and denote the index set of the leaders by $\mathcal{I} = \{1, 2, \dots, m\}$. Suppose each cluster in subnetwork N_2 has to track the leader of one cluster in N_1 , and different clusters in N_2 are supposed to track different leaders. Suppose the mapping relationship between the corresponding two clusters, which have a tracking relation for the common leader, is denoted as $\phi : \{1, 2, \dots, m\} \rightarrow \{\phi_1, \phi_2, \dots, \phi_m\}$, where $\{\phi_1, \phi_2, \dots, \phi_m\}$ is an ordered set, and $\phi_1, \phi_2, \dots, \phi_m$ is an rearrangement of $1, 2, \dots, m$. Then the node set of network N_2 can be denoted by $\tilde{V} = V_{\phi_1} \cup V_{\phi_2} \cup \dots \cup V_{\phi_m}$, and similarly, $V_{\phi_p} \cap V_{\phi_q} = \emptyset$ ($p \neq q$). Furthermore, V_k and V_{ϕ_k} are arranged to track the same leader node v_k^* . To the subnetwork N_1 , let \hat{i} denote the subscript of the index set of the cluster to which the i th node belongs, that is, $v_i \in V_{\hat{i}}$. It is obvious that $\hat{i} \in \mathcal{I}$, and the number of clusters is the same with the number of leaders. The followers of the p th leader are the nodes in $V_p, p \in \mathcal{I}$. Let $\tilde{V}_p \subseteq V_p$ be set of nodes, which can receive information from other clusters, that is, for any node $v_i \in \tilde{V}_p$, there

exists at least one node $v_j \in V \setminus V_p$ such that $a_{ij} \neq 0$. Node v_i is called as the inter-act agent if $v_i \in \tilde{V}_i$, while v_i is called as the intra-act agent if $v_i \in V_i \setminus \tilde{V}_i$. To N_2 , the corresponding notations, \tilde{V}_{ϕ_p} and y_i , are defined similarly.

The leader of each cluster for system (1) and (2) is described by

$$\dot{s}_k(t) = f(s_k(t)), \quad k = 1, 2, \dots, m, \tag{3}$$

where $s_k(t)$ is the state of v_k^* .

The main aim of the paper is to impose suitable control protocols $u_i(t)$ on system (1) and (2) such that the clustered consensus tracking problem with an aperiodic intermittent communication can be solved. In order to obtain the main results, the following definitions and assumption are necessary.

Definition 1. The multi-agent system with (1)–(3) is said to achieve bipartite cluster consensus (BCC) if the solutions of (1) and (3) satisfy $\lim_{t \rightarrow \infty} \|x_i(t) - s_i(t - \tau_i)\| = 0$ and $\lim_{t \rightarrow \infty} \|y_i(t) - s_i(t)\| = 0, i = 1, 2, \dots, N$.

As shown in Fig. 1, one can view the intermittent transmission of parts of information flows as two layers: the first communication layer can be indicated by the links between $v_k^* (k = 2, \dots, m)$ and subnetwork N_1 ; the second layer of communication links are designed to exist between v_k^* and network N_2 . According to these understanding, the notion of layered intermittence is proposed as follows.

Definition 2 [Layered intermittence]. Based on the bipartite clustered structure, the communication graph G generated by system (1)–(3) is called layered intermittence if the communication in G can be partitioned into two steps (see Fig. 1): the communication links of the first layer exist over the time period $[t_k, t_k + \delta_k)$ and the links of second layer exist over the period $[t_k + \delta_k, t_{k+1})$, where $[t_k, t_{k+1}), k \in \mathbb{N}_+$ is an infinite sequence of uniformly bounded, nonoverlapping time intervals, and δ_k is the length of communication time of the first layer, the communication links of two layers exist alternatively between the two time period.

Assumption 1. Assume that there exists a constant $\eta > 0$ such that for any vectors $x, y \in \mathbb{R}^n$, vector function f satisfies $(x - y)^T (f(x) - f(y)) \leq \eta(x - y)^T (x - y)$.

Lemma 1. (See [13].) If node v_i is an intra-act agent of V_i , i.e., $v_i \in V_i \setminus \tilde{V}_i$, then $\sum_{j=1}^N l_{ij} s_j(t) = 0$.

