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On the new Hyers–Ulam–Rassias stability of the generalized cubic set-valued mapping in the incomplete normed spaces

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Abstract. We present a novel generalization of the Hyers–Ulam–Rassias stability definition to study a generalized cubic set-valued mapping in normed spaces. In order to achieve our goals, we have applied a brand new fixed point alternative. Meanwhile, we have obtained a practicable example demonstrating the stability of a cubic mapping that is not defined as stable according to the previously applied methods and procedures.

Keywords: stability, orthogonal set, cubic mapping, fixed point, incomplete metric space.

1 Introduction and literature reviews

The study for the set-valued dynamics in Banach spaces has been developed in the last decades. The pioneering published papers by Aumann [2] and Debreu [9] were inspired by some problems arising in the control theory and mathematical economics. We refer to the articles by Arrow and Debreu [1], McKenzie [29], and the survey by Hess [18].

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The stability of functional equations was first introduced by Ulam [38] in 1940. He proposed the following problem: Given a group G_1, a metric group (G_2, d), and a positive number ϵ, does there exist a $\delta > 0$ such that if a mapping $f : G_1 \to G_2$ satisfies the inequality $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G_1$, then there is a homomorphism $T : G_1 \to G_2$ such that $d(f(x), T(x)) < \epsilon$ for all $x \in G_1$. If the answer is positive, we say that the homomorphisms from G_1 to G_2 are stable. In 1941, Hyers [19] gave a partial solution of Ulam's problem for the case of approximate additive mappings under the assumption that G_1 and G_2 are Banach spaces. In 1978, a generalized version of the theorem of Hyers by considering the stability problem with unbounded Cauchy differences was given by Rassias [36]. This phenomenon of stability that was introduced by Rassias [36] is called the Hyers–Ulam–Rassias stability of functional equations.

Theorem 1. *Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\big\|f(x+y) - f(x) - f(y)\big\| \leqslant \epsilon\big(\|x\|^p + \|y\|^p\big),$$

where ϵ and p are constants with $\epsilon > 0$, and $p \neq 1$. Then there exists a unique additive mapping $T : E \to E'$ such that in the case of $p < 1$,

$$\big\|f(x) - T(x)\big\| \leqslant \frac{2\epsilon}{2 - 2^p}\|x\|^p \quad \forall x \in E,$$

while in the case of $p > 1$,

$$\big\|f(x) - T(x)\big\| \leqslant \frac{2\epsilon}{2^p - 2}\|x\|^p \quad \forall x \in E.$$

The solution to this problem was obtained by Gajda [13] for $p > 1$, and the problem for $p < 1$ was solved by Rassias [36]. Rassias and Semrl [37] proved that the stability does not occur for $p = 1$. The result of the Rassias theorem was generalized by Forti [11] and Gávruta [14], who permitted the Cauchy difference to become arbitrary unbounded.

The stability problems of several functional equations have been extensively investigated by many mathematicians. The results of these kinds of problems have been extensively studied. We refer, for instance, to [6, 12, 15–17, 19, 27] and also [11, 13, 14, 22, 25, 36, 37] and references therein.

A stability problem of Ulam for the cubic functional equation

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x) \tag{1}$$

was established by Jun and Kim [22] for mapping $f : E_1 \to E_2$, where E_1 is a normed space, and E_2 a Banach space. Also, they solved the stability problem of Ulam for the generalized *Euler–Lagrange-type cubic functional equation*

$$\begin{aligned} f(ax+y) + f(x+ay) = {} & (a+1)(a-1)^2\big[f(x) + f(y)\big] \\ & + a(a+1)f(x+y) \end{aligned}$$

for fixed integer a with $a \neq 0, \pm 1$, and

$$f(ax+by)+f(bx+ay)=(a+b)(a-b)^2\big[f(x)+f(y)\big] + ab(a+b)f(x+y)$$

for fixed integers a, b with $a \neq 0$, $b \neq 0$, and $a \pm b \neq 0$, and the equations being equivalent to (1). Afterwards, referring to [7], Chu et al. extended the cubic functional equation to the following generalized form:

$$\begin{aligned} &f\left(\sum_{j=1}^{n-1} x_j + 2x_n\right) + f\left(\sum_{j=1}^{n-1} x_j - 2x_n\right) + \sum_{j=1}^{n-1} f(2x_j) \\ &\quad = 2f\left(\sum_{j=1}^{n-1} x_j\right) + 4\sum_{j=1}^{n-1}\big(f(x_j+x_n)+f(x_j-x_n)\big), \end{aligned} \tag{2}$$

where $n \geqslant 2$ is an integer, and they also investigated the Hyers–Ulam stability. Moreover, in [25], Jung and Chang investigated a generalized Hyers–Ulam–Rassias stability for a cubic functional equation by using the fixed point alternative. The first systematic study of the iterative methods in the stability of mappings is due to Isac and Rassias [20].

The stability of the set-valued functional equations has been widely examined by a number of authors (see [8, 21, 30–32]), and the Hyers–Ulam stability of the set-valued functional equations was proved in [21, 26, 28]. Also, there are many interesting stability results concerning this problem (see [8, 23, 24]).

Quite recently, Eshaghi et al. [10] and M. Ramezani et al. [35] introduced the notion of orthogonal sets and gave a real generalization of the Banach fixed point theorem in incomplete metric spaces. The main result of [10] is the following theorem.

Theorem 2. (*See* [10].) *Let* $(X, \perp, d)$ *be an O-complete orthogonal metric space* (*not necessarily complete metric space*) *and* $0 < \lambda < 1$*. Let* $f : X \to X$ *be O-continuous,* $\perp$*-contraction with Lipschitz constant* λ*, and* $\perp$*-preserving. Then* f *has a unique fixed point* $x^* \in X$*. Also,* f *is a Picard operator, that is,* $\lim_{n\to\infty} f^n(x) = x^*$ *for all* $x \in X$*.*

For more details about the orthogonal space, we refer the reader to [3–5, 10, 34, 35].

The aim of this paper is to offer a new generalized Hyers–Ulam–Rassias stability result for the functional equation (2) for the set-valued mappings in normed spaces, which are not necessarily Banach spaces, by using the fixed point alternative [10] as in [3]. Examplewise, we present a special case of our results, which is a real extension of the previous results as of this literature.

At first, we recall some basic definitions and our main tools.

Definition 1. (See [10].) Let $X \neq \emptyset$ and $\perp \subseteq X \times X$ be a binary relation. If $\perp$ satisfies the following condition

$$\exists x_0 \in X\colon \quad (\forall y,\ y \perp x_0) \quad \text{or} \quad (\forall y,\ x_0 \perp y),$$

then $\perp$ is called an orthogonal relation, and the pair $(X, \perp)$ – an orthogonal set (briefly, *O-set*).

Note that in the above definition, we say that x_0 is an orthogonal element. Also, we say that elements $x, y \in X$ are $\perp$-comparable either $x \perp y$ or $y \perp x$.

Definition 2. (See [10, 35].) Let $(X, \perp)$ be O-set. A sequence $\{x_n\}$ is called

(i) an orthogonal sequence (briefly, *O-sequence*) if

$$(\forall n,\ x_n \perp x_{n+1}) \quad \text{or} \quad (\forall n,\ x_{n+1} \perp x_n);$$

(ii) an strongly orthogonal sequence (briefly, *SO-sequence*) if

$$(\forall n, k,\ x_n \perp x_{n+k}) \quad \text{or} \quad (\forall n, k,\ x_{n+k} \perp x_n).$$

Every SO-sequence is an O-sequence. But the converse is not true in general.

Definition 3. (See [10, 35].) Let $(X, \perp, d)$ be an orthogonal metric space ($(X, \perp)$ is an O-set, and $(X, \perp)$ a metric space). X is

(i) orthogonal complete (briefly, *O-complete*) if every Cauchy O-sequence is convergent;
(ii) strongly orthogonal complete (briefly, *SO-complete*) if every Cauchy SO-sequence is convergent.

