Nonlinear Analysis: Modelling and Control, Vol. 26, No. 4, 642–660
https://doi.org/10.15388/namc.2021.26.24178

Large deviations for stochastic Kuramoto–Sivashinsky equation with multiplicative noise

Gregory Amali Paul Rose[a,1], **Murugan Suvinthra**[a], **Krishnan Balachandran**[b]

[a]Department of Applied Mathematics, Bharathiar University,
Coimbatore - 641 046, India
amalipaulrose@gmail.com; suvinthra@gmail.com

[b]Department of Mathematics, Bharathiar University,
Coimbatore - 641 046, India
kb.maths.bu@gmail.com

Received: March 28, 2020 / **Revised:** January 30, 2021 / **Published online:** July 1, 2021

Abstract. The Kuramoto–Sivashinsky equation is a nonlinear parabolic partial differential equation, which describes the instability and turbulence of waves in chemical reactions and laminar flames. The aim of this work is to prove the large deviation principle for the stochastic Kuramoto–Sivashinsky equation driven by multiplicative noise. To establish the large deviation principle, the weak convergence approach is used, which relies on proving basic qualitative properties of controlled versions of the original stochastic partial differential equation.

Keywords: large deviations, stochastic partial differential equations, weak convergence, uniform Laplace principle.

1 Introduction

The theory of large deviations studies the exponential decay of probabilities in certain random systems. It has been applied in a wide variety of areas, which includes statistical mechanics, nonlinear dynamics, information theory, queueing systems, communication networks, biology and in engineering. The background, motivation and fundamental results in this area can be found in [10, 11, 13, 29]. Large deviation theory for small noise stochastic differential equation (SDE) has been extensively studied, which was introduced by Friedlin and Wentzell [18], who established the large deviations for such SDEs driven by finitely many Brownian motions. To understand the basics of this theory, consider a k-dimensional SDE of the form

$$\mathrm{d}X^\varepsilon(t) = b\big(X^\varepsilon(t)\big)\,\mathrm{d}t + \sqrt{\varepsilon}a\big(X^\varepsilon(t)\big)\,\mathrm{d}W_t, \quad X^\varepsilon(0) = x^\varepsilon, \quad t \in [0, T],$$

[1]Corresponding author.

with coefficients a, b satisfying suitable regularity properties and W_t a finite dimensional standard Brownian motion. If $x^\varepsilon \to x^0$ as $\varepsilon \to 0$, then $X^\varepsilon \xrightarrow{\mathrm{P}} X^0$ in $C([0,T];\mathbb{R}^k)$, where X^0 solves the equation $\dot{x} = b(x)$ with initial data x^0. The Friedlin–Wentzell theory describes the path asymptotics of probabilities of large deviations of the solution of the SDE away from X^0 as $\varepsilon \to 0$. The proofs of large deviation principle (LDP) using the aforesaid theory have relied on first approximating the original problem by time-discretization so that LDP can be shown on the resulting simpler problems via contraction principle and then showing that LDP holds in the limit using exponential probability estimates that are specific to the model under study. Later, Dupuis and Ellis [13] have combined weak convergence methods to the stochastic control approach developed earlier by Fleming [16] to the large deviations theory.

The Kuramoto–Sivashinsky (KS) equation was derived by Kuramoto and Tsuzuki [21] in phase turbulence of wave fronts in reaction–diffusion systems. The equation was also developed by Sivashinsky [25] in higher space dimensions to model small thermal diffusive instabilities in laminar flame fronts of gaseous combustible mixtures. The KS equation arises in a broad spectrum of contexts describing the behavioral aspects in thin film flows of long waves, unstable drift waves driven by electron collision, instability in thin film hydrodynamics, bright spots formation by a self forcing in optics and many other fields. The solution of the KS equation that arises in the modeling of surface erosion via ion sputtering in amorphous materials on a bounded interval subject to a random forcing term was investigated in [12]. A class of nonlocal stochastic Kuramoto–Sivashinsky (SKS) equations driven by Poisson random measures, and the existence and uniqueness of weak solution were studied in [3]. The KS equation with random forcing term is analyzed in [15], which provides sufficient conditions for existence and uniqueness of invariant measures using a Markovian semigroup.

Large deviations for weak solution of a nonlocal SKS equation with small additive noise perturbations using the contraction principle was deliberated in [2]. A control problem for a one-dimensional nonlinear parabolic system of Kuramoto–Sivashinsky–Korteweg de Vries equation coupled to a heat equation based on a Carleman estimate for the linearized system is addressed in [8]. Consider the stochastic Kuramoto–Sivashinsky equation with multiplicative noise

$$\begin{aligned} &\mathrm{d}u^\varepsilon + \bigl(\Delta^2 u^\varepsilon + \Delta u^\varepsilon + \operatorname{div} f\bigl(u^\varepsilon\bigr)\bigr)\,\mathrm{d}t = \sqrt{\varepsilon}\sigma\bigl(t,x,u^\varepsilon\bigr)\,\mathrm{d}W_t, \quad x \in D,\ t > 0, \\ &u^\varepsilon|_{\partial D} = \Delta u^\varepsilon|_{\partial D} = 0, \quad t > 0, \qquad u^\varepsilon(x,0) = \xi(x), \quad x \in D. \end{aligned} \tag{1}$$

Here D is a bounded domain in $\mathbb{R}^n$ with a smooth boundary, W_t is a multiplicative noise defined in a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbf{P})$, and $\xi \in L^2(D)$ is the given initial condition. The global well-posedness for (1) in $C([0,T], L^2(D))$ follows from [30].

In this paper the LDP is established for the law of solutions of (1) by using the weak convergence approach. Existing LDP results for various stochastic dynamical systems can be found in [1, 5, 17, 23, 24, 26–28] and many more. The main advantage of using the weak convergence approach is that it do not require any time discretization and exponential probability estimates as in the Friedlin–Wentzell technique, which are tedious in infinite dimensional setting, where these estimates are needed with metrics on exotic function spaces. In fact, a more strengthened form, which is the uniform LDP, where the uniformity is with respect to the initial condition, is established here. In many applications, stable equilibrium or periodic behaviour is critical to a well functioning system. In stochastic dynamical systems with a small noise, it is of interest to understand its effect on the dynamics, especially, near stable equilibrium points and periodic orbits of the corresponding deterministic system. While solutions of the deterministic system that start near a stable equilibrium point or periodic orbit may remain near the equilibrium point or periodic orbit for all time, solutions of the small noise stochastic system will eventually exit any bounded domain that contains the equilibrium point or periodic orbit. The uniform LDP over compact sets for finite-dimensional diffusions has been used by Friedlin and Wentzell [18] to study the exit time and exit place asymptotics for bounded subsets of $\mathbb{R}^n$. The perspective of uniform large deviations over compact sets used in this work will be fruitful to the study of asymptotic properties of invariant measures and exit times from suitable bounded domains as $\varepsilon \to 0$. Here the uniform Laplace principle is proved, which is equivalent to the LDP for Polish space-valued random elements. The sufficient conditions are obtained by proving certain qualitative properties such as existence, uniqueness and tightness of the analogous controlled processes and convergence to its limiting zero noise equation, which leads to a simple, short and more straightforward proof as compared to the other methods.

Outline of the paper. In Section 2, we introduce the SKS equation under study, its existence and uniqueness results and state some of the estimates for the corresponding Green's kernel. In Section 3, we state the LDP, the uniform Laplace principle and sufficient conditions to prove it using variational representations for appropriate family of measurable maps. In Section 4, we introduce the controlled and skeleton equations and establish their existence and uniqueness. Section 5 is devoted for proving the main theorem. The LDP is established by proving tightness and convergence of the controlled process using Prohorov's theorem. Unless otherwise noted, we adopt the following notation throughout this paper. Denote the expectation with respect to $\mathbf{P}$ by $\mathbf{E}$. The notation $\doteq$ means definition. $C(T)$ denotes a constant depending on parameter say T, and C is a constant depending on no specific parameter. The precise value of such constants may change from one line to the other. Also, the $L^p(D)$ norm is denoted by $\|h(t,\cdot)\|_{L^p}$ for a function $h(t,x)$ with respect to the variable $x \in D$.

