Nonlinear Analysis: Modelling and Control, Vol. 26, No. 5, 861–883
https://doi.org/10.15388/namc.2021.26.24177

A mathematical model of population dynamics for the internet gaming addiction*

Hiromi Seno

Department of Computer and Mathematical Sciences,
Graduate School of Information Sciences, Tohoku University,
Sendai, Miyagi 980-8579, Japan

seno@math.is.tohoku.ac.jp

Received: September 17, 2020 / **Revised:** February 25, 2021 / **Published online:** September 1, 2021

Abstract. As the number of internet users appears to steadily increase each year, Internet Gaming Disorder (IGD) is bound to increase as well. The question how this increase will take place, and what factors have the largest impact on this increase, naturally arises. We consider a system of ordinary differential equations as a simple mathematical model of the population dynamics about the internet gaming. We assume three stages about the internet gamer's state: moderate, addictive, and under treatment. The transition of the gamer's state between the moderate and the addictive stages is significantly affected by the social nature of internet gaming. As the activity of social interaction gets higher, the gamer would be more likely to become addictive. With the inherent social reinforcement of internet game, the addictive gamer would hardly recontrol his/herself to recover to the moderate gamer. Our result on the model demonstrates the importance of earlier initiation of a system to check the IGD and lead to some medical/therapeutic treatment. Otherwise, the number of addictive gamers would become larger beyond the socially controllable level.

Keywords: population dynamics, mathematical model, addiction, gaming disorder.

1 Introduction

World Health Organization [32] acknowledged addiction to internet gaming as a real disorder, called by *Gaming disorder, predominantly online*, frequently mentioned now as Internet Gaming Disorder (IGD), which is generally defined as "Persistent and recurrent use of the internet to engage in games, often with other players, leading to clinically significant impairment or distress" [7], although the concept of IGD has been questioned [12, 22, 30]. The criteria for an individual to be classified with IGD are given by Diagnostic and Statistical Manual of Mental Disorders (DSM-5) ([1]; albeit not intended

*This work was supported in part by JSPS KAKENHI grant No. 18K03407.

for clinical use; also see [22]):

- Preoccupation with internet games becomes the dominant activity in daily life;
- Withdrawal symptoms when removing the internet (e.g., irritability, anxiety, or sadness, no physical signs of pharmacological withdrawal);
- Need to spend increasing amounts of time with internet gaming (tolerance);
- Lacking control of internet gaming;
- Loss of interest in previous hobbies and entertainment;
- Continued excessive use despite knowledge of psychosocial problems;
- Deception of family members, therapists, or others about the amount of internet gaming;
- Escape from or relief of negative mood (e.g., feeling helpless, guilty, anxious);
- Loss of important aspects of life (e.g., significant relationships, jobs, or educational/career opportunities).

As the number of internet users appears to steadily increase each year [31], IGD is bound to increase as well [4, 7], whereas the present public attention would be so weak as to aware its harmfulness in the public health [3], and the prevalence estimates vary significantly according to the criteria and the cultural factors [15, 21, 25]. The exact size and nature of IGD is still waiting for the definitive investigations even after a number of researches on the consequences of excessive and addictive gaming [8, 12, 23, 24]. The question how this increase will take place, and what factors have the largest impact on this increase, naturally arises. According to statistical studies on children and adolescents [22], as well as on adults [28], a prevailing reason why internet gamers transition into addiction is the social aspect of the internet game (also see [11, 17]). It may be the significant factor to distinguish the internet gaming addiction from the other addictive behaviors related to the internet use [4,8]. As stated in [22], “Internet and role-playing games possess more addictive potential than offline games because of their inherent social reinforcements”. It is suggested that the social nature of online games “reinforces gaming instead of criticizing it”. In other words, social interaction between gamers may be an underlying mechanism to the increase of IGD.

In this paper, we consider a simple mathematical model with a system of ordinary differential equations for the population dynamics about the internet gaming, especially focusing on the social nature of online gaming which would be regarded as a factor to reinforce the transition to the addictive gaming. Although our mathematical model may seem to be an application of some models about the transmission of infectious disease like the previous models on the addition (see, for example, [9, 16, 29] on the online gaming addiction, [2, 20] on the drug addiction), such an introduction of the effect from the social nature of online gaming in the population dynamics makes the mathematical model distinctive from that for the epidemic dynamics of a transmissible disease (as for the mathematical modeling and models about the epidemic dynamics, for example, see [5, 10, 13, 19]). Further, our model is not on the state transition of individual who are craving to play the online game, for example, like that by [6], but it is on an epidemiological dynamics of addictive gamers in a community. We will not intend to propose any control or treatment policy for the IGD in a community, whereas we will try to present a basic mathematical model and its mathematical nature which will be expected useful to advance some future theoretical/mathematical researches on the IGD, especially from the epidemiologically interdisciplinary viewpoint.

2 Assumptions

Gamer's states

We consider the three stages about the internet gamer's state: moderate, addictive, and under treatment (see Fig. 1). The moderate state means the stage in which the gamer can control him/herself about playing the internet game. In contrast, the addictive state means the stage in which the gamer plays the internet game pathologically (i.e., without properly controlling him/herself) in the sense following to DSM-5 [1].

We suppose that there are some chances for the addictive gamer to have a medical or/and therapeutic treatment after being identified his/her addictive gaming by him/herself or by some others near him/her. It is assumed that under the treatment, the gamer has no effective contact to the other gamers on web, so that such a gamer has no significant contribution to the population dynamics like an isolated infective individual in the epidemic dynamics.

Transition between moderate and addictive states

The transition of the gamer's state between the moderate and the addictive stages is significantly affected by the social nature of internet gaming [22]. Therefore, we assume that the state transition depends on the activity of social interaction between gamers on web. As the activity of social interaction gets higher, the gamer would be more likely to become addictive. With the inherent social reinforcement of internet game, the addictive gamer would hardly recontrol his/herself to recover to the moderate gamer.

Relapse gaming by the gamer under treatment

We assume the possibility that the gamer under treatment relapses into gaming. This means stopping the treatment and restarting gaming. In one case the effective treatment may result in the gamer's stopping gaming (i.e., becoming a nongamer), while in the other case, it may not.

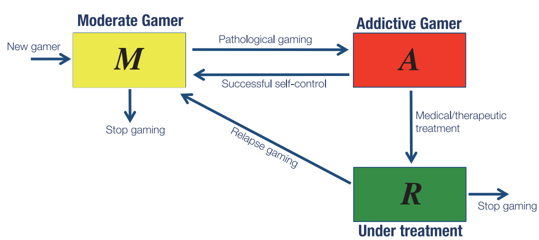

Figure 1. General scheme of the state transition of internet gamer in our modeling.

3 General model

With the assumptions described in the previous section, we consider the following model with a system of ordinary differential equations governing the population dynamics according to the modeling indicated by Fig. 2 about the state transition of internet gamer:

$$\begin{aligned}\frac{\mathrm{d}M}{\mathrm{d}t} &= \lambda - f(M,A)M + g(M,A)A - \mu_{\mathrm{M}}M + \rho R,\\ \frac{\mathrm{d}A}{\mathrm{d}t} &= f(M,A)M - g(M,A)A - \sigma A,\\ \frac{\mathrm{d}R}{\mathrm{d}t} &= \sigma A - \rho R - \mu_{\mathrm{R}}R,\end{aligned} \tag{1}$$

where $M = M(t)$, $A = A(t)$, and $R = R(t)$ are, respectively, the population sizes of moderate gamer, addictive gamer, and gamer under treatment at time t. Parameters λ, μ_{M}, ρ, σ, and μ_{R} are positive constants, while $f(M,A)$ and $g(M,A)$ are functions of M and A.

The parameter λ gives the recruitment flux of new internet gamers. In this modeling, we assume it a positive constant independent of the gamer population, although it could be related to the population size of gamer or internet user. The parameters μ_{M} and μ_{R} are the rate of stopping gaming for moderate gamer and that for the gamer under treatment about the addiction. The parameter ρ is the rate of relapsing into gaming after the treatment. For a simplicity, in this preliminary work, we assume that such a relapsing gamer could be categorized in the moderate gamer, even though such a gamer would relapse into the addiction easier than the moderate gamer. The parameter σ is the rate of identifying the addiction and beginning the treatment. We can regard the value of σ as indicating a result derived from the amount of effort (e.g., budget and human power) and the effectiveness of the treatment operation.

The transition rate from the moderate to the addictive gamer $f(M,A)$ and the inverse transition rate $g(M,A)$ in (1) are assumed to satisfy the following conditions according to their nonnegative meanings and the modeling assumptions given in the previous section:

(i) $f(x,y)$ and $g(x,y)$ are positive and differentiable on $[0,\infty)\times[0,\infty)$;
(ii) $f(x,y)$ is *increasing* in terms of x and y;
(iii) $g(x,y)$ is *decreasing* in terms of x and y.

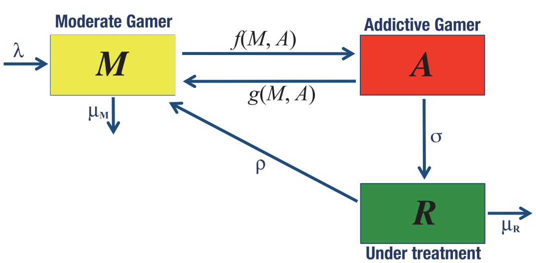

Figure 2. The state transition of internet gamer in our modeling.