It is easy to see that every complete metric space is O-complete and every O-complete metric space is SO-complete. In [3, 35], the authors proved that the converse is not true in general.

Definition 4. (See [10, 35].) Let $(X, \perp, d)$ be an orthogonal metric space. Then $f: X \to X$ is

(i) orthogonal continuous (briefly, *O-continuous*) at $a \in X$ if for each O-sequence $\{a_n\}$ in X, $a_n \to a$ implies $f(a_n) \to f(a)$.
(ii) strongly orthogonal continuous (briefly, *SO-continuous*) at $a \in X$ if for each SO-sequence $\{a_n\}$ in X, $a_n \to a$ implies $f(a_n) \to f(a)$.

Also, f is O-continuous (SO-continuous) on X if f is O-continuous (SO-continuous) in each $a \in X$.

It is obvious that every continuous mapping is O-continuous and every O-continuous mapping is SO-continuous, but the converse is not hold in general (see [3, 35]).

Definition 5. (See [3].) Let $(X, \perp)$ be an O-set. A mapping $f : X \to X$ is said to be $\perp$-preserving if $f(x) \perp f(y)$ whenever $x \perp y$ and $x, y \in X$.

Theorem 3. *Let $(X, \perp, d)$ be an SO-complete orthogonal metric space (not necessarily complete metric space) and $0 < \lambda < 1$. Let $f : X \to X$ be SO-continuous, $\perp$-preserving, and $\perp$-contraction with Lipschitz constant λ. Then f has a unique fixed point $x^* \in X$. Also, f is a Picard operator, that is, $\lim_{n\to\infty} f^n(x) = x^*$ for all $x \in X$.*

Proof. The proof of this result uses the same ideas in Theorem 3.11 of [10], and it suffices to replace the O-sequence by SO-sequence. □

By the aforementioned results we can conclude that Theorem 3 is a real generalization of Theorem 2. So, in the next steps, we are going to prove the stability of functional equation (2) in the SO-complete normed spaces.

2 An incomplete distance on subsets of a set

Before introducing the main results, we recall some notations and definitions.

Let $(X, \|\cdot\|)$ be a normed space (not necessarily a Banach space), and let $\perp$ be an orthogonal relation on X such that $(X, \perp, d)$ is an orthogonal metric space, where d is the induced metric by $\|\cdot\|$.

We say that $\perp$ is $\mathbb{R}$-preserving whenever $x \perp y$ implies $rx \perp ry$ for each $r \in \mathbb{R}$. See the next example.

Example 1. Let $X = \mathbb{R}$, and let two relations $\perp_1$ and $\perp_2$ on X be defined as

$$x \perp_1 y \quad \Longleftrightarrow \quad xy \in \{x, y\} \quad \text{and} \quad x \perp_2 y \quad \Longleftrightarrow \quad \exists k \in \mathbb{Z}\colon\ y = kx.$$

It is obvious that an orthogonal element of $(X, \perp_1)$ and $(X, \perp_2)$ is zero. However, $(X, \perp_1)$ is not $\mathbb{R}$-preserving. To see this, if $x = 1$, $y = 2$, and $r = 3$, then $x \perp_1 y$, while $3x \not\perp_1 3y$. Notice that it is easy to see that $(X, \perp_2)$ is $\mathbb{R}$-preserving.

Let $C_{\mathrm{cb}}(X)$ be the set of all nonempty, closed, convex, and bounded subsets of X. Consider the addition and the scalar multiplication as follows:

$$C + C' = \{x + x'\colon\ x \in C,\ x' \in C'\} \quad \text{and} \quad \lambda C = \{\lambda x\colon\ x \in C\},$$

where $C, C' \in C_{\mathrm{cb}}(X)$ and $\lambda \in \mathbb{R}$. One can show that

$$\lambda C + \lambda C' = \lambda(C + C') \quad \text{and} \quad (\lambda + \mu)C = \lambda C + \mu C$$

for all $\lambda, \mu \in \mathbb{R}$ and $C, C' \in C_{\mathrm{cb}}(X)$. We consider $\mathcal{H}^+$ on pairs of elements in $C_{\mathrm{cb}}(X)$ by

$$\mathcal{H}^+(C, C') = \frac{1}{2}\bigl(\rho(C, C') + \rho(C', C)\bigr),$$

where $\rho(C, C') = \sup_{x \in C} D(x, C')$ and $D(x, C') = \inf\{d(x, y)\colon\ y \in C'\}$. Pathak and Shahzad in [33] proved that $\mathcal{H}^+$ is a metric on $C_{\mathrm{cb}}(X)$. We define the relation $\oplus$ on $C_{\mathrm{cb}}(X)$ as

$$C \oplus C' = \overline{C + C'}.$$

The following proposition can be proved from some properties of the distance $\mathcal{H}^+$.

Proposition 1. (*See* [33].) *For any* $C, C', K, K' \in C_{\mathrm{cb}}(X)$ *and* $\lambda > 0$, *the following properties hold*:

(i) $\mathcal{H}^+(\{a\}, \{b\}) = \|a - b\|$;
(ii) $\mathcal{H}^+(C \oplus C', K \oplus K') \leqslant \mathcal{H}^+(C, K) + \mathcal{H}^+(C', K')$;
(iii) $\mathcal{H}^+(\lambda C, \lambda K) = \lambda \mathcal{H}^+(C, K)$;

(iv) $\mathcal{H}^+(C \oplus \{x\}, C' \oplus \{x\}) = \mathcal{H}^+(C, C')$ *for all* $x \in X$;

(v) $\mathcal{H}^+(C, C') = \inf\{r > 0\colon\ C \subset \delta(C', r_1),\ C' \subset \delta(C, r_2),\ r = (r_1 + r_2)/2\}$, *where* $\delta(C, r) = \{x \in X\colon\ D(x, C) < r\}$ *for all positive real number* r;

(vi) $\mathcal{H}^+(C, C') = \inf\{r > 0\colon\ C \subset \overline{\delta}(C', r_1),\ C' \subset \overline{\delta}(C, r_2),\ r = (r_1 + r_2)/2\}$, *where* $\overline{\delta}(C, r)$ *is the closure of* $\delta(C, r)$ *for each positive real number* r.

Given $A, B \in C_{\mathrm{cb}}(X)$. We define the relation $\perp^*$ between A and B as follows:

$$A \perp^* B \quad \Longleftrightarrow \quad \forall a \in A,\ b \in B, \quad a \perp b.$$

If x_0 is an orthogonal element of $(X, \perp)$, then the singleton $\{x_0\}$ is an orthogonal element for $(C_{\mathrm{cb}}(X), \perp^*)$.

Theorem 4. *If* $(X, d, \perp)$ *is an SO-complete* (*not necessarily complete*) *metric space, then* $(C_{\mathrm{cb}}(X), \oplus, \mathcal{H}^+)$ *with orthogonal relation* $\perp^*$ *is SO-complete.*

Proof. Let $\{A_n\}$ be a Cauchy SO-sequence in $(C_{\mathrm{cb}}(X), \oplus, \mathcal{H}^+)$. We need to show that $\{A_n\}$ converges to some element in $C_{\mathrm{cb}}(X)$.

Let A be the set of limit points of sequences $\{a_n\}$ with $a_n \in A_n$ for all $n \in \mathbb{N}$. Our aim is to prove that $A \in C_{\mathrm{cb}}(X)$ and $\{A_n\}$ converges to A. To see end, let us to divide the proof in the following steps.

Step 1: A is closed. Let $a \in \overline{A}$. Definition of A ensures that we can choose the sequence $\{a_k\}$ in A converging to a. This leads to for all $k \in \mathbb{N}$, there exists $\{y_n^k\}_n$ in X such that for any $n \in \mathbb{N}$, $y_n^k \in A_n$ and $y_n^k \to a_k$ as $n \to \infty$. Let $\{n_k\}$ be a strictly increasing sequence of positive integers such that for any $k \in \mathbb{N}$, $\|y_{n_k}^k - a_k\| < 1/2^k$. We observe that

$$\left\|y_{n_k}^k - a\right\| \leqslant \left\|y_{n_k}^k - a_k\right\| + \|a_k - a\|.$$

As $k \to \infty$, the right-hand of above inequality converges to zero, which implies $a \in A$.