3 Main result

Note that $l_{ij} = -a_{ij}, i \neq j$, and $l_{ii} = \sum_{j=1, j \neq i}^N a_{ij}; \bar{l}_{ij} = -\bar{a}_{ij}, i \neq j$, and $\bar{l}_{ii} = \sum_{j=1, j \neq i}^N \bar{a}_{ij}$, therefore, (1) can be rewritten by

$$\dot{x}_i(t) = f(x_i(t)) - c \sum_{v_j \in V} l_{ij} x_j(t) + u_i(t), \quad v_i \in V. \tag{4}$$

Let the system error in each cluster of N_1 be

$$e_i(t) = x_i(t) - s_i(t - \tau_i). \tag{5}$$

Then, combining (3),(4) and (5), one has

$$\begin{aligned} \dot{e}_i(t) &= f(x_i(t)) - f(s_i(t - \tau_i)) \\ &\quad - c \sum_{j=1}^N l_{ij} e_j(t) - c \sum_{j=1}^N l_{ij} s_j(t - \tau_j) + u_i(t). \end{aligned} \tag{6}$$

Similarly, in N_2 , the error is described by

$$\bar{e}_i(t) = y_i(t) - s_i(t)$$

and

$$\dot{\bar{e}}_i(t) = f(y_i(t)) - f(s_i(t)) - c \sum_{j=1}^N \bar{l}_{ij} \bar{e}_j(t) - c \sum_{j=1}^N \bar{l}_{ij} s_j(t) + u_i(t).$$

Due to the network structure and Lemma 1, the control input $u_i(t)$ can be designed with intermittent effects as follows:

$$u_i(t) = \begin{cases} u_{i1}(t), & v_i \in V_i, \\ u_{i2}(t), & v_i \in V_{\phi_i}, \end{cases} \tag{7}$$

$$u_{i1}(t) = \begin{cases} -cd_{i1}(t)e_i(t) + c \sum_{j=1}^N l_{ij} s_j(t - \tau_j), & v_i \in \tilde{V}_i, \\ -cd_{i1}(t)e_i(t), & v_i \in V_i \setminus \tilde{V}_i \wedge \deg(v_i)_{in} = 0, \\ 0 & \text{otherwise,} \end{cases}$$

$$u_{i2}(t) = \begin{cases} -cd_{i2}(t)\bar{e}_i(t) + c \sum_{j=1}^N \bar{l}_{ij} s_j(t), & v_i \in \tilde{V}_{\phi_i}, \\ -cd_{i2}(t)\bar{e}_i(t), & v_i \in V_{\phi_i} \setminus \tilde{V}_{\phi_i} \wedge \deg(v_i)_{in} = 0, \\ 0 & \text{otherwise,} \end{cases}$$

$$d_{i1}(t) = \begin{cases} d_i, & t \in [t_k, t_k + \delta_k), \\ 0, & t \in [t_k + \delta_k, t_{k+1}), \end{cases} \quad d_{i2}(t) = \begin{cases} 0, & t \in [t_k, t_k + \delta_k), \\ \bar{d}_i, & t \in [t_k + \delta_k, t_{k+1}), \end{cases}$$

where d_i and \bar{d}_i are positive feed back control gains, $\delta_k \in (0, t_{k+1} - t_k)$ is the control time length of $d_{i1}(t)$, and it is also the rest time length of $d_{i2}(t)$.

To simplify the description, denote the three sorts of nodes in V_i by $V_1^i = \tilde{V}_i$, $V_2^i = \{v_i: v_i \in V_i \setminus \tilde{V}_i, \deg(v_i)_{in} = 0\}$, $V_3^i = V_i \setminus (V_1^i \cup V_2^i)$. To $d_{i1}(t)$, in the period with intermittent effect, let $D = \text{diag}\{d_{11}(t), d_{21}(t), \dots, d_{N1}(t)\}$ with $d_{i1}(t) = d_i > 0$ for $v_i \in (V_1^i \cup V_2^i)$, otherwise, $d_{i1}(t) = 0$; in the period without intermittent effect, $D = \mathbf{O}_{N \times N}$. To $d_{i2}(t)$, in the period with intermittent effect, let $\bar{D} = \text{diag}\{d_{12}(t), d_{22}(t), \dots, d_{N2}(t)\}$ with $d_{i2}(t) = \bar{d}_i > 0$ for $v_i \in (V_1^i \cup V_2^i)$, otherwise, $d_{i2}(t) = 0$. In the period without the intermittent effect, $\bar{D} = \mathbf{O}_{N \times N}$.

Remark 1. Through the form of (7), because of the weakly connected graph structure, each node can either receive information from other nodes or send information to others.

