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Asymptotic analysis of Sturm–Liouville problem with nonlocal integral-type boundary condition

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Abstract. In this study, we obtain asymptotic formulas for eigenvalues and eigenfunctions of the one-dimensional Sturm–Liouville equation with one classical-type Dirichlet boundary condition and integral-type nonlocal boundary condition. We investigate solutions of special initial value problem and find asymptotic formulas of arbitrary order. We analyze the characteristic equation of the boundary value problem for eigenvalues and derive asymptotic formulas of arbitrary order. We apply the obtained results to the problem with integral-type nonlocal boundary condition.

Keywords: Sturm–Liouville problem, nonlocal integral condition, asymptotics of eigenvalues and eigenfunctions.

1 Introduction

Consider the following one-dimensional Sturm–Liouville equation:

$$-u''(t) + q(t)u(t) = \lambda u(t), \quad t \in (0,1), \tag{1}$$

where the real-valued function $q \in C[0,1]$; $\lambda = s^2$ is a complex spectral parameter, and $s = x + \imath y$; $x, y \in \mathbb{R}$.

Remark 1. In this article, $s \in \mathbb{C}_s := \mathbb{R}_s \cup \mathbb{C}_s^+ \cup \mathbb{C}_s^-$, where $\mathbb{R}_s := \mathbb{R}_s^- \cup \mathbb{R}_s^+ \cup \mathbb{R}_s^0$, $\mathbb{R}_s^- := \{s = x + \imath y \in \mathbb{C}\colon x = 0,\ y > 0\}$, $\mathbb{R}_s^+ := \{s = x + \imath y \in \mathbb{C}\colon x > 0,\ y = 0\}$, $\mathbb{R}_s^0 := \{s = 0\}$, $\mathbb{C}_s^+ := \{s = x + \imath y \in \mathbb{C}\colon x > 0,\ y > 0\}$ and $\mathbb{C}_s^- := \{s = x + \imath y \in \mathbb{C}\colon x > 0,\ y < 0\}$. Then a map $\lambda = s^2$ is the bijection between $\mathbb{C}_s$ and $\mathbb{C}_\lambda := \mathbb{C}$ [21].

In this study, we will investigate nonlocal eigenvalue problems, which consist of equation (1) on $[0,1]$ with one classical (local) Boundary Condition (BC)

$$u(0) = 0, \tag{2}$$

another integral-type Nonlocal Boundary Condition (NBC)

$$u(1)=\gamma\int_0^{\xi} u(t)\,\mathrm{d}t,\quad \xi\in(0,1],\quad \text{(Case 1)} \tag{3}$$

or

$$u(1)=\gamma\int_{\xi}^{1} u(t)\,\mathrm{d}t,\quad \xi\in[0,1),\quad \text{(Case 2)}, \tag{4}$$

where $\gamma\in\mathbb{R}$. Both these BC we can write as

$$u(1)=\gamma\int_{\alpha}^{\beta} u(t)\,\mathrm{d}t, \tag{5}$$

where $\alpha=0$, $\beta=\xi$ in Case 1, and $\alpha=\xi$, $\beta=1$ in Case 2. We note that $0\leqslant\alpha<\beta\leqslant 1$.

The first work on Boundary Value Problems (BVPs) with nonlocal integral-type BCs belongs to J.R. Cannon [2]. These kinds of BCs together with a parabolic equation arise, for example, in the study of the process of heat transmission in a thin heated rod when the part of the rod adjoining one of its ends [7]. Parabolic equations with NBCs are also encountered in the study of the heat conduction within linear thermo elasticity [3,4].

Eigenvalues and eigenfunctions of BVPs with integral-type NBCs and discrete case have been investigated in [1,6,8,10,11,16,19]. Structure of eigenvalues of multi-point BVPs were presented in [5,13,14]. The spectrum structure of one-dimensional differential operator with nonlocal conditions and of the difference operator, corresponding to it, has been exhaustively investigated in [15]. A more comprehensive list can be found in the survey article [20].

Spectral asymptotics of eigenvalues and eigenfunctions of SLPs with Bitsadze–Samarskii-type NBC

$$u(1)=\gamma u(\xi),\quad \xi\in(0,1), \tag{6}$$

where $\gamma\in\mathbb{R}$, have been investigated recently [17,18]. In [17], for sufficiently large k and $|\gamma|<1$, it is derived that the asymptotic expansions

$$s_k=x_k+\mathcal{O}\bigl(k^{-1}\bigr),\qquad u_k(t)=-\frac{\sin(x_k t)}{x_k}+\mathcal{O}\bigl(k^{-2}\bigr) \tag{7}$$

are valid for eigenvalues and eigenfunctions, respectively, for the SLP (1)–(2), (6), where x_k, $k\in\mathbb{N}$, are the positive roots of $\sin x-\gamma\sin(\xi x)=0$. Under the condition $q\in C^1[0,1]$, it is obtained that the asymptotic formulas

$$s_k=x_k+Q_1(x_k)x_k^{-1}+\mathcal{O}\bigl(k^{-2}\bigr), \tag{8}$$

$$u_k(t)=-\frac{\sin(x_k t)}{x_k}+\bigl(Q(t)-tQ_1(x_k)\bigr)\frac{\cos(x_k t)}{x_k^2}+\mathcal{O}\bigl(k^{-3}\bigr) \tag{9}$$

are valid for eigenvalues and eigenfunctions, respectively, for the SLP (1)–(2), (6). Here

$$Q_1(s) := \frac{Q(1)\cos s - \gamma Q(\xi)\cos(\xi s)}{\cos s - \gamma\xi\cos(\xi s)}, \qquad Q(t) = \frac{1}{2}\int_0^t q(\tau)\,\mathrm{d}\tau. \tag{10}$$

In [18] the authors consider the equation with retarded argument

$$-u''(t) + q(t)u\bigl(t - \varDelta(t)\bigr) = \lambda u(t), \quad t \in (0,1), \tag{11}$$

together with the BCs (2), (6), where the real-valued function $q(t) \in C[0,1]$; the real-valued function $\varDelta(t) \geqslant 0$ is continuous on $[0,1]$, $\lambda = s^2$ is a complex spectral parameter. They calculate the asymptotics of eigenvalues and eigenfunctions. To speak more precise, under the conditions $q \in C^1[0,1]$, $\varDelta''(t)$ exist and bounded in $[0,1]$, $\varDelta'(t) \leqslant 1$ in $[0,1]$, $\varDelta(0) = 0$ and $|\gamma| < 1$ they find the asymptotic formulas

$$\begin{aligned} s_k &= x_k + Q_1(x_k)x_k^{-1} + \mathcal{O}\bigl(k^{-2}\bigr), \\ u_k(t) &= -\frac{\sin(x_k t)}{x_k} + A(x_k,t)\frac{\sin(x_k t)}{{x_k}^2} + \bigl(B(x_k,t) - tQ_1(x_k)\bigr)\frac{\cos(x_k t)}{x_k^2} \\ &\quad + \mathcal{O}\bigl(k^{-3}\bigr) \end{aligned}$$

for eigenvalues and eigenfunctions, respectively, for the SLP (11), (2), (6), where

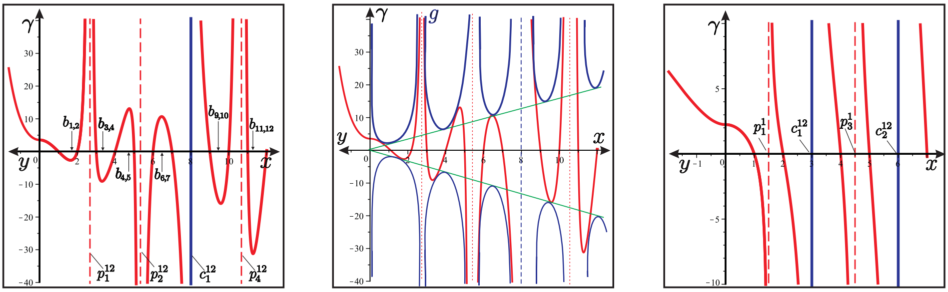
$$\begin{aligned} A(s,t) &:= \frac{1}{2}\int_0^t q(\tau)\sin\bigl(s\varDelta(\tau)\bigr)\,\mathrm{d}\tau, \qquad B(s,t) := \frac{1}{2}\int_0^t q(\tau)\cos\bigl(s\varDelta(\tau)\bigr)\,\mathrm{d}\tau, \\ Q_1(s) &:= \frac{A(s,1)\sin s + B(s,1)\cos s - \gamma A(s,\xi)\sin(\xi s) - \gamma B(s,\xi)\cos(\xi s)}{\cos s - \gamma\xi\cos(\xi s)}. \end{aligned}$$

In both of these studies, it is proven that $\cos x_k - \gamma\xi\cos(\xi x_k) \neq 0$. Furthermore, in these articles the authors prove the simplicity and countability of eigenvalues and show that all eigenvalues are real.

The article is organized as follows. The statement of the problem and a literature review are given in Section 1. In Sections 1–3, notation and definitions used in the paper are stated. In Section 2, some results about the case $q \equiv 0$ are presented. In Section 3, we write the fundamental solutions of the Initial Value Problem (IVP) and find formulas for their asymptotics. In Section 4, we analyze the characteristic equation of the BVP (1)–(2), (5). In Section 5, we investigate the distribution of eigenvalues and obtain asymptotic formulas for eigenvalues and eigenfunctions. Also, we calculate normalized eigenfunctions.

2 Properties of a spectrum in the case $q \equiv 0$

In the case $q(t) \equiv 0$ the spectrum of problems (1)–(3) and (1)–(2), (4) have countably many eigenvalues [11, 16]. A unique negative eigenvalue exists for $\gamma > 2/\xi^2$ in Case 1



(a) Case 1, $\xi = 3/4$ (b) Case 1, $\xi = 3/4$ (c) Case 2, $\xi = 1/3$

Figure 1. Real Characteristic Functions $\gamma(s)$ for $s = \pi x$, $x > 0$, and $s = \imath\pi y$, $y \geqslant 0$. (b) $g = \pi x/(1-\cos(\pi\xi x))$ – amplitude modulation function.

and $\gamma > 2/(1-\xi^2)$ in Case 2. Also, $\lambda = 0$ is eigenvalue if and only if $\gamma = 2/\xi^2$ in Case 1 and $\gamma = 2/(1-\xi^2)$ in Case 2.