2 Stochastic Kuramoto–Sivashinsky equation

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbf{P})$ be a filtered probability space, D is a bounded domain in $\mathbb{R}^n$ with smooth boundary. Let W_t, $t \geqslant 0$, be a $L^2(D)$-valued Wiener process adapted to the filtration $\{\mathcal{F}_t\}$, $t \geqslant 0$. Recall the stochastic Kuramoto–Sivashinsky equation (SKS) indexed by $\varepsilon > 0$ as

$$\begin{aligned} &\mathrm{d}u^\varepsilon + \left(\Delta^2 u^\varepsilon + \Delta u^\varepsilon + \operatorname{div} f\left(u^\varepsilon\right)\right)\mathrm{d}t = \sqrt{\varepsilon}\sigma\left(t,x,u^\varepsilon\right)\mathrm{d}W_t, \quad x \in D,\ t > 0, \\ &u^\varepsilon\big|_{\partial D} = \Delta u^\varepsilon\big|\partial D = 0, \quad t > 0, \qquad u^\varepsilon(x,0) = \xi(x), \quad x \in D. \end{aligned} \tag{2}$$

f is a given vector function, σ is the noise intensity, which is a continuous $L^2(D)$-valued random field, and the initial condition $\xi \in L^2(D)$. Let $\{\lambda_k\}_{k=1}^{\infty}$ be the sequence of eigenvalues of the operator $-\Delta$ on D subject to the homogeneous Dirichlet boundary condition, where multiple eigenvalues are counted in their multiplicities. Let $\{\phi_k\}_{k=1}^{\infty}$ be the corresponding sequence of eigenfunctions chosen to form an orthonormal basis of $L^2(D)$. Since $\lim_{k\to\infty} \lambda_k(\lambda_k - 1) = \infty$, choose a $c \geqslant 0$ and fix it such that $\mu_k \doteq \lambda_k(\lambda_k - 1) + c > 0$ for all $k \in \mathbb{N}$. Let

$$
\begin{aligned}
G(t,x,y) &= \sum_{k=1}^{\infty} \phi_k(x)\phi_k(y)\mathrm{e}^{-\mu_k t}, \quad x, y \in \bar{D},\ t > 0, \\
G(0,x,y) &= \delta(x-y), \quad x, y \in \bar{D},
\end{aligned}
\tag{3}
$$

where $G(t,x,y)$ is the Green's function of the linear partial differential equation $\partial_t u + \Delta^2 u + \Delta u + cu = 0$ (in D).

The equivalent stochastic integral equation of (2) using the Green's function and Duhamel's formula is given by

$$
\begin{aligned}
u^{\varepsilon}(t,x) &= \int_D G(t,x,y)\,\xi(y)\,\mathrm{d}y + c\int_0^t \int_D G(t-s,\,x,\,y)\,u^{\varepsilon}(s,y)\,\mathrm{d}y\,\mathrm{d}s \\
&\quad + \int_0^t \int_D \nabla G(t-s,\,x,\,y)\,f\big(u^{\varepsilon}(s,y)\big)\,\mathrm{d}y\,\mathrm{d}s \\
&\quad + \sqrt{\varepsilon}\int_0^t \int_D G(t-s,\,x,\,y)\,\sigma\big(u^{\varepsilon}(s,y)\big)\,\mathrm{d}y\,\mathrm{d}W_s(y).
\end{aligned}
$$

Here we use $\sigma(u^{\varepsilon}(s,y))$ in the place of $\sigma(s,y,u^{\varepsilon}(s,y))$ for the sake of simplicity.

There exists a complete normalized orthogonal basis $\{e_k\}_{k=1}^{\infty}$ of $L^2(D)$, a sequence of positive numbers $\{c_k\}_{k=1}^{\infty}$ satisfying $\sum_{k=1}^{\infty} c_k^2 < \infty$ and a sequence of independent, identically distributed standard Brownian motions w_t^k ($k = 1, 2, \dots$) such that

$$
W_t(x,\omega) = \sum_{k=1}^{\infty} c_k w_t^k(\omega) e_k(x),
$$

and let

$$
r(x,y) = \sum_{k=1}^{\infty} c_k^2 e_k(x) e_k(y).
$$

Then $\iint_{D\times D} |r(x,y)|^2\,\mathrm{d}x\,\mathrm{d}y = \sum_{k=1}^{\infty} c_k^4 < \infty$, i.e., $r \in L^2(D\times D)$. Also, $r(x,y) = r(y,x)$, and $\iint_{D\times D} r(x,y)\varphi(x)\varphi(y)\,\mathrm{d}x\,\mathrm{d}y \geqslant 0$ for any $\varphi \in L^2(D)$. Thus, there exists a positive semidefinite self-adjoint Hilbert–Schmidt operator R, which is the covariance operator of the Wiener process W_t on $L^2(D)$ with kernel $r(x,y)$ given by

$$(R\varphi)(x) = \int_D r(x,y)\varphi(y)\,\mathrm{d}y, \quad \varphi \in L^2(D).$$

The following are the assumptions made on the nonlinearity f, noise intensity σ and kernel function $r(x,y)$ of the trace class operator.

Assumption A. $f(0) = 0$, and there exists constants $C > 0$ and $p \geqslant 1$ such that for $u, v \in \mathbb{R}$,

$$\bigl|f(u) - f(v)\bigr| \leqslant C\bigl(1 + |u| + |v|\bigr)^{p-1}|u - v|.$$

Assumption B. There exists constant $C > 0$ such that for $u, v \in \mathbb{R}$ and $t \geqslant 0$, $x \in D$,

$$\bigl|\sigma(t,x,u)\bigr| \leqslant C\bigl(1 + |u|\bigr),$$
$$\bigl|\sigma(t,x,u) - \sigma(t,x,v)\bigr| \leqslant C\bigl(|u - v|\bigr).$$

Assumption C. The kernel function r is in $D \times D$, so that there exists a constant $C > 0$ such that for $x, y \in D$,

$$r(x,y) \leqslant C.$$

The existence and uniqueness of a solution to (2) follow from the theorem stated below [30, Thm. 1.1].

Theorem 1 [Existence and uniqueness of the solution]. *Let Assumptions* A–C *be satisfied. Suppose further that* $1 \leqslant p \leqslant 2$ *for* $1 \leqslant n \leqslant 5$ *and* $1 \leqslant p < 1 + 6/n$ *for* $n \geqslant 6$. *Then problem* (2) *is globally well posed in* $L^2(D \times \Omega)$. *More precisely, for any* $\mathcal{F}_0$*-measurable* $\xi \in L^2(D \times \Omega)$*, problem* (2) *has a unique solution* u^ε *such that for any* $T > 0$, $u^\varepsilon \in L^2(\Omega, C([0,T], L^2(D)))$*, and the solution map* $\xi \mapsto u^\varepsilon$ *is a Lipschitz continuous map from* $L^2(D \times \Omega)$ *to* $L^2(\Omega, C([0,T], L^2(D)))$.

Using Assumption A, it can be ensured that for any $u, v \in L^2(D)$,

$$\|f(u)\|_{L^{2/p}} \leqslant C(1 + \|u\|_{L^2}^p), \tag{4}$$

$$\bigl\|f(u) - f(v)\bigr\|_{L^{2/p}} \leqslant C\bigl(1 + \|u\|_{L^2} + \|v\|_{L^2}\bigr)^{p-1}\|u - v\|_{L^2}, \tag{5}$$

$$\bigl\|f(u) - f(v)\bigr\|_{L^{2/p}}^{2/p} \leqslant C\|u - v\|_{L^2}^{2/p}\bigl(1 + \|u\|_{L^2} + \|v\|_{L^2}\bigr)^{2(p-1)/p}. \tag{6}$$

Also, from Assumptions B and C, for any $u, v \in L^2(D)$, $x \in D$, $t \geqslant 0$, we get

$$\|u\|_R \leqslant C\|u\|_{L^2},$$

$$\bigl\|\sigma(t,x,u)\bigr\|_R^2 \leqslant C\bigl(1 + \|u\|_{L^2}^2\bigr), \tag{7}$$

$$\bigl\|\sigma(t,x,u) - \sigma(t,x,v)\bigr\|_R^2 \leqslant C\|u - v\|_{L^2}^2. \tag{8}$$

For any $T > 0$, let X_T be the set of $L^2(D)$-valued $\mathcal{F}_t$-adapted continuous random processes u on $L^2(\Omega, C([0,T], L^2(D)))$ such that the norm

$$\|u\|_{X_T} = \Big(\mathbf{E} \sup_{0\leqslant t\leqslant T} \big\|u(t)\big\|_{L^2}^2\Big)^{1/2}$$

is finite. Then it is apparent that $(X_T, \|\cdot\|_{X_T})$ is a Banach space. If $u(t)$ is the solution of (2) in the finite interval $[0,T]$, then there exists a corresponding $C > 0$ such that

$$\mathbf{E}\big(\big\|u(T)\big\|_{L^2}^2\big) \leqslant C(T), \tag{9}$$

which follows from [30].