Therefore, $V_3^i = \{v_i: v_i \in V_i \setminus \tilde{V}_i, \text{deg}(v_i)_{\text{in}} \neq 0\}$. In the cluster V_i , the three sorts of nodes have been controlled by different laws, that is, for $v_i \in (V_1^i \cup V_2^i)$, the first term, which has the factor of $d_{i1}(t)$, is an intermittent effect for making the agents achieve the consensus in the same cluster. The second term employed to V_1^i is aimed at counteracting the interaction among clusters. To $u_{i2}(t)$, similar analysis is omitted here.

By the analysis above, the controlled system (4) with layered intermittence can be written as follows:

$$\dot{x}_i(t) = \begin{cases} f(x_i(t)) - c \sum_{v_j \in V} l_{ij} x_j(t) - cd_i e_i(t) + c \sum_{j=1}^N l_{ij} s_j(t), & v_i \in V_1^i, \\ f(x_i(t)) - c \sum_{v_j \in V} l_{ij} x_j(t) - cd_i e_i(t), & v_i \in V_2^i, \\ f(x_i(t)) - c \sum_{v_j \in V} l_{ij} x_j(t), & v_i \in V_3^i, \end{cases} \quad (81)$$

if $t \in [t_k, t_k + \delta_k)$;

$$\dot{x}_i(t) = \begin{cases} f(x_i(t)) - c \sum_{v_j \in V} l_{ij} x_j(t) + c \sum_{j=1}^N l_{ij} s_j(t), & v_i \in V_1^i, \\ f(x_i(t)) - c \sum_{v_j \in V} l_{ij} x_j(t), & V_2^i \cup V_3^i, \end{cases} \quad (82)$$

if $t \in [t_k + \delta_k, t_{k+1})$.

Similarly, (2) with intermittence can be written as

$$\dot{y}_i(t) = \begin{cases} f(y_i(t)) - c \sum_{v_j \in V_{\phi_i}} \bar{l}_{ij} y_j(t) + c \sum_{j=1}^N \bar{l}_{ij} s_j(t), & v_i \in \bar{V}_1^i, \\ f(y_i(t)) - c \sum_{v_j \in V_{\phi_i}} \bar{l}_{ij} y_j(t), & v_i \in \bar{V}_2^i \cup \bar{V}_3^i, \end{cases} \quad (91)$$

if $t \in [t_k, t_k + \delta_k)$;

$$\dot{y}_i(t) = \begin{cases} f(y_i(t)) - c \sum_{v_j \in V_{\phi_i}} \bar{l}_{ij} y_j(t) - c \bar{d}_i \bar{e}_i(t) + c \sum_{j=1}^N \bar{l}_{ij} s_j(t), & v_i \in \bar{V}_1^i, \\ f(y_i(t)) - c \sum_{v_j \in V_{\phi_i}} \bar{l}_{ij} y_j(t) - c \bar{d}_i \bar{e}_i(t), & v_i \in \bar{V}_2^i, \\ f(y_i(t)) - c \sum_{v_j \in V_{\phi_i}} \bar{l}_{ij} y_j(t), & v_i \in \bar{V}_3^i, \end{cases} \quad (92)$$

if $t \in [t_k + \delta_k, t_{k+1})$.

Based on the bipartite clustered tracking model with (3), (8) and (9), one can derive the following theorem.

Theorem 1. Under Assumption 1, the BCC for system (3), (8) and (9) can be achieved if there exist an infinite sequence of uniformly bounded, nonoverlapping time intervals $[t_k, t_{k+1})$, $k \in \mathbb{N}_+$, with $t_1 = 0$, satisfying following conditions:

- (i) $\eta I_N - cL_s - D < 0$ and $\eta I_N - c\bar{L}_s - \bar{D} < 0$,
- (ii) $\frac{\beta}{\alpha + \beta}(t_{k+1} - t_k) < \delta_k < \frac{\bar{\beta}}{\bar{\alpha} + \bar{\beta}}(t_{k+1} - t_k)$, $[t_k, t_{k+1})$, $k \in \mathbb{N}_+$,
 where $\alpha = -2\lambda_{\max}(H_1)$, $H_1 = \eta I_N - c(L + D)_s$, and $\beta = 2\lambda_{\max}(H_2)$, $H_2 = \eta I_N - cL_s$. $\bar{\alpha}$ and $\bar{\beta}$ are defined similarly in the proof.