Step 2: A is convex. Let $a, b \in A$ and $0 < t < 1$. Take two sequences $\{a_n\}$ and $\{b_n\}$ such that for each $n \in \mathbb{N}$, $a_n, b_n \in A_n$ and $a_n \to a$ and $b_n \to b$ as $n \to \infty$. Since for any $n \in \mathbb{N}$, A_n is a convex set, then $ta_n + (1 - t)b_n \in A_n$. The closeness of A implies that $ta + (1 - t)b \in A$.

Step 3: A is nonempty. We observe from $\{A_n\}$ is a Cauchy sequence that there exists a strictly increasing sequence $\{n_k\}$ such that for all $k \in \mathbb{N}$ and $n, m \geqslant n_k$, $\mathcal{H}^+(A_n, A_m) < 1/2^{k+1}$. Definition of $\mathcal{H}^+$ ensures that for each $k \in \mathbb{N}$, there exists $a_{n_{k+1}} \in A_{n_{k+1}}$ for which $\|a_{n_k} - a_{n_{k+1}}\| < 1/2^k$. This results show that the sequence $\{a_{n_k}\}_k$ is Cauchy.

On the other hand, since $\{A_n\}$ is an SO-sequence, it follows that

$$(\forall k, m,\ a_{n_k} \perp a_{n_{k+m}}) \quad \text{or} \quad (\forall k, m,\ a_{n_{k+m}} \perp a_{n_k}).$$

Therefore, $\{a_{n_k}\}$ is a Cauchy SO-sequence in X. Since X is SO-complete and $a_{n_k} \in A_{n_k}$ for each $k \in \mathbb{N}$, the conclusion follows easily.

Step 4: $\lim_{n\to\infty} \mathcal{H}^+(A_n, A) = 0$. Fix $\epsilon > 0$. There exists a positive integer N_1 such that for all $m > n \geqslant N_1$, $\mathcal{H}^+(A_n, A_m) < \epsilon$. By definition of $\mathcal{H}^+$ and condition (v) of

Proposition 1 we see that for all $m > n \geqslant N_1$, $A_n \subset \delta(A_m, \epsilon_1)$ and $A_m \subset \delta(A_n, \epsilon_2)$, where $\epsilon = (\epsilon_1 + \epsilon_2)/2$. Let $a \in A$ and $\{a_i\}$ be a sequence such that $a_i \in A_i$ for all i and $\{a_i\}$ converges to a. We observe that for all $i > n \geqslant N_1$, $a_i \in \delta(A_n, \epsilon_2)$, and the continuity of D implies that $a \in \overline{\delta}(A_n, \epsilon_2)$ for all $n \geqslant N_1$. This results show that $A \subset \overline{\delta}(A_n, \epsilon_2)$ for all $n \geqslant N_1$.

On the other hand, we can choose a positive integer N_2 such that for all $n, m \geqslant N_2$, $\mathcal{H}^+(A_m, A_n) < \epsilon_1/4$ and a strictly increasing sequence $\{n_i\}$ of positive integers such that $n_1 > N_2$ and $\mathcal{H}^+(A_{n_i}, A_{n_{i+1}}) < \epsilon_1/2^{i+2}$ for all $i \in \mathbb{N}$.

Assume $n \geqslant N_2$ and $y \in A_n$. It follows from $\mathcal{H}^+(A_n, A_{n_1}) < \epsilon_1/4$ that there is $a_{n_1} \in A_{n_1}$ for which $\|y - a_{n_1}\| \leqslant \epsilon_1/2$. Similarly, for each i, since $\mathcal{H}^+(A_{n_i}, A_{n_{i+1}}) < \epsilon_1/2^{i+2}$, then there is $a_{n_{i+1}} \in A_{n_{i+1}}$ for which $\|a_{n_i} - a_{n_{i+1}}\| < \epsilon_1/2^{i+1}$. We easily see that $\{a_{n_i}\}$ is a Cauchy sequence. Arguing in the Step 3, we obtain that $\{a_{n_i}\}$ is an SO-sequence in X and so converges to an element $a \in X$. Moreover, for all $i \in \mathbb{N}$,

$$\|y - a\| \leqslant \|y - a_{n_1}\| + \|a_{n_1} - a_{n_2}\| + \cdots + \|a_{n_{i-1}} - a_{n_i}\| + \|a_{n_i} - a\|.$$

For large enough numbers of i, $\|y - a\| < \epsilon_1$, which implies that $y \in \delta(A, \epsilon_1)$, and hence, $A_n \subset \delta(A, \epsilon_1)$ for all $n \geqslant N_2$.

Now, take $N = \max\{N_1, N_2\}$, then condition (vi) of Proposition 1 ensures that $\mathcal{H}^+(A_n, A) < \epsilon$ for each $n \geqslant N$. This completes the proof of Step 4. $\square$

3 New generalized Hyers–Ulam–Rassias stability

Throughout this section, we assume $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are two normed spaces. Also, $\perp$ and $\perp^*$ are the same orthogonal relations on Y and $C_{\mathrm{cb}}(Y)$ as defined in the previous section, respectively. We consider the relation $\perp$ as $\mathbb{R}$-persevering and d as the metric induced by $\|\cdot\|_Y$.

Definition 6. Let $f : X \to C_{\mathrm{cb}}(Y)$ be a set-valued mapping.

(i) The n-dimensional cubic set-valued functional equation is defined by

$$\begin{aligned} &f\left(\sum_{j=1}^{n-1} x_j + 2x_n\right) \oplus f\left(\sum_{j=1}^{n-1} x_j - 2x_n\right) \oplus \sum_{j=1}^{n-1} f(2x_j) \\ &\quad = 2f\left(\sum_{j=1}^{n-1} x_j\right) \oplus 4\sum_{j=1}^{n-1} \big(f(x_j + x_n) \oplus f(x_j - x_n)\big) \end{aligned}$$

for every $x_1, \dots, x_n \in X$, where $n \geqslant 2$ is an integer.

(ii) Every solution of the n-dimensional cubic set-valued functional equation is called an n-dimensional cubic set-valued mapping.

Theorem 5. *Let $n \geqslant 2$ be an integer, $m \in \{1, \dots, n-1\}$ and $(Y, d, \perp)$ be an SO-complete metric space (not necessarily a complete metric space). Assume that $f : X \to C_{\mathrm{cb}}(Y)$ is a set-valued mapping such that $f(x/2^r)$ and $f(x)/8^r$ are $\perp^*$-comparable for each $x \in X$*

and $r \in \mathbb{N}$, and also, there exist two functions $\phi : X^n \to [0,\infty)$ and $\alpha : [0,\infty) \to [0,1)$ satisfying the following conditions:

$$\begin{aligned}&\mathcal{H}^+\Biggl(f\Biggl(\sum_{j=1}^{n-1} x_j + 2x_n\Biggr) \oplus f\Biggl(\sum_{j=1}^{n-1} x_j - 2x_n\Biggr) \oplus \sum_{j=1}^{n-1} f(2x_j),\\ &\quad 2f\Biggl(\sum_{j=1}^{n-1} x_j\Biggr) \oplus 4\sum_{j=1}^{n-1}\bigl(f(x_j + x_n) \oplus f(x_j - x_n)\bigr)\Biggr)\\ &\leqslant \phi(x_1,\dots,x_n)\end{aligned} \tag{3}$$

for all $x_1,\dots,x_n \in X$ and also

(A1) $\limsup_{t\to s^+} \alpha(t) < 1$ *for all $s \geqslant 0$*;
(A2) *For all $x_1,\dots,x_n \in X$,*

$$\phi\biggl(\frac{x_1}{2},\dots,\frac{x_n}{2}\biggr) \leqslant \frac{1}{8}\alpha\bigl(\phi(x_1,\dots,x_n)\bigr)\phi(x_1,\dots,x_n);$$

(A3) *For all $x \in X$,*

$$\alpha\Biggl(\phi\Biggl(\biggl(\underbrace{\frac{x}{2},\dots,\frac{x}{2}}_{m\ terms},0,\dots,0\biggr)\Biggr)\Biggr) \leqslant \alpha\bigl(\phi(\underbrace{x,\dots,x}_{m\ terms},0,\dots,0)\bigr).$$

Then there exist an n-dimensional cubic set-valued mapping $F : X \to C_{\mathrm{cb}}(Y)$ and a subset X^* in X with $\operatorname{card}(X^*) > 1$ such that for some positive real number $L < 1$, we have

$$\mathcal{H}^+\bigl(f(x),F(x)\bigr) \leqslant \frac{L}{m(1-L)}\phi(\underbrace{x,\dots,x}_{m\ terms},0,\dots,0) \tag{4}$$

for all $x \in X^*$. In particular, if $X^* = X$, then the mapping F is unique.