2.1 Estimates for the Green's function

The following lemma gathers several estimates for integrals of space (respectively time) increments of G, the results are the same as those in [7, Lemma 1.8] that are deduced from the explicit formulation (3) of G by using similar arguments.

Lemma 1. *For $\gamma < 4 - n$ and $\gamma \leqslant 2, \gamma' < 1 - n/4$, there exists $C > 0$ such that for $t > t'$, $x, y, z \in D$,*

$$\int_0^t \int_D \big|G(t-s,\, x,\, y) - G(t-s, z, y)\big|^2 \,\mathrm{d}y \,\mathrm{d}s \leqslant C|x-z|^\gamma,$$

$$\int_0^{t'} \int_D \big|G(t-s,\, x,\, y) - G(t'-s, x, y)\big|^2 \,\mathrm{d}y \,\mathrm{d}s \leqslant C|t-t'|^{\gamma'},$$

$$\int_{t'}^t \int_D \big|G(t-s,\, x,\, y)\big|^2 \,\mathrm{d}y \,\mathrm{d}s \leqslant |t-t'|^{\gamma'}.$$

The lemma below gives the upper estimates on the Green's function proved in [14], which are similar to those in [7].

Lemma 2. *Let G be the Green's function defined by* (3). *Then there exist positive constants c_1 and c_2 such that for any $t \in [0,T]$, any $x, y \in D$ and any multi-index $\alpha \in Z_+^n$ the following inequalities are satisfied*:

$$\big|G(t,x,y)\big| \leqslant c_1 t^{-n/4} \exp\big(-c_2|x-y|^{4/3} t^{-1/3}\big), \tag{10}$$

$$\big|\partial_y^\alpha G(t,x,y)\big| \leqslant c_1 t^{(-n+|\alpha|)/4} \exp\big(-c_2|x-y|^{4/3} t^{-1/3}\big), \tag{11}$$

$$\big|\partial_t G(t,x,y)\big| \leqslant c_1 t^{(-n+4)/4} \exp\big(-c_2|x-y|^{4/3} t^{-1/3}\big). \tag{12}$$

Lemma 3. *Let $q \geqslant 1$. Define the linear operator J by*

$$\begin{aligned} J(v)(t,x) &\doteq \int_0^t \int_D H(r,t;x,y)\, v(r,y)\,\mathrm{d}y\,\mathrm{d}r, \quad t \in [0,T],\ x \in D, \\ J(v)(t,x) &\doteq 0, \quad x \notin D, \end{aligned} \tag{13}$$

for $v \in L^\infty([0,T], L^q(D))$, provided the integral exists. Let $\zeta_\varepsilon(t,x)$ be a sequence of random fields on $[0,T] \times D$ such that almost surely

$$\|\zeta_\varepsilon(t,\cdot)\|_{L^2} \leqslant \theta_\varepsilon, \quad t \in [0,T],$$

where θ_ε is a finite random variable for every ε. Assume that θ_ε is bounded in probability, i.e.,

$$\lim_{c\to\infty} \sup_\varepsilon \mathbf{P}(\theta_\varepsilon \geqslant C) = 0.$$

Then the sequence $J(\zeta_\varepsilon)$ is uniformly tight in $C([0,T], L^2(D))$.

The above lemma can be verified for the operator J as defined in (13) when $H(s,t;x,y) \doteq G(t-s,\, x,\, y)$ or $\partial_y G(t-s,\, x, y)$ using estimates (10)–(12) analogous to [**?**, Cor. 3.2].

Lemma 4. (*See* [30, *Lemma* 2.1].) *Let $\varphi \in L^2(D)$. Then for any $\alpha \in Z_+^n$, we have*

$$\left\| \partial_x^\alpha \int_D G(t,x,y)\, \varphi(y)\,\mathrm{d}y \right\|_{L^2} \leqslant C t^{-|\alpha|/4} \|\varphi\|_{L^2}, \quad t > 0.$$

Lemma 5. (*See* [30, *Lemma* 2.2].) *Let $\sigma_t(t,x,\omega)$ be a continuous $L^2(D)$-valued random field satisfying the condition*

$$\mathbf{E} \int_0^T \|\sigma_t\|_R^2 \,\mathrm{d}t < \infty, \quad \|\sigma_t\|_R^2 = \int_D r(x,x) \bigl|\sigma_t(t,x,\omega)\bigr|^2 \,\mathrm{d}x.$$

Then we have the following estimate:

$$\mathbf{E}\left(\sup_{0\leqslant t\leqslant T} \left\| \int_0^t \int_D G(t-s,\, x,\, y)\sigma_t \,\mathrm{d}y\,\mathrm{d}W_s(y) \right\|_{L^2}^2 \right) \leqslant C\mathbf{E}\left(\int_0^T \|\sigma_t\|_R^2 \,\mathrm{d}t \right).$$

3 Large deviation principle

In this section, we state some standard definitions and results from the theory of large deviations and some of the results presented in [4,5]. In particular, we state the variational representation formulas for appropriate family of measurable maps, and subsequently, we state the uniform Laplace principle, which is equivalent to the uniform LDP under a Polish space. Here the uniformity is with respect to the initial condition. Using the weak convergence approach from [5], the uniform Laplace principle can be proved under two main assumptions using the basic qualitative properties (existence, uniqueness and tightness) for the controlled and zero noise versions of the original process.

Let $\{X^\varepsilon,\, \varepsilon > 0\} \equiv \{X^\varepsilon\}$ be a family of random variables defined on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ that takes values in a Polish space (i.e., a complete separable metric space) E. The theory of large deviations is concerned with events A for which probability $\mathbf{P}\ (X \in A)$ converges to zero exponentially fast as $\varepsilon \to 0$. The exponential decay rate of such probabilities is typically expressed in terms of a "rate function".

Definition 1 [Rate function]. A function $I : E \to [0, \infty]$ is called a rate function on E if for each $M < \infty$, the level set $\{x \in E\colon\ I(x) \leqslant M\}$ is a compact subset of E.

Definition 2 [Large deviation principle]. The sequence $\{X^\varepsilon\}$ is said to satisfy the large deviation principle on E with rate function I if the following conditions hold:

1. *Large deviation upper bound.* For each closed subset F of E,
$$\limsup_{\varepsilon\to 0} \varepsilon \log \mathbf{P}\bigl(X^\varepsilon \in F\bigr) \leqslant -\inf_{x\in F} I(x).$$
2. *Large deviation lower bound.* For each open subset G of E,
$$\liminf_{\varepsilon\to 0} \varepsilon \log \mathbf{P}\bigl(X^\varepsilon \in G\bigr) \geqslant -\inf_{x\in G} I(x).$$

In many problems, exponential estimates on functions are of interest and are more general than indicator functions of closed or open sets. This induced the study of the Laplace principle.

Lemma 6 [Laplace principle]. *Given h a bounded continuous function mapping $[0, 1]$ into $\mathbb{R}$,*
$$\lim_{n\to\infty} \frac{1}{n} \log \int_0^1 \exp\bigl[-nh(x)\bigr]\, \mathrm{d}x = -\min_{x\in[0,1]} h(x).$$

An important consequence of the LDP, which involve the asymptotic behavior of certain expectations as stated below is proved by Varadhan [29], which generalizes the Laplace principle.

Lemma 7 [Varadhan's lemma]. (*See* [13].) *Assume that the sequence $\{X^\varepsilon,\, \varepsilon > 0\}$ of random variables defined on a probability space and taking values in a Polish space E satisfies the LDP on E with rate function I. Then for all bounded continuous functions h mapping E into $\mathbb{R}$,*
$$\lim_{\varepsilon\to 0} \varepsilon \log \mathbf{E}\biggl\{\exp\biggl[-\frac{1}{\varepsilon} h\bigl(X^\varepsilon\bigr)\biggr]\biggr\} = -\inf_{x\in E} \bigl\{h(x) + I(x)\bigr\}.$$

Lemma 8 [Bryc's converse]. (*See* [13].) *The Laplace principle implies the LDP with the same rate function. More precisely, if I is a rate function on E and the limit*

$$\lim_{\varepsilon\to 0}\varepsilon\log\mathbf{E}\left\{\exp\left[-\frac{1}{\varepsilon}h\left(X^\varepsilon\right)\right]\right\} = -\inf_{x\in E}\left\{h(x)+I(x)\right\}$$

is valid for all bounded continuous functions h, then $\{X^\varepsilon,\ \varepsilon>0\}$ satisfies the LDP on E with rate function I.