Proof. Consider the following Lyapunov function candidate:

$$V(t) = \frac{1}{2} \sum_{i=1}^N e_i^T(t) e_i(t).$$

Denote $e(t) = (e_1^T(t), e_2^T(t), \dots, e_N^T(t))^T$, then the time derivative of $V(t)$ along the trajectories of (6) can be derived as follows.

(i) When $t \in [t_k, t_k + \delta_k)$, $k \in \mathbb{N}_+$,

$$\begin{aligned} \dot{V}(t) &= \sum_{i=1}^N e_i^T(t) \dot{e}_i(t) \\ &= \sum_{i=1}^N e_i^T(t) (f(x_i(t)) - f(s_i(t))) - c \sum_{i=1}^N \sum_{j=1}^N l_{ij} e_i^T(t) e_j(t) \\ &\quad - c \sum_{v_i \in (V_1^i \cup V_2^i)} d_i e_i^T(t) e_i(t) \\ &\leq e^T(t) [(\eta I_N - c(L + D))_s \otimes I_n] e(t). \end{aligned}$$

Let $H_1 = \eta I_N - c(L + D)_s$, and denote $\lambda_1 = \lambda_{\max}(H_1)$ as the largest eigenvalue of H_1 . By condition (i), $\lambda_1 < 0$, it can be derived that

$$\dot{V}(t) \leq e^T(t) (H_1 \otimes I_n) e(t) \leq \lambda_1 e^T(t) e(t) = 2\lambda_1 V(t) = -\alpha V(t),$$

where $\alpha = -2\lambda_1 > 0$, then we have

$$V(t) \leq V(t_k) e^{-\alpha(t-t_k)}.$$

(ii) When $t \in [t_k + \delta_k, t_{k+1})$, $k \in \mathbb{N}_+$,

$$\begin{aligned} \dot{V}(t) &= \sum_{i=1}^N e_i^T(t) \dot{e}_i(t) \\ &= \sum_{i=1}^N e_i^T(t) (f(x_i(t)) - f(s_i(t))) - c \sum_{i=1}^N \sum_{j=1}^N l_{ij} e_i^T(t) e_j(t) \\ &\leq e(t)^T [(\eta I_N - cL_s) \otimes I_n] e(t). \end{aligned}$$

Let $H_2 = \eta I_N - cL_s$, and denote $\lambda_2 = \lambda_{\max}(H_2)$. By condition (i), $\lambda_2 > 0$, we have

$$\dot{V}(t) \leq e^T(t) (H_2 \otimes I_n) e(t) \leq \lambda_2 e^T(t) e(t) = 2\lambda_2 V(t) = \beta V(t),$$

where $\beta = 2\lambda_2 > 0$, therefore, it can be derived that

$$V(t) \leq V(t_k + \delta_k) e^{\beta(t-t_k-\delta_k)}.$$

By the derivation above it can be obtained that

$$V(t_2) \leq V(t_1 + \delta_1)e^{\beta(t_2 - t_1 - \delta_1)} \leq V(t_1)e^{-\alpha\delta_1 + \beta(t_2 - t_1 - \delta_1)} = V(t_1)e^{-\gamma_1},$$

where $\gamma_1 = \alpha\delta_1 - \beta(t_2 - t_1 - \delta_1)$. According to condition (ii), one has $\gamma_1 > 0$. By recursion, for any positive integer k , the following inequality holds:

$$V(t_{k+1}) \leq V(t_1)e^{-\sum_{j=1}^k \gamma_j},$$

where $\gamma_j = \alpha\delta_j - \beta(t_{j+1} - t_j - \delta_j) > 0, j = 1, 2, \dots, k$. For any $t > 0$, there exists $l \in \mathbb{N}$ such that $t_{l+1} \leq t < t_{l+2}$. Furthermore, since $\{t_k, t_{k+1}\}, k \in \mathbb{N}_+$, is a uniformly bounded and nonoverlapping time sequence, let $\xi_{\max} = \max_{k \in \mathbb{N}_+} \{t_{k+1} - t_k\}$ and $\gamma = \min_{j \in \mathbb{N}_+} \{\gamma_j\}$. Therefore, it can be derived that

$$\begin{aligned} V(t) &\leq V(t_{l+1})e^{\xi_{\max}\beta} \leq e^{\xi_{\max}\beta} V(t_1)e^{-\sum_{j=1}^l \gamma_j} \leq e^{\xi_{\max}\beta} V(t_1)e^{-l\gamma} \\ &\leq e^{\xi_{\max}\beta} V(0)e^{-\gamma t/\xi_{\max}} = \zeta_0 e^{-\zeta_1 t}, \end{aligned}$$

where $\zeta_0 = e^{\xi_{\max}\beta} V(0)$ and $\zeta_1 = \gamma/\xi_{\max}$. It can be acquired that the states of agents in each cluster exponentially converges to that of the leader, this implies that $\lim_{t \rightarrow \infty} \|x_i(t) - s_i(t - \tau_i)\| = 0 (i = 1, 2, \dots, N)$ holds under controller (7), that is, the clustered system with (3) and (4) can achieve the expected delayed cluster consensus.