With these assumptions for functions f and g, we can get the following lemma with respect to the positiveness of variables M, A, and R:

Lemma 1. *For any nonnegative initial condition* $(M(0), A(0), R(0))$*, the solution* $(M(t), A(t), R(t))$ *of* (1) *is positive for any finite* $t > 0$*.*

This lemma can be easily proved by showing that $\mathrm{d}M/\mathrm{d}t|_{M=0} > 0$ for any finite $t \geqslant 0$, $\mathrm{d}A/\mathrm{d}t|_{A=0} > 0$ and $\mathrm{d}R/\mathrm{d}t|_{R=0} > 0$ for any finite $t > 0$, whereas we do not describe the detail here. This indicates that the solution $(M(t), A(t), R(t))$ cannot reach 0 at any $t > 0$.

Model without the treatment state

As for today's actual situation, we can set $\sigma \approx 0$ because there has not yet any practical treatment policy in any country. So let us consider here the following model with $\sigma = 0$ and $R \equiv 0$ for (1):

$$\begin{aligned} \frac{\mathrm{d}M}{\mathrm{d}t} &= \lambda - f(M, A)M + g(M, A)A - \mu_{\mathrm{M}} M, \\ \frac{\mathrm{d}A}{\mathrm{d}t} &= f(M, A)M - g(M, A)A. \end{aligned} \tag{2}$$

For this system, similarly with Lemma 1, we can show that for any nonnegative initial condition $(M(0), A(0))$, the solution $(M(t), A(t))$ of (2) is positive for any finite $t > 0$.

For this 2-dimensional model, we can get the general result about the condition for the local stability of an equilibrium state $(M, A) = (M^*, A^*)$ with $\mathrm{d}M/\mathrm{d}t = \mathrm{d}A/\mathrm{d}t = 0$ supposing its existence. From (2) we can easily find the following equations about the equilibrium state (M^*, A^*) when it exists:

$$M^* = \frac{\lambda}{\mu_{\mathrm{M}}}, \quad f(M^*, A^*)M^* - g(M^*, A^*)A^* = 0. \tag{3}$$

As seen from the second equation in (3), the existence depends on the detail of functions f and g. Suppose its existence now leaving aside the condition about the functions for it. The Jacobi matrix for this equilibrium state of (1) becomes

$$J(M^*, A^*) := \begin{pmatrix} -\varPhi^*_{\mathrm{M}} - \mu_{\mathrm{M}} & -\varPhi^*_{\mathrm{A}} \\ \varPhi^*_{\mathrm{M}} & \varPhi^*_{\mathrm{A}} \end{pmatrix}, \tag{4}$$

where

$$\begin{aligned} \varPhi^*_{\mathrm{M}} &:= f(M^*, A^*) + f_M(M^*, A^*)M^* - g_M(M^*, A^*)A^* \geqslant 0, \\ \varPhi^*_{\mathrm{A}} &:= f_A(M^*, A^*)M^* - g_A(M^*, A^*)A^* - g(M^*, A^*), \end{aligned} \tag{5}$$

$f_M(M^*, A^*) := \partial f(M, A)/\partial M|_{(M,A)=(M^*,A^*)}$, etc. From the above Jacobi matrix $J(M^*, A^*)$ for the equilibrium state (M^*, A^*) we can get the following result about its local stability:

Lemma 2. *When a positive equilibrium state* $(M, A) = (M^*, A^*)$ *exists for system* (1)*, it is locally asymptotically stable if* $\Phi_{\mathrm{A}}^* < 0$*, whereas it is unstable if* $\Phi_{\mathrm{A}}^* > 0$.

At the same time, it can be easily shown that the eigenvalues of the Jacobi matrix (4) are necessarily real and that one of them is necessarily negative. Therefore, the locally asymptotically stable positive equilibrium state (M^*, A^*) is a stable node, while the unstable one is a saddle. This result indicates that any oscillatory behavior does not appear near the positive equilibrium state (M^*, A^*).

4 A simple model

In this section, we consider one of the mathematically simplest version of (1) with the following f and g:

$$f(M, A) = \beta(M + A), \qquad g(M, A) = g_0\,\mathrm{e}^{-\gamma(M+A)},$$

where parameters β and γ indicate the strength of the effect from the social interaction with the other gamers on web about the transition between the moderate and the addictive states: as each value gets larger, the effect becomes stronger. Especially the parameter γ introduces the strength of the reinforcement of additive gaming since its larger value causes the lower possibility to escape from the gaming addiction.

In this modeling, the transition between the moderate and the addictive states is assumed to be simply affected by the total number of gamers. For the simplification of the model, we now ignore the difference between the moderate gamer and the addictive one with respect to the contribution to the social interaction among the gamers on web, whereas the addictive gamer could be regarded as more active than the moderate one in terms of the social interaction. $g_0 = g(0, 0) > 0$ means the transition rate from the addictive to the moderate being without any effect from the inherent social reinforcement of online game (i.e., when $\gamma = 0$).

Now we have the following model as one of the mathematically simplest version of (1):

$$\begin{aligned}
\frac{\mathrm{d}M}{\mathrm{d}t} &= \lambda - \beta(M + A)M + g_0\,\mathrm{e}^{-\gamma(M+A)}A - \mu_{\mathrm{M}}M + \rho R,\\
\frac{\mathrm{d}A}{\mathrm{d}t} &= \beta(M + A)M - g_0\,\mathrm{e}^{-\gamma(M+A)}A - \sigma A,\\
\frac{\mathrm{d}R}{\mathrm{d}t} &= \sigma A - \rho R - \mu_{\mathrm{R}}R.
\end{aligned} \tag{6}$$

For the mathematical convention, let us apply the following transformation of variables and parameters for (6):

$$\begin{gathered}
\tau = \mu_{\mathrm{M}}t, \quad \widetilde{M} = \frac{M}{\lambda/\mu_{\mathrm{M}}}, \quad \widetilde{A} = \frac{A}{\lambda/\mu_{\mathrm{M}}}, \quad \widetilde{R} = \frac{R}{\lambda/\mu_{\mathrm{M}}}, \quad \widetilde{\beta} = \frac{\lambda}{\mu_{\mathrm{M}}^2}\beta,\\
\widetilde{g}_0 = \frac{g_0}{\mu_{\mathrm{M}}}, \quad \widetilde{\gamma} = \frac{\lambda}{\mu_{\mathrm{M}}}\gamma, \quad \widetilde{\rho} = \frac{\rho}{\mu_{\mathrm{M}}}, \quad \widetilde{\sigma} = \frac{\sigma}{\mu_{\mathrm{M}}}, \quad \widetilde{\mu} = \frac{\mu_{\mathrm{R}}}{\mu_{\mathrm{M}}}.
\end{gathered} \tag{7}$$

Then we get the following system with nondimensionalized variables and parameters:

$$\begin{aligned}\frac{\mathrm{d}\widetilde{M}}{\mathrm{d}\tau} &= 1 - \widetilde{\beta}(\widetilde{M} + \widetilde{A})\widetilde{M} + \widetilde{g}_0\, \mathrm{e}^{-\widetilde{\gamma}(\widetilde{M}+\widetilde{A})}\, \widetilde{A} - \widetilde{M} + \widetilde{\rho}\widetilde{R},\\ \frac{\mathrm{d}\widetilde{A}}{\mathrm{d}\tau} &= \widetilde{\beta}(\widetilde{M} + \widetilde{A})\widetilde{M} - \widetilde{g}_0\, \mathrm{e}^{-\widetilde{\gamma}(\widetilde{M}+\widetilde{A})}\, \widetilde{A} - \widetilde{\sigma}\widetilde{A},\\ \frac{\mathrm{d}\widetilde{R}}{\mathrm{d}\tau} &= \widetilde{\sigma}\widetilde{A} - \widetilde{\rho}\widetilde{R} - \widetilde{\mu}\widetilde{R}.\end{aligned} \tag{8}$$

4.1 Model without the treatment state

We consider the following system without the treatment state derived from (8) with $\widetilde{\sigma} = 0$ and $\widetilde{R} \equiv 0$:

$$\begin{aligned}\frac{\mathrm{d}\widetilde{M}}{\mathrm{d}\tau} &= 1 - \widetilde{\beta}(\widetilde{M} + \widetilde{A})\widetilde{M} + \widetilde{g}_0\, \mathrm{e}^{-\widetilde{\gamma}(\widetilde{M}+\widetilde{A})}\, \widetilde{A} - \widetilde{M},\\ \frac{\mathrm{d}\widetilde{A}}{\mathrm{d}\tau} &= \widetilde{\beta}(\widetilde{M} + \widetilde{A})\widetilde{M} - \widetilde{g}_0\, \mathrm{e}^{-\widetilde{\gamma}(\widetilde{M}+\widetilde{A})}\, \widetilde{A}.\end{aligned} \tag{9}$$

The positive equilibrium state $(\widetilde{M}^*, \widetilde{A}^*)$ for this model satisfies the following equalities:

$$\widetilde{M}^* = 1, \qquad \frac{\widetilde{g}_0}{\widetilde{\beta}} = \left(1 + \frac{1}{\widetilde{A}^*}\right) \mathrm{e}^{\widetilde{\gamma}(1+\widetilde{A}^*)}. \tag{10}$$

Note that $\widetilde{M} = \widetilde{M}^* = 1$ corresponds to $M = M^* := \lambda/\mu_{\mathrm{M}}$. Then from these equalities we can obtain the following result about the existence of the positive equilibrium state $(\widetilde{M}^*, \widetilde{A}^*)$ given by (10) for (9):

Lemma 3. *The positive equilibrium state* $(\widetilde{M}^*, \widetilde{A}^*)$ *given by* (10) *for* (9) *exists when and only when the following condition holds*:

$$\frac{\widetilde{g}_0}{\widetilde{\beta}} \geqslant \left(1 + \frac{1}{\alpha}\right) \mathrm{e}^{\widetilde{\gamma}(1+\alpha)} \quad \textit{with } \alpha := -\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{1}{\widetilde{\gamma}}}. \tag{11}$$

The proof is given in Appendix A. This result indicates it necessary for the existence of the positive equilibrium state $(\widetilde{M}^*, \widetilde{A}^*)$ that $\widetilde{g}_0/\widetilde{\beta} > 1$ (see Fig. 3).