From Varadhan's lemma, together with Bryc's converse of Varadhan's lemma, the Laplace principle and LDP are equivalent for Polish space-valued random elements [13]. In view of this equivalence the rest of this work will be concerned with the study of the Laplace principle. In fact, we will study a somewhat strengthened notation, namely, a uniform Laplace principle, which will be stated in Theorem 2.

Let $E_0 = L^2(D)$ be the space of the initial condition, and let $E = C([0,T], L^2(D))$ be the space of solutions, which are Polish spaces. The initial condition ξ takes values in the compact subsets of E_0. Let $\mathcal{G}^\varepsilon : E_0\times C([0,T]\times D,\mathbb{R}) \to E$ be a family of measurable maps. The solution map of (2) is defined as

$$X^{\varepsilon,\xi} \doteq \mathcal{G}^\varepsilon(\xi, \sqrt{\varepsilon}W_t).$$

Then $X^{\varepsilon,\xi}$ are Polish space-valued random elements. Let $\nu : \Omega\times[0,T]\to L^2(D)$ be an $L^2(D)$-valued predictable process. Define

$$\mathcal{P}_2 \doteq \left\{\nu\colon \int_0^T \left\|\nu(s)\right\|_{L^2}^2\,\mathrm{d}s<\infty\ \text{a.s.}\right\},$$

$$S_N \doteq \left\{\phi\in \mathrm{L}^2\left([0,T]\times D\right)\colon \int_{[0,T]\times D}\phi^2(s,y)\,\mathrm{d}y\,\mathrm{d}s\leqslant N\right\},\quad N\in\mathbb{N}.$$

The set of bounded deterministic controls is given by the Polish space S_N, which is a compact metric space endowed with the weak topology on $L^2([0,T]\times D)$ as in [5]. The set of bounded stochastic controls is given by

$$\mathcal{P}_2^N \doteq \left\{\nu\in\mathcal{P}_2\colon \nu(\omega)\in S_N\ \ \mathbf{P}\text{-a.s.}\right\}. \tag{14}$$

For $\nu\in \mathrm{L}^2([0,T]\times D)$, we define

$$\mathrm{Int}\,\nu(t,x) \doteq \int_0^t\int_0^x \nu(s,y)\,\mathrm{d}y\,\mathrm{d}s.$$

The following condition is the standing assumption of Theorem 2.

Assumption. There exists a measurable map $\mathcal{G}^0 : E_0\times C([0,T]\times D,\mathbb{R})\to E$ such that:

(S1) For every $M < \infty$ and compact set $K \subset E_0$, the set

$$\Gamma_{M,K} \doteq \bigl\{\mathcal{G}^0(\xi, \operatorname{Int}\nu)\colon\ \nu \in S_M,\ \xi \in K\bigr\}$$

is a compact subset of E.

(S2) Consider $M < \infty$ and let $\{\nu^\varepsilon\} \in \mathcal{P}_2^M$ and $\{\xi^\varepsilon\} \subset E_0$ such that $\xi^\varepsilon \to \xi$, $\nu^\varepsilon \to \nu$ in distribution as $\varepsilon \to 0$. Then

$$\mathcal{G}^\varepsilon\bigl(\xi^\varepsilon,\ \sqrt{\varepsilon}W_t + \operatorname{Int}\nu^\varepsilon\bigr) \to \mathcal{G}^0(\xi, \operatorname{Int}\nu)$$

in distribution as $\varepsilon \to 0$.

The following theorem states the uniform Laplace principle for the family $\{X^{\varepsilon,\xi}\}$.

Theorem 2. (*See* [5, *Thm.* 7].). *Let* $\mathcal{G}^0 : E_0 \times C([0,T]\times L^2(D), \mathbb{R}) \to E$ *be a measurable map satisfying* (S1) *and* (S2). *For* $\xi \in E_0$ *and* $f \in E$, *define*

$$I_\xi(f) \doteq \inf_{\{\beta \in L^2([0,T]\times D)\colon\ f \doteq \mathcal{G}^0(\xi, \operatorname{Int}\beta)\}} \left\{\frac{1}{2}\int_0^T \bigl\|\beta(s)\bigr\|_{L^2}^2\,\mathrm{d}s\right\}.$$

Suppose that for all $f \in E$, $\xi \mapsto I_\xi(f)$ *is a lower semicontinuous map from* E_0 *to* $[0,\infty]$. *Then, for all* $\xi \in E_0$, $f \mapsto I_\xi(f)$ *is a rate function on* E, *and the family* $\{I_\xi,\ \xi \in E_0\}$ *of rate functions has compact level sets on compacts. Furthermore, the family* $\{X^{\varepsilon,\xi}\}$ *satisfies the Laplace principle on* E *with rate function* I_ξ *uniformly in* ξ *on compact subsets of* E_0.

4 Controlled and the skeleton equations

The solution map for the SKS equation (2) is $u^\varepsilon = \mathcal{G}^\varepsilon(\xi, \sqrt{\varepsilon}W_t)$. For a control $\nu \in \mathcal{P}_2$, $u_\xi^{\varepsilon,\nu} = \mathcal{G}^\varepsilon(\xi,\ \sqrt{\varepsilon}W_t + \operatorname{Int}\nu)$ is the solution map of the stochastic controlled equation

$$\begin{aligned}
&\mathrm{d}u^\varepsilon + \bigl(\Delta^2 u^\varepsilon + \Delta u^\varepsilon + \operatorname{div} f\bigl(u^\varepsilon\bigr)\bigr)\,\mathrm{d}t \\
&\quad = \sqrt{\varepsilon}\sigma\bigl(t,x,u^\varepsilon\bigr)\,\mathrm{d}W_t + \sigma\bigl(t,x,u^\varepsilon\bigr)\nu(t,x)\,\mathrm{d}t, \quad x \in D,\ t > 0, \\
&u^\varepsilon|_{\partial D} = \Delta u^\varepsilon|\partial D = 0, \quad t > 0, \qquad u^\varepsilon(x,0) = \xi(x), \quad x \in D.
\end{aligned} \tag{15}$$

The mild solution of the stochastic controlled equation (15) is

$$\begin{aligned}
u^{\varepsilon,\nu}(t,x) &= \int_D G(t,x,y)\,\xi(y)\,\mathrm{d}y + c\int_0^t\int_D G(t-s,\ x,\ y)\,u^{\varepsilon,\nu}(s,y)\,\mathrm{d}y\,\mathrm{d}s \\
&\quad + \int_0^t\int_D \nabla G(t-s,\ x,\ y)\,f\bigl(u^{\varepsilon,\nu}(s,y)\bigr)\,\mathrm{d}y\,\mathrm{d}s
\end{aligned}$$

$$+ \sqrt{\varepsilon} \int_0^t \int_D G(t-s,\, x,\, y)\, \sigma\big(u^{\varepsilon,\nu}(s,y)\big)\, \mathrm{d}y\, \mathrm{d}W_s(y)$$
$$+ \int_0^t \int_D G(t-s,\, x,\, y)\, \sigma\big(u^{\varepsilon,\nu}(s,y)\big)\nu(s,y)\, \mathrm{d}y\, \mathrm{d}s, \tag{16}$$

whereas $u_\xi^{0,\nu} = \mathcal{G}^0(\xi, \mathrm{Int}\,\nu)$ is the solution map of the skeleton equation

$$\begin{aligned} &\mathrm{d}u^0 + \big(\Delta^2 u^0 + \Delta u^0 + \operatorname{div} f\big(u^0\big)\big)\, \mathrm{d}t = \sigma\big(t, x, u^0\big)\nu(t,x)\, \mathrm{d}t, \quad x \in D,\ t > 0, \\ &u^0\big|_{\partial D} = \Delta u^0\big|\partial D = 0, \quad t > 0, \qquad u^0(x,0) = \xi(x), \quad x \in D, \end{aligned} \tag{17}$$

whose mild solution is

$$\begin{aligned} u^{0,\nu}(t,x) = {}& \int_D G(t,x,y)\, \xi(y)\, \mathrm{d}y + c \int_0^t \int_D G(t-s,\, x,\, y)\, u^{0,\nu}(s,y)\, \mathrm{d}y\, \mathrm{d}s \\ &+ \int_0^t \int_D \nabla G(t-s,\, x,\, y)\, f\big(u^{0,\nu}(s,y)\big)\, \mathrm{d}y\, \mathrm{d}s \\ &+ \int_0^t \int_D G(t-s,\, x,\, y)\, \sigma\big(u^{0,\nu}(s,y)\big)\nu(s,y)\, \mathrm{d}y\, \mathrm{d}s. \end{aligned}$$