To N_2 , use similar notations with Remark 1, set $\bar{V}_1^i = \tilde{V}_{\phi_i}, \bar{V}_2^i = \{v_i: v_i \in V_{\phi_i} \setminus \tilde{V}_{\phi_i} \text{ and } \deg(v_i)_{\text{in}} = 0\}$ and denote $\bar{V}_3^i = V_{\phi_i} \setminus (\bar{V}_1^i \cup \bar{V}_2^i)$. Thus, $y_i(t)$ denote the state of the i th node v_i , where $v_i \in V_{\phi_i}$ and $V_{\phi_i} = \bar{V}_1^i \cup \bar{V}_2^i \cup \bar{V}_3^i$.

For achieving the cluster consensus in N_2 , consider the following Lyapunov function candidate:

$$\bar{V}(t) = \frac{1}{2} \sum_{i=1}^N \bar{e}_i^T(t) \bar{e}_i(t).$$

Then it can be obtained that

(i) When $t \in [t_k, t_k + \delta_k)$,

$$\begin{aligned} \dot{\bar{V}}(t) &= \sum_{i=1}^N \bar{e}_i^T(t) \dot{\bar{e}}_i(t) \\ &= \sum_{i=1}^N \bar{e}_i^T(t) (f(x_i(t)) - f(s_i(t))) - c \sum_{i=1}^N \sum_{j=1}^N \bar{l}_{ij} \bar{e}_i^T(t) \bar{e}_j(t) \\ &\leq e^T(t) [(\eta I_N - c\bar{L}) \otimes I_n] e(t) = e^T(t) [(\eta I_N - c\bar{L}_s) \otimes I_n] e(t). \end{aligned}$$

Denote $\bar{H}_1 = \eta I_N - c\bar{L}_s$ and $\lambda_{\max}(\bar{H}_1) = \bar{\lambda}_1$. By condition (i), $\bar{\lambda}_1 > 0$, we have

$$\dot{\bar{V}}(t) \leq e^T(t) (\bar{H}_1 \otimes I_n) e(t) \leq \bar{\lambda}_1 e^T(t) e(t) = 2\bar{\lambda}_1 \bar{V}(t) = \bar{\alpha} \bar{V}(t),$$

where $\bar{\alpha} = 2\bar{\lambda}_1 > 0$.

Therefore, it can be derived that

$$\bar{V}(t) \leq \bar{V}(t_k)e^{\bar{\alpha}(t-t_k)}.$$

(ii) When $t \in [t_k + \delta_k, t_{k+1})$, $k \in N_+$,

$$\begin{aligned} \dot{\bar{V}}(t) &= \sum_{i=1}^N e_i^T(t) \dot{e}_i(t) \\ &= \sum_{i=1}^N e_i^T(t) (f(x_i(t)) - f(s_i(t))) - c \sum_{i=1}^N \sum_{j=1}^N \bar{l}_{ij} e_i^T(t) e_j(t) \\ &\quad - c \sum_{v_i \in (V_1^i \cup V_2^i)} \bar{d}_i e_i^T(t) e_i(t) \\ &\leq e^T(t) [(\eta I_N - c(\bar{L} + \bar{D}))_s \otimes I_n] e(t). \end{aligned}$$

Let $\bar{H}_2 = \eta I_N - c(\bar{L} + \bar{D})_s$ and denote $\lambda_{\max}(\bar{H}_2) = \bar{\lambda}_2$. By condition (i), $\bar{\lambda}_2 < 0$, we have

$$\dot{\bar{V}}(t) \leq e^T(t) (\bar{H}_2 \otimes I_n) e(t) \leq \bar{\lambda}_2 e^T(t) e(t) = 2\bar{\lambda}_2 \bar{V}(t) = -\bar{\beta} \bar{V}(t),$$

where $\bar{\beta} = -2\bar{\lambda}_2 > 0$, which gives

$$\bar{V}(t) \leq \bar{V}(t_k + \delta_k) e^{-\bar{\beta}(t-t_k-\delta_k)}.$$

Similar to the former part of derivation, by condition (ii) one can derive that $\dot{\bar{V}}(t) \leq 0$, the equality holds if and only if $\bar{e}(t) = 0$ for $k = 2, \dots, m$. Hence, by Lyapunov stability theory, $\lim_{t \rightarrow \infty} \|y_i(t) - s_i(t)\| = 0$ holds, $k = 2, \dots, m$.