This result can be expressed in the following different way useful for our argument (Appendix A):

Lemma 4. *The positive equilibrium state* $(\widetilde{M}^*, \widetilde{A}^*)$ *given by* (10) *for* (9) *exists when and only when the following condition holds*:

$$\frac{\widetilde{g}_0}{\widetilde{\beta}} > 1 \quad \textit{and} \quad \widetilde{\gamma} \leqslant \widetilde{\gamma}_c := \frac{1}{\alpha_c(1 + \alpha_c)}, \tag{12}$$

where α_c *is the unique positive root of the following equation*:

$$\frac{\widetilde{g}_0}{\widetilde{\beta}} = \left(1 + \frac{1}{\alpha_c}\right) \mathrm{e}^{1/\alpha_c}. \tag{13}$$

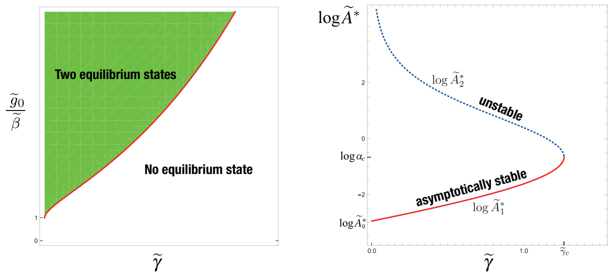

Figure 3. Parameter region for the existence of equilibrium states given by Lemmas 3 and 5 and the bifurcation diagram of $\widetilde{A}^*$ in terms of $\widetilde{\gamma}$. The bifurcation diagram is numerically drawn for $\widetilde{g}_0/\widetilde{\beta} = 20.0$ when $\alpha_c = 0.519996$ and $\widetilde{\gamma}_c = 1.26519$ defined by (12) and (13). There is no equilibrium state if $\widetilde{\gamma} > \widetilde{\gamma}_c$. For the detail, see the main text.

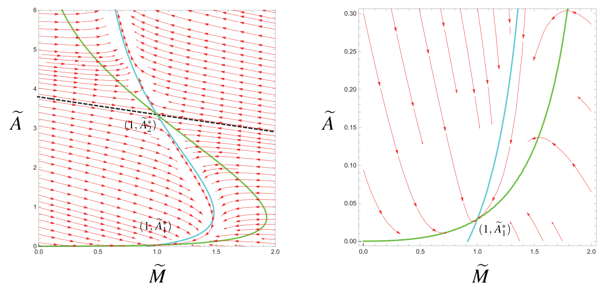

Figure 4. Numerically drawn vector flows, nullclines, and separatrix for (9) when there exist two equilibrium states $(\widetilde{M}^*, \widetilde{A}^*) = (1, \widetilde{A}_1^*)$ and $(1, \widetilde{A}_2^*)$ with parameter values $\widetilde{\beta} = 0.1$, $\widetilde{g}_0 = 10.0$, and $\widetilde{\gamma} = 1.0$. The dashed curve in the left figure shows the separatrix. The right figure is the magnification near $(1, \widetilde{A}_1^*)$ in the left one.

Further, from this lemma we can find the following additional result as proved together in Appendix A (see Fig. 3):

Lemma 5. *If and only if $\widetilde{g}_0/\widetilde{\beta} > 1$ and $\widetilde{\gamma} < \widetilde{\gamma}_c$, there exist two different positive equilibrium states $(1, \widetilde{A}_1^*)$ and $(1, \widetilde{A}_2^*)$ such that*

$$\widetilde{A}_0^* := \frac{1}{\widetilde{g}_0/\widetilde{\beta} - 1} < \widetilde{A}_1^* < \alpha_c < \alpha < \widetilde{A}_2^*. \tag{14}$$

Then, immediately from this lemma and Lemma 2 with $\Phi_{\mathrm{A}}^* = \widetilde{\beta}(\widetilde{\gamma}\widetilde{A}^{*2} + \widetilde{\gamma}\widetilde{A}^* - 1)/\widetilde{A}^*$ for (9) and (10) as shown also in Appendix A, we can obtain the following result about the stability of positive equilibrium state (see Figs. 3 and 4):

Lemma 6. *When there exist two different positive equilibrium states $(1, \widetilde{A}_1^*)$ and $(1, \widetilde{A}_2^*)$ defined in Lemma 5, the equilibrium state $(1, \widetilde{A}_1^*)$ is locally asymptotically stable, while $(1, \widetilde{A}_2^*)$ is unstable.*

As shown by Lemma 3 and seen in Fig. 3, there is no equilibrium state when condition (11) or, equivalently, (12) is unsatisfied. In such case, we have the following behavior of system (9), as numerically demonstrated by Fig. 5(b):

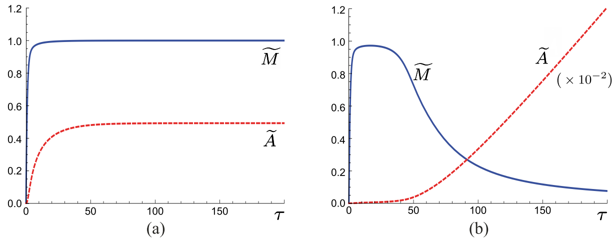

Figure 5. Numerically obtained temporal variations of $(\widetilde{M}, \widetilde{A})$ by (9): (a) $\widetilde{\gamma} = 0.8$; (b) $\widetilde{\gamma} = 1.0$. Commonly, $(\widetilde{M}(0), \widetilde{A}(0)) = (0.0, 0.0)$, $\widetilde{\beta} = 0.1$, $\widetilde{g}_0 = 1.0$, $\widetilde{\gamma}_c = 0.832529$ (defined by (11) and (13); numerically estimated).

Lemma 7. *When condition* (11) *or, equivalently,* (12) *is unsatisfied, we have* $\widetilde{M} \to 0$ *and* $\widetilde{A} \to \infty$ *as* $\tau \to \infty$ *for any nonnegative initial condition.*

The proof of this lemma is given in Appendix B.

Consequently, we have the following theorem about model (9) from these arguments (Appendix C):

Theorem 1. *For model* (9) *without any treatment for the addictive gamer, when the initial population size of addictive gamer is sufficiently small, the population size of addictive gamer can be bounded at a low level if and only if the following condition is satisfied*:

$$\gamma \frac{g_0}{\beta} \geqslant \left(-\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{1}{\gamma M^*}}\right)^{-2} \exp\left[\left(-\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{1}{\gamma M^*}}\right)^{-1}\right], \qquad (15)$$

where $M^* := \lambda/\mu_{\mathrm{M}}$. *Otherwise, it increases unboundedly.*

Mathematically, even when the positive equilibrium state exists, the population size $\widetilde{A}$ can increase unboundedly (i.e., diverge) if the initial condition $(\widetilde{M}(0), \widetilde{A}(0))$ is in the region beyond the separatrix shown in Fig. 4. However, such initial condition requires a sufficiently large value of $\widetilde{A}(0)$, so that it is unreasonable from the meaning of our model because such a value cannot be realized without passing the state with its small value. For this reason, we can give the above conclusive theorem about the possibility to bound the population size of addictive gamer.

4.2 Model with the treatment state

As for model (8) with the treatment state, the equilibrium state $(\widetilde{M}^*, \widetilde{A}^*, \widetilde{R}^*)$ satisfies the following equations with $\widetilde{G}^* := \widetilde{M}^* + \widetilde{A}^*$:

$$\widetilde{M}^* = 1 - \frac{\widetilde{\sigma}\,\widetilde{\mu}}{\widetilde{\rho} + \widetilde{\mu}}\,\widetilde{A}^*, \quad \text{that is,} \quad \widetilde{G}^* = \left(1 - \frac{\widetilde{\sigma}\,\widetilde{\mu}}{\widetilde{\rho} + \widetilde{\mu}}\right)\widetilde{A}^* + 1, \qquad (16_1)$$

$$\widetilde{A}^* = \frac{\widetilde{G}^*}{\widetilde{G}^* + \widetilde{\sigma}/\widetilde{\beta} + (\widetilde{g}_0/\widetilde{\beta})\mathrm{e}^{-\widetilde{\gamma}\widetilde{G}^*}}\,\widetilde{G}^*, \qquad (16_2)$$

$$\widetilde{R}^* = \frac{\widetilde{\sigma}}{\widetilde{\rho} + \widetilde{\mu}}\,\widetilde{A}^*. \qquad (16_3)$$

We can prove the following lemma about the existence (Appendix D):

Lemma 8. *There necessarily exists at least one positive equilibrium state* $(\widetilde{M}^*, \widetilde{A}^*, \widetilde{R}^*)$ *for* (8).