4.1 The rate function

Let $h \in C([0,T], L^2(D))$ for every $t \in [0,T]$ and $x \in D$. Define the following rate function:

$$I_\xi(h) \doteq \frac{1}{2} \inf_\nu \int_0^T \int_D \nu^2(s,y)\, \mathrm{d}y\, \mathrm{d}s, \tag{18}$$

where the infimum is taken over all $\nu \in L^2([0,T] \times D)$ such that

$$\begin{aligned} h(t,x) = {}& \int_D G(t,x,y)\, \xi(y)\, \mathrm{d}y + c \int_0^t \int_D G(t-s,\, x,\, y)\, h(s,y)\, \mathrm{d}y\, \mathrm{d}s \\ &+ \int_0^t \int_D \nabla G(t-s,\, x,\, y)\, f\big(h(s,y)\big)\, \mathrm{d}y\, \mathrm{d}s \\ &+ \int_0^t \int_D G(t-s,\, x,\, y)\, \sigma\big(h(s,y)\big)\nu(s,y)\, \mathrm{d}y\, \mathrm{d}s. \end{aligned}$$

4.2 Existence and uniqueness of the controlled process

The following theorem establishes the existence and uniqueness of the controlled process using Girsanov's theorem [9, Thm. 10.14].

Theorem 3 [Existence and uniqueness of the controlled process]. *Let $\mathcal{G}^\varepsilon$ denote the solution mapping, and let $\nu \in \mathcal{P}_2^N$ for some $N \in \mathbb{N}$, where $\mathcal{P}_2^N$ is as defined in* (14)*. For $\varepsilon > 0$ and $\xi \in E_0$, define*

$$u_\xi^{\varepsilon,\nu} = \mathcal{G}^\varepsilon\bigl(\xi,\ \sqrt{\varepsilon}W_t + \operatorname{Int}\nu\bigr).$$

Then $u_\xi^{\varepsilon,\nu}$ is the unique solution of (15)*.*

Proof. For a fixed $\nu \in \mathcal{P}_2^N$, define the measure $Q^{\nu,\varepsilon}$ by

$$\frac{\mathrm{d}Q^{\nu,\varepsilon}}{\mathrm{d}\mathbf{P}} \doteq \exp\left\{-\frac{1}{\sqrt{\varepsilon}}\int_0^T\int_D \nu(s,y)\,\mathrm{d}y\,\mathrm{d}W_s(y) - \frac{1}{2\varepsilon}\int_0^T\int_D \nu^2(s,y)\,\mathrm{d}y\,\mathrm{d}s\right\}.$$

Then $Q^{\nu,\varepsilon}$ is probability measure on $(\Omega, \mathcal{F}, Q^{\nu,\varepsilon})$ and is equivalent to $\mathbf{P}$. By Girsanov's theorem [9, Thm. 10.14], the process $\widetilde{W} \doteq W_t + \varepsilon^{-1/2}\operatorname{Int}\nu$ is a Wiener process under the measure $Q^{\nu,\varepsilon}$. By Theorem 1, $u_\xi^{\varepsilon,\nu}$ is the unique solution of (2) with W_t replaced by $\widetilde{W}$ under the measure $Q^{\nu,\varepsilon}$. This is precisely the same as (15) on the probability space $(\Omega, \mathcal{F}, Q^{\nu,\varepsilon})$. By the equivalence of measures $u_\xi^{\varepsilon,\nu}$ is the unique solution of (15) under the probability measure $(\Omega, \mathcal{F}, \mathbf{P})$, and the proof is complete. $\square$

Theorem 4 [Existence and uniqueness of the skeleton]. *Fix $\xi \in E_0$ and $\nu \in \mathcal{P}_2$. Then there exists a unique function $u_\xi^{0,\nu} \in C([0,T], L^2(D))$, which is the mild solution to the skeleton equation* (17)*.*

The proof of this theorem is almost same to that of Theorem 1 and hence omitted.

5 Main theorem

The following theorem is the main contribution of this paper, which establishes the uniform LDP for the law of solutions $\{u^\varepsilon\}$ of the SKS equation (2).

Theorem 5 [Main theorem]. *The family of solutions of* (2) *given by $\{u^\varepsilon = \mathcal{G}^\varepsilon(\xi, \sqrt{\varepsilon}W_t),\ \varepsilon \in (0,1)\}$ satisfies the LDP on the Polish space E with rate function I_ξ given by* (18) *uniformly for ξ in compact subsets of E_0.*

As mentioned earlier, in a Polish space, the uniform Laplace principle is equivalent to the uniform LDP. In view of Theorem 2, it suffices to prove conditions (S1) and (S2). To accomplish this purpose, define a measurable map $\beta : [0,1) \to [0,1)$ such that $\beta(r) \to \beta(0) = 0$ as $r \to 0$.

The following theorem will be used to obtain the weak convergence of a sequence of controlled processes by using the tightness arguments.

Theorem 6 [Prohorov's theorem]. (*See* [19, *Thm.* 14.3].) *For any sequence of random elements $\{X_n\}$, $n = 1, 2, \dots$, in a metric space S, tightness implies relative compactness in distribution, and the two conditions are equivalent when S is separable and complete.*

Lemma 9. (*See* [6].) *Let $\xi \in L^\rho(D)$, $\rho \geqslant 2$. Then $(t \to G(t)\xi)$ belongs to $C([0,T], L^\rho(D))$, and $\xi \to \{t \to G(t)\xi\}$ is a continuous map in ξ.*

Theorem 7 [Convergence of the controlled process]. *Let $M < \infty$, and suppose that $\xi^\varepsilon \to \xi$, $\nu^\varepsilon \to \nu$ in distribution as $\varepsilon \to 0$ with $\{\nu^\varepsilon\} \subset \mathcal{P}_2^M$. Then $u_{\xi^\varepsilon}^{\beta(\varepsilon),\nu^\varepsilon} \to u_\xi^{0,\nu}$ in distribution.*

Proof. From (16) the solution of the controlled process is given by

$$u_{\xi^\varepsilon}^{\beta(\varepsilon),\nu^\varepsilon}(t,x) = \sum_{l=1}^{5} Z_l^\varepsilon,$$

where

$$\begin{aligned}
Z_1^\varepsilon &= \int_D G(t,x,y)\,\xi^\varepsilon(y)\,\mathrm{d}y,\\
Z_2^\varepsilon &= \int_0^t\int_D G(t-s,\,x,\,y)\,u_{\xi^\varepsilon}^{\beta(\varepsilon),\nu^\varepsilon}(s,y)\,\mathrm{d}y\,\mathrm{d}s,\\
Z_3^\varepsilon &= \int_0^t\int_D \nabla G(t-s,\,x,\,y)\,f\bigl(u_{\xi^\varepsilon}^{\beta(\varepsilon),\nu^\varepsilon}(s,y)\bigr)\,\mathrm{d}y\,\mathrm{d}s,\\
Z_4^\varepsilon &= \sqrt{\beta(\varepsilon)}\int_0^t\int_D G(t-s,\,x,\,y)\,\sigma\bigl(u_{\xi^\varepsilon}^{\beta(\varepsilon),\nu^\varepsilon}(s,y)\bigr)\,\mathrm{d}y\,\mathrm{d}W_s(y),\\
Z_5^\varepsilon &= \int_0^t\int_D G(t-s,\,x,\,y)\,\sigma\bigl(u_{\xi^\varepsilon}^{\beta(\varepsilon),\nu^\varepsilon}\bigr)\nu^\varepsilon(s,y)\,\mathrm{d}y\,\mathrm{d}s.
\end{aligned}$$

The proof is carried out in two steps.

Step 1: Tightness. We show the tightness of Z_l^ε for $l = 1, 2, 3, 4, 5$ in $C([0,T], L^2(D))$. Since $\xi^\varepsilon \in L^2(D)$, the tightness of Z_1^ε follows from the Lemma 9.