Let us define a *Constant Eigenvalue* (CE) as the eigenvalue λ, which does not depend on the parameter $\gamma \in \mathbb{R}$ for fixed ξ. In [11] the spectrum and eigenfunctions with BC (2) and integral-type BCs (3) and (4) were investigated for the case $q(t) \equiv 0$. Constant eigenvalues exist only for rational numbers $\xi = m/n \in [0,1]$, and those eigenvalues $\lambda_k = \pi^2 q_k^2$, $k \in \mathbb{N}$, are given by: $q_k = nk$ for $m \in \mathbb{N}_{\text{even}}$ and $q_k = 2nk$ for $m \in \mathbb{N}_{\text{odd}}$ in Case 1; $q_k = nk$ for $n - m \in \mathbb{N}_{\text{even}}$ and $q_k = 2nk$ for $n - m \in \mathbb{N}_{\text{odd}}$ in Case 2. So, all CE are positive.

All nonconstant (that depend on the parameter $\gamma \in \mathbb{R}$) eigenvalues $\lambda = s^2$, $s \in \mathbb{C}_s$, are γ-points of the Characteristic Function $\gamma : \mathbb{C}_s \to \mathbb{R}$ [21]

$$\gamma(s) = \frac{s \sin s}{2\sin^2(\xi s/2)} = \frac{s\ \sin s}{1-\cos(\xi s)} \qquad \text{(Case 1)}, \qquad (12)$$

$$\gamma(s) = \frac{s \sin s}{2\sin((1+\xi)s/2)\sin((1-\xi)s/2)} = \frac{s\ \sin s}{\cos(\xi s) - \cos s} \qquad \text{(Case 2)}. \qquad (13)$$

So, for fixed $\gamma \in \mathbb{R}$, the roots of this meromorphic function describe nonconstant eigenvalues. The graphs of CF on $\mathbb{R}_s$ are presented in Fig. 1(a) in Case 1 and Fig. 1(c) in Case 2.

In Case 2, all nonconstant eigenvalues are real and simple [11, 12]. All poles of CF belong to one of the families of the first order poles:

$$\mathcal{P}_\xi^1 = \left\{ p_k^1 = \frac{2\pi k}{1+\xi},\ k \in \mathbb{N} \right\}, \qquad \mathcal{P}_\xi^2 = \left\{ p_l^2 := \frac{2\pi l}{1-\xi},\ l \in \mathbb{N} \right\}.$$

If $\xi \notin \mathbb{Q}$, then $\mathcal{P}_\xi^1 \cap \mathcal{P}_\xi^2 = \emptyset$.

Lemma 1. (*See* [11].) *If $\xi = m/n \in \mathbb{Q}$, then in Case 2, points $p_j^{12} = c_j^{12} = \pi q_j$, $j \in \mathbb{N}$, where $q_j = nj$ for $n - m \in \mathbb{N}_{even}$ or $q_j = 2nj$ for $n - m \in \mathbb{N}_{odd}$ are the first-order poles of CF and CE points. A set of these points is intersection of $\mathcal{P}_\xi^1$ and $\mathcal{P}_\xi^2$.*

So, all poles of CF are of the first order. We can enumerate all poles in nondecreasing order: p_k, $k \in \mathbb{N}$. If a pole is CE point, then we write it twice $p_k = p_{k+1}$, else $p_k < p_{k+1}$. Additionaly, we denote $p_0 = 0$. Then in Case 2, we can enumerate positive eigenvalues $\lambda_k = x_k^2$, $k \in \mathbb{N}$, where $x_k \in (p_{k-1}, p_k)$ for nonconstant eigenvalues, and $x_k = p_k = p_{k+1}$ for CE. Note that $x_k = \pi k$, $k \in \mathbb{N}$, in the case $\gamma = 0$ and $|x_k - \pi k| < \pi$ for all k.

In Case 1, nonconstant eigenvalues can be complex [11, 12]. All poles of CF belong to the family

$$\mathcal{P}_\xi^{12} = \left\{ p_k^{12} = \frac{2\pi k}{\xi},\ k \in \mathbb{N} \right\}.$$

If $\xi \notin \mathbb{Q}$, then all poles are of the second order.

Lemma 2. (See [11].) *If $\xi = m/n \in \mathbb{Q}$, then in Case 1, points $p_j^{12} = c_j^{12} = \pi q_j$, $j \in \mathbb{N}$, where $q_j = mnj$ for $m \in \mathbb{N}_{even}$ or $q_j = 2mnj$ for $m \in \mathbb{N}_{odd}$ are the first-order poles of CF and CE points, else we have the second-order poles.*

Lemma 3. *Let x_k, $k \in \mathbb{N}$, be eigenvalues of problem* (1)–(3) *in the case $q \equiv 0$. Then exists $K \in \mathbb{N}$ such that for fixed $\gamma \in \mathbb{R}$, all eigenvalues x_k, $k \geqslant K$, are positive, simple and $x_k \in (\pi k - \pi, \pi k + \pi)$, i.e. $|x_k - \pi k| < \pi$ for all $k \geqslant K$.*

Proof. For not simple positive eigenvalues we have

$$\gamma \cos(\xi x) = \gamma - x \sin x, \qquad \xi\gamma \sin(\xi x) = x \cos x + \sin x.$$

From this system we get

$$\xi^2\gamma^2 = \xi^2(\gamma - x \sin x)^2 + (x \cos x + \sin x)^2.$$

Then we estimate

$$\begin{aligned} |\gamma| &\geqslant |\gamma| \cdot |\sin x| \\ &= \frac{x(\xi^{-2} - (\xi^{-2} - 1)\sin^2 x)}{2} + \frac{\xi^{-2}(\sin(2x) + \frac{\sin^2 x}{x})}{2} \\ &\geqslant \frac{x}{2} - \xi^{-2}. \end{aligned}$$

So, all eigenvalues in the angle $|\gamma| < x/\pi$ for $x \geqslant 8\xi^{-2}$ are positive and simple. CE points are the first-order poles of CF. Eigenvalues corresponding to these points are positive and simple. Since CF has zeros at points πk, $k \in \mathbb{N}$, we have $|x_k - \pi k| < \pi$. $\square$

3 Solutions of initial value problem and their asymtotics

Let $\lambda = s^2$, $s \in \mathbb{C}_s$, and $\omega_s(t)$ be a solution of equation (1) satisfying the initial conditions

$$\omega_s(0) = 0, \qquad \omega_s'(0) = -1. \tag{14}$$

According to [9, Chap. I, Thm. 1.1], this IVP determine a unique solution of (1) on $[0, 1]$. The function $\omega_s(t) = \omega(t, s)$ is an analytic (holomorphic) function of s. We will use notation for derivatives $\omega_s'(t) := \partial\omega(t,s)/\partial t$, $(\omega_s)_s^{(l)}(t,s) := \partial^l\omega(t,s)/\partial s^l$, $(\omega_s')_s^{(l)}(t,s) := \partial^{l+1}\omega(t,s)/\partial t\partial s^l$.

Lemma 4. (*See* [17].) *Let* $\omega_s(t)$ *be a solution of IVP* (1), (14). *Then the following integral equation holds*:

$$\omega_s(t) = -\frac{1}{s}\sin(st) + \frac{1}{s}\int_0^t q(\tau)\sin\bigl(s(t-\tau)\bigr)\omega_s(\tau)\,\mathrm{d}\tau. \tag{15}$$

We will use notation for integrals ($k \in \mathbb{N}_0 := \{0\} \cup \mathbb{N}$):

$$\begin{aligned}
I_s^k(t,q,f) &= \int_0^t q(\tau)(t-\tau)^k \sin\bigl(s(t-\tau)\bigr) f(\tau)\,\mathrm{d}\tau,\\
J_s^k(t,q,f) &= \int_0^t q(\tau)(t-\tau)^k \cos\bigl(s(t-\tau)\bigr) f(\tau)\,\mathrm{d}\tau,\\
\tilde{I}_s^k(t,q,f) &= \int_0^t q(\tau)(t-\tau)^k \sin\bigl(s(t-\tau)\bigr)\mathrm{e}^{-|y|(t-\tau)} f(\tau)\,\mathrm{d}\tau,\\
\tilde{J}_s^k(t,q,f) &= \int_0^t q(\tau)(t-\tau)^k \cos\bigl(s(t-\tau)\mathrm{e}\bigr)\mathrm{e}^{-|y|(t-\tau)} f(\tau)\,\mathrm{d}\tau.
\end{aligned}$$

We note that $I_s^k(t,q,f) = I_s^0(t,q(\tau)(t-\tau)^k,f)$, $J_s^k(t,q,f) = J_s^0(t,q(\tau)(t-\tau)^k,f)$. Then we rewrite equation (15) as

$$\omega_s(t) = -\sin(st)s^{-1} + I_s^0(t,q,\omega_s)s^{-1}. \tag{16}$$

Taking derivative with respect to t and s in (15), we get

$$\omega_s'(t) = -\cos(st) + J_s^0(t,q,\omega_s), \tag{17}$$

$$(\omega_s)_s'(t,s) = -t\cos(st)s^{-1} + \bigl(I_s^0\bigl(t,q,(\omega_s)_s'\bigr) + J_s^1(t,q,\omega_s) - \omega_s(t)\bigr)s^{-1}. \tag{18}$$

For derivatives of the second order, we get formulas

$$(\omega_s')_s'(t,s) = t\sin(st) + J_s^0\bigl(t,q,(\omega_s)_s'\bigr) - I_s^1(t,q,\omega_s), \tag{19}$$

$$\begin{aligned}
(\omega_s)_s''(t,s) = t^2\sin(st)s^{-1} + \bigl(&I_s^0\bigl(t,q,(\omega_s)_s''\bigr) + 2J_s^1\bigl(t,q,(\omega_s)_s'\bigr)\\
&- I_s^2(t,q,\omega_s) - 2(\omega_s)_s'(t)\bigr)s^{-1}.
\end{aligned} \tag{20}$$

Remark 2. The following formulas

$$(\omega_s)_s^{(l)}(t,s) = -\frac{\partial^l \sin(st)}{\partial s^l}s^{-1} - l(\omega_s)_s^{(l-1)}s^{-1} + s^{-1}\sum_{j=0}^{l}(-1)^{\lfloor j/2\rfloor}\binom{l}{j}\mathcal{I}_s^j\big(t,q,(\omega_s)_s^{(l-j)}\big),$$

$$(\omega_s')_s^{(l)}(t,s) = -\frac{\partial^l \cos(st)}{\partial s^l} + \sum_{j=0}^{l}(-1)^{\lfloor j/2+1/2\rfloor}\binom{l}{j}\mathcal{J}_s^j\big(t,q,(\omega_s)_s^{(l-j)}\big), \quad l\in\mathbb{N},$$

are valid, where $\mathcal{I}_s^{2k-2} = I_s^{2k-2}$, $\mathcal{I}_s^{2k-1} = J_s^{2k-1}$, $\mathcal{J}_s^{2k-2} = J_s^{2k-2}$, $\mathcal{J}_s^{2k-1} = I_s^{2k-1}$, $k\in\mathbb{N}$.