This result shows a prominent difference from the case without the effect of the treatment operation in which there is a parameter region such that there is no equilibrium state.

Further, for the number of equilibrium states, we can prove the following lemmas (Appendix E):

Lemma 9. *There exists only one positive equilibrium state* $(\widetilde{M}^*, \widetilde{A}^*, \widetilde{R}^*)$ *for* (8) *when one of the following four conditions is satisfied*:

$$\frac{\widetilde{\sigma}\widetilde{\mu}}{\widetilde{\rho}+\widetilde{\mu}} \geqslant \min\left\{1, \frac{\widetilde{g}_0}{\widetilde{\beta}}\widetilde{\gamma}\mathrm{e}^{-\widetilde{\gamma}}\right\}, \tag{17_1}$$

$$Q(x_i) \leqslant \frac{\widetilde{\sigma}\widetilde{\mu}}{\widetilde{\rho}+\widetilde{\mu}} < \min\left\{1, \frac{\widetilde{g}_0}{\widetilde{\beta}}\widetilde{\gamma}\mathrm{e}^{-\widetilde{\gamma}}\right\}, \tag{17_2}$$

$$\frac{\widetilde{\sigma}\widetilde{\mu}}{\widetilde{\rho}+\widetilde{\mu}} < \min\left\{1, \frac{\widetilde{g}_0}{\widetilde{\beta}}\widetilde{\gamma}\mathrm{e}^{-\widetilde{\gamma}}, Q(0), Q(x_i)\right\}, \tag{17_3}$$

where x_i *is the unique positive value satisfying that*

$$\frac{\widetilde{g}_0}{\widetilde{\beta}}\frac{\widetilde{\gamma}^2}{2}\left(\frac{2}{\widetilde{\gamma}} - x_i\right)\mathrm{e}^{-\widetilde{\gamma}(x_i+1)} = \frac{\widetilde{\sigma}\widetilde{\mu}}{\widetilde{\rho}+\widetilde{\mu}},$$

which is defined only unless condition (17_1) *holds, and*

$$Q(x) := \frac{\widetilde{\beta}}{2\widetilde{\beta}(x+1)+1+\widetilde{\rho}/\widetilde{\mu}}\left\{1 - \frac{\widetilde{g}_0}{\widetilde{\beta}}(1-\widetilde{\gamma}x)\mathrm{e}^{-\widetilde{\gamma}(x+1)}\right\}.$$

This result shows the sufficient condition that there is only one positive equilibrium state $(\widetilde{M}^*, \widetilde{A}^*, \widetilde{R}^*)$ for (8). By the arguments to prove the above lemma and subsequently obtained results described in Appendix E, the following result gives the necessary condition that there are three positive equilibrium states for (8):

Lemma 10. *There is a parameter region such that system* (8) *has three positive equilibrium states only if the following condition is satisfied*:

$$Q(0) < \frac{\widetilde{\sigma}\widetilde{\mu}}{\widetilde{\rho}+\widetilde{\mu}} < \min\left\{1, \frac{\widetilde{g}_0}{\widetilde{\beta}}\widetilde{\gamma}\mathrm{e}^{-\widetilde{\gamma}}, Q(x_i)\right\}. \tag{18}$$

As seen by the numerical result in Fig. 6, condition (18) is actually necessary for the existence of three positive equilibrium states, and such parameter region is truly included inside of the region given by (18).

Consequently, taking account of numerical results for (8), we present now the following result about the stability of equilibrium state for (8) (see Fig. 7):

Lemma 11. *For a certain parameter region, system* (8) *has three equilibrium states of which one is unstable and the other two are locally asymptotically stable. For the other parameter region, system* (8) *has only one positive equilibrium, which is globally asymptotically stable.*

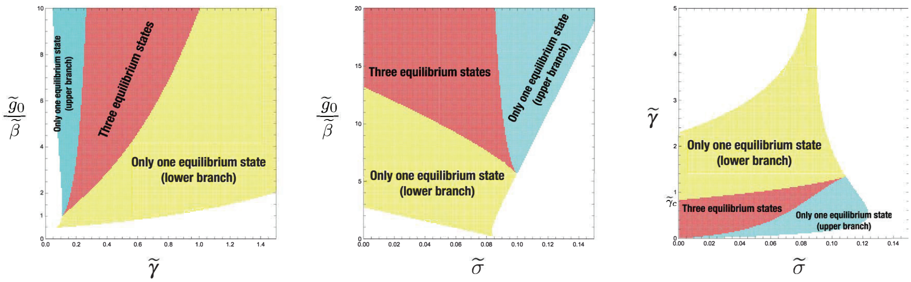

Figure 6. Parameter region about the existence of equilibrium states. The painted region is numerically obtained by (18) of Lemma 10, while the region where three equilibrium states exist is numerically identified (refer the last part of Appendix E). For any region out of the region of three equilibrium states, including that without being painted, there exists only one equilibrium state. In these numerical calculations, $\widetilde{\sigma} = 0.05$ for $(\widetilde{\gamma}, \widetilde{g}_0/\widetilde{\beta})$-space; $\widetilde{\gamma} = 1.0$ for $(\widetilde{\sigma}, \widetilde{g}_0/\widetilde{\beta})$-space; $\widetilde{g}_0/\widetilde{\beta} = 10.0$ for $(\widetilde{\sigma}, \widetilde{\gamma})$-space. Commonly, $\widetilde{\beta} = 0.1$ and $\widetilde{\rho}/\widetilde{\mu} = 1.0$. For the detail, see the main text.

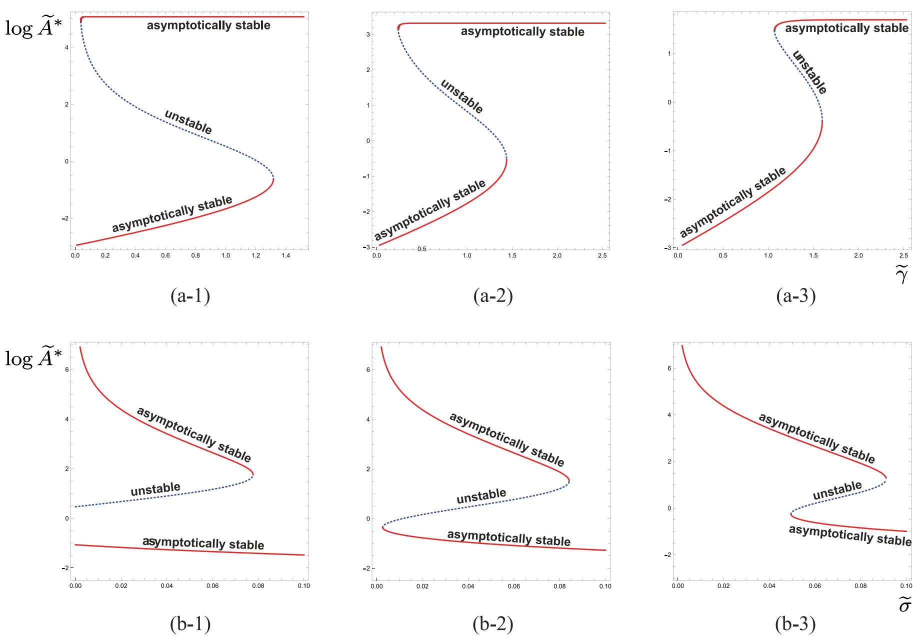

Figure 7. Numerically drawn bifurcation diagram of $\log \widetilde{A}^*$ in terms of $\widetilde{\gamma}$ and $\widetilde{\sigma}$. (a-1) $\widetilde{\sigma} = 0.01$; (a-2) $\widetilde{\sigma} = 0.03$; (a-3) $\widetilde{\sigma} = 0.05$ with $\widetilde{\beta} = 0.05$, $\alpha_c = 0.519996$, and $\widetilde{\gamma}_c = 1.26519$ (defined by (11) and (13)). (b-1) $\widetilde{\gamma} = 0.7 < \widetilde{\gamma}_c$; (b-2) $\widetilde{\gamma} = 0.84$; (b-3) $\widetilde{\gamma} = 1.0 > \widetilde{\gamma}_c$ with $\widetilde{\beta} = 0.1$, $\alpha_c = 0.704641$, and $\widetilde{\gamma}_c = 0.832529$. Commonly, $\widetilde{g}_0 = 1.0$ and $\widetilde{\rho}/\widetilde{\mu} = 1.0$.