Therefore, the multi-tracking for N_2 can be solved, thus the BCC can be achieved under controller (7). □

Remark 2. It is obvious that if the graph of a networked system is undirected and connected, then the node with zero in-degree does not exist, and the control protocol (7) can be simplified as follows:

$$\begin{aligned} u_i(t) &= \begin{cases} u_{i1}(t), & v_i \in V_{\hat{i}}, \\ u_{i2}(t), & v_i \in V_{\phi_{\hat{i}}}, \end{cases} & (10) \\ u_{i1}(t) &= \begin{cases} -cd_{i1}(t)e_i(t) + c \sum_{j=1}^N l_{ij} s_j(t), & v_i \in \tilde{V}_{\hat{i}}, \\ 0, & v_i \in V_{\hat{i}} \setminus \tilde{V}_{\hat{i}}, \end{cases} \\ u_{i2}(t) &= \begin{cases} -cd_{i2}(t)e_i(t) + c \sum_{j=1}^N l_{ij} s_j(t), & v_i \in \tilde{V}_{\phi_{\hat{i}}}, \\ 0, & v_i \in V_{\phi_{\hat{i}}} \setminus \tilde{V}_{\phi_{\hat{i}}}, \end{cases} \\ d_{i1}(t) &= \begin{cases} d_i, & t \in [t_k, t_k + \delta_k), \\ 0, & t \in [t_k + \delta_k, t_{k+1}), \end{cases} & d_{i2}(t) = \begin{cases} 0, & t \in [t_k, t_k + \delta_k), \\ \bar{d}_i, & t \in [t_k + \delta_k, t_{k+1}). \end{cases} \end{aligned}$$

Then the following corollary can be derived.

Corollary 1. Assume that the graph of multi-agent system with (1)–(3) is undirected and connected. Under Assumption 1, the system can achieve the BCC under control protocol (10) if the following conditions hold for any time intervals $[t_k, t_{k+1})$, $k \in \mathbb{N}_+$:

- (i) $\eta I_N - c(L + D) < 0$ and $\eta I_N - c(\bar{L} + \bar{D}) < 0$,
- (ii) $\frac{\beta}{\alpha + \beta}(t_{k+1} - t_k) < \delta_k < \frac{\bar{\beta}}{\bar{\alpha} + \bar{\beta}}(t_{k+1} - t_k)$.

The form of this system is similar to Theorem 1, and the proof is omitted here.

Remark 3. MAS related problems have been widely analyzed in many fields, such as sensor networks [15], neural networks [9, 20, 30, 31], etc. Since different networks may contain the same or similar graph structures, the study of one network may shed light on other related networks. One may consider the similarity and do some enlightening works in the future research.

4 Numerical simulations

In this section, numerical results are given to verify the effectiveness of criteria of Theorem 1. A MAS with 16 nodes and four clusters is considered, and the communication graph of the model is shown in Fig. 2.

The two black nodes denote the leaders, and the blue nodes represent the followers. The indexes of the nodes are labelled with an ascending sequence from cluster V_1 to V_2 and cluster V_{ϕ_1} to V_{ϕ_2} (here we use node set to present the corresponding clusters). The dotted ellipses represent the clusters, and the green dashed boxes represent the subnetworks. The red and yellow arrows denote the pinning effects that the k th leader

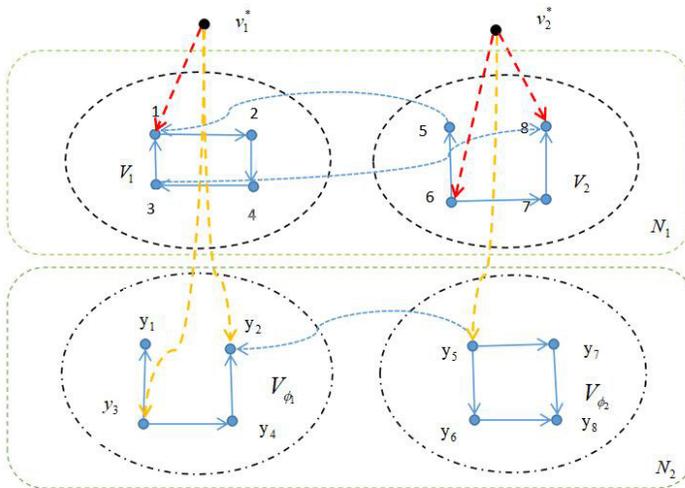


Figure 2. An example of the bipartite clustered network.