Lemma 5. *Let $s\in\mathbb{C}_s$. Then there exists $q_0>0$ such that for $|s|\geqslant q_0$, the asymptotic formulas*

$$\omega_s(t) = -\sin(st)s^{-1} + \mathcal{O}\big(s^{-2}\mathrm{e}^{|y|t}\big), \tag{21}$$

$$(\omega_s)_s'(t,s) = -t\cos(st)s^{-1} + \mathcal{O}\big(s^{-2}\mathrm{e}^{|y|t}\big), \tag{22}$$

$$(\omega_s)_s''(t,s) = t^2\sin(st)s^{-1} + \mathcal{O}\big(s^{-2}\mathrm{e}^{|y|t}\big), \tag{23}$$

$$\omega_s'(t) = -\cos(st) + \mathcal{O}\big(s^{-1}\mathrm{e}^{|y|t}\big), \tag{24}$$

$$(\omega_s')_s'(t,s) = t\sin(st) + \mathcal{O}\big(s^{-1}\mathrm{e}^{|y|t}\big), \tag{25}$$

are valid. These formulas hold uniformly for $0\leqslant t\leqslant 1$.

Proof. Put $\omega_s(t) = \mathrm{e}^{|y|t}F_s(t)$, $(\omega_s)_s'(t,s) = \mathrm{e}^{|y|t}G_s(t)$, $(\omega_s)_s''(t,s) = \mathrm{e}^{|y|t}H_s(t)$, $\omega_s'(t) = \mathrm{e}^{|y|t}K_s(t)$ and $(\omega_s')_s'(t,s) = \mathrm{e}^{|y|t}L_s(t)$. Then from (16)–(20) we obtain

$$F_s(t) = -\sin(st)\mathrm{e}^{-|y|t}s^{-1} + \tilde{I}_s^0(t,q,F_s)s^{-1}, \tag{26}$$

$$G_s(t) = -t\cos(st)\mathrm{e}^{-|y|t}s^{-1} + \big(\tilde{I}_s^0(t,q,G_s) + \tilde{J}_s^1(t,q,F_s) - F_s(t)\big)s^{-1}, \tag{27}$$

$$H_s(t) = +t^2\sin(st)\mathrm{e}^{-|y|t}s^{-1} + \big(\tilde{I}_s^0(t,q,H_s) + 2\tilde{J}_s^1(t,q,G_s) - \tilde{I}_s^2(t,q,F_s) - 2G_s(t)\big)s^{-1}, \tag{28}$$

$$K_s(t) = -\cos(st)\mathrm{e}^{-|y|t} + \tilde{J}_s^0(t,q,F_s), \tag{29}$$

$$L_s(t) = t\sin(st)\mathrm{e}^{-|y|t} + \tilde{J}_s^0(t,q,G_s) - \tilde{I}_s^1(t,q,F_s). \tag{30}$$

Let $\mu_s = \max_{0\leqslant t\leqslant 1}|F_s(t)|$, $\nu_s = \max_{0\leqslant t\leqslant 1}|G_s(t)|$, $\sigma_s = \max_{0\leqslant t\leqslant 1}|H_s(t)|$, $\varkappa_s = \max_{0\leqslant t\leqslant 1}|K_s(t)|$, $\kappa_s = \max_{0\leqslant t\leqslant 1}|L_s(t)|$ and $q_0 := 2\int_0^1|q(\tau)|\,\mathrm{d}\tau$. Since

$$|\sin s|\,\mathrm{e}^{-|y|} \leqslant \frac{1}{2}\big(\mathrm{e}^{y}+\mathrm{e}^{-y}\big)\,\mathrm{e}^{-|y|} = \frac{1}{2}\big(\mathrm{e}^{|y|}+\mathrm{e}^{-|y|}\big)\,\mathrm{e}^{-|y|} \leqslant 1,$$

$|\cos s|\mathrm{e}^{-|y|} \leqslant 1$ and $t \leqslant 1$, from (26)–(28) we have

$$\mu_s \leqslant \frac{|s|^{-1}q_0\mu_s}{2} + |s|^{-1}, \qquad \nu_s \leqslant \frac{|s|^{-1}q_0\nu_s}{2} + |s|^{-1}\left(1 + \frac{q_0\mu_s}{2} + \mu_s\right),$$
$$\sigma_s \leqslant \frac{|s|^{-1}q_0\sigma_s}{2} + |s|^{-1}\left(1 + q_0\nu_s + \frac{q_0\mu_s}{2} + 2\nu_s\right).$$

If $|s| \geqslant q_0$, then

$$\mu_s \leqslant 2|s|^{-1} = \mathcal{O}\big(s^{-1}\big), \qquad \nu_s \leqslant |s|^{-1}(2 + q_0\mu_s + 2\mu_s) = \mathcal{O}\big(s^{-1}\big), \tag{31}$$
$$\sigma_s \leqslant |s|^{-1}(2 + 2q_0\nu_s + q_0\mu_s + 4\nu_s) = \mathcal{O}\big(s^{-1}\big). \tag{32}$$

It follows from (29), (30) that $\varkappa_s \leqslant 1+q_0\mu_s/2 = \mathcal{O}(1)$, $\kappa_s \leqslant 1+q_0(\mu_s+\nu_s)/2 = \mathcal{O}(1)$. So, we prove asymptotic formulas

$$\omega_s(t),\ (\omega_s)'_s(t,s),\ (\omega_s)''_s(t,s) = \mathcal{O}\big(s^{-1}\mathrm{e}^{|y|t}\big), \qquad \omega'_s(t),\ (\omega'_s)'_s(t,s) = \mathcal{O}\big(\mathrm{e}^{|y|t}\big).$$

Now, substituting formulas (31)–(32) into the integrals of (26)–(30), we obtain

$$F_s(t) = -\sin(st)s^{-1}\mathrm{e}^{-|y|t} + \mathcal{O}\big(s^{-2}\big), \qquad K_s(t) = -\cos(st)\mathrm{e}^{-|y|t} + \mathcal{O}\big(s^{-1}\big),$$
$$G_s(t) = -t\cos(st)s^{-1}\mathrm{e}^{-|y|t} + \mathcal{O}\big(s^{-2}\big), \qquad L_s(t) = t\sin(st)\mathrm{e}^{-|y|t} + \mathcal{O}\big(s^{-1}\big),$$
$$H_s(t) = t^2\sin(st)s^{-1}\mathrm{e}^{-|y|t} + \mathcal{O}\big(s^{-2}\big).$$

Lemma is proved. □

Remark 3. $q_0 := 2\int_0^1 |q(\tau)|\,\mathrm{d}\tau$.

Remark 4. The asymptotic formulas (21)–(24) were proved in [17].

Remark 5. The asymptotic formulas

$$(\omega_s)^{(l)}_s(t,s) = (-1)^l t^l \cos\left(st + \frac{1}{2}\pi(1-l)\right)s^{-1} + \mathcal{O}\big(s^{-2}\mathrm{e}^{|y|t}\big),$$
$$(\omega'_s)^{(l)}_s(t,s) = (-1)^{l-1} t^l \cos\left(st - \frac{1}{2}\pi l\right) + \mathcal{O}\big(s^{-1}\mathrm{e}^{|y|t}\big)$$

are valid for $l = 0, 1, \ldots$. The proof is the same as in Lemma 5.