Proof. We denote by S_0 the set

$$S_0 = \bigl\{g : X \to C_{\mathrm{cb}}(Y) \bigm| g(0) \text{ is a singelton set}\bigr\}$$

and the generalized metric $\mathcal{D}$ on S_0 as follows:

$$\mathcal{D}(h,g) = \inf\bigl\{M > 0 \bigm| \mathcal{H}^+\bigl(h(x),g(x)\bigr) \leqslant M\phi(\underbrace{x,\dots,x}_{m\text{ terms}},0,\dots,0)\ \forall x \in X\bigr\}.$$

Consider the set $S = \{g \in S_0 \mid \mathcal{D}(g,f) < \infty\}$. Putting $x_j = 0$ ($j = 1,2,\dots,m$) in (A2) yields that $\phi(0,\dots,0) = 0$, and by using (3) we observe that $f(0) = \{0\}$. Hence S is a nonempty set.

Now, let $T : S \to S_0$ be a function as given by $Tg(x) = 8g(x/2)$ for all $x \in X$. We must show that T is a self-adjoint mapping, that is, $T(S) \subseteq S$. To see this, put $x_j = 0$

$(j = 1, \dots, m)$ and $x_{m+1} = x_{m+2} = \dots = x_n = 0$ in inequality (3). Since the range of f is convex and applying (A2), we have

$$\mathcal{H}^+\big(f(mx) \oplus f(mx) \oplus mf(2x), 2f(mx) \oplus 8mf(x)\big) \leqslant \phi(\underbrace{x, \dots, x}_{m \text{ terms}}, 0, \dots, 0),$$

and so,

$$\mathcal{H}^+\big(mf(2x), 8mf(x)\big) \leqslant \phi(\underbrace{x, \dots, x}_{m \text{ terms}}, 0, \dots, 0) \tag{5}$$

for all $x \in X$. Dividing by $8m$ in (5), we get

$$\mathcal{H}^+\left(\frac{f(2x)}{8}, f(x)\right) \leqslant \frac{1}{8m}\phi(\underbrace{x, \dots, x}_{m \text{ terms}}, 0, \dots, 0) \tag{6}$$

for all $x \in X$. Replacing x by $x/2$ in (6) and applying (A2), we have

$$\begin{aligned}
\mathcal{H}^+\left(f(x), 8f\left(\frac{x}{2}\right)\right) &\leqslant \frac{1}{m}\phi\Bigg(\underbrace{\frac{x}{2}, \dots, \frac{x}{2}}_{m \text{ terms}}, 0, \dots, 0\Bigg) \\
&\leqslant \frac{1}{8m}\alpha\big(\phi(\underbrace{x, \dots, x}_{m \text{ terms}}, 0, \dots, 0)\big)\phi(\underbrace{x, \dots, x}_{m \text{ terms}}, 0, \dots, 0) \\
&\leqslant \frac{1}{8m}\phi(\underbrace{x, \dots, x}_{m \text{ terms}}, 0, \dots, 0)
\end{aligned} \tag{7}$$

for all $x \in X$. This ensures that $\mathcal{D}(Tf, f) \leqslant 1/(8m)$. On the other hand, if $g \in S$, definition of $\mathcal{D}$ conclude that $\mathcal{D}(Tg, Tf) \leqslant D(g, f)$, and the triangle inequality implies that $\mathcal{D}(Tg, f) \leqslant D(Tg, Tf) + \mathcal{D}(Tf, f) < \infty$, that is, $Tg \in S$. Consider

$$O(x) = \big\{f(x), (Tf)(x), \big(T^2 f\big)(x), \big(T^3 f\big)(x), \dots\big\}$$

for all $x \in X$. Define the relation $\perp_S$ on S as the following:

$$g \perp_S h \quad \Longleftrightarrow \quad \big(\big\{g(x), h(x)\big\} \cap O(x) \neq \emptyset \text{ or } g(x) \perp^* h(x)\big) \quad \forall x \in X.$$

It follows from Theorem 4 that $(S, \mathcal{D}, \perp_S)$ is an SO-complete metric space. Since the relation $\perp$ is $\mathbb{R}$-preserving, definition of $\perp_S$ and $\perp^*$ imply that T is $\perp_S$-preserving. By using the hypothesis we obtain

$$\left(f\left(\frac{x}{2^r}\right) \perp^* \frac{f(x)}{8^r}\right) \quad \text{or} \quad \left(\frac{f(x)}{8^r} \perp^* f\left(\frac{x}{2^r}\right)\right)$$

for all $x \in X$ and $r \in \mathbb{N}$. From $\mathbb{R}$-preserving of $\perp$ and definition of T we get

$$\big(\big(T^r f\big)(x) \perp^* f(x)\big) \quad \text{or} \quad \big(f(x) \perp^* \big(T^r f\big)(x)\big)$$

for all $x \in X$ and $r \in \mathbb{N}$. This means that

$$\bigl(T^r f \perp_S f\bigr) \quad \text{or} \quad \bigl(f \perp_S T^r f\bigr)$$

for all $r \in \mathbb{N}$. It follows from $\perp_S$-preserving of T that

$$\bigl(\forall r, s \in \mathbb{N},\ T^{r+s} f \perp_S T^r f\bigr) \quad \text{or} \quad \bigl(\forall r, s \in \mathbb{N},\ T^r f \perp_S T^{r+s} f\bigr).$$

That is, $\{T^r f\}$ and consequently $\{(T^r f)(x)\}$ for all $x \in X$ are SO-sequences in S and $C_{\mathrm{cb}}(X)$, respectively. In order to show that the SO-sequence $\{T^r f\}$ is Cauchy, replacing x by $x/2^r$ and multiplying by 8^r in (7) and using (A2) and (A3), we get

$$\mathcal{H}^+\bigl(\bigl(T^{r+1} f\bigr)(x), \bigl(T^r f\bigr)(x)\bigr) \leqslant \bigl[\alpha\bigl(\phi(\underbrace{x, \dots, x}_{m \text{ terms}}, 0, \dots, 0)\bigr)\bigr]^r \phi(\underbrace{x, \dots, x}_{m \text{ terms}}, 0, \dots, 0)$$

for all $x \in X$ and $r \in \mathbb{N}$. Considering

$$L_x = \alpha\bigl(\phi(\underbrace{x, \dots, x}_{m \text{ terms}}, 0, \dots, 0)\bigr),$$

we obtain that

$$\begin{aligned}
&\mathcal{H}^+\bigl(\bigl(T^s f\bigr)(x), \bigl(T^r f\bigr)(x)\bigr) \\
&\quad \leqslant \sum_{i=r}^{s-1} \mathcal{H}^+\bigl(\bigl(T^{i+1} f\bigr)(x), \bigl(T^i f\bigr)(x)\bigr) \leqslant \sum_{i=r}^{s-1} L_x^i \phi(\underbrace{x, \dots, x}_{m \text{ terms}}, 0, \dots, 0) \\
&\quad = \frac{L_x^r(1 - L_x^{s-1})}{1 - L_x} \phi(\underbrace{x, \dots, x}_{m \text{ terms}}, 0, \dots, 0)
\end{aligned}$$