put on the agents of certain topological properties of subnetwork N_1 and N_2 , respectively ($k = 1, 2$). The blue dotted arrows denote the directed information links between two clusters. The blue solid arrow lines represent the communication links inside the clusters.

In view of controller (7), one can see that the nodes labelled by 1 and 8 belong to V_1^i , the nodes labelled by 6 and y_5 belong to V_2^i , and the other nodes belong to V_3^i . For simplicity, it is assumed that each node is a one-dimensional dynamical system and set $c = 1$. The nonlinear function f is described by

$$f(x_i) = 0.25 \sin x_i + 0.5 \tanh x_i + 1,$$

where $x_i \in R$. The coupling weights a_{ij} are chosen as follows:

$$\begin{aligned} a_{13} &= 0.5, & a_{15} &= 0.7, & a_{21} &= 1.5, & a_{34} &= 0.6, & a_{42} &= 1, \\ a_{56} &= 1.5, & a_{76} &= 1.1, & a_{87} &= 1.2, & a_{83} &= 1.6, & & \\ \bar{a}_{13} &= 1.8, & \bar{a}_{4,3} &= 1.6, & \bar{a}_{2,4} &= 2.3, & \bar{a}_{2,5} &= 2.8, & \bar{a}_{7,5} &= 4.5, \\ \bar{a}_{6,5} &= 2.3, & \bar{a}_{8,6} &= 2.8, & \bar{a}_{8,7} &= 0.8. & & & & \end{aligned}$$

The pinning coupling weights for the pinned nodes of the subnetworks are selected as follows:

$$\begin{aligned} d_1 &= 8, & d_6 &= 6, & d_8 &= 6 \\ \bar{d}_2 &= 7, & \bar{d}_3 &= 8, & \bar{d}_5 &= 5. \end{aligned}$$

In view of Theorem 1, one has

$$\begin{aligned} \lambda_1 &= -0.22 < 0, & \alpha &= 0.44, & \lambda_2 &= 0.4 > 0, & \beta &= 0.8 \\ \bar{\lambda}_1 &= 0.26 > 0, & \bar{\alpha} &= 0.52, & \bar{\lambda}_2 &= -1.2 < 0, & \bar{\beta} &= 2.4. \end{aligned}$$

Thus, choose

$$0.65 \approx \frac{\beta}{\alpha + \beta} < \frac{\delta_k}{t_{k+1} - t_k} < \frac{\bar{\beta}}{\bar{\alpha} + \bar{\beta}} \approx 0.82,$$

then condition (ii) holds.

Thus, all the conditions in Theorem 1 have been satisfied. Therefore, the BCC can be achieved in multi-agent system with (1), (2) and (3) under controller (7). The states of all agents are shown in Figs. 3–5 with initial conditions

$$\begin{aligned} x_1(0) &= 5.8, & x_2(0) &= 2.2, & x_3(0) &= 1.8, & x_4(0) &= 9.2, \\ x_5(0) &= 8.5, & x_6(0) &= 17, & x_7(0) &= 14.5, & x_8(0) &= -3, \\ y_1(0) &= 10.5, & y_2(0) &= 7.1, & y_3(0) &= 7.5, & y_4(0) &= 13, \\ y_5(0) &= 5.8, & y_6(0) &= 1.8, & y_7(0) &= 2.4, & y_8(0) &= -0.1, \\ s_1(0) &= 5.7, & s_2(0) &= 3.8. \end{aligned}$$

In this example, the error of BCC in subnetwork N_1 is denoted by $e_i(t), i = 1, 2, \dots, 8$, and the trajectories are showed in Figs. 5 and 9. The error of BCC in N_2 is denoted by $E_i(t), i = 1, 2, \dots, 8$, and they are showed in Figs. 6 and 10.