Corollary 1. *Let* $x \in \mathbb{R}$, $\delta \in \mathbb{R}$, $q \in C[0,1]$. *If* $s = x + \delta$, $\delta = \mathcal{O}(x^{-1})$, *then we have the following formulas*:

$$\omega_s(t) = -\sin(xt)x^{-1} + \mathcal{O}\big(x^{-2}\big), \qquad \omega'_s(t) = -\cos(xt) + \mathcal{O}\big(x^{-1}\big). \tag{33}$$

Proof. We substitute expressions (21)–(23) ($y = 0$) into formula

$$\omega_s(t) = \omega(t,\, x+\delta) = \omega_x(t) + (\omega_s)'_s(t,x)\delta + (\omega_s)''_s(t,\, x+\theta\delta)\frac{\delta^2}{2}, \quad \theta \in [0,1],$$

and get

$$\begin{aligned}\omega_s(t) &= -\sin(xt)x^{-1} + \mathcal{O}(x^{-2}) + \big(-t\cos(xt)x^{-1} + \mathcal{O}(x^{-2})\big)\delta + \mathcal{O}(\delta^2)\\ &= -\sin(xt)x^{-1} + \mathcal{O}(x^{-2}).\end{aligned}$$

Analogously, we prove the second formula

$$\omega_s'(t) = \omega'(t,\, x+\delta) = \omega_x'(t) + \mathcal{O}(\delta) = -\cos(xt) + \mathcal{O}(x^{-1}). \qquad \square$$

Lemma 6. *Let* $f \in C^1[a,b]$, $t \in [a,b] \subset [0,1]$. *Then the following asymptotic formulas*

$$\int_a^b f(\tau)\mathrm{e}^{\pm \imath s\tau}\,\mathrm{d}\tau = \mathcal{O}(s^{-1}\mathrm{e}^{|y|b}), \tag{34}$$

$$\int_a^b f(\tau)\cos(s\tau)\,\mathrm{d}\tau = \mathcal{O}(s^{-1}\mathrm{e}^{|y|b}), \qquad \int_a^b f(\tau)\sin(s\tau)\,\mathrm{d}\tau = \mathcal{O}(s^{-1}\mathrm{e}^{|y|b}), \tag{35}$$

$$\int_a^b f(\tau)\cos(2s\tau - st)\,\mathrm{d}\tau = \mathcal{O}(s^{-1}\mathrm{e}^{3|y|b}),$$

$$\int_a^b f(\tau)\sin(2s\tau - st)\,\mathrm{d}\tau = \mathcal{O}(s^{-1}\mathrm{e}^{3|y|b})$$

are valid.

Proof. We use integration by parts formula

$$\begin{aligned}\int_a^b f(t)\mathrm{e}^{\pm \imath st}\,\mathrm{d}t &= \frac{f(b)\mathrm{e}^{\pm \imath sb} - f(a)\mathrm{e}^{\pm \imath sa}}{\pm \imath s} - \frac{1}{\pm \imath s}\int_a^b \mathrm{e}^{\pm \imath st}\mathrm{e}^{-|y|t} f'(t)\mathrm{e}^{|y|t}\,\mathrm{d}t\\ &= \mathcal{O}(s^{-1}\mathrm{e}^{|y|b}) + \mathcal{O}(s^{-1}\mathrm{e}^{|y|a}) + \mathcal{O}(s^{-1}\mathrm{e}^{|y|b}) = \mathcal{O}(s^{-1}\mathrm{e}^{|y|b}).\end{aligned}$$

The other four formulas follow from formula (34). $\square$

For real s ($y = 0$, $a = 0$, $b = t \in [0,1]$), we can find formulas (35) in [9, 22].

Let $f \in C^r[0,1]$, $t \in [0,1]$, $r > 1$. Then we generalize the last two asymptotic formulas in Lemma 6:

$$\begin{aligned}\int_0^t f(\tau)\cos(2s\tau - st)\,\mathrm{d}\tau = &-\sum_{i=1}^{r-1} \frac{f^{(i-1)}(t) - (-1)^i f^{(i-1)}(0)}{(2s)^i}\cos\left(st + \frac{\pi i}{2}\right)\\ &+ \mathcal{O}(s^{-r}\mathrm{e}^{3|y|t}),\end{aligned} \tag{36}$$

$$\begin{aligned}\int_0^t f(\tau)\sin(2s\tau - st)\,\mathrm{d}\tau = &-\sum_{i=1}^{r-1} \frac{f^{(i-1)}(t) + (-1)^i f^{(i-1)}(0)}{(2s)^i}\sin\left(st + \frac{\pi i}{2}\right)\\ &+ \mathcal{O}(s^{-r}\mathrm{e}^{3|y|t}).\end{aligned} \tag{37}$$

For proof, we use integration by parts formula and Lemma 6.

Under the condition that $q \in C^r[0,1]$, $r \in \mathbb{N}$, the more exact asymptotic formulas may be obtained

$$\omega_s(t) = -\sum_{j=1}^{r+1} p_j(t)\cos\biggl(st + \frac{1}{2}\pi j\biggr)s^{-j} + \mathcal{O}\bigl(s^{-(r+2)}\mathrm{e}^{(r+2)|y|t}\bigr), \tag{38}$$

$$\omega_s'(t) = -\sum_{j=0}^{r} \bar{p}_j(t)\cos\biggl(st + \frac{1}{2}\pi j\biggr)s^{-j} + \mathcal{O}\bigl(s^{-(r+1)}\mathrm{e}^{(r+2)|y|t}\bigr), \tag{39}$$

$$(\omega_s)_s'(t) = -\sum_{j=1}^{r+1} p_j^1(t)\cos\biggl(st + \frac{1}{2}\pi(j-1)\biggr)s^{-j} + \mathcal{O}\bigl(s^{-(r+2)}\mathrm{e}^{(r+2)|y|t}\bigr), \tag{40}$$

where $p_1(t) = -1$, $\bar{p}_0(t) = 1$, $p_1^1(t) = t$.

Now we derive formulas for p_j, $j = \overline{2, r+1}$. We can use the mathematical induction. Let us substitute

$$\begin{aligned} \omega_s(t) &= -\sum_{j=1}^{r} p_j(t)\cos\biggl(st + \frac{\pi}{2}j\biggr)s^{-j} + \mathcal{O}\bigl(s^{-(r+1)}\mathrm{e}^{(r+1)|y|t}\bigr) \\ &= -\sum_{j=2}^{r+1} p_{j-1}(t)\sin\biggl(st + \frac{\pi}{2}j\biggr)s^{-j+1} + \mathcal{O}\bigl(s^{-(r+1)}\mathrm{e}^{(r+1)|y|t}\bigr) \end{aligned} \tag{41}$$

into integral $s^{-1}I_s^0(t, q, \omega_s)$ in right-hand side of (16):

$$\begin{aligned} &\sum_{j=2}^{r+1} \frac{-1}{s^j} \int_0^t q(\tau)p_{j-1}(\tau)\sin(st - s\tau)\sin\biggl(s\tau + \frac{\pi}{2}j\biggr)\,\mathrm{d}\tau \\ &\quad + \mathcal{O}\bigl(s^{-(r+2)}\mathrm{e}^{(r+2)|y|t}\bigr). \end{aligned}$$

Then we rewrite the sum

$$\begin{aligned} &\sum_{j=2}^{r+1} \frac{1}{2} \int_0^t q(\tau)p_{j-1}(\tau)\,\mathrm{d}\tau \cos\biggl(st + \frac{\pi}{2}j\biggr)s^{-j} \\ &\quad - \sum_{j=2}^{r+1} \frac{\cos(\frac{\pi}{2}j)}{2s^j} \int_0^t q(\tau)p_{j-1}(\tau)\cos(2s\tau - st)\,\mathrm{d}\tau \\ &\quad + \sum_{j=2}^{r+1} \frac{\sin(\frac{\pi}{2}j)}{2s^j} \int_0^t q(\tau)p_{j-1}(\tau)\sin(2s\tau - st)\,\mathrm{d}\tau \end{aligned}$$

and apply (36)–(37) for $p_{j-1} \in C^{r-j+2}$, $j = \overline{2, r+1}$:

$$\sum_{j=2}^{r+1} \frac{1}{2} \int_0^t q(\tau) p_{j-1}(\tau)\, \mathrm{d}\tau \cos\biggl(st + \frac{\pi}{2} j\biggr) s^{-j} + \mathcal{O}\bigl(s^{-(r+2)} \mathrm{e}^{(r+2)|y|t}\bigr)$$
$$+ \sum_{j=2}^{r+1} \sum_{i=1}^{r-j+1} \frac{\cos(\frac{\pi}{2} j)}{2s^j} \cdot \frac{(qp_{j-1})^{(i-1)}(t) - (-1)^i (qp_{j-1})^{(i-1)}(0)}{(2s)^i} \cos\biggl(st + \frac{\pi}{2} i\biggr)$$
$$- \sum_{j=2}^{r+1} \sum_{i=1}^{r-j+1} \frac{\sin(\frac{\pi}{2} j)}{2s^j} \cdot \frac{(qp_{j-1})^{(i-1)}(t) + (-1)^i (qp_{j-1})^{(i-1)}(0)}{(2s)^i} \sin\biggl(st + \frac{\pi}{2} i\biggr).$$

We look for terms near $s^{-(r+1)}$, i.e. $i + j = r + 1$,

$$\frac{1}{2} \int_0^t q(\tau) p_r(\tau)\, \mathrm{d}\tau \cos\biggl(st + \frac{\pi(r+1)}{2}\biggr)$$
$$+ \sum_{j=2}^{r} \frac{(qp_{j-1})^{(r-j)}(t) + (-1)^{r-j} (qp_{j-1})^{(r-j)}(0)}{2^{r-j+2}} \cos\biggl(st + \frac{\pi(r-j+1)}{2}\biggr) \cos\frac{\pi j}{2}$$
$$- \sum_{j=2}^{r} \frac{(qp_{j-1})^{(r-j)}(t) - (-1)^{r-j} (qp_{j-1})^{(r-j)}(0)}{2^{r-j+2}} \sin\biggl(st + \frac{\pi(r-j+1)}{2}\biggr) \sin\frac{\pi j}{2}$$

or

$$\frac{1}{2} \int_0^t q(\tau) p_r(\tau)\, \mathrm{d}\tau \cos\biggl(st + \frac{\pi(r+1)}{2}\biggr) + \sum_{j=2}^{r} \frac{(qp_{j-1})^{(r-j)}(t)}{2^{r-j+2}} \cos\biggl(st + \frac{\pi(r+1)}{2}\biggr)$$
$$+ \sum_{j=2}^{r} \frac{(-1)^r (qp_{j-1})^{(r-j)}(0)}{2^{r-j+2}} \cos\biggl(st + \frac{\pi(r+1)}{2}\biggr) = -p_{r+1}(t) \cos\biggl(st + \frac{\pi(r+1)}{2}\biggr).$$

So, we prove recursive formula

$$p_{i+1}(t) = -\frac{1}{2} \int_0^t q(\tau) p_i(\tau)\, \mathrm{d}\tau - \sum_{j=2}^{i} \frac{(qp_{j-1})^{(i-j)}(t) + (-1)^i (qp_{j-1})^{(i-j)}(0)}{2^{i-j+2}} \tag{42}$$

for $i = \overline{1, r}$ and $p_1(t) = -1$. This formula shows that $p_j \in C^{r-j+3}[0,1]$, $p_{j-1} q \in C^{r-j+4}[0,1]$, $j = \overline{2, r+1}$. So, the application of formulas (36)–(37) was correct. We note that $p_j \in C^2[0,1]$ for all j.