for all $x \in X$ and $r, s \in \mathbb{N}$ with $r < s$. Since $L_x < 1$, letting $r, s \to \infty$ in the above inequality, we deduce that the sequence $\{(T^r f)(x)\}$ is a Cauchy sequence for each $x \in X$. By SO-completeness of $C_{\mathrm{cb}}(Y)$ we obtain that for every $x \in X$, there exists an element $F(x) \in C_{\mathrm{cb}}(Y)$, which is a limit point of $\{(T^r f)(x)\}$. That is, $F : X \to C_{\mathrm{cb}}(Y)$ is well defined and given by

$$F(x) = \lim_{r \to \infty} \bigl(T^r f\bigr)(x) = \lim_{r \to \infty} 8^r f\left(\frac{x}{2^r}\right) \tag{8}$$

for all $x \in X$. On the other hand, since $\limsup_{t \to 0^+} \alpha(t) < 1$, then there exist $\lambda \in (0, \infty]$ and $0 < L < 1$ such that $\alpha(t) \leqslant L$ for all $0 \leqslant t < \lambda$. Put

$$X^* = \bigl\{x \in X \bigm| \phi(\underbrace{x, \dots, x}_{m \text{ terms}}, 0, \dots, 0) < \lambda\bigr\}.$$

It follows from $\phi(0, \dots, 0) = 0$ that $0 \in X^*$. Also, if x_0 is an arbitrary nonzero point of X, then by using (A2) we can easily see that

$$\phi\Biggl(\underbrace{\frac{x_0}{2^r}, \dots, \frac{x_0}{2^r}}_{m \text{ terms}}, 0, \dots, 0\Biggr) \leqslant \frac{1}{8^r} \phi(\underbrace{x_0, \dots, x_0}_{m \text{ terms}}, 0, \dots, 0).$$

So, there exists a natural number r_0 for which

$$\phi\biggl(\underbrace{\frac{x_0}{2^{r_0}},\dots,\frac{x_0}{2^{r_0}}}_{m\text{ terms}},0,\dots,0\biggr) < \lambda,$$

and this means that $x_0/2^{r_0}$ belongs to X^*. This implies that $\operatorname{card}(X^*) > 1$. Now, we replace X by X^* in definition of S_0. For $g, h \in S$, we have the following implications:

$$\begin{aligned}
&D(g,h) < K\\
&\quad\Longrightarrow\quad \mathcal{H}^+\bigl(g(x),h(x)\bigr) \leqslant K\phi(\underbrace{x,\dots,x}_{m\text{ terms}},0,\dots,0),\quad x\in X^*,\\
&\quad\Longrightarrow\quad 8\mathcal{H}^+\biggl(g\biggl(\frac{x}{2}\biggr),h\biggl(\frac{x}{2}\biggr)\biggr) \leqslant K8\phi\biggl(\underbrace{\frac{x}{2},\dots,\frac{x}{2}}_{m\text{ terms}},0,\dots,0\biggr),\quad x\in X^*,\\
&\quad\Longrightarrow\quad \mathcal{H}^+\biggl(8g\biggl(\frac{x}{2}\biggr),8h\biggl(\frac{x}{2}\biggr)\biggr) \leqslant K\alpha\bigl(\phi(\underbrace{x,\dots,x}_{m\text{ terms}},0,\dots,0)\bigr)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\times\phi(\underbrace{x,\dots,x}_{m\text{ terms}},0,\dots,0),\quad x\in X^*,\\
&\quad\Longrightarrow\quad \mathcal{H}^+\biggl(8g\biggl(\frac{x}{2}\biggr),8h\biggl(\frac{x}{3}\biggr)\biggr) \leqslant KL\phi(\underbrace{x,\dots,x}_{m\text{ terms}},0,\dots,0),\quad x\in X^*,\\
&\quad\Longrightarrow\quad \mathcal{H}^+\bigl(Tg(x),Th(x)\bigr) \leqslant KL\phi(\underbrace{x,\dots,x}_{m\text{ terms}},0,\dots,0),\quad x\in X^*,\\
&\quad\Longrightarrow\quad \mathcal{D}(Tg,Th) \leqslant KL.
\end{aligned}$$

Hence, we see that $\mathcal{D}(Tg,Th) \leqslant L\mathcal{D}(g,h)$ for all $g,h\in S$. It follows from $L<1$ that T is a contraction. Consequently, T is an SO-continuous mapping and is a contraction on $\perp_S$-comparable elements with Lipschitz constant L. Since $(S,\mathcal{D},\perp_S)$ is SO-complete and T is also $\perp_S$-preserving, then from Theorem 3 we conclude that T has a unique fixed point and T is a Picard operator. This means that the sequence $\{T^r f\}$ is convergent to the fixed point of T. It follows from (8) that F is a unique fixed point of T. Moreover,

$$\begin{aligned}
\mathcal{D}(F,f) &\leqslant \mathcal{D}(F,TF)+\mathcal{D}(TF,Tf)+\mathcal{D}(Tf,f)\\
&\leqslant L\mathcal{D}(F,f)+\mathcal{D}(Tf,f).
\end{aligned}$$

Therefore, $\mathcal{D}(F,f) \leqslant \mathcal{D}(Tf,f)/(1-L)$. Relation (7) ensures that inequality (4) holds.

Finally, we need to show that F is an n-dimensional cubic set-valued mapping. To this end, let $x_1,\dots,x_n$ be fixed elements of X. Since $\{\phi(x_1/2^r,\dots,x_n/2^r)\}$ is a nonnegative and decreasing sequence, then there is $\tau \geqslant 0$ for which $\phi(x_1/2^r,\dots,x_n/2^r) \to \tau$ as $r\to\infty$. Taking into account (A1), we have $\limsup_{t\to\tau^+}\alpha(t) < 1$, so there exist $\delta > 0$ and $\nu < 1$ such that for all $t\in[\tau,\,\tau+\delta)$, $\alpha(t) < \nu$. Consider the positive integer N such that for all $r \geqslant N$, $\phi(x_1/2^r, \dots, x_n/2^r) \in [\tau, \tau + \delta)$. By virtue of (3) we obtain

$$\begin{aligned}
&\mathcal{H}^+\Bigg(F\Bigg(\sum_{j=1}^{n-1} x_j + 2x_n\Bigg) \oplus F\Bigg(\sum_{j=1}^{n-1} x_j - 2x_n\Bigg) \oplus \sum_{j=1}^{n-1} F(2x_j), \\
&\quad 2F\Bigg(\sum_{j=1}^{n-1} x_j\Bigg) \oplus 4\sum_{j=1}^{n-1}\big(F(x_j + x_n) \oplus F(x_i - x_n)\big)\Bigg) \\
&\leqslant \lim_{r\to\infty} 8^r \phi\bigg(\frac{x_1}{2^r}, \dots, \frac{x_n}{2^r}\bigg) \\
&\leqslant \lim_{r\to\infty} \prod_{i=0}^{r-1} \alpha\bigg(\phi\bigg(\frac{x_1}{2^i}, \dots, \frac{x_n}{2^i}\bigg)\bigg)\phi(x_1, \dots, x_n) \\
&= \lim_{r\to\infty} \prod_{i=N}^{r-1} \nu \cdot \prod_{i=0}^{N-1} \alpha\bigg(\phi\bigg(\frac{x_1}{2^i}, \dots, \frac{x_n}{2^i}\bigg)\bigg)\phi(x_1, \dots, x_n) \\
&\leqslant \lim_{r\to\infty} \nu^{r-N} \cdot \prod_{i=0}^{N-1} \alpha\bigg(\phi\bigg(\frac{x_1}{2^i}, \dots, \frac{x_n}{2^i}\bigg)\bigg)\phi(x_1, \dots, x_n) = 0.
\end{aligned}$$