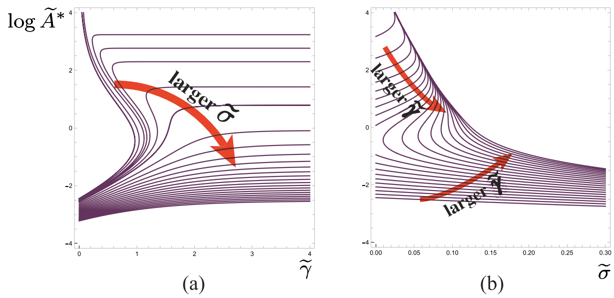

Figure 8. Parameter dependence of the bifurcation curve of $\log \widetilde{A}^*$. Refer to Figs. 3, 6, and 7. Numerically drawn with $\widetilde{\beta} = 0.08$, $\widetilde{g}_0 = 1.0$, $\widetilde{\rho}/\widetilde{\mu} = 1.0$; (a) $\widetilde{\sigma} = 0.0{-}1.0$; (b) $\widetilde{\gamma} = 0.0{-}2.0$. For these numerical calculations, $\alpha_c = 0.633507$ and $\widetilde{\gamma}_c = 0.966335$ (defined by (11) and (13)).

Although the mathematical proof for the stability of each equilibrium state is the open problem, every numerical calculation for (8) supports this result, which moreover appears consistent with the result for model (9) without the treatment state as shown in the following part. Figure 7 shows the numerically drawn bifurcation diagram of the equilibrium states for (8). As indicated by Lemma 11 and the bifurcation diagram in Fig. 7, there is a parameter region with which a bistable situation occurs.

Further, from numerical results and those lemmas on the existence and the stability of equilibrium state we can get the following important result:

Theorem 2. *For model* (6)*, the effect of the treatment operation results in the necessary existence of an asymptotically stable equilibrium state, which the system eventually approaches.*

This result indicates that the unbounded increase of the population size of addictive gamer never occurs with an effective treatment operation, whereas it can occur without any treatment operation as shown in the previous part.

As expected from Figs. 6 and 7, our numerical results clearly indicate the continuity of the structure in the stability of equilibrium states according to systems (8) and (9). Actually, as shown by numerical calculations given in Fig. 8, the curve to determine the shape of branches in the bifurcation diagram in terms of parameter $\widetilde{\gamma}$ or $\widetilde{\sigma}$ depends on the parameter continuously. Taking account of those numerical results given in Fig. 7, we can imply from Figs. 6 and 8 that there exists a critical value for $\widetilde{\sigma}$ beyond which the bistable situation does not appear. In other words, we can get the following result:

Theorem 3. *For model* (8) *with sufficiently large value of $\widetilde{\sigma}$, the equilibrium value of the population size of addictive gamer depends continuously on the parameters $\widetilde{g}_0/\widetilde{\beta}$ and $\widetilde{\gamma}$.*

This result means that, under a sufficiently effective treatment operation (with sufficiently large value of σ), the change in the effect from the social interaction with the other gamers on web is reflected continuously to the population size of addictive gamer. If the treatment operation is poorly effective (with rather small value of σ), the change in the effect from the social interaction would cause a drastic change in the population size of addictive gamer. Especially, it is likely that when the effect from the social interaction through gaming becomes stronger (with larger value of β or γ), the population size of addictive gamer would show an explosive increase.

5 Concluding remarks

As already seen from Figs. 6 and 7(b) for a small value of $\widetilde{\sigma}$, the treatment operation with a low efficiency may not suppress the explosive increase of the addictive gamers. Figure 9(a) numerically demonstrates such a case. Only when the treatment operation has a sufficiently high efficiency, it would be successful to suppress the population size of addictive gamer at a certain level as demonstrated by Fig. 9(b). Especially in the former case of the failure to suppress the increase of the addictive gamers, the addictive gamers under treatment are increasing in time as well. This implies that such a situation would potentially cause the breakdown of the treatment operation.

The effectiveness of a treatment operation strongly depends on the inherent *social* reinforcements of online gaming. As indicated by Figs. 6 and 7(b) about the $\widetilde{\gamma}$-dependence of the equilibrium states, the stronger social reinforcement makes the effective treatment operation harder because it requires the larger value of $\widetilde{\sigma}$, that is, the higher efficiency of the treatment operation. Although such effect of the inherent social reinforcements of online gaming were indicated by many previous works [4, 8, 11, 17, 22, 28], no mathematical model on the population dynamics for the addictive gamer involved the strength of social reinforcement to discuss its contribution to the dynamical consequence. From our analysis on a simple model involving its strength it is implied that there could exist a case with a hysteresis nature of the population dynamics, that is, with a bistable situation. It indicates a possibility of a drastic increase of addictive gamers, which would be hardly able to be return to the previous lower level once it occurs. Previous mathematical models extendedly applied those in epidemic dynamics in [9, 16, 29] could not indicate such hysteresis nature of the population dynamics for the gaming addiction. Therefore, the existence of such a hysteresis nature would essentially depend on the nonlinear relation of social reinforcement to the population of addictive gamers.

Because of the possible existence of a bistable situation shown in Fig. 7, our model demonstrates that the conclusion of a treatment operation could significantly depend on when the treatment operation starts. A numerical example is given in Fig. 10. As we have seen, the population size of addictive gamer alternatively approaches a certain saturating level or unboundedly grows when no treatment operation is conducted. The latter is the serious case to be resolved, so that a treatment operation has to be conducted. In such a case, if the treatment operation starts too late, the population size of addictive gamer may have been beyond a critical level so that it continues to increase and approaches a certain high saturating level, which corresponds to the upper branch of the bistable equilibrium states (see Fig. 10(b)). In contrast, if the treatment operation starts sufficiently early, the population size of addictive gamer may be successfully let to a certain low level (see Fig. 10(a)). Therefore, the effectiveness of a treatment operation is determined not only by the effort to conduct it but also by the timing of its start.

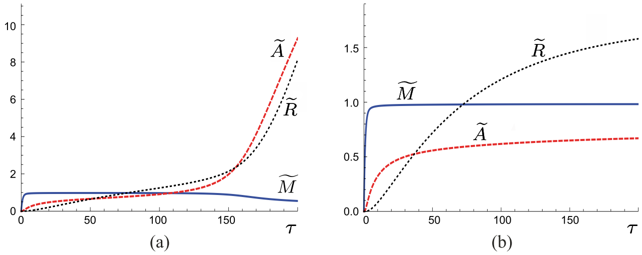

Figure 9. Numerically obtained temporal variations of $(\widetilde{M}, \widetilde{A}, \widetilde{R})$ by (8): (a) $\widetilde{\sigma} = 0.04$; (b) $\widetilde{\sigma} = 0.05$. Commonly, $(\widetilde{M}(0), \widetilde{A}(0), \widetilde{R}(0)) = (0.0, 0.0, 0.0)$, $\widetilde{\beta} = 0.1$, $\widetilde{g}_0 = 1.0$, $\widetilde{\gamma} = 1.0$, $\widetilde{\rho} = 0.01$, $\widetilde{\mu} = 0.01$. Refer to Fig. 7(b-3).

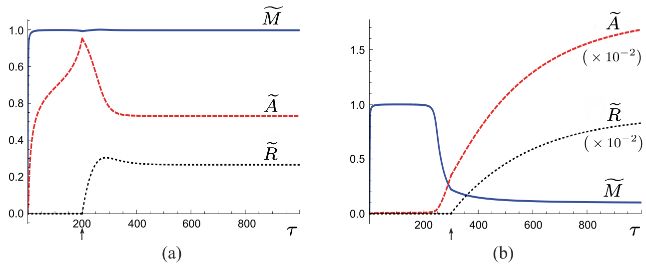

Figure 10. Numerically obtained temporal variations of $(\widetilde{M}, \widetilde{A}, \widetilde{R})$ by (8) with a moment $\tau = \tau_1$ before which there is no treatment ($\widetilde{\sigma} = 0$) and at which the treatment starts with $\widetilde{\sigma} = 0.01$: (a) $\tau_1 = 200$; (b) $\tau_1 = 300$. Commonly, $(\widetilde{M}(0), \widetilde{A}(0), \widetilde{R}(0)) = (0.0, 0.0, 0.0)$, $\widetilde{\beta} = 0.1$, $\widetilde{g}_0 = 1.0$, $\widetilde{\gamma} = 0.84$, $\widetilde{\rho} = 0.01$, $\widetilde{\mu} = 0.01$, $\alpha_c = 0.704641$, $\widetilde{\gamma}_c = 0.832529$. Refer to Fig. 7(b-2).

Consequently, in order to suppress an explosive increase of addictive gamers, it is necessary to conduct a treatment operation not only high efficient but also executed at sufficiently early stage of its increase. Otherwise, the population size of addictive gamer would show an explosive increase to reach a level beyond the capacity of such a treatment. While the interdisciplinary and crossover researches are still necessary on the IGD as mentioned in a variety of recent works [11,12,14,18,24–27], it would be important to start a certain practical operation to make its treatment as early as possible for the prospective suppression of its outbreak in future.

Although we did not intend to propose any control or treatment policy for the IGD in a community, we tried to present a basic mathematical model and its mathematical nature, which would be useful to advance some future theoretical/mathematical researches on the IGD, especially from the epidemiologically interdisciplinary viewpoint. We expect that like the development of mathematical modeling and model for the epidemic dynamics of transmissible disease, the mathematical/theoretical works on problems about the IGD will be necessarily developed soon or later.

Acknowledgment. The author appreciates the valuable comments by anonymous reviewers and the editor to improve and complete the manuscript.