For example, $p_2(t) = Q(t) := \int_0^t q(\tau)\, \mathrm{d}\tau / 2$. It is obvious that the function $Q(t)$ is bounded for $0 \leqslant t \leqslant 1$.

Let us substitute (41) into integral $J_s^0(t,q,\omega_s)$ in right-hand side of (17). Then we get formula

$$\bar{p}_i(t) = \frac{1}{2}\int_0^t q(\tau)p_i(\tau)\,\mathrm{d}\tau - \sum_{j=2}^{i} \frac{(qp_{j-1})^{(i-j)}(t) - (-1)^i(qp_{j-1})^{(i-j)}(0)}{2^{i-j+2}} \tag{43}$$

for $i=\overline{0,r}$. If we add (42) and (43), we get

$$\bar{p}_i(t) + p_{i+1}(t) = -\sum_{j=2}^{i}\frac{(qp_{j-1})^{(i-j)}(t)}{2^{i-j+1}} = p_i'(t).$$

So, more simple formula $\bar{p}_i(t) = p_i'(t) - p_{i+1}(t)$ may be used for calculation $\bar{p}_i(t)$, $i=\overline{1,r}$ (we note that $\bar{p}_0 = 1$). This formula can be proved directly, but formula (43) is useful independently. We use notation $p_j^0(t) := p_j(t)$, $\bar{p}_{j-1}^{\,0}(t) := \bar{p}_{j-1}(t)$, $j=\overline{1,r+1}$, too.

If we substitute (38) and (40) into integrals $I_s^0(t,q,(\omega_s)_s')$ and $J_s^1(t,q,\omega_s) = J_s^0(t,q(\tau)(t-\tau),\omega_s)$, then from (18) we get recursive formula ($p_1^1(t)=t$)

$$\begin{aligned} p_{i+1}^1(t) = &-\frac{1}{2}\int_0^t q(\tau)p_i^1(\tau)\,\mathrm{d}\tau - \sum_{j=2}^{i}\frac{(qp_{j-1}^1)^{(i-j)}(t) - (-1)^i(qp_{j-1}^1)^{(i-j)}(0)}{2^{i-j+2}} \\ &+\frac{1}{2}\int_0^t \tilde{q}(\tau)p_i(\tau)\,\mathrm{d}\tau - \sum_{j=2}^{i}\frac{(\tilde{q}p_{j-1})^{(i-j)}(t) - (-1)^i(\tilde{q}p_{j-1})^{(i-j)}(0)}{2^{i-j+2}} \\ &- p_i(t), \quad i=\overline{1,r}, \quad \tilde{q}(\tau) := \tilde{q}(t,\tau) = q(\tau)(t-\tau). \end{aligned} \tag{44}$$

For example, $p_2^1(t) = 1 - tQ(t)$. We see that $p_2^1(t) = -p_1 - tp_2$. Using the mathematical induction and formulas (44) and (42), we can prove simple formula $p_i^1(t) = (1-i)p_{i-1}(t) - tp_i(t)$, $i=\overline{2,r+1}$.

Lemma 7. *Let $s\in\mathbb{C}_s$ and $q\in C^r[0,1]$. Then for $|s|\geqslant q_0$, the asymptotic formulas*

$$(\omega_s)_s^{(l)}(t,s) = -\sum_{j=1}^{r+1} p_j^l(t)\cos\left(st + \frac{\pi}{2}(j-l)\right)s^{-j} + \mathcal{O}\left(s^{-(r+2)}\mathrm{e}^{(r+2)|y|t}\right), \tag{45}$$

$$(\omega_s')_s^{(l)}(t,s) = -\sum_{j=0}^{r} \bar{p}_j^l(t)\cos\left(st + \frac{\pi}{2}(j-l)\right)s^{-j} + \mathcal{O}\left(s^{-(r+1)}\mathrm{e}^{(r+2)|y|t}\right) \tag{46}$$

are valid for $l\in\mathbb{N}_0$, where $p_i^k(t) = (1-i)p_{i-1}^{k-1}(t) - tp_i^{k-1}(t)$, $i=\overline{1,r+1}$, $\bar{p}_0^k(t) = -t\bar{p}_0^{k-1}(t)$, $\bar{p}_i^k(t) = (1-i)\bar{p}_{i-1}^{k-1}(t) - t\bar{p}_i^{k-1}(t)$, $i=\overline{1,r}$, $k\in\mathbb{N}$, $\bar{p}_i^0(t) = p_i^{0\prime}(t) - p_{i+1}^0(t)$, $i=\overline{1,r}$, $\bar{p}_0^0(t)=1$, and $p_j^0(t)$ is calculated by (42).

Proof. We prove (45) formula in the case $l=0,1$ and (46) formula in the case $l=0$. The other cases we can prove by mathematical induction by l. □

For example, $p_1^l(t) = (-1)^{l+1} t^l$, $\bar{p}_0^l(t) = (-1)^l t^l$, $l \in \mathbb{N}_0$ (see Remark 5, too).

Corollary 2. *Let $x \in \mathbb{R}$, $\delta \in \mathbb{R}$, $q \in C^r[0,1]$, $Q_j(x)$, $j = \overline{1,r}$, are bounded functions. If $s = x + \delta$,*

$$\delta = \sum_{j=1}^{r} Q_j(x) x^{-j} + \mathcal{O}\bigl(x^{-(r+1)}\bigr), \tag{47}$$

then we have the following formulas:

$$\omega_s(t) = \sum_{j=1}^{r+1} R_j(t,x) x^{-j} + \mathcal{O}\bigl(x^{-(r+2)}\bigr), \qquad \omega_s'(t) = \sum_{j=0}^{r} \bar{R}_j(t,x) x^{-j} + \mathcal{O}\bigl(x^{-(r+1)}\bigr).$$

From previous results we have $R_1(t,x) = -\sin(xt)$, $\bar{R}_0(t,x) = -\cos(xt)$. Now we derive general formula for $R_j(t,x)$, $j = \overline{1, r+1}$. We substitute expressions (45), (47) into formula

$$\begin{aligned} \omega_s(t) &= \omega(t,\, x+\delta) \\ &= \sum_{l=0}^{r+1} (\omega_s)_s^{(l)}(t,x) \frac{\delta^l}{l!} + (\omega_s)_s^{(r+2)}(t,\, x+\theta\delta) \frac{\delta^{r+2}}{(r+2)!}, \quad \theta \in [0,1], \end{aligned}$$

and get the following expression for $\omega_s(t)$:

$$-\sum_{l=0}^{r+1} \left(\sum_{j=1}^{r+1} p_j^l(t) \cos\left(xt + \frac{\pi}{2}(j-l)\right) x^{-j} \right) \frac{1}{l!} \left(\sum_{i=1}^{r} Q_i(x) x^{-i} \right)^l + \mathcal{O}\bigl(x^{-(r+2)}\bigr).$$

From binomial formula we have ($n_i \geqslant 0$, $i = \overline{1,r}$)

$$\left(\sum_{i=1}^{r} Q_i(x) x^{-i} \right)^l = \sum_{n_1+\cdots+n_r=l} \frac{l!}{n_1! \cdots n_r!} Q_1^{n_1}(x) \cdots Q_r^{n_r}(x) x^{-(n_1+2n_2+\cdots+rn_r)}.$$

Collecting terms near $x^{-(r+1)}$ (i.e. $j + n_1 + 2n_2 + \cdots + rn_r = r+1$), we get

$$\begin{aligned} R_{r+1}(t,x) = -\sum_{\substack{n_1+\cdots+n_r=l,\ j>0,\\ j+n_1+2n_2+\cdots+rn_r=r+1}} &\frac{1}{n_1! \cdots n_r!} p_j^l(t) \cos\left(xt + \frac{\pi}{2}(j-l)\right) \\ &\times Q_1^{n_1}(x) \cdots Q_r^{n_r}(x). \end{aligned} \tag{48}$$

For $\bar{R}(t,x)$, we get formula

$$\begin{aligned} \bar{R}_r(t,x) = -\sum_{\substack{n_1+\cdots+n_r=l,\ j\geqslant 0\\ j+n_1+2n_2+\cdots+rn_r=r}} &\frac{1}{n_1! \cdots n_r!} \bar{p}_j^l(t) \cos\left(xt + \frac{\pi}{2}(j-l)\right) \\ &\times Q_1^{n_1}(x) \cdots Q_r^{n_r}(x). \end{aligned} \tag{49}$$

We write explicit formulas in the case $q \in C^1[0,1]$:

$$\begin{aligned} &p_2(t) = p_2^0(t) = Q(t), &&\bar{p}_1(t) = p_1'(t) - p_2(t) = -Q(t), \\ &p_2^l(t) = (-t)^{l-1}\bigl(l - tQ(t)\bigr), &&\bar{p}_1^l(t) = -(-t)^l Q(t), \quad l \in \mathbb{N}, \end{aligned}$$

$$R_2(t,x) = p_2(t)\cos(xt) - p_1^1(t)Q_1(x)\cos(xt) = \big(Q(t) - tQ_1(x)\big)\cos(xt),$$
$$\bar{R}_1(t,x) = \bar{p}_1(t)\sin(xt) - \bar{p}_0^1(t)Q_1(x)\sin(xt) = -\big(Q(t) - tQ_1(x)\big)\sin(xt).$$

We formulate these results in the next two statements.