When $l = 2$, let $\zeta_{1\varepsilon}(t,x) \doteq u_{\xi^\varepsilon}^{\beta(\varepsilon),\nu^\varepsilon}(t,x)$. Then

$$\bigl\|\zeta_{1\varepsilon}(t,\cdot)\bigr\|_{L^2} \leqslant \sup_{0\leqslant t\leqslant T}\bigl\|u_{\xi^\varepsilon}^{\beta(\varepsilon),\nu^\varepsilon}(t,\cdot)\bigr\|_{L^2}.$$

Using (9), we have

$$\sup_{\varepsilon\in(0,1)} \mathbf{E}\Bigl(\sup_{0\leqslant t\leqslant T}\|u_{\xi^\varepsilon}^{\beta(\varepsilon),\nu^\varepsilon}(t,\cdot)\|_{L^2}^2\Bigr) \leqslant C(T).$$

Using Chebyshev's inequality, for a constant $R > 0$, we obtain

$$\sup_{\varepsilon\in(0,1)} \mathbf{P}\Big(\sup_{0\leqslant t\leqslant T} \big\| u_{\xi^\varepsilon}^{\beta(\varepsilon),\nu^\varepsilon}(t,\cdot)\big\|_{L^2} \geqslant R\Big) \leqslant \frac{C(T)}{R^2}.$$

Therefore,

$$\lim_{R\to\infty} \sup_{\varepsilon\in(0,1)} \mathbf{P}\Big(\sup_{0\leqslant t\leqslant T} \big\| u_{\xi^\varepsilon}^{\beta(\varepsilon),\nu^\varepsilon}(t,\cdot)\big\|_{L^2} \geqslant R\Big) = 0. \tag{19}$$

Hence, by using Lemma 3 the sequence $J(\zeta_{1\varepsilon}) = Z_2^\xi$ is uniformly tight in $C([0,T], L^2(D))$.

When $l = 3$, let $\zeta_{2\varepsilon}(t,x) \doteq f(u_{\xi^\varepsilon}^{\beta(\varepsilon),\nu^\varepsilon}(t,x))$. Then

$$\big\|\zeta_{2\varepsilon}(t,\cdot)\big\|_{L^2} \leqslant \sup_{0\leqslant t\leqslant T} \big\| f\big(u_{\xi^\varepsilon}^{\beta(\varepsilon),\nu^\varepsilon}(t,\cdot)\big)\big\|_{L^2}.$$

Using (4) and (9), we have

$$\sup_{\varepsilon\in(0,1)} \mathbf{E}\Big(\sup_{0\leqslant t\leqslant T} \big\| f(u_{\xi^\varepsilon}^{\beta(\varepsilon),\nu^\varepsilon}(t,\cdot))\big\|_{L^2}^2\Big) \leqslant C(T). \tag{20}$$

Also, when $l = 5$, let $\zeta_{3\varepsilon}(t,x) \doteq \sigma(u_{\xi^\varepsilon}^{\beta(\varepsilon),\nu^\varepsilon})\nu^\varepsilon(t,x)$. Then

$$\big\|\zeta_{3\varepsilon}(t,\cdot)\big\|_{L^2} \leqslant \sup_{0\leqslant t\leqslant T} \big\| \sigma\big(u_{\xi^\varepsilon}^{\beta(\varepsilon),\nu^\varepsilon}\big)\nu^\varepsilon(t,\cdot)\big\|_{L^2}.$$

Using (7), (9) and properties of controls, we have

$$\sup_{\varepsilon\in(0,1)} \mathbf{E}\Big(\sup_{0\leqslant t\leqslant T} \big\| \sigma\big(u_{\xi^\varepsilon}^{\beta(\varepsilon),\nu^\varepsilon}\big)\nu^\varepsilon(t,\cdot)\big\|_{L^2}^2\Big) \leqslant C(T). \tag{21}$$

Similar to (19), using Chebyshev's inequality, (20) and (21), we obtain the tightness of $J(\zeta_{2\varepsilon}) = Z_3^\xi$ and $J(\zeta_{3\varepsilon}) = Z_5^\xi$ in $C([0,T], L^2(D))$ from Lemma 3.

When $l = 4$, to show the tightness of Z_4^ξ, using [20, Thm. 4.10], it suffices to show the following two conditions. For any (t,x) and $(t',y) \in [0,T]\times D$, $t > t'$,

$$\lim_{R\to\infty} \sup_{\varepsilon\in(0,1)} \mathbf{P}\big(\big|Z_4^\varepsilon(t,x)\big| > R\big) = 0, \quad R > 0,$$

$$\lim_{\rho\to 0} \sup_{\varepsilon\in(0,1)} \mathbf{P}\Big(\sup_{|t-t'|+|x-y|\leqslant\rho} \big|Z_4^\varepsilon(t,x) - Z_4^\varepsilon(t',y)\big| > R\Big) = 0.$$

Using the Itô isometry, (7), (9) and Lemma 1, we obtain

$$\begin{aligned}\mathbf{E}\big(\big|Z_4^\varepsilon(t,x)\big|^2\big) &\leqslant C\mathbf{E}\Bigg(\int_0^t\int_D \big|G(t-s,\,x,\,y)\big|^2 \big\|\sigma\big(u_{\xi^\varepsilon}^{\beta(\varepsilon),\nu^\varepsilon}\big)(s,\cdot)\big\|_R^2 \,\mathrm{d}y\,\mathrm{d}s\Bigg) \\ &\leqslant C(T),\end{aligned}$$

which by Chebyshev's inequality leads to

$$\lim_{R\to\infty}\sup_{\varepsilon\in(0,1)}\mathbf{P}\bigl(\bigl|Z_4^\varepsilon(t,x)\bigr|>R\bigr)\leqslant\lim_{R\to\infty}\sup_{\varepsilon\in(0,1)}\frac{\mathbf{E}(|Z_4^\varepsilon(t,x)|^2)}{R^2}=0. \tag{22}$$

Similarly,

$$\mathbf{E}\bigl(\bigl|Z_4^\varepsilon(t,x)-Z_4^\varepsilon(t',y)\bigr|^2\bigr)\leqslant C\bigl(|t-t'|^{\gamma'}+|x-y|^{\gamma}\bigr)$$

from which, using Chebyshev's inequality, we get

$$\begin{aligned}&\lim_{\rho\to0}\sup_{\varepsilon\in(0,1)}\mathbf{P}\Bigl(\sup_{|t-t'|+|x-y|\leqslant\rho}\bigl|Z_4^\varepsilon(t,x)-Z_4^\varepsilon(t',y)\bigr|>R\Bigr)\\&\quad\leqslant\lim_{\rho\to0}\sup_{\varepsilon\in(0,1)}\frac{1}{R^2}\mathbf{E}\Bigl(\sup_{|t-t'|+|x-y|\leqslant\rho}\bigl|Z_4^\varepsilon(t,x)-Z_4^\varepsilon(t',y)\bigr|^2\Bigr)=0. \end{aligned}\tag{23}$$

Hence, from (22) and (23) the sequence Z_4^ε is tight in $C([0,T],L^2(D))$.

Step 2: Convergence. With the proof of tightness of Z_l^ε for $l=1,2,3,4,5$ at hand, by Prohorov's theorem a subsequence can be extracted along which each of the aforementioned processes and $u_{\xi^\varepsilon}^{\beta(\varepsilon),\nu^\varepsilon}$ converge in distribution in $C([0,T],L^2(D))$. Let Z_l^0 and $u_\xi^{0,\nu}$ denote the respective limits. We will show that

$$\begin{aligned}Z_1^0&=\int_D G(t,x,y)\,\xi(y)\,\mathrm{d}y,\qquad Z_2^0=\int_0^t\int_D G(t-s,\,x,\,y)\,u_\xi^{0,\nu}(s,y)\,\mathrm{d}y\,\mathrm{d}s,\\Z_3^0&=\int_0^t\int_D \nabla G(t-s,\,x,\,y)\,f\bigl(u_\xi^{0,\nu}(s,y)\bigr)\,\mathrm{d}y\,\mathrm{d}s,\qquad Z_4^0=0,\\Z_5^0&=\int_0^t\int_D G(t-s,\,x,\,y)\,\sigma\bigl(u_\xi^{0,\nu}(s,y)\bigr)\nu(s,y)\,\mathrm{d}y\,\mathrm{d}s.\end{aligned}$$

Case 1. The case $l=1$ follows from Lemma 9.