Appendix A: Proof of Lemmas 3–6

To get the condition for the existence of a positive equilibrium state $(\widetilde{M}^*, \widetilde{A}^*)$, it is necessary and sufficient to consider the existence of a positive $\widetilde{A}^*$, which satisfies the second equation of (10). The value $\widetilde{A}^*$ is given by the positive root of the equation $\widetilde{g}_0/\widetilde{\beta} = \varphi_1(x) := (1 + 1/x)\,\mathrm{e}^{\widetilde{\gamma}(1+x)}$ in terms of x if it exists. It can be easily found that the graph of $\varphi_1(x)$ is convex with the unique extremal minimum at $x = \alpha$, where α is defined by (11), that is, the unique positive root of quadratic equation $h(x) := \widetilde{\gamma}x^2 + \widetilde{\gamma}x - 1 = 0$, and $\lim_{x\to 0+} \varphi_1(x) = \lim_{x\to\infty} \varphi_1(x) = \infty$ as shown by Fig. 11. Note that $\varphi_1'(x) = h(x)\,\mathrm{e}^{\widetilde{\gamma}(1+x)}/x^2$. Hence, if and only if the first inequality of (11) is satisfied, a positive $\widetilde{A}^*$ such as to satisfy the second equation of (10) exists. This argumentation proves Lemma 3.

On the other hand, the second equation of (10) can be rewritten as follows:

$$\widetilde{\gamma} = \varphi_2(\widetilde{A}^*) := \frac{1}{1 + \widetilde{A}^*} \log \frac{\widetilde{g}_0/\widetilde{\beta}}{1 + 1/\widetilde{A}^*}.$$

Therefore, to get the condition for the existence of a positive equilibrium state $(\widetilde{M}^*, \widetilde{A}^*)$, it is necessary and sufficient to consider the existence of a positive root for the equation $\widetilde{\gamma} = \varphi_2(x)$ in terms of x. At first, it is easily see that, if $\widetilde{g}_0/\widetilde{\beta} \leqslant 1$, $\varphi_2(x) < 0$ for any $x > 0$, so that there is no positive root for the equation $\widetilde{\gamma} = \varphi_2(x)$ since $\widetilde{\gamma} > 0$. Thus, for the existence of a positive root, it is necessary that $\widetilde{g}_0/\widetilde{\beta} > 1$. Then, under the condition that $\widetilde{g}_0/\widetilde{\beta} > 1$, it can be easily found that the graph of $\varphi_2(x)$ is unimodal with the unique extremal maximum at $x = \alpha_c$, where α_c is the unique positive root of (13), and $\lim_{x\to 0+} \varphi_2(x) = -\infty$, $\lim_{x\to\infty} \varphi_2(x) = 0$ as shown by Fig. 11. Note that

$$\varphi_2'(x) = \frac{1}{(1+x)^2}\left(\frac{1}{x} - \log \frac{\widetilde{g}_0/\widetilde{\beta}}{1 + 1/x}\right).$$

Consequently, if and only if the inequality $\widetilde{\gamma} \leqslant \widetilde{\gamma}_c := \varphi_2(\alpha_c)$ is satisfied, the equation $\widetilde{\gamma} = \varphi_2(x)$ in terms of x has a positive root, that is, a positive $\widetilde{A}^*$ such as to satisfy the second equation of (10) exists. From (13) we can easily derive the following simpler expression of the extremal maximum $\widetilde{\gamma}_c$ of φ_2, too:

$$\widetilde{\gamma}_c = \varphi_2(\alpha_c) = \frac{1}{1 + \alpha_c} \log \frac{\widetilde{g}_0/\widetilde{\beta}}{1 + 1/\alpha_c} = \frac{1}{\alpha_c(1 + \alpha_c)}.$$

This argumentation proves Lemma 4.

Further, from the above arguments for Lemma 4 we note that there are different two positive roots for the equation $\widetilde{\gamma} = \varphi_2(x)$ in terms of x if and only if $\widetilde{g}_0/\widetilde{\beta} > 1$ and $\widetilde{\gamma} < \widetilde{\gamma}_c$. As seen from the graph of φ_2 given in Fig. 11, those two positive roots $\widetilde{A}_1^*$ and $\widetilde{A}_2^*$ satisfy the inequality $\widetilde{A}_0^* < \widetilde{A}_1^* < \alpha_c < \widetilde{A}_2^*$, where $\widetilde{A}_0^*$ is the unique positive root for the equation $\varphi_2(x) = 0$, and can be easily obtained as given in (14). Similarly, as seen

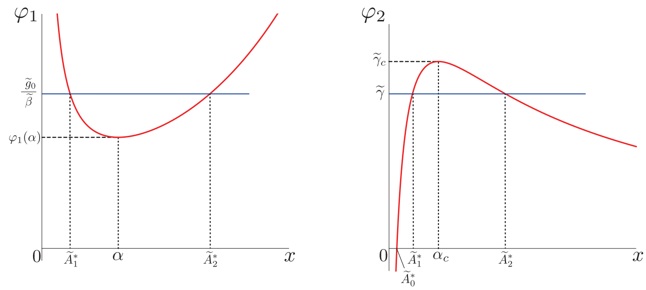

Figure 11. Graphs of $\varphi_1(x)$ and $\varphi_2(x)$ used in the argument about the existence of the equilibrium state. For the detail, see Appendix A.

from the graph of φ_1 given in Fig. 11, the inequality $\widetilde{A}_1^* < \alpha < \widetilde{A}_2^*$ must be satisfied. Besides, from the condition that $\widetilde{\gamma} < \widetilde{\gamma}_c$ we have $\widetilde{\gamma}\alpha_c^2 + \widetilde{\gamma}\alpha_c - 1 < 0$, that is,

$$\alpha_c < -\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{1}{\widetilde{\gamma}}} = \alpha. \tag{A.1}$$

These arguments is the proof for Lemma 5.

Next, we will apply Lemma 2 for those two positive roots $\widetilde{A}_1^*$ and $\widetilde{A}_2^*$ to investigate their local stability. For the equilibrium state (10) of the 2-dimensional system (9), we have $\Phi_{\mathrm{A}}^* = \widetilde{\beta}(\widetilde{\gamma}\widetilde{A}^{*2} + \widetilde{\gamma}\widetilde{A}^* - 1)/\widetilde{A}^*$ from the definition given in (5), and we can rewrite it to $\Phi_{\mathrm{A}}^* = \widetilde{\beta}h(\widetilde{A}^*)/\widetilde{A}^*$ with the function h defined in the above arguments for Lemma 3. The function $h(x)$ satisfies that $h(x) < 0$ for $0 < x < \alpha$ and $h(x) > 0$ for $x > \alpha$. Therefore, since $0 < \widetilde{A}_1^* < \alpha < \widetilde{A}_2^*$ from Lemma 5, we find that $h(\widetilde{A}_1^*) < 0$ and $h(\widetilde{A}_2^*) > 0$. As a result, we have $\Phi_{\mathrm{A}}^* < 0$ for $A^* = \widetilde{A}_1^*$ and $\Phi_{\mathrm{A}}^* > 0$ for $A^* = \widetilde{A}_2^*$. Consequently, from Lemma 2 we can obtain Lemma 6.

Appendix B: Proof of Lemma 7

Lemma 7 can be proved as follows: At first, let us consider the case that $\widetilde{\beta}/\widetilde{g}_0 \geqslant 1$ when condition (11) is unsatisfied. Then we find that

$$\begin{aligned}\frac{\mathrm{d}\widetilde{M}}{\mathrm{d}\tau}\bigg|_{\widetilde{M}\geqslant 1} &\leqslant \widetilde{g}_0(1+\widetilde{A})\biggl[-\frac{\widetilde{\beta}}{\widetilde{g}_0} + \frac{\widetilde{A}}{1+\widetilde{A}}\,\mathrm{e}^{-\widetilde{\gamma}\,(1+\widetilde{A})}\biggr]\\ &\leqslant \widetilde{g}_0(1+\widetilde{A})\biggl[-1 + \frac{\widetilde{A}}{1+\widetilde{A}}\,\mathrm{e}^{-\widetilde{\gamma}(1+\widetilde{A})}\biggr] < 0\end{aligned}$$

for any positive $\widetilde{A}$. Next, let us consider the case that $\widetilde{\gamma} > \widetilde{\gamma}_c$. In this case, we can note that the inequality

$$\frac{\widetilde{\beta}}{\widetilde{g}_0} > \zeta_c := \frac{\alpha_c}{1+\alpha_c}\,\mathrm{e}^{-\gamma(1+\alpha_c)}$$

holds. Further, it can be easily proved that the inequality $(x/(1+x))\mathrm{e}^{-\widetilde{\gamma}(1+x)} \leqslant \zeta_c$ holds for any $x > 0$. Thus, we have

$$\frac{\mathrm{d}\widetilde{M}}{\mathrm{d}\tau}\bigg|_{\widetilde{M}\geqslant 1} \leqslant \widetilde{g}_0(1+\widetilde{A})\biggl[-\frac{\widetilde{\beta}}{\widetilde{g}_0} + \frac{\widetilde{A}}{1+\widetilde{A}}\,\mathrm{e}^{-\widetilde{\gamma}(1+\widetilde{A})}\biggr]$$
$$< \widetilde{g}_0(1+\widetilde{A})\biggl[-\zeta_c + \frac{\widetilde{A}}{1+\widetilde{A}}\,\mathrm{e}^{-\widetilde{\gamma}(1+\widetilde{A})}\biggr] \leqslant 0$$

for any positive $\widetilde{A}$. As a consequence, we find that, when condition (11) is unsatisfied, $\mathrm{d}\widetilde{M}/\mathrm{d}\tau|_{\widetilde{M}\geqslant 1} < 0$ for any $\tau > 0$. This means that $\widetilde{M}$ is monotonically decreasing in terms of τ as long as $\widetilde{M} \geqslant 1$.