Lemma 8. *Let $s \in \mathbb{C}_s$ and $q \in C^1[0,1]$. Then for $|s| \geqslant q_0$, the asymptotic formulas*

$$\omega_s(t) = -\sin(st)s^{-1} + Q(t)\cos(st)s^{-2} + \mathcal{O}\big(s^{-3}\mathrm{e}^{3|y|t}\big),$$
$$(\omega_s)'_s(t,s) = -t\cos(st)s^{-1} + \big(1 - tQ(t)\big)\sin(st)s^{-2} + \mathcal{O}\big(s^{-3}\mathrm{e}^{3|y|t}\big),$$
$$(\omega_s)''_s(t,s) = t^2\sin(st)s^{-1} + t\big(2 - tQ(t)\big)\cos(st)s^{-2} + \mathcal{O}\big(s^{-3}\mathrm{e}^{3|y|t}\big),$$
$$\omega'_s(t) = -\cos(st) - Q(t)\sin(st)s^{-1} + \mathcal{O}\big(s^{-2}\mathrm{e}^{3|y|t}\big),$$
$$(\omega'_s)'_s(t,s) = t\sin(st) - tQ(t)\cos(st)s^{-1} + \mathcal{O}\big(s^{-2}\mathrm{e}^{3|y|t}\big)$$

are valid.

Corollary 3. *Let $x \in \mathbb{R}$, $\delta \in \mathbb{R}$, $q \in C^1[0,1]$, $Q_1(x)$ is bounded function. If $s = x + \delta$, $\delta = Q_1(x)x^{-1} + \mathcal{O}(x^{-2})$, then we have the following formulas*:

$$\omega_s(t) = -\sin(xt)x^{-1} + \big(Q(t) - tQ_1(x)\big)\cos(xt)x^{-2} + \mathcal{O}\big(x^{-3}\big),$$
$$\omega'_s(t) = -\cos(xt) - \big(Q(t) - tQ_1(x)\big)\sin(xt)x^{-1} + \mathcal{O}\big(x^{-2}\big).$$

4 Characteristic equation for problem with integral condition

Substituting $\omega_s(t)$ into (5), we get the characteristic equation

$$h(s) := \omega_s(1) - \gamma\int_\alpha^\beta \omega_s(t)\,\mathrm{d}t = 0. \tag{50}$$

The set of eigenvalues of the BVP (1), (2), (5) coincides with the set $\{\lambda\colon\ \lambda = s^2,\ h(s) = \omega_s(1) - \gamma\int_\alpha^\beta \omega_s(t)\,\mathrm{d}t = 0\}$. The function h is analytic function of parameter $s \in \mathbb{C}_s$, and

$$h^{(l)}(s) := (\omega_s)_s^{(l)}(1,s) - \gamma\int_\alpha^\beta (\omega_s)_s^{(l)}(t,s)\,\mathrm{d}t, \quad l \in \mathbb{N}_0. \tag{51}$$

Substituting (45) into (51), we get

$$h^{(l)}(s) = \gamma\sum_{j=1}^{r}\left(\int_\alpha^\beta p_j^l(t)\cos\left(st + \frac{\pi}{2}(j-l)\right)\mathrm{d}t\, s^{-j}\right) - \sum_{j=1}^{r+1} p_j^l(1)\cos\left(s + \frac{\pi}{2}(j-l)\right)s^{-j} + \mathcal{O}\big(s^{-(r+2)}\mathrm{e}^{(r+2)|y|}\big),$$

where $p_j^l \in C^{r-j+3}[0,1]$.

Remark 6. In the case $r = 0$ the last term is $\mathcal{O}(s^{-2}\mathrm{e}^{|y|})$.

If $f \in C^r[0,1]$, $[a,b] \subset [0,1]$, $c \in \mathbb{R}$, then we use integration by parts formula and have

$$\int_a^b f(t)\cos(st+c)\,\mathrm{d}t = \sum_{i=1}^{r} \frac{f^{(i-1)}(b)\cos(bs+c-\frac{\pi}{2}i) - f^{(i-1)}(a)\cos(as+c-\frac{\pi}{2}i)}{(-1)^{i-1}s^i} + \mathcal{O}\bigl(s^{-(r+1)}\mathrm{e}^{|y|}\bigr).$$

So, we derive

$$h^{(l)}(s) = -\sum_{j=1}^{r+1} p_j^l(1)\cos\biggl(s+\frac{\pi}{2}(j-l)\biggr)s^{-j} + \mathcal{O}\bigl(s^{-(r+2)}\mathrm{e}^{(r+2)|y|}\bigr) + \gamma\sum_{j=1}^{r}\sum_{i=1}^{r-j+1} \frac{{p_j^l}^{(i-1)}(\beta)\cos(\beta s+\frac{\pi(j-l-i)}{2}) - {p_j^l}^{(i-1)}(\alpha)\cos(\alpha s+\frac{\pi(j-l-i)}{2})}{(-1)^{i-1}s^{i+j}}.$$

We look for terms near $s^{-(r+1)}$, i.e. $i + j = r + 1$,

$$h_{r+1}^l(s) := -p_{r+1}^l(1)\cos\biggl(s+\frac{\pi}{2}(r+1-l)\biggr) + \gamma\sum_{j=1}^{r} \frac{{p_j^l}^{(r-j)}(\beta)\cos(\beta s+\frac{\pi(2j-l-r-1)}{2}) - {p_j^l}^{(r-j)}(\alpha)\cos(\alpha s+\frac{\pi(2j-l-r-1)}{2})}{(-1)^{r-j}}.$$

Thus, the next lemma immediately follows from results in the above.

Lemma 9. *Let $s \in \mathbb{C}_s$ and $q \in C^r[0,1]$. Then for $|s| \geqslant q_0$, the asymptotic formula*

$$h^{(l)}(s) = \sum_{j=1}^{r+1} h_j^l(s)s^{-j} + \mathcal{O}\bigl(s^{-(r+2)}\mathrm{e}^{(r+2)|y|}\bigr), \quad l \in \mathbb{N}_0, \tag{52}$$

is valid.

Corollary 4. *If $q \in C[0,1]$, then we have (see Remark 6, too)*

$$h^{(l)}(s) = h_1^l(s)s^{-1} + \mathcal{O}\bigl(s^{-2}\mathrm{e}^{|y|}\bigr), \quad l \in \mathbb{N}_0, \tag{53}$$

where

$$h_1^l(s) = -\cos\biggl(s+\frac{\pi}{2}(l-1)\biggr).$$

Corollary 5. *If $q \in C^1[0,1]$, then we have*

$$h^{(l)}(s) = h_1^l(s)s^{-1} + h_2^l(s)s^{-2} + \mathcal{O}\bigl(s^{-3}\mathrm{e}^{3|y|}\bigr), \quad l \in \mathbb{N}_0, \tag{54}$$

where

$$h_2^l(s) = \bigl(Q(1)-l\bigr)\cos\biggl(s+\frac{\pi}{2}l\biggr) - \gamma\beta^l\cos\biggl(\beta s+\frac{\pi}{2}l\biggr) + \gamma\alpha^l\cos\biggl(\alpha s+\frac{\pi}{2}l\biggr).$$

For example, $h_1^0(s) = -\sin s$, $h_2^0(s) = Q(1)\cos s - \gamma\cos(\beta s) + \gamma\cos(\alpha s)$, $h_1^1(s) = -\cos s$, $h_1^1(s) = -\cos s$.

Analytic functions $H(s) := -h(s)s$, $M(s) := -h(s)s^2$ have the same nonzero roots as function h.

Corollary 6. *If $q \in C[0,1]$, then we have*

$$H(s) = -h_1^0(s) + \mathcal{O}\bigl(s^{-1}\mathrm{e}^{|y|}\bigr), \tag{55}$$

$$H'(s) = -h_1^1(s) - h_1^0 s^{-1} + \mathcal{O}\bigl(s^{-1}\mathrm{e}^{|y|}\bigr) = -h_1^1(s) + \mathcal{O}\bigl(s^{-1}\mathrm{e}^{|y|}\bigr), \tag{56}$$

$$M(s) = -h_1^0(s)s + \mathcal{O}\bigl(\mathrm{e}^{|y|}\bigr), \tag{57}$$

$$M'(s) = -h_1^1(s)s - 2h_1^0(s) + \mathcal{O}\bigl(\mathrm{e}^{|y|}\bigr) = -h_1^1(s)s + \mathcal{O}\bigl(\mathrm{e}^{|y|}\bigr). \tag{58}$$

Remark 7. The asymptotic formula (53) for the integral condition (5) are simpler than in the case Bitsadze–Samarskii NBC (6) (see [17]). For Bitsadze–Samarskii NBC,

$$h(s) := \omega_s(1) - \gamma\omega_s(\xi), \qquad h^{(l)}(s) = (\omega_s)_s^{(l)}(1,s) - \gamma(\omega_s)_s^{(l)}(\xi,s), \quad l \in \mathbb{N}.$$

Substituting (45) into these expressions, we get

$$h^{(l)}(s) = -\sum_{j=1}^{r+1}\biggl(p_j^l(1)\cos\biggl(s + \frac{\pi}{2}(j-l)\biggr) - \gamma p_j^l(\xi)\cos\biggl(\xi s + \frac{\pi}{2}(j-l)\biggr)\biggr)s^{-j} + \mathcal{O}\bigl(s^{-(r+2)}\mathrm{e}^{(r+2)|y|}\bigr).$$

We look for terms near $s^{-(r+1)}$

$$h_{r+1}^l(s) := -p_{r+1}^l(1)\cos\biggl(s + \frac{\pi}{2}(r+1-l)\biggr) + \gamma p_{r+1}^l(\xi)\cos\biggl(\xi s + \frac{\pi}{2}(r+1-l)\biggr).$$

So, Remark 9 is valid in the case Bitsadze–Samarskii NBC with above defined $h_j^l(s)$, $j = \overline{1, r+1}$, and we get two corollaries.