Therefore, F is an n-dimensional cubic set-valued mapping as desired. □

Corollary 1. *Let $n \geqslant 2$ be an integer and $m \in \{1, \dots, n-1\}$. Let Y be a Banach space and $f : X \to Y$ be a mapping such that there exists a function $\phi : X^n \to [0, \infty)$ satisfying*

$$\begin{aligned}
&\Bigg\| f\Bigg(\sum_{j=1}^{n-1} x_j + 2x_n\Bigg) + f\Bigg(\sum_{j=1}^{n-1} x_j - 2x_n\Bigg) + \sum_{j=1}^{n-1} f(2x_j) \\
&\quad - 2f\Bigg(\sum_{j=1}^{n-1} x_j\Bigg) - 4\sum_{j=1}^{n-1}\big(f(x_j + x_n) - f(x_j - x_n)\big)\Bigg\|_Y \\
&\leqslant \phi(x_1, \dots, x_n)
\end{aligned}$$

for all $x_1, \dots, x_n \in X$. If there exists a positive real number $L < 1$ such that

$$\phi(x_1, \dots, x_n) \leqslant \frac{1}{8} L\phi(2x_1, \dots, 2x_n) \tag{9}$$

for all $x_1, \dots, x_n \in X$, then there exists a unique n-dimensional cubic mapping $F : X \to Y$, which satisfies the inequality

$$\big\|f(x) - F(x)\big\|_Y \leqslant \frac{L}{m(1-L)} \phi(\underbrace{x, \dots, x}_{m\ terms}, 0, \dots, 0)$$

for all $x \in X$. The mapping F is given by

$$F(x) = \lim_{r\to\infty} 8^r f\bigg(\frac{x}{2^r}\bigg) \quad \forall x \in X.$$

Proof. For every $y_1, y_2 \in Y$, define $y_1 \perp y_2$ if and only if $\|y_1\|_Y \leqslant \|y_2\|_Y$. It is clear that $(Y, \perp)$ is an O-set. Moreover, we can consider $(Y, d, \perp)$ as a closed subset of $(C_{\mathrm{cb}}(Y), \mathcal{H}^+, \perp^*)$, which d is the metric induced by $\|\cdot\|_Y$. Since Y is a Banach space, so $(Y, d, \perp)$ is an SO-complete metric space. From definition of $\perp$ follows that

$$\left[\forall x \in X,\ r \in \mathbb{N},\ f\left(\frac{x}{2^r}\right) \perp \frac{f(x)}{8^r}\right]$$

or

$$\left[\forall x \in X,\ r \in \mathbb{N},\ \frac{f(x)}{8^r} \perp f\left(\frac{x}{2^r}\right)\right].$$

It is enough to pick $\alpha(t) = L$ for all $t \in [0, \infty)$. The result is an immediate consequence of Theorem 5. □

Theorem 6. *Let $n \geqslant 2$ be an integer, $m \in \{1, \dots, n-1\}$, and $(Y, d, \perp)$ be an SO-complete metric space (not necessarily complete metric space). Suppose that $f : X \to C_{\mathrm{cb}}(Y)$ is a set-valued mapping such that $f(2^r x)$ and $8^r f(x)$ are $\perp^*$-comparable for each $x \in X$ and $r \in \mathbb{N}$, and there exists a function $\phi : X^n \to [0, \infty)$ satisfying equation (3) of Theorem 5 and the following property:*

- (B1) *$\phi(x_1, \dots, x_n) = 0$ if and only if $x_j = 0$ for all $j \in \{1, \dots, n\}$, and $\{\phi(2^r x_1, \dots, 2^r x_n)\}$ is an increasing sequence for all $x_1, \dots, x_n \in X$ that are not all zero. Also,*

$$\Bigl\{\phi\bigl(\underbrace{2^r x_0, \dots, 2^r x_0}_{m\ \text{terms}}, 0, \dots, 0\bigr)\Bigr\}$$

is an unbounded sequence for some $x_0 \in X$.

If $\alpha : [0, \infty) \to [0, 1)$ is a mapping, which satisfies relation (A1) of Theorem 5 and the following conditions:

- (B2) *For all $x_1, \dots, x_n \in X$ that are not all zero,*

$$\phi(2x_1, \dots, 2x_n) \leqslant 8\alpha\bigl(\bigl[\phi(x_1, \dots, x_n)\bigr]^{-1}\bigr)\phi(x_1, \dots, x_n);$$

- (B3) *For every nonzero element x of X,*

$$\alpha\bigl(\bigl[\phi(\underbrace{2x, \dots, 2x}_{m\ \text{terms}}, 0, \dots, 0)\bigr]^{-1}\bigr) \leqslant \alpha\bigl(\bigl[\phi(\underbrace{x, \dots, x}_{m\ \text{terms}}, 0, \dots, 0)\bigr]^{-1}\bigr).$$

Then there exist an n-dimensional cubic set-valued mapping $F : X \to C_{\mathrm{cb}}(Y)$ and a subset X^* in X with $\mathrm{card}(X^*) > 1$ such that for some positive real number $L < 1$, we have

$$\mathcal{H}^+\bigl(f(x), F(x)\bigr) \leqslant \frac{1}{1-L}\frac{1}{8m}\phi(\underbrace{x, \dots, x}_{m\ \text{terms}}, 0, \dots, 0) \tag{10}$$

for all $x \in X^*$. Moreover, if $X^* = X$, then F is unique.

Proof. By the same reasoning as in the proof of Theorem 5, there exist $\lambda \in (0,\infty]$ and $0 < L < 1$ such that $\alpha(t) \leqslant L$ for each $0 \leqslant t < \lambda$. Set

$$X^* := \bigl\{x \in X \bigm| x \neq 0,\ \bigl[\phi(\underbrace{x,\dots,x}_{m \text{ terms}},0,\dots,0)\bigr]^{-1} < \lambda\bigr\} \cup \{0\}.$$

As a result of (B1), we can easily see that for some $x_0 \in X$, the sequence

$$\bigl\{\bigl[\phi(\underbrace{2^r x_0,\dots,2^r x_0}_{m \text{ terms}},0,\dots,0)\bigr]^{-1}\bigr\}$$

is a decreasing sequence which converges to zero. This concludes that $\operatorname{card}(X^*) > 1$. By the same argument of Theorem 5 one can show that the mapping $T : S \to S$ defined by $Tg(x) = g(2x)/8$ for all $x \in X$ is a $\perp_S$-preserving mapping and is a contraction with Lipschitz constant L on X^*. Define $F : X \to C_{\mathrm{cb}}(Y)$ by $F(x) = \lim_{r\to\infty} f(2^r x)/8^r$ for all $x \in X$. Replacing X^* by X in definition of S_0 and applying Theorem 3, we obtain that F is a unique fixed point of T. It follows from (6) that $\mathcal{D}(f,Tf) \leqslant 1/(8m)$ and so

$$\mathcal{D}(f,F) \leqslant \mathcal{D}(f,Tf) + \mathcal{D}(Tf,TF) \leqslant \mathcal{D}(f,Tf) + L\mathcal{D}(f,F),$$

and consequently,

$$\mathcal{D}(f,F) \leqslant \frac{1}{1-L}\mathcal{D}(f,Tf) \leqslant \frac{1}{1-L}\frac{1}{8m}.$$

That is, inequality (10) holds. To show that the function F is an n-dimensional set-valued mapping on X, let $x_1,\dots,x_n$ be fixed elements of X, which are not all zero. Since

$$\bigl\{\bigl[\phi(\underbrace{2^r x,\dots,2^r x}_{m \text{ terms}},0,\dots,0)\bigr]^{-1}\bigr\}$$

is a nonnegative and decreasing sequence, so the rest of the proof is similar to the proof of Theorem 5. $\square$