If $\widetilde{M} \to 1$ as $\tau \to \infty$, this means that M converges to the equilibrium value 1 as $\tau \to \infty$. That is, system (9) converges to the equilibrium state $(1, \overline{A}_1^*)$ or $(1, \overline{A}_2^*)$. This is contradictory from Lemma 3 because we are considering here the case when condition (11) is unsatisfied. Hence, there necessarily exists a finite time $\tau_1 \geqslant 0$ such that the value of $\widetilde{M}$ is less than 1 for any $\tau > \tau_1$. Once $\widetilde{M}$ is less than 1, $\widetilde{M}$ cannot reach or become beyond 1, that is, $\widetilde{M}$ is bounded by 1 because $\mathrm{d}\widetilde{M}/\mathrm{d}\tau|_{\widetilde{M}\geqslant 1} < 0$ for any $\tau > 0$. Further, since

$$\frac{\mathrm{d}\widetilde{M}}{\mathrm{d}\tau}\bigg|_{\widetilde{M}=0,\,\widetilde{A}<\infty} = 1 + \widetilde{g}_0\,\mathrm{e}^{-\widetilde{\gamma}\widetilde{A}}\widetilde{A} > 0,$$

we consequently have $\widetilde{M}(\tau) \in (0,1)$ for any $\tau > 0$ with any nonnegative initial condition.

From (9), on the other hand, the temporal variation of $\widetilde{G} := \widetilde{M} + \widetilde{A}$ is governed by the following ordinary differential equation:

$$\frac{\mathrm{d}\widetilde{G}}{\mathrm{d}\tau} = 1 - \widetilde{M}. \tag{B.1}$$

From the above argument about the temporal variation of $\widetilde{M}$ we find that when condition (11) is unsatisfied, there exists a finite time $\tau_1 \geqslant 0$ such that the right side of (B.1) is always positive for any $\tau > \tau_1$ with any nonnegative initial condition. This means that $\widetilde{G}$ is monotonically increasing for $\tau > \tau_1$, while $\widetilde{M}$ is bounded by 1.

Suppose now that $\widetilde{G}$ converges to a certain finite value. Then $\widetilde{M}$ must converge to 1 since $\mathrm{d}\widetilde{G}/\mathrm{d}\tau \to 0$ as $t \to \infty$. This is contradictory to the previous arguments showing that $\widetilde{M}(\tau) < 1$ for any $\tau > \tau_1$. Therefore, we find that $\widetilde{G}$ is monotonically increasing unboundedly in terms of $\tau > \tau_1$, that is, $\widetilde{G}$ becomes positively infinite as $\tau \to \infty$. Subsequently, this means that $\widetilde{A} = \widetilde{G} - \widetilde{M}$ becomes positively infinite as $\tau \to \infty$ because $\widetilde{M}$ is bounded from upper as shown in the above arguments.

When $\widetilde{A} \to \infty$ as $\tau \to \infty$, it is impossible that $\widetilde{M}$ converges to a finite positive value in $(0,1]$ as $\tau \to \infty$. This is because $\mathrm{d}\widetilde{M}/\mathrm{d}\tau \to -\infty$ as $\tau \to \infty$ from (9) in such a case, which is contradictory to the convergence of $\widetilde{M}$. Therefore, we conclude that $\widetilde{M} \to 0$ when $\widetilde{A} \to \infty$ as $\tau \to \infty$. Consequently, Lemma 7 has been proved.

Appendix C: Proof of Theorem 1

Let us consider the function $\varphi_3(x) := (1 + 1/x)\mathrm{e}^{1/x}$, which appears in (13) of Lemma 4. It can be easily found that $\varphi_3(x)$ is monotonically decreasing for $x > 0$ with $\lim_{x\to 0+} \varphi_3(x) = \infty$, $\lim_{x\to\infty} \varphi_3(x) = 1$, and $\varphi_3(x) > 1$ for any $x > 0$.

The condition that $\widetilde{\gamma} \leqslant \widetilde{\gamma}_c$ in (12) is equivalent to that $\alpha_c \leqslant \alpha$ as shown by (A.1) in Appendix A. Therefore, with the function φ_3, we have $\varphi_3(\alpha_c) \geqslant \varphi_3(\alpha)$, which can be regarded as equivalent to $\alpha_c \leqslant \alpha$, that is, to $\widetilde{\gamma} \leqslant \widetilde{\gamma}_c$. Then, since $\varphi_3(\alpha_c) = \widetilde{g}_0/\widetilde{\beta}$ from (13), we finally derive the condition that $\widetilde{g}_0/\widetilde{\beta} \geqslant \varphi_3(\alpha)$ is equivalent to that $\widetilde{\gamma} \leqslant \widetilde{\gamma}_c$ and, subsequently, to (12) since necessarily $\varphi_3(\alpha) > 1$. From Lemma 4 this means that, if and only if the condition $\widetilde{g}_0/\widetilde{\beta} \geqslant \varphi_3(\alpha)$ is satisfied, the positive equilibrium state $(\widetilde{M}^*, \widetilde{A}^*)$ given by (10) for (9) exists. With the definition of α in (11) and the parameter transformation (7), the inequality $\widetilde{g}_0/\widetilde{\beta} \geqslant \varphi_3(\alpha)$ can be written as (15).

Therefore, from Lemma 6 and the vector direction in the phase plane of $(\widetilde{M}, \widetilde{A})$, as shown in Fig. 4, $\widetilde{A}$ asymptotically converges to $\widetilde{A}_1^*$ as $t \to \infty$ from sufficiently small initial value $\widetilde{A}(0)$ if and only if condition (15) is satisfied, so that the equilibrium states $(1, \widetilde{A}_1^*)$ and $(1, \widetilde{A}_2^*)$ exist.

Appendix D: Proof of Lemma 8

From (16_1), for the existence of a positive equilibrium, it is necessary that $\widetilde{A}^* < (\widetilde{\rho} + \widetilde{\mu})/(\widetilde{\sigma}\widetilde{\mu})$. When $\widetilde{s} := \widetilde{\sigma}\widetilde{\mu}/(\widetilde{\rho} + \widetilde{\mu}) \neq 1$, equation ($16_1$) can be rewritten by

$$\widetilde{A}^* = \frac{\widetilde{G}^* - 1}{1 - \widetilde{s}}. \tag{D.1}$$

First, let us consider the case of $\widetilde{s} > 1$. In this case, from (D.1) it is necessary that $\widetilde{G}^* < 1$ for the existence of the positive equilibrium state $(\widetilde{M}^*, \widetilde{A}^*, \widetilde{R}^*)$. The right side of (D.1) is monotonically decreasing in terms of $\widetilde{G}^*$, while the right side of (16_2) is monotonically increasing in terms of $\widetilde{G}^*$ and always positive less than $\widetilde{G}^*$. Besides, substituting $1/\widetilde{s}$ for $\overline{G}^*$, the right side of (16_2) is smaller than that of (D.1) because then the right side of (D.1) becomes $1/\widetilde{s}$, while the right side of (16_2) is less than $1/\widetilde{s}$. Further, the right side of (16_2) is positive for $\widetilde{G}^* = 1$, while the right side of (D.1) is 0 for $\widetilde{G}^* = 1$. Therefore, it is proved that equations (16_2) and (D.1) determine the unique solution $(\widetilde{A}^*, \widetilde{G}^*)$ such that $0 < \widetilde{A}^* < 1/\widetilde{s} < \widetilde{G}^* < 1$. This result means that the unique positive equilibrium state $(\widetilde{M}^*, \widetilde{A}^*, \widetilde{R}^*)$ exists when $\widetilde{s} > 1$.