Corollary 7 [Bitsadze–Samarskii NBC]. *If $q \in C[0,1]$, then we have formula* (53)*, where $h_1^l(s) = -\sin(s + \pi l/2) + \gamma\xi^l\sin(\xi s + \pi l/2)$. For example, $h_1^0(x) = -\sin s + \gamma\sin(\xi s)$, $h_1^1(x) = -\cos s + \gamma\xi\cos(\xi s)$.*

Corollary 8 [Bitsadze–Samarskii NBC]. *If $q \in C^1[0,1]$, then we have formula* (54)*, where $h_2^l(s) = (Q(1) - l)\cos(s + \pi l/2) + \gamma\xi^{l-1}(l - \xi Q(\xi))\cos(\xi s + \pi l/2)$. For example, $h_2^0(x) = Q(1)\cos s - \gamma Q(\xi)\cos(\xi s)$.*

Let us consider real eigenvalues. In this case, (52) is valid with $s = x \in \mathbb{R}$ ($y = 0$), and functions h_j^l, $j = \overline{1, r+1}$, $l \in \mathbb{N}_0$, are bounded. We investigate equation $h(x+\delta) = 0$ (or $M(x+\delta) = -x^2 h(x+\delta) = 0$), $\delta \in \mathbb{R}$, with additional condition $|h_1^1(x)| \geqslant \varkappa > 0$.

Lemma 10. *If $q \in C[0,1]$ and $\delta = o(1)$, then the following asymptotic formula is valid*:

$$\delta = -h_1^0(x)\big(h_1^1(x)\big)^{-1}\delta\cos\delta(\sin\delta)^{-1} + \mathcal{O}\big(x^{-1}\big). \tag{59}$$

Proof. From formula (57) we have

$$\begin{aligned}\sin x\cos\delta - \cos x\sin\delta = -\sin(x+\delta) = h_1^0(x+\delta) = (x+\delta)^{-1}\mathcal{O}(1)\\ = \mathcal{O}\big(x^{-1}\big).\end{aligned}$$

Since $h_1^1(x) = -\cos x$ we have $h_1^1(x)\sin\delta = -h_1^0(x)\cos\delta + \mathcal{O}(x^{-1})$. We multiply this equality by $\delta(\sin\delta)^{-1}(h_1^1(x))^{-1}$ (using condition $|h_1^1(x)| \geqslant \varkappa > 0$) and get (59). □

Corollary 9. *If $h_1^0(x) = 0$, then $\delta = \mathcal{O}(x^{-1})$.*

Lemma 11. *If $q \in C^r[0,1]$ and $\delta = o(1)$, $h_1^0(x) = 0$, then asymptotic formula*

$$\delta = \sum_{j=1}^{r} Q_j(x)x^{-j} + \mathcal{O}\big(x^{-(r+1)}\big) \tag{60}$$

is valid, where $Q_j(x)$, $j = \overline{1,r}$, are bounded functions.

Proof. Formula (60) is valid for $r = 0$. So, $\delta = \mathcal{O}(x^{-1})$. If $r > 0$, then substituting (54) into equality

$$0 = h(x+\delta) = h(x) + h'(x)\delta + h''(x)\frac{\delta^2}{2} + \mathcal{O}\big(\delta^3\big),$$

we have

$$-h_1^1(x)x^{-1}\delta = h_1^0(x)x^{-1} + h_2^0(x)x^{-2} + \mathcal{O}\big(x^{-3}\big) = h_2^0(x)x^{-2} + \mathcal{O}\big(x^{-3}\big),$$

i.e. $\delta = Q_1(x)x^{-1} + \mathcal{O}(x^{-2})$, where

$$Q_1(x) = -h_2^0(x)\big(h_1^1(x)\big)^{-1}. \tag{61}$$

We derive formulas for Q_j, $j = \overline{2,r}$, $r \geqslant 2$. We can use the mathematical induction. Suppose that $\delta = \sum_{j=1}^{r-1} Q_j(x)x^{-j} + \mathcal{O}(x^{-r})$. Substituting (52) expression in the case $y = 0$ into equality

$$0 = h(x+\delta) = h(x) + \delta\sum_{i=0}^{r} h^{(i+1)}(x)\frac{\delta^i}{(i+1)!} + \mathcal{O}\big(\delta^{r+2}\big),$$

we get

$$\delta x^{-1}\sum_{i=0}^{r}\sum_{j=0}^{r}\frac{h_{j+1}^{i+1}(x)}{i+1}\frac{\delta^i}{i!}x^{-j} = -\sum_{j=2}^{r+1} h_j^0(x)x^{-j} + \mathcal{O}\big(x^{-(r+2)}\big)$$

or

$$Z(x)\delta = -\sum_{j=1}^{r+1} h_j^0(x)x^{-j} + \mathcal{O}\bigl(x^{-(r+2)}\bigr)$$
$$= h_1^1(x)x^{-1}\left(\sum_{j=1}^{r} h_j(x)x^{-j} + \mathcal{O}\bigl(x^{-(r+1)}\bigr)\right),$$

where $h_j(x) := -h_{j+1}^0(x)(h_1^1(x))^{-1}$, $1 \leqslant j \leqslant r$,

$$Z(x) := x^{-1}\sum_{i=0}^{r}\sum_{j=0}^{r} h_{j+1}^{i+1}(x)\left(\sum_{l=1}^{r-1} Q_l(x)x^{-l}\right)^i \frac{x^{-j}}{(i+1)!}$$
$$= x^{-1}\sum_{i,j=0}^{r} h_{j+1}^{i+1}(x)\sum_{n_1+\cdots+n_{r-1}=i}\frac{Q_1^{n_1}(x)\cdots Q_{r-1}^{n_{r-1}}(x)}{(i+1)n_1!\cdots n_{r-1}!}x^{-(j+n_1+2n_2+\cdots+(r-1)n_{r-1})}$$
$$= h_1^1(x)x^{-1}\left(1 - \sum_{k=1}^{r-1} z_k(x)x^{-k} + \mathcal{O}\bigl(x^{-r}\bigr)\right),$$

For z_k, $k = \overline{1, r-1}$, we have expressions

$$z_k(x) = \sum_{\substack{n_1+\cdot d+n_{r-1}=i,\ j\geqslant 0\\ j+n_1+2n_2+\cdots+(r-1)n_{r-1}=k}} -h_{j+1}^{i+1}(x)(h_1^1(x))^{-1}\frac{Q_1^{n_1}(x)\cdots Q_{r-1}^{n_{r-1}}(x)}{(i+1)n_1!\cdots n_{r-1}!}. \quad (62)$$

So,

$$\delta = \sum_{j=1}^{r} h_j(x)x^{-j}\cdot\left(1 - \sum_{k=1}^{r-1} z_k(x)x^{-k} + \mathcal{O}\bigl(x^{-r}\bigr)\right)^{-1} + \mathcal{O}\bigl(x^{-(r+1)}\bigr)$$
$$= \sum_{j=1}^{r}\sum_{l=0}^{r} h_j(x)x^{-j}\left(\sum_{k=1}^{r-1} z_k(x)x^{-k}\right)^l + \mathcal{O}\bigl(x^{-(r+1)}\bigr)$$
$$= \sum_{j=1}^{r}\sum_{l=0}^{r} h_j(x)\sum_{n_1+\cdots+n_{r-1}=l}\frac{l!}{n_1!\cdots n_{r-1}!}\cdot\frac{z_1^{n_1}(x)\cdots z_{r-1}^{n_{r-1}}(x)}{x^{j+n_1+2n_2+\ldots+(r-1)n_{r-1}}} + \mathcal{O}\bigl(x^{-(r+1)}\bigr).$$

Collecting terms near x^{-r} (i.e. $j + n_1 + 2n_2 + \ldots + (r-1)n_{r-1} = r$), we get

$$Q_r(x) = \sum_{\substack{n_1+\cdots+n_{r-1}=l,\ j>0\\ j+n_1+2n_2+\cdots+(r-1)n_{r-1}=r}} \frac{l!}{n_1!\cdots n_{r-1}!}\frac{-h_{j+1}^0(x)}{h_1^1(x)}z_1^{n_1}(x)\cdots z_{r-1}^{n_{r-1}}(x). \quad (63)$$

Lemma is proved. □

Corollary 10. *If* $q \in C^2[0,1]$*, then*

$$Q_2(x) = -h_3^0(x)\big(h_1^1(x)\big)^{-1} - h_2^0(x)\big(h_1^1(x)\big)^{-1} z_1(x), \qquad (64)$$
$$z_1(x) = -h_2^1(x)\big(h_1^1(x)\big)^{-1} - \frac{1}{2}h_1^2(x)\big(h_1^1(x)\big)^{-1} Q_1(x).$$

Corollary 11 [Integral NBC]. *If* $q \in C^1[0,1]$*, then*

$$Q_1(x) = Q(1) - \gamma\frac{\cos(\beta x) - \cos(\alpha x)}{\cos x}. \qquad (65)$$

Corollary 12 [Bitsadze–Samarskii NBC]. *If* $q \in C^1[0,1]$*, then*

$$Q_1(x) = \frac{Q(1)\cos x - \gamma Q(\xi)\cos(\xi x)}{\cos x - \gamma\xi\cos(\xi x)}. \qquad (66)$$

Formula (66) was proved in [17].

5 Spectral asymptotics for eigenvalues and eigenfunctions for problem with integral condition

In this section, we investigate eigenvalues for SLP (1), (2), (5).