Corollary 2. *Let $n \geqslant 2$ be an integer and Y be a Banach space. Suppose that $f : X \to Y$ is a mapping such that there exists a function $\phi : X^n \to [0,\infty)$ satisfying conditions* (B1) *of Theorem* 6 *and, in addition,*

$$\begin{aligned}
&\Biggl\| f\Biggl(\sum_{j=1}^{n-1} x_j + 2x_n\Biggr) + f\Biggl(\sum_{j=1}^{n-1} x_j - 2x_n\Biggr) + \sum_{j=1}^{n-1} f(2x_j) \\
&\qquad - 2f\Biggl(\sum_{j=1}^{n-1} x_j\Biggr) - 4\sum_{j=1}^{n-1}\bigl(f(x_j+x_n) - f(x_j-x_n)\bigr)\Biggr\|_Y \\
&\quad \leqslant \phi(x_1,\dots,x_n)
\end{aligned}$$

for all $x_1,\dots,x_n \in X$. If there exists a positive real number $L < 1$ such that

$$\phi(2x_1,\dots,2x_n) \leqslant 8L\phi(x_1,\dots,x_n)$$

for all $x_1, \dots, x_n \in X$, then for every $m \in \{1, \dots, n-1\}$, there exists a unique n-dimensional cubic mapping $F : X \to Y$, which satisfies the inequality

$$\big\|f(x) - F(x)\big\|_Y \leqslant \frac{1}{1-L}\frac{1}{8m}\phi(\underbrace{x, \dots, x}_{m \text{ terms}}, 0, \dots, 0)$$

for all $x \in X$. The mapping F is given by

$$F(x) = \lim_{r\to\infty} \frac{f(2^r x)}{8^r} \quad \forall x \in X.$$

Proof. Take the same metric d and the orthogonal relation of Corollary 1. By the same argument of Corollary 1 one can show that $(Y, d, \perp)$ is an SO-complete metric space and $f(2^r x)$ and $8^r f(x)$ are $\perp$-comparable for each $x \in X$ and $r \in \mathbb{N}$. Putting $\alpha(t) = L$ for all $t \in [0, \infty)$ and applying Theorem 6, we can easily obtain the results. $\square$

Corollary 3. *Suppose that Y is a Banach space and $\theta \geqslant 0$ and $p \neq 3$ are fixed. Assume that $f : X \to Y$ is a function satisfies the functional inequality*

$$\begin{aligned}
&\Bigg\|f\left(\sum_{j=1}^{n-1} x_j + 2x_n\right) + f\left(\sum_{j=1}^{n-1} x_j - 2x_n\right) + \sum_{j=1}^{n-1} f(2x_j) \\
&\qquad - 2f\left(\sum_{j=1}^{n-1} x_j\right) - 4\sum_{j=1}^{n-1}\big(f(x_j + x_n) - f(x_j - x_n)\big)\Bigg\|_Y \\
&\quad\leqslant \theta \sum_{j=1}^{n} \|x_j\|_X^p
\end{aligned} \tag{11}$$

for all $x_1, \dots, x_n \in X$. Then there exists a unique n-dimensional cubic mapping $F : X \to Y$ such that the inequality

$$\big\|f(x) - F(x)\big\|_Y \leqslant \frac{8\theta}{2^p - 8}\|x\|_X^p \tag{12}$$

holds for all $x \in X$, where $p > 3$, or the inequality

$$\big\|f(x) - F(x)\big\|_Y \leqslant \frac{\theta}{8 - 2^p}\|x\|_X^p \tag{13}$$

holds for all $x \in X$, where $p < 3$.

Proof. Take the same metric d and the orthogonal relation of Corollary 1. By the same argument of Corollary 1 one can show that $(Y, d, \perp)$ is an SO-complete metric space. Moreover, definition of $\perp$ ensures that $f(x/2^r)$ and $f(x)/8^r$ are $\perp$-comparable for each $x \in X$ and $r \in \mathbb{N}$. Similarly, $f(2^r x)$ and $8^r f(x)$ are $\perp$-comparable for each $x \in X$ and $r \in \mathbb{N}$.

We define $\phi(x_1,\dots,x_n)=\theta\sum_{j=1}^{n}\|x_j\|_X^p$ for each $x_1,\dots,x_n\in X$. It follows that

$$\phi\left(\frac{x_1}{2},\dots,\frac{x_n}{2}\right)\leqslant\frac{1}{2^3}\frac{1}{2^{p-3}}\phi(x_1,\dots,x_n)$$

for all $x_1,\dots,x_n\in X$, where $p>3$. Set $\alpha(t)=1/2^{p-3}$ for all $t\in[0,\infty)$. This ensures that $X^*=X$ and relations (A1) and (A3) of Theorem 5 hold. Applying Theorem 5, we see that inequality (4) holds with $L=1/2^{p-3}$, which yields inequality (12). On the other hand, the function ϕ satisfies properties (B1), (B2) and also

$$\phi(2x_1,\dots,2x_n)\leqslant 2^3 2^{p-3}\phi(x_1,\dots,x_n)$$

for all $x_1,\dots,x_n\in X$, where $p<3$. Putting $\alpha(t)=2^{p-3}$ for every $t\in[0,\infty)$, it is easily seen that $X^*=X$ and conditions (A1) and (B3) are hold. Employing Theorem 6, we see that inequality (10) holds with $L=1/2^{3-p}$. This implies inequality (13). $\square$

The next example shows that Theorem 6 is a real extension of Corollary 1.

Example 2. Let $n\geqslant 2$ be an integer and $m\in\{1,\dots,n-1\}$, and Y be a Banach space. Let $\{\tau_p\}$ be a sequence defined by $\tau_0=0$, $\tau_1=1$, and $\tau_p=p+1/p$ for all natural number p with $p\geqslant 2$. It is easy to see that $\{\tau_p\}$ is a strictly increasing sequence of real numbers. Suppose that $f:X\to Y$ is a mapping satisfying

$$\begin{aligned}&\Bigg\|f\left(\sum_{j=1}^{n-1}x_j+2x_n\right)+f\left(\sum_{j=1}^{n-1}x_j-2x_n\right)+\sum_{j=1}^{n-1}f(2x_j)\\&\qquad-2f\sum_{j=1}^{n-1}x_j-4\sum_{j=1}^{n-1}\big(f(x_j+x_n)-f(x_j-x_n)\big)\Bigg\|_Y\\&\quad\leqslant\phi(x_1,\dots,x_n)\end{aligned}$$

for all $x_1,\dots,x_n\in X$. Define a mapping $\phi:X^n\to[0,\infty)$ by

$$\phi(x_1,\dots,x_n)=\begin{cases}\tau_p\sum_{j=1}^{n}\|x_j\|_X^3, & \sum_{j=1}^{n}\|2x_j\|_X^3-\sum_{j=1}^{n}\|x_j\|_X^3>1,\\ & \text{and } p \text{ is the smallest natural number such that}\\ & \sum_{j=1}^{n}\|x_j\|_X^3<\tau_p<\sum_{j=1}^{n}\|2x_j\|_X^3,\\ 0 & \text{otherwise}\end{cases}$$

and the function $\alpha:[0,\infty)\to[0,1)$ as

$$\alpha(t)=\begin{cases}\frac{\tau_{p-1}}{\tau_p}, & p \text{ is the smallest natural number such that } t\leqslant\tau_p,\\ 0 & \text{otherwise.}\end{cases}$$

Then the following hold:

(i) For every $s\geqslant 0$, $\limsup_{t\to s^+}\alpha(t)<1$.