Next, let us consider the case of $\widetilde{s} = 1$. Then we have $\widetilde{M}^* = 1 - \widetilde{A}^*$ from (16_1), and $\widetilde{R}^* = \widetilde{\sigma}\widetilde{A}^*/(\widetilde{\rho} + \widetilde{\mu})$ from (16_3). From (16_2) we can derive the following unique $\widetilde{A}^*$:

$$A^* = \frac{1}{1 + \widetilde{\sigma}/\widetilde{\beta} + (\widetilde{g}_0/\widetilde{\beta})\mathrm{e}^{-\widetilde{\gamma}}} < 1$$

since now $\widetilde{G}^* = \widetilde{M}^* + \widetilde{A}^* = 1$. So the positive equilibrium state $(\widetilde{M}^*, \widetilde{A}^*, \widetilde{R}^*)$ is uniquely determined in this case. These arguments show that system (8) has only one positive equilibrium state $(\widetilde{M}^*, \widetilde{A}^*, \widetilde{R}^*)$ if $\widetilde{s} \geqslant 1$, that is, if

$$\frac{\widetilde{\sigma}\widetilde{\mu}}{\widetilde{\rho} + \widetilde{\mu}} \geqslant 1. \tag{D.2}$$

When $\widetilde{s} < 1$, the situation is not simple. In this case, from (D.1) it is necessary that $1 < \widetilde{G}^* < 1/\widetilde{s}$ for the existence of the positive equilibrium state $(\widetilde{M}^*, \widetilde{A}^*, \widetilde{R}^*)$. In the following argument, we will investigate the equation $\mathscr{F}(x) = 1$ with $\widetilde{s} < 1$ in terms of $x := \widetilde{G}^* - 1 > 0$, where the function $\mathscr{F}(x)$ is derived from (16_2) and (D.1) defined by

$$\mathscr{F}(x) := \frac{\widetilde{g}_0}{\widetilde{\beta}} x\, \mathrm{e}^{-\widetilde{\gamma}(x+1)} + \widetilde{s}\,x^2 + \left(2\widetilde{s} + \frac{\widetilde{\sigma}}{\widetilde{\beta}} - 1\right)x + \widetilde{s}. \tag{D.3}$$

If and only if the equation $\mathscr{F}(x) = 1$ has a positive root less than $1/\widetilde{s} - 1$, there exists a positive equilibrium state $(\widetilde{M}^*, \widetilde{A}^*, \widetilde{R}^*)$ for (8), which is uniquely determined by the root. Since the function $\mathscr{F}(x)$ is of C^∞-class for any x satisfying that $\mathscr{F}(0) = \widetilde{s} < 1$ and

$$\mathscr{F}\left(\frac{1}{\widetilde{s}} - 1\right) = \left(\frac{1}{\widetilde{s}} - 1\right)\left(\frac{\widetilde{g}_0}{\widetilde{\beta}}\, \mathrm{e}^{-\widetilde{\gamma}/\widetilde{s}} + \frac{\widetilde{\sigma}}{\widetilde{\beta}}\right) + 1 > 1,$$

it is immediately seen that there exists at least one positive root less than $1/\widetilde{s} - 1$ for the equation $\mathscr{F}(x) = 1$. This means that there exists at least one positive equilibrium state $(\widetilde{M}^*, \widetilde{A}^*, \widetilde{R}^*)$ in this case. Therefore, as a consequence from these arguments, we have Lemma 8.

Appendix E: Proof of Lemmas 9 and 10

We have already proved that, if condition (D.2) in Appendix D is satisfied, there exists only one positive equilibrium state $(\widetilde{M}^*, \widetilde{A}^*, \widetilde{R}^*)$ for (8). Hence, in this appendix, we will argue how many positive equilibrium states exist for (8) when

$$\widetilde{s} := \frac{\widetilde{\sigma}\widetilde{\mu}}{\widetilde{\rho} + \widetilde{\mu}} < 1. \tag{E.1}$$

To consider the number of positive equilibrium states for (8), let us consider how many different extrema the function $\mathscr{F}(x)$ defined by (D.3) in Appendix D has when $\widetilde{s} < 1$. This is because the number of positive roots less than $1/\widetilde{s} - 1$ for the equation $\mathscr{F}(x) = 1$ corresponds to the number of positive equilibrium states for (8) as explained in Appendix D.

It can be easily shown that the function $\mathscr{F}(x)$ has *at most two local extrema for* $x > 0$ since $\mathscr{F}(x)$ is of C^∞-class for any x satisfying that $\mathscr{F}(0) = \widetilde{s} < 1$, $\lim_{x\to\infty} \mathscr{F}(x) = \infty$, and that it has at most one inflection point. It is clear that the equation $\mathscr{F}(x) = 1$ has only one positive root when the function $\mathscr{F}(x)$ has no local extremum or only one local extremum for $x > 0$ because $\mathscr{F}(0) = \widetilde{s}$ and $\mathscr{F}(1/\widetilde{s} - 1) > 1$. $\mathscr{F}(x)$ is monotonically increasing in terms of x when it has no local extremum for $x > 0$. When $\mathscr{F}(x)$ has only one local extremum for $x > 0$, it has a value $\mathscr{F}(x_e) < \mathscr{F}(0) = \widetilde{s} < 1$ at the local extremal point $x = x_e$, and then $\mathscr{F}(x)$ is less than $\widetilde{s} < 1$ for $x \in (0, x_0)$ with a certain positive x_0 such that $\mathscr{F}(x_0) = \widetilde{s}$, while it is greater than $\widetilde{s}$ and monotonically increasing in the interval $(x_0, 1/\widetilde{s} - 1)$.

We can easily find that, if $\mathscr{F}''(0) \geqslant 0$, that is, if

$$\widetilde{s} \geqslant \frac{\widetilde{g}_0}{\widetilde{\beta}}\widetilde{\gamma}\mathrm{e}^{-\widetilde{\gamma}}, \tag{E.2}$$

then necessarily $\mathscr{F}''(x) > 0$ for any $x > 0$. Hence, if $\mathscr{F}'(0) \geqslant 0$, then $\mathscr{F}'(x) > 0$ for any $x > 0$, so that $\mathscr{F}(x)$ is monotonically increasing. This means that in this case, there exists only one positive equilibrium state for (8) because $\mathscr{F}(0) = \widetilde{s}$ and $\mathscr{F}(1/\widetilde{s} - 1) > 1$. If $\mathscr{F}'(0) < 0$, $\mathscr{F}(x)$ has only one local minimum at $x = x_e > 0$ with $\mathscr{F}(x_e) < \widetilde{s} < 1$, so that, also in this case, there exists only one positive equilibrium state for (8). As a result, under condition (E.2), there exists only one positive equilibrium state for (8). As there exists only one positive equilibrium state for (8) when condition (D.2) in Appendix D or (E.2) in this appendix under condition (E.1) is satisfied, we consequently obtain condition (17_1) of Lemma 9.

Next, let us consider the case that $\mathscr{F}''(0) < 0$, that is,

$$\widetilde{s} < \frac{\widetilde{g}_0}{\widetilde{\beta}}\widetilde{\gamma}\mathrm{e}^{-\widetilde{\gamma}}. \tag{E.3}$$

Then it can be easily shown that there exists a unique positive x_i such that $\mathscr{F}''(x_i) = 0$. Besides, it is shown that $x_i < 2/\widetilde{\gamma}$ and that $\mathscr{F}''(x) < 0$ for $x < x_i$, and $\mathscr{F}''(x) > 0$ for $x > x_i$. If $\mathscr{F}'(x_i) \geqslant 0$, it can be easily seen that $\mathscr{F}'(x) \geqslant 0$ for $x > 0$, so that there exists only one positive equilibrium state for (8) since $\mathscr{F}(x)$ is monotonically increasing for $x > 0$. From (E.1), (E.3), and these arguments we have condition (17_2) of Lemma 9. Only when $\mathscr{F}''(0) < 0$ and $\mathscr{F}'(x_i) < 0$, we need to investigate some different cases with respect to the number of the extrema for $\mathscr{F}(x)$.

If $\mathscr{F}''(0) < 0$, $\mathscr{F}'(x_i) < 0$, and $\mathscr{F}'(0) \leqslant 0$, we can easily see that there exists a unique local minimal point $x = x_e$ and that $\mathscr{F}(x)$ is less than $\mathscr{F}(0) = \widetilde{s} < 1$ for $x \in (0, x_0)$ with a certain positive x_0 such that $\mathscr{F}(x_0) = \widetilde{s}$, while it is greater than $\widetilde{s}$ and monotonically increasing in the interval $(x_0, 1/\widetilde{s} - 1)$. Therefore, there exists only one positive equilibrium state for (8) in this case. Consequently, from (E.1), (E.3), and these arguments we have condition (17_3) of Lemma 9.

Finally, from the arguments in Appendix D and this appendix we find the following necessary and sufficient condition that there are two extrema for $\mathscr{F}(x)$ for $x > 0$:

$$\widetilde{s} < 1, \quad \mathscr{F}''(0) < 0, \quad \mathscr{F}'(x_i) < 0, \quad \text{and} \quad \mathscr{F}'(0) > 0. \tag{E.4}$$

Hence, condition (E.4) is necessary for the existence of three positive equilibrium states for (8), which is shown in Lemma 10.

The necessary and sufficient condition for the existence of three positive equilibrium states for (8) is such that the equation $\mathscr{F}(x) = 1$ has three different positive roots in the interval $(1/\widetilde{s} - 1,\, \infty)$. Even under condition (E.4) necessary for it, it is still possible that an extremum of $\mathscr{F}(x)$ is located in the interval $[1/\widetilde{s} - 1,\, \infty)$. In such a case, the equation $\mathscr{F}(x) = 1$ has only one positive root in $(x_0,\, 1/\widetilde{s} - 1)$. When $\mathscr{F}(x)$ has two extrema for $x > 0$ if and only if condition (E.4) is satisfied, there are necessarily two positive roots of $\mathscr{F}'(x) = 0$, x_1 and x_2 $(> x_1)$ for $x > 0$. Then, if and only if $\mathscr{F}(x_1) > 1 > \mathscr{F}(x_2)$ and $x_2 < 1/\widetilde{s} - 1$, the equation $\mathscr{F}(x) = 1$ has three different positive roots in $(x_0,\, 1/\widetilde{s} - 1)$. This argument is used to numerically identify the parameter region where three equilibrium states exist in Fig. 6.

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