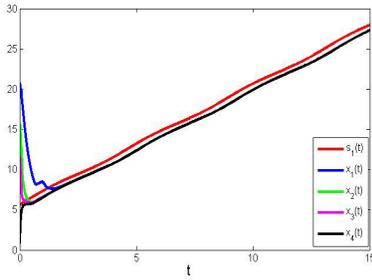


Figure 3. The state trajectories in cluster V_1 .

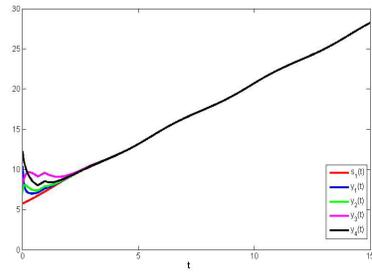


Figure 4. The state trajectories in cluster V_{ϕ_1} .

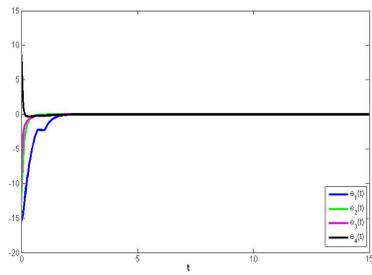


Figure 5. The change of errors in cluster V_1 .

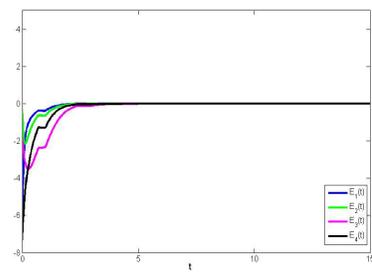


Figure 6. The change of errors in cluster V_{ϕ_1} .

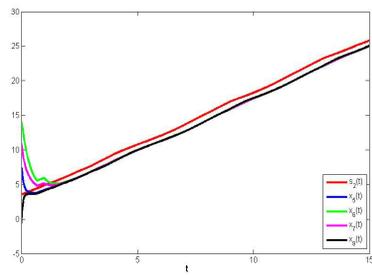


Figure 7. The state trajectories in cluster V_2 .

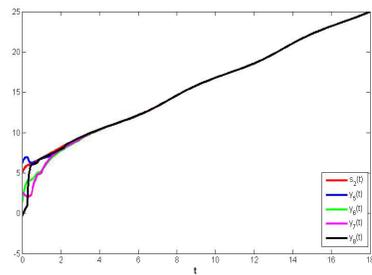


Figure 8. The state trajectories in cluster V_{ϕ_2} .

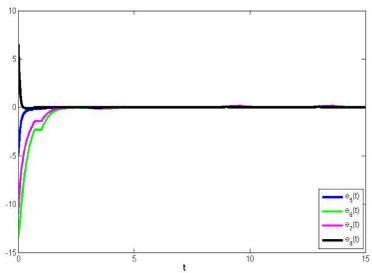


Figure 9. The change of errors in cluster V_2 .

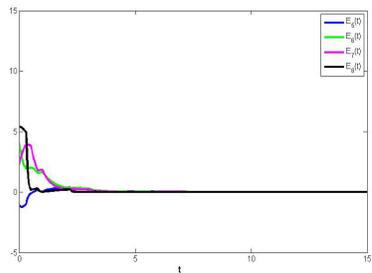


Figure 10. The change of errors in cluster V_{ϕ_2} .

It can be seen that the cluster consensus problem is indeed solved and the numerical simulation verify the theoretical results well.

Remark 4. Due to the form of controller (7), it can be seen that the nodes 1, 8 and y_2 has the access to receive information from other clusters, and the first law of u_{i1} or u_{i2} should be applied to these agents. The node 6 and y_3, y_5 have no information flows in from other agents of the whole network, so it must be pinned and controlled by the second law of u_{i1} and u_{i2} , respectively. The other follower nodes in the network have no access to receive information from the clusters they do not belong, but they have information flows in from inside the clusters, therefore, they should be controlled by the third law of u_{i1} or u_{i2} .

5 Conclusion

In this paper, according to the properties of nodes in the clustered structure, a new pinning scheme with intermittent effect has been established. Due to the notion of bipartite clustered network, some novel switching MASs with intermittent pattern have been established, and the cluster consensus problem named BCC has been studied through multitasking approach. Several sufficient conditions for the problem have been derived. Finally, the effectiveness of the theoretical results has been proved by a numerical example. There may exist lots of works related to bipartite cluster consensus deserving further research, for instance, BCC with adaptive control, impulsive control, BCC with the factor of fixed time [4], etc., and one may consider some of these problems in the future research.

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