(ii) For every $x \in X$,

$$\alpha\biggl(\phi\biggl(\underbrace{\frac{x}{2},\dots,\frac{x}{2}}_{m \text{ terms}},0,\dots,0\biggr)\biggr) \leqslant \alpha\bigl(\phi(\underbrace{x,\dots,x}_{m \text{ terms}},0,\dots,0)\bigr).$$

(iii) For every $x_1,\dots,x_n \in X$,

$$\phi\biggl(\frac{x_1}{2},\dots,\frac{x_n}{2}\biggr) \leqslant \frac{1}{8}\alpha\bigl(\phi(x_1,\dots,x_n)\bigr)\phi(x_1,\dots,x_n).$$

(iv) For every positive real number s, there exist a constant $L \in (0,1)$ and an n-dimensional cubic mapping $F : X \to Y$ such that

$$\bigl\|F(x) - f(x)\bigr\|_Y \leqslant \frac{L}{m(1-L)}\phi(\underbrace{x,\dots,x}_{m \text{ terms}},0,\dots,0)$$

for all x with $\|x\|_X \leqslant s$.

Proof. Take the same metric d and the orthogonal relation of Corollary 1. By the same argument of Corollary 1 one can show that $(Y,d,\perp)$ is an SO-complete metric space and $f(x/2^r)$ and $f(x)/(8^r)$ are $\perp$-comparable for each $x \in X$ and $r \in \mathbb{N}$.

Let us take $x_1,\dots,x_n \in X$ and $(\sum_{j=1}^n \|x_j\|_X^3) - \sum_{j=1}^n \|x_j/2\|_X^3 > 1$, and let p be the smallest natural number such that $\sum_{j=1}^n \|x_j/2\|_X^3 < \tau_p < \sum_{j=1}^n \|x_j\|_X^3$. Then

$$\phi\biggl(\frac{x_1}{2},\dots,\frac{x_n}{2}\biggr) = \tau_p \sum_{j=1}^n \biggl\|\frac{x_j}{2}\biggr\|_X^3.$$

We observe that

$$\sum_{j=1}^n \|2x_j\|_X^3 - \sum_{j=1}^n \|x_j\|_X^3 > 8.$$

This follows that there exists $k_0 \in \mathbb{N}$, which

$$\sum_{j=1}^n \|x_j\|_X^3 < \tau_{k_0} < \sum_{j=1}^n \|2x_j\|_X^3.$$

Assume that k is the smallest natural number satisfying the above condition. Clearly, $k > p$ and

$$\phi(x_1,\dots,x_n) = \tau_k \sum_{j=1}^n \|x_j\|_X^3.$$

Now, we suppose that q is the smallest natural number that $\tau_k \sum_{j=1}^n \|x_j\|_X^3 \leqslant \tau_q$, then $\alpha(\phi(x_1,\dots,x_n)) = \tau_{q-1}/\tau_q$. Since $\sum_{j=1}^n \|x_j\|_X^3 > 1$, then $\tau_k < \tau_q$, and we conclude

$\tau_p/\tau_k < \tau_p/\tau_{p+1} < \tau_{q-1}/\tau_q$. This implies that

$$\phi\left(\frac{x_1}{2},\ldots,\frac{x_n}{2}\right) = \tau_p \sum_{j=1}^{n} \left\|\frac{x_j}{2}\right\|_X^3 = \frac{1}{8}\tau_p \sum_{j=1}^{n} \|x_j\|_X^3 \leqslant \frac{1}{8}\frac{\tau_{q-1}}{\tau_q}\tau_k \sum_{j=1}^{n} \|x_j\|_X^3$$
$$= \frac{1}{8}\alpha\big(\phi(x_1,\ldots,x_n)\big)\phi(x_1,\ldots,x_n).$$

That is, condition (i) holds. From definition α it is easily seen that α is a nondecreasing mapping.

Finally, it follows from $\limsup_{t\to s^+}\alpha(t) = 0$ that for every $s > 0$, there exists $L < 1$ such that

$$\alpha\big(\phi(\underbrace{x,\ldots,x}_{m \text{ terms}},0,\ldots,0)\big) < L$$

for all x with $\|x\|_X \leqslant s$. By the same proof of Theorem 5 we prove (iv). $\square$

Notice that there is no $L < 1$ such that inequality (9) holds, and hence, the stability of f does not imply by Corollary 1.

Now, we observe in the following example that our results go further than the stability on Banach spaces.

Example 3. Let $\theta \geqslant 0$ and $p \neq 3$ be given. Consider $Y = C([0,1],\mathbb{R})$ (the set all of continuous functions on $[0,1]$) with norm $\|h\|_Y = (\int_0^1 |h(x)|^s \,\mathrm{d}x)^{1/s} = \|h\|_s$, where $1 < s < \infty$. Suppose that $f : X \to Y$ is a mapping satisfying inequality (11) and the following condition:

$$\exists\gamma > 0: \quad f\left(\frac{x}{2}\right) = \frac{\gamma}{8} f(x), \quad x \in X. \tag{14}$$

Then there exists a unique n-dimensional cubic mapping $F : X \to Y$ such that inequality (12) holds for all $x \in X$, where $p > 3$, or inequality (13) holds for all $x \in X$, where $p < 3$.

Proof. Let q be the conjugate of s, i.e., $1/s + 1/q = 1$. For all $h, g \in Y$, define

$$h \perp g \iff \int_0^1 h(x)g(x)\,\mathrm{d}x = \left(\int_0^1 h(x)^s \,\mathrm{d}x\right)^{1/s}\left(\int_0^1 g(x)^q \,\mathrm{d}x\right)^{1/q}$$
$$= \|h\|_s\|g\|_q$$

and $d(h,g) = \|h-g\|_Y$. We claim that $(Y,\perp,d)$ is an SO-complete metric space. Indeed, let $\{h_n\}$ be a Cauchy SO-sequence in Y, and for all $n,k \in \mathbb{N}$, $h_n \perp h_{n+k}$. The relation $\perp$ ensures that for all $n \in \mathbb{N}$,

$$\exists\lambda_n \neq 0: \quad h_n^s = \lambda_n h_{n+1}^q \text{ a.e.} \quad \text{or} \quad h_{n+1}^q = \lambda_n h_n^s \text{ a.e.} \tag{15}$$

We distinguish two cases.

Case 1. There exists a subsequence $\{h_{n_k}\}$ of $\{h_n\}$ such that $h_{n_k} = 0$ a.e. for all k. This implies that $h_n \to 0 \in X$.

Case 2. For all sufficiently large $n \in \mathbb{N}$, $h_n \neq 0$. Take $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$, $h_n \neq 0$. It follows from (15) that for all $n \geqslant n_0$, there exists $\lambda_n \neq 0$ for which $h_n = \lambda_n h_{n_0}^{s/q}$. It leads to

$$|\lambda_n - \lambda_m| \big\| h_{n_0}^{s/q} \big\|_p = \big\| \lambda_n h_{n_0}^{s/q} - \lambda_m h_{n_0}^{s/q} \big\|_p = \| h_n - h_m \|_p$$

for each $m, n \geqslant n_0$. As $n \to \infty$, the right-hand side of the above inequality tends to 0. Therefore, $\{\lambda_n\}$ is a Cauchy sequence in $\mathbb{R}$. Assume that $\lambda_n \to \lambda$ as $n \to \infty$. Put $h = \lambda h_{n_0}^{s/q}$. It follows that $h \in Y$ and for all $n \geqslant n_0$,

$$\| h_n - h \|_s = \big\| \lambda_n h_{n_0}^{s/q} - \lambda h_{n_0}^{s/q} \big\| = |\lambda_n - \lambda| \big\| h_{n_0}^{s/q} \big\|_s.$$

This implies that $h_n \to h$ as $n \to \infty$. Note that the case $h_{n+k} \perp h_n$ for all $n, k \in \mathbb{N}$ is in a similar way.

By virtue of (14) and definition of $\perp$ we obtain that $f(x/2^r)$ and $f(x)/8^r$ are $\perp$-comparable elements for each $x \in X$ and $r \in \mathbb{N}$. Moreover, putting $x := rx$ in (14), we can also see that $f(2^r x)$ and $8^r f(x)$ are $\perp$-comparable elements in Y for all $x \in X$ and $r \in \mathbb{N}$. The rest of the proof is similar to the proof of Corollary 3. $\square$

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