Case 2. When $l=2$, we invoke the Skorokhod representation theorem [22] and thus assume the almost sure convergence on a larger common probability space. Denote the right-hand side of Z_2^0 by $\bar Z_2^0$. Using Lemma 4 with $\alpha=0$, we have

$$\begin{aligned}&\bigl\|Z_2^\varepsilon-\bar Z_2^0\bigr\|_{X_T}^2\\&\quad\leqslant\mathbf{E}\Biggl(\sup_{0\leqslant t\leqslant T}\Biggl\|\int_0^t\int_D G(t-s,\,x,\,y)\bigl(u_{\xi^\varepsilon}^{\beta(\varepsilon),\nu^\varepsilon}-u_\xi^{0,\nu}\bigr)(s,y)\,\mathrm{d}y\,\mathrm{d}s\Biggr\|_{L^2}^2\Biggr)\\&\quad\leqslant CT\mathbf{E}\Bigl(\sup_{0\leqslant t\leqslant T}\bigl\|u_{\xi^\varepsilon}^{\beta(\varepsilon),\nu^\varepsilon}-u_\xi^{0,\nu}\bigr\|_{L^2}^2\Bigr),\end{aligned}$$

and thus converges to zero as $\varepsilon \to 0$ since $u_{\xi^\varepsilon}^{\beta(\varepsilon),\nu^\varepsilon} \to u_\xi^{0,\nu}$ by the fact that the limit is unique, and $\bar{Z}_2^0$ is a continuous random field using Theorem 4. Hence we conclude that $Z_2^0 = \bar{Z}_2^0$.

Case 3. When $l = 3$, denote the right-hand side of Z_3^0 by $\bar{Z}_3^0$ and invoke Skorokhod representation theorem as before. By using Lemma 4 with $|\alpha| = 1$ and (6) we have

$$\begin{aligned}
&\left\|Z_3^\varepsilon - \bar{Z}_3^0\right\|_{X_T}^2 \\
&\quad = \mathbf{E}\left(\sup_{0\leqslant t\leqslant T}\left\|\int_0^t\int_D \nabla G(t-s,\,x,\,y)\left[f\left(u_{\xi^\varepsilon}^{\beta(\varepsilon),\nu^\varepsilon}\right)(s,y) - f\left(u_\xi^{0,\nu}\right)(s,y)\right]\mathrm{d}y\,\mathrm{d}s\right\|_{L^2}^2\right) \\
&\quad \leqslant CT^{3/2}\mathbf{E}\sup_{0\leqslant t\leqslant T}\left\|u_{\xi^\varepsilon}^{\beta(\varepsilon),\nu^\varepsilon} - u_\xi^{0,\nu}\right\|_{L^2}^2.
\end{aligned}$$

Using a similar argument as in the previous case, $Z_3^0 = \bar{Z}_3^0$, and the right-hand side converges to zero as $\varepsilon \to 0$.

Case 4. When $l = 4$, let

$$\mathcal{T} = \int_0^t\int_D G(t-s,\,x,\,y)\,\sigma\left(u_{\xi^\varepsilon}^{\beta(\varepsilon),\nu^\varepsilon}(s,y)\right)\mathrm{d}y\,\mathrm{d}W_s(y).$$

As proven before, since $\mathcal{T}$ is tight, using [5, Lemma 3], Z_4^ε converges in probability in $C([0,T]\times L^2(D))$, which implies the corresponding convergence in $C([0,T],L^2(D))$.

Case 5. When $l = 5$, let us denote the right-hand side of Z_5^0 by $\bar{Z}_5^0$. As $\nu^\varepsilon \to \nu$ in distribution as $\varepsilon \to 0$ and since S_M endowed with weak topology is a Polish space, from the Skorokhod representation theorem there exists a probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \{\bar{\mathcal{F}}_t\}, \bar{\mathbf{P}})$. On this basis, a sequence of independent Brownian motions $\bar{W}_t$ and also a family of $\bar{\mathcal{F}}_t$-predictable processes $\{\bar{\nu}^\varepsilon\colon \varepsilon > 0\}$, $\bar{\nu}$ taking values in S_M $\mathbf{P}$-a.s. such that the joint law of $(\nu^\varepsilon, \nu, W_t)$ under $\mathbf{P}$ coincides with that of $(\bar{\nu}^\varepsilon, \bar{\nu}, \bar{W}_t)$ under $\bar{\mathbf{P}}$, and

$$\lim_{\varepsilon\to 0}\int_0^t\int_D \left(\bar{\nu}^\varepsilon(s,y) - \bar{\nu}(s,y)\right)g(s,y)\,\mathrm{d}y\,\mathrm{d}s = 0, \quad g \in L^2\left([0,T]\times D\right),\ \bar{\mathbf{P}}\text{-a.s.}$$

In what follows, we will write ν^ε, ν instead of $\bar{\nu^\varepsilon}$, $\bar{\nu}$ for simplicity:

$$\begin{aligned}
Z_5^\varepsilon - \bar{Z}_5^0 &= \int_0^t\int_D G(t-s,\,x,\,y)\left[\sigma\left(u_{\xi^\varepsilon}^{\beta(\varepsilon),\nu^\varepsilon}(s,y)\right)\nu^\varepsilon(s,y) - \sigma\left(u_\xi^{0,\nu}(s,y)\right)\nu(s,y)\right]\mathrm{d}y\,\mathrm{d}s \\
&= \int_D G(t-s,\,x,\,y)\,\sigma\left(u_{\xi^\varepsilon}^{\beta(\varepsilon),\nu^\varepsilon}(s,y)\right)\left[\nu^\varepsilon(s,y) - \nu(s,y)\right]\mathrm{d}y\,\mathrm{d}s \\
&\quad + \int_0^t\int_D G(t-s,\,x,\,y)\left[\sigma\left(u_{\xi^\varepsilon}^{\beta(\varepsilon),\nu^\varepsilon}(s,y)\right) - \sigma\left(u_\xi^{0,\nu}(s,y)\right)\right]\nu(s,y)\,\mathrm{d}y\,\mathrm{d}s \\
&= I_{51}^\varepsilon + I_{52}^\varepsilon.
\end{aligned}$$

Next, we will show that

$$\lim_{\varepsilon\to 0}\mathbf{E}\Bigl(\sup_{0\leqslant t\leqslant T}\bigl\|I_{51}^{\varepsilon}\bigr\|_{L^2}^2\Bigr)=0. \tag{24}$$

For any $(t,x)\in[0,T]\times D$, from assumption B and Lemma 1 we have

$$\int_0^T\int_D G^2(t-s,\,x,\,y)\,\sigma^2\bigl(u_{\xi^\varepsilon}^{\beta(\varepsilon),\nu^\varepsilon}(s,y)\bigr)\,\mathrm{d}y\,\mathrm{d}s\leqslant C(T)<\infty,$$

which implies that the function $\{G(t-s,\,x,\,y)\sigma(u_{\xi^\varepsilon}^{\beta(\varepsilon),\nu^\varepsilon}(s,y))\colon (s,y)\in[0,T]\times D\}$ takes values in S_M for some $M\in\mathbb{N}$. Since $\nu^\varepsilon\to\nu$ weakly in S_M, we get

$$\lim_{\varepsilon\to 0} I_{51}^{\varepsilon}=0\quad\text{a.s.}$$

Using Hölder's inequality and the fact that $\nu^\varepsilon,\nu\in S_M$, we get

$$\left|\int_D G(t-s,\,x,\,y)\sigma\bigl(u_{\xi^\varepsilon}^{\beta(\varepsilon),\nu^\varepsilon}(s,y)\bigr)\bigl[\nu^\varepsilon(s,y)-\nu(s,y)\bigr]\,\mathrm{d}y\,\mathrm{d}s\right|\leqslant C(T,M),$$

which implies that I_{51}^{ε} is uniformly bounded. Hence, by the dominated convergence theorem we obtain (24). Using (8), (9) and properties of controls, we get

$$\begin{aligned}\mathbf{E}\Bigl(\sup_{0\leqslant t\leqslant T}\bigl\|I_{52}^{\varepsilon}\bigr\|_{L^2}^2\Bigr)&\leqslant CM\mathbf{E}\Biggl(\sup_{0\leqslant t\leqslant T}\int_0^t\bigl\|u_{\xi^\varepsilon}^{\beta(\varepsilon),\nu^\varepsilon}-u_{\xi}^{0,\nu}\bigr\|_{L^2}^2\,\mathrm{d}s\Biggr)\\&\leqslant CTM\mathbf{E}\Bigl(\sup_{0\leqslant t\leqslant T}\bigl\|u_{\xi^\varepsilon}^{\beta(\varepsilon),\nu^\varepsilon}-u_{\xi}^{0,\nu}\bigr\|_{L^2}^2\Bigr).\end{aligned}\tag{25}$$

The RHS of (25) converges to zero since $u_{\xi^\varepsilon}^{\beta(\varepsilon),\nu^\varepsilon}\to u_{\xi}^{0,\nu}$ as $\varepsilon\to 0$.