Lemma 12. *The real eigenvalues of the SLP* (1)*,* (2)*,* (5) *are bounded from below.*

Proof. Set $\tilde{H}(y) := \imath^3 H(\imath y)$, $y > 0$. Then

$$\tilde{H}(y) = \sinh y + \mathcal{O}\big(y^{-1}\mathrm{e}^y\big) = \frac{\mathrm{e}^y}{2} - \frac{\mathrm{e}^{-y}}{2} + \mathcal{O}\big(y^{-1}\mathrm{e}^y\big).$$

It is clear that $\lim_{y\to+\infty}\tilde{H}(y) = +\infty$. Then there exists a $y_0 > 0$ such that $\tilde{H}(y) \neq 0$ for $y > y_0$. Therefore, we get $H(\imath y) = \imath\tilde{H}(y) \neq 0$ for $y > y_0$. Accordingly, $-y_0^2 \leqslant \lambda$ for negative λ. □

Corollary 13. *The number of negative eigenvalues of problem* (1)*,* (2)*,* (5) *are finite* (*maybe zero*)*.*

Lemma 13. *The function* $H: \mathbb{R} \to \mathbb{R}$ *has at least one positive root in the interval* $((k-1/2)\pi,\ (k+1/2)\pi)$ *for large* k*.*

Proof. If $s = x$, $0 < x \in \mathbb{R}$, then $y = 0$. In this case, formulas (55) is

$$H(x) = \sin x + \mathcal{O}\big(x^{-1}\big), \qquad (67)$$

We have $|\mathcal{O}(x^{-1})| < 1$ for large x. The function $\sin x$ takes its local maximum points at $M_k = (2k-3/2)\pi$, $k \in \mathbb{N}$, and its local minimum points at $m_k = (2k-1/2)\pi$, $k \in \mathbb{N}$. Thus, from Intermediate value theorem at least one root of the function $H(x)$ lies in each interval $((k-1/2)\pi,\ (k+1/2)\pi)$, $K < k \in \mathbb{N}$, for large K. So, we have infinite (countable) number positive roots of equation $H(x) = 0$. □

Corollary 14. *The SLPs* (1), (2), (5) *have infinitely many* (*countable*) *positive eigenvalues.*

Remark 8. The function $\sin s$ has the same property, but only one root is in the interval $((k-1/2)\pi,\ (k+1/2)\pi)$, $k \in \mathbb{N}$.

Let us denote domain $D_k = \{s\colon |x| \leqslant a_k = (k+1/2)\pi,\ |y| \leqslant a_k\}$, $D_{sk} = \mathbb{C}_s \cap D_k$, $k \in \mathbb{N}$, and a contour $\Gamma_{sk} = \mathbb{C}_s \cap \partial D_k$. Then we have $|s| \geqslant 3\pi/2$ on Γ_{sk}, $k \in \mathbb{N}$.

Remark 9. The corresponding contour $\Gamma_{\lambda k}$ in the plane $\mathbb{C}_\lambda = \mathbb{C}$ will be the boundary of the domain $D_{\lambda k}$.

Lemma 14. *There exists $q_1 > 0$ such that all eigenvalues of problems* (1)–(2), (5) *in the domains $\{s \in \mathbb{C}_s\colon |s| > q_1\}$ are positive and, more precisely, there exists only one positive root of function $H(s)$ in each interval $((k-1/2)\pi,\ (k+1/2)\pi)$ for sufficiently large k.*

Proof. On the vertical part of contour $s = a_k + \imath y$, $y \in [-a_k, a_k]$, $k \in \mathbb{N}$, $\mathrm{Re}(\sin s) = \sin a_k \cosh y$. We estimate

$$|\sin s| \geqslant \bigl|\mathrm{Re}(\sin s)\bigr| \geqslant |\sin a_k| \cosh y = \cosh y = \frac{\mathrm{e}^{|y|} + \mathrm{e}^{-|y|}}{2} \geqslant \frac{\mathrm{e}^{|y|}}{4}.$$

On the remaining part of contour $y = \pm a_k$, $0 \leqslant x \leqslant a_k$, we estimate

$$|\sin s| = \sqrt{\sinh^2 y + \sin^2 x} \geqslant \sinh |y| = \frac{\mathrm{e}^{|y|} - \mathrm{e}^{-|y|}}{2} \geqslant \frac{\mathrm{e}^{|y|}}{4}.$$

So, we have $|\sin s| \geqslant \mathrm{e}^{|y|}/4$ on Γ_{sk} for sufficiently large k.

From formula (55) $H(s) = \sin s + \mathcal{O}(s^{-1}\mathrm{e}^{|y|})$. Hence, we have $|\mathcal{O}(s^{-1}\mathrm{e}^{|y|})| \leqslant c_1|s|^{-1}\mathrm{e}^{|y|} < \mathrm{e}^{|y|}/4 \leqslant |\sin s|$ on the contours Γ_{sk} for sufficiently large k. Therefore, by Rouché theorem it follows that the number of zeros of $H(s) = \sin s + \mathcal{O}(s^{-1}\mathrm{e}^{|y|})$ and $\sin s$ are the same inside Γ_{sk} for sufficiently large k.

In the domain between contours $\Gamma_{s,k-1}$ and Γ_{sk}, there is exactly one positive root of the function $\sin s$ (see Remark 8). The function H has one root in this domain for sufficiently large k. But interval $((k-1/2)\pi,\ (k+1/2)\pi)$ belongs to this domain. So, the single root of H in this domain is positive. □

This lemma clarifies Lemma 13.

Corollary 15. *The function $H : \mathbb{R} \to \mathbb{R}$ has one positive root in the interval $((k-1/2)\pi,\ (k+1/2)\pi)$ for large k.*

We can enumerate the zeros of H as s_k, $k \in \mathbb{N}$. The first zeros can be complex numbers or not simple. From Corollary 15 we have that s_k are positive for sufficiently large k. Now we will investigate the distribution of these positive eigenvalues of problem (1)–(3), and we leave out the note about sufficiently large k. Now we consider only real positive $s = x > 0$. Since $s_k \in ((k-1/2)\pi,\ (k+1/2)\pi)$, we have $s_k \sim x_k := \pi k$ (as $k \to \infty$) and $h_1^0(x_k) = 0$, $h_1^1(x_k) = -\cos(x_k) = (-1)^{k+1}$, $|h_1^1(x_k)| = 1$.

Let us denote $\delta_k = s_k - x_k$. The functions H and $\sin s$ are analytic. So, from (67) we have

$$s_k = x_k + o(1) \quad \text{or} \quad \delta_k = o(1) \quad (\text{as } k \to \infty). \tag{68}$$

Theorem 1. *Let* $q \in C[0,1]$*. For eigenvalues* $\lambda_k = s_k^2$ *and eigenfunctions* u_k *of problem* (1)–(2)*,* (5)*, the asymptotic formulas*

$$s_k = x_k + \mathcal{O}\bigl(k^{-1}\bigr), \qquad u_k(t) = -\sin(x_k t)x_k^{-1} + \mathcal{O}\bigl(k^{-2}\bigr) \tag{69}$$

are valid for sufficiently large k*.*

Proof. For our problem, $h_1^0(s) = -\sin s$, $h_1^1(s) = -\cos s$ and $h_1^0(x_k) = 0$, $h_1^1(x_k) = (-1)^{k+1} = \pm 1$. We have $\delta_k = o(1)$. So, all conditions of Lemma 10 are valid, and from Corollary 9 it follows $\delta_k = \mathcal{O}(x_k^{-1}) = \mathcal{O}(k^{-1})$.

Then we apply Corollary 1 and get

$$u_k = \omega_{s_k}(t) = -\sin(x_k t)x_k^{-1} + \mathcal{O}\bigl(x_k^{-2}\bigr) = -\sin(x_k t)x_k^{-1} + \mathcal{O}\bigl(k^{-2}\bigr). \qquad \square$$

Remark 10. Normalized eigenfunctions are

$$v_k(t) = \sqrt{2}\sin(x_k t) + \mathcal{O}\bigl(k^{-1}\bigr).$$

Theorem 2. *Let* $q \in C^r[0,1]$*. For eigenvalues* $\lambda_k = s_k^2$ *and eigenfunctions* u_k *of problem* (1)–(2)*,* (5)*, the asymptotic formulas*

$$s_k = x_k + \sum_{j=1}^{r} Q_j(x_k)x_k^{-j} + \mathcal{O}\bigl(k^{-(r+1)}\bigr), \tag{70}$$

$$u_k(t) = \sum_{j=1}^{r+1} R_j(t, x_k)x_k^{-j} + \mathcal{O}\bigl(k^{-(r+2)}\bigr) \tag{71}$$

are valid for sufficiently large k*.*

Proof. We have $\delta_k = \mathcal{O}(k^{-1}) = o(1)$ (see Theorem 1). So, all conditions of Lemma 11 are valid, and it follows

$$\delta_k = \sum_{j=1}^{r} Q_j(x_k)x_k^{-j} + \mathcal{O}\bigl(k^{-(r+1)}\bigr).$$

Then we apply Corollary 2 and get

$$u_k = \omega_{s_k}(t) = \sum_{j=1}^{r+1} R_j(t, x_k)x_k^{-j} + \mathcal{O}\bigl(k^{-(r+2)}\bigr). \qquad \square$$

Corollary 16. *If* $q \in C^1[0,1]$*, then the asymptotic formulas*

$$s_k = x_k + Q_1(x_k)x_k^{-1} + \mathcal{O}(k^{-2}),$$
$$u_k(t) = -\sin(x_k t)x_k^{-1} + \bigl(Q(t) - tQ_1(x_k)\bigr)\cos(x_k t)x_k^{-2} + \mathcal{O}\bigl(k^{-3}\bigr)$$

are valid for sufficiently large k*, where*

$$Q_1(x_k) = Q(1) + (-1)^{k+1}\gamma\bigl(\cos(\beta x_k) - \cos(\alpha x_k)\bigr).$$

Remark 11. In this case, normalized eigenfunctions are

$$v_k(t) = \sqrt{2}\sin(x_k t) + \sqrt{2}\left(\frac{\sin(2x_k)\sin(x_k t)}{4} - R_2(t, x_k)\right)x_k^{-1} + \mathcal{O}\bigl(k^{-2}\bigr).$$

6 Conclusion

In this paper the spectrum, existence of solutions and spectral properties of eigenfunctions for a SLP with one integral-type NBC was investigated. The considered problem differs from the classical (local) one-dimensional SLP with BCs in that it contains a NBC in two cases. Therefore, it is not obvious how to apply the classical methods of theory to such type BVPs. Therefore, suggesting own approach and modifying the techniques of classical Sturm theory, we obtained asymptotic formulas for eigenvalues and normalized eigenfunctions. The results obtained in this work can be extended to two- or higher-dimensional SLPs and to higher-order differential equations. Furthermore, asymptotics of eigenvalues and eigenfunctions of the same differential equation but with different NBCs such as eigenvalue-parameter dependent NBCs can be also investigated.

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