Therefore, from (24) and (25) we get

$$\bigl\|Z_5^{\varepsilon}-\bar{Z}_5^{0}\bigr\|_{X_T}^2\to 0\quad\text{as }\varepsilon\to 0.$$

Again, using the fact that the limit is unique and $\bar{Z}_5^0$ is a continuous random field, it can be concluded that $Z_5^0=\bar{Z}_5^0$. Hence, from the uniqueness Theorem 4 it follows that along a subsequence, the controlled process converges to the skeleton equation. □

Verification of assumption (S1). Assumption (S1) follows from Theorem 4 and applying Theorem 7 with $\beta=0$.

Verification of assumption (S2). Assumption (S2) follows by applying Theorem 7 with $\beta(r)=r$, $r\in[0,1)$.

This concludes the proof of main theorem.

6 Conclusion

Thus, the uniform LDP is proved in a precise manner, where the proofs rely on some basic qualitative properties like existence, uniqueness, tightness of the controlled and zero noise analogues of the original system. The future research directions would include studies of exit time problems using the uniform LDP, moderate deviation principle and central limit theorem for the SKS equation. Further, one can focus upon the analysis of well-posedness, exit time, LDP, moderate deviations for the SKS equation and other similar SPDEs perturbed by a Levy noise.

References

1. G. Amali Paul Rose, M. Suvinthra, K. Balachandran, Large deviation principle for stochastic generalized KdV equation with additive noise, *Indian Journal of Industrial and Applied Mathematics*, **10**(2):178–193, 2019, `https://doi.org/10.5958/1945-919X.2019.00033.1`.
2. L. Bo, Y. Jiang, Large deviation for the nonlocal Kuramoto-Sivashinsky SPDE, *Nonlinear Anal. Theory Methods Appl.*, **82**:100–114, 2013, `https://doi.org/10.1016/j.na.2013.01.005`.
3. L. Bo, K. Shi, Y. Wang, On a nonlocal stochastic Kuramoto-Sivashinsky equation with jumps, *Stoch. Dynam.*, **7**(4):439–457, 2007, `https://doi.org/10.1142/S0219493707002104`.
4. A. Budhiraja, P. Dupuis, A variational representation for positive functionals of infinite dimensional Brownian motion, *Probab. Math. Stat.*, **20**(1):39–61, 2000, `https://www.math.uni.wroc.pl/~pms/files/20.1/Abstract/20.1.3.abs.pdf`.
5. A. Budhiraja, P. Dupuis, V. Maroulas, Large deviations for infinite dimensional stochastic dynamical systems, *Ann. Probab.*, **36**(4):1390–1420, 2008, `https://doi.org/10.1214/07-AOP362`.
6. C. Cardon - Weber, Large deviations for a Burgers'-type SPDE, *Stoch. Proc. Appl.*, **84**(1):53–70, 1999, `https://doi.org/10.1016/S0304-4149(99)00047-2`.
7. C. Cardon - Weber, Cahn - Hilliard stochastic equation: existence of the solution and of its density, *Bernoulli*, **7**(5):777–816, 2001, `https://doi.org/10.2307/3318542`.
8. E. Cerpa, A. Mercado, A.F. Pazoto, On the boundary control of a parabolic system coupling KS-KdV and heat equations, *Scientia Series A: Mathematical Sciences*, **22**:55–74, 2012, `http://scientia.mat.utfsm.cl/archivos/vol22/articulo6Vol2012Rev.pdf`.
9. G. Da Prato, J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, Cambridge University Press, Cambridge, 1992, `https://doi.org/10.1017/CBO9780511666223`.
10. A. Dembo, O. Zeitouni, *Large Deviations Techniques and Applications*, Springer - Verlag, New York, 2010, `https://doi.org/10.1007/978-3-642-03311-7`.
11. J.D. Deuschel, D.W. Stroock, *Large Deviations*, Academic Press, San Diego, 1989, `https://doi.org/10.1090/chel/342`.

12. J. Duan, V.J. Ervin, On the stochastic Kuramoto-Sivashinsky equation, *Nonlinear Anal. Theory Methods Appl.*, **44**(2):205–216, 2001, `https://doi.org/10.1016/S0362-546X(99)00259-X`.

13. P. Dupuis, R.S. Ellis, *A Weak Convergence Approach to the Theory of Large Deviations*, John Wiley & Sons, New York, 1997, `https://doi.org/10.1002/9781118165904`.

14. S.D. Èĭdel′man, N.V. Ivašisen, Investigation of the Green's matrix of a homogeneous parabolic boundary value problem, *Tr. Mosk. Mat. Obs.*, **23**:179–234, 1970, `http://www.mathnet.ru/php/archive.phtml?wshow=paper&jrnid=mmo&paperid=242&option_lang=eng`.

15. B. Ferrario, Invariant measures for a stochastic Kuramoto-Sivashinsky equation, *Stoch. Anal. Appl.*, **26**(2):379–407, 2008, `https://doi.org/10.1080/07362990701857335`.

16. W.H. Fleming, A stochastic control approach to some large deviations problems, in C. Dolcetta, W.H. Fleming, T. Zolezzi (Eds.), *Recent Mathematical Methods in Dynamic Programming: Proceedings of the conference held in Rome, Italy, March 26-28, 1984*, Springer - Verlag, Berlin, 1985, pp. 52–66, `https://doi.org/10.1007/BFb0074780`.

17. M. Foondun, L. Setayeshgar, Large deviations for a class of semilinear stochastic partial differential equations, *Stat. Probab. Lett.*, **121**:143–151, 2017, `https://doi.org/10.1016/j.spl.2016.10.019`.

18. M.I. Friedlin, A.D. Wentzell, *Random Perturbations of Dynamical Systems*, Springer, New York, 1984, `https://doi.org/10.1007/978-1-4684-0176-9`.

19. O. Kallenberg, *Foundations of Modern Probability*, Springer, New York, 2002, `https://doi.org/10.1007/978-1-4757-4015-8`.

20. I. Karatzas, S.E. Shreve, *Brownian Motion and Stochastic Calculus*, Springer, New York, 1998, `https://doi.org/10.1007/978-1-4612-0949-2`.

21. Y. Kuramoto, T. Tsuzuki, On the formation of dissipative structures in reaction-diffusion systems: Reductive perturbation approach, *Prog. Theor. Phys.*, **54**(3):687–699, 1975, `https://doi.org/10.1143/PTP.54.687`.

22. H.J. Kushner, P. Dupuis, *Numerical Methods for Stochastic Control Problems in Continuous Time*, Springer, New York, 1992, `https://doi.org/10.1007/978-1-4684-0441-8`.

23. V. Maroulas, Large deviations for infinite-dimensional stochastic system with jumps, *Mathematika*, **57**(1):175–192, 2011, `https://doi.org/10.1112/S0025579310001282`.

24. L. Setayeshgar, Large deviations for a stochastic Burgers' equation, *Commun. Stoch. Anal.*, **8**(2):141–154, 2014, `https://doi.org/10.31390/cosa.8.2.01`.

25. G.I. Sivashinsky, Nonlinear analysis of hydrodynamic instability in laminar flames - I. Derivation of basic equations, *Acta Astronaut.*, **4**(11–12):1177–1206, 1977, `https://doi.org/10.1016/0094-5765(77)90096-0`.

26. S.S. Sritharan, P. Sundar, Large deviations for the two-dimensional Navier-Stokes equations with multiplicative noise, *Stoc. Proc. Appl.*, **116**(11):1636–1659, 2006, `https://doi.org/10.1016/j.spa.2006.04.001`.

27. M. Suvinthra, K. Balachandran, Large deviations for the stochastic predator-prey model with nonlinear functional response, *J. Appl. Probab.*, **54**(2):507–521, 2017, `https://doi.org/10.1017/jpr.2017.14`.

28. M. Suvinthra, S.S. Sritharan, K. Balachandran, Large deviations for stochastic tidal dynamics equation, *Commun. Stoch. Anal.*, **9**(4):477–502, 2015, https://doi.org/10.31390/cosa.9.4.04.
29. S.R.S. Varadhan, *Large Deviations and Applications*, SIAM, Philadelphia, 1984, https://dx.doi.org/10.1137/1.9781611970241.
30. W. Wu, S. Cui, J. Duan, Global well-posedness of the stochastic generalized Kuramoto-Sivashinsky equation with multiplicative noise, *Acta Math. Appl. Sin. English Series*, **34**(3):566–584, 2018, https://doi.org/10.1007/s10255-018-0769-3.