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Synchronization of chaotic delayed systems via intermittent control and its adaptive strategy*

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Abstract. In this paper the problem of synchronization for delayed chaotic systems is considered based on aperiodic intermittent control. First, delayed chaotic systems are proposed via aperiodic adaptive intermittent control. Next, to cut down the control gain, a new generalized intermittent control and its adaptive strategy is introduced. Then, by constructing a piecewise Lyapunov auxiliary function and making use of piecewise analysis technique, some effective and novel criteria are obtained to ensure the global synchronization of delayed chaotic systems by means of the designed control protocols. At the end, two examples with numerical simulations are provided to verify the effectiveness of the theoretical results proposed scheme.

Keywords: delayed chaotic systems, aperiodic intermittent control, synchronization, adaptive strategy.

1 Introduction

Chaos is an interesting nonlinear phenomenon and has been known for a rather long time in the mathematics, physics, engineering and other related fields [5]. Nevertheless, duo to its sensitive dependence on initial conditions, chaos was believed in a long time to be

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neither predictable nor controllable. However, a great many researchers from various fields have made insightful investigations on this subject, and many important and fundamental results have been reported.

Chaos synchronization is a contemporary topic in nonlinear science with its applications to diverse areas such as secure communications, biological chemical reactions and so on. In general, in order to synchronize two or more nonlinear dynamical systems, it is natural to address the proper control approaches. Up to now, many typical and useful control methods have been proposed such as adaptive control [9, 25], feedback control [6,17], pinning control [10,18], intermittent control [19,23] and impulsive control [7,13].

Recently, discontinuous control approaches such as impulsive, switched [20] and intermittent controls have attracted increasing attention since they are commonly used in transportation, manufacturing and communication engineering. For instance, in communications, intermittent control scheme is usually used as a central means of transmitting information between transmitter and receiver in order to realize synchronization. However, discontinuous control dynamical systems are governed by complex mathematical models displaying rather irregular dynamical behaviors with interesting challenges. Generally speaking, intermittent control strategy is composed of control time and the rest time in turn. And intermittent control is activated during the work time and is off during the rest time.

Compared with impulsive control, intermittent control is easier to be implemented because it may be more reasonable to realise control process in some time intervals other than some time instants in practice control application. The intermittent controller has been triumphant applied to stabilize and synchronize neural networks [26, 27], complex networks [10, 18, 19, 23], chaotic systems [9, 11] in recent decades. Nevertheless, periodic intermittent control may be inadequate in the practical application and may cause unreasonable and unnecessary results. For instance, the generation of wind power is emblematical aperiodically intermittent. Therefore, for the real applications and the theoretical analysis, it is momentous to consider the synchronization problem of chaotic systems under aperiodically intermittent control scheme.

As everyone knows, adaptive control scheme are designed under control objective because of the characteristics of considered system [1, 4, 15, 16, 22, 28]. The strong merit of adaptive control is that the control parameters can automatically adjust themselves according to some appropriate updating laws. Adaptive consensus control for a class of nonlinear multiagent time-delay systems using neural networks was investigated in [4]. In [1], adaptive synchronization of fractional-order memristor-based neural networks with time delay was studied. In [15], Li and Hu proposed Pinning adaptive and impulsive synchronization of fractional-order complex dynamical networks. Pinning synchronization of complex directed dynamical networks under decentralized adaptive strategy for aperiodically intermittent control was investigated in [28].

It is well known that due to the finite switching speeds of transmission and spreading, time delay is a very typical phenomenon in some fields and may lead to undesirable dynamic behaviors such as oscillation and instability behavior. Therefore, it is extraordinary essential to consider the influence of time delay on the dynamical behavior for systems. Recently, a great many of results for delays dynamical systems have been obtained [2, 3, 8]. However, to the best of our knowledge, the synchronization problem for delayed dynamical systems under intermittent adaptive control, until now, receives few attentions. We will devote our efforts to this problem in this letter.

Motivated by the above discussion, synchronization of nonlinear dynamical systems with time-varying delays is investigated via a novel and generalized adaptive intermittent control protocol. The main contributions in this paper can be summarized as follows. The first one is that the time-varying delays is taken into account for the chaos systems, which may be more consistent with the real world case. When dealing with delays, it is difficult to construct a piecewise Lyapunov auxiliary function and use analytical techniques. The second one is that adaptive intermittent control strategy is adopted to synchronize nonlinear dynamical systems. The controller is an economic and realistic method for network and is an extension of periodic intermittent adaptive control strategy. The third one is that by constructing a piecewise Lyapunov auxiliary function and making use of piecewise analysis technique, some effective and novel criteria are obtained to ensure the global synchronization of delayed chaotic systems by means of the designed control protocols. At the end, two numerical examples with are provided to verify the effectiveness of the theoretical results proposed scheme.

The rest of this paper is organized as follows. In Section 2, model of nonlinear dynamical systems with time-varying delays and preliminaries are given. Synchronization of the considered model under the aperiodically intermittent control and its adaptive strategy is investigated in Section 3. In Section 4 the effectiveness and feasibility of the developed methods are presented by two numerical examples. Finally, some conclusions are obtained in Section 5.

Notations. Let $\mathbb{R}^n$ be the space of n-dimensional real column vectors. $\|x\|$ denotes a vector norm defined by $\|x\| = (\sum_{i=1}^{n} x_i^2)^{1/2}$, where $x = (x_1, \dots, x_n)^{\mathrm{T}} \in \mathbb{R}^n$. $\mathbb{R}^{n\times n}$ denotes the set of all $n \times n$ real matrix. $\mathbf{I}_n$ is the identity matrix with n dimensions. For a square matrix $\mathbf{A}$, $\mathbf{A}^{\mathrm{T}}$ denotes its transpose, $\lambda_{\max}(\mathbf{A})$ and $\lambda_{\min}(\mathbf{A})$ are the maximum and minimum eigenvalues of matrix $\mathbf{A}$. For a real matrix $\mathbf{M} \in \mathbb{R}^{N\times N}$, write $\mathbf{M} > 0$ ($\mathbf{M} < 0$) if $\mathbf{M}$ is positive (negative) definite, and $\mathbf{M} \geqslant 0$ ($\mathbf{M} \leqslant 0$) if $\mathbf{M}$ is semipositive (seminegative) definite. $\mathscr{I}$ be the set of all natural numbers. Z^+ be the set of all nonnegative integers.

2 Preliminaries

In this section a class of delayed nonlinear chaotic systems is considered. The differential equations are described by

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{f}\big(\mathbf{x}(t)\big) + \mathbf{C}\mathbf{f}\big(\mathbf{x}\big(t - \tau(t)\big)\big) + \mathbf{J}, \tag{1}$$

where $\mathbf{x}(t) = (x_1(t), x_2(t), \dots, x_n(t))^{\mathrm{T}} \in \mathbb{R}^n$ denotes the state vector, $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear vector function, the time-varying delay $\tau(t)$ may be unknown but is bounded by known constant, i.e., $0 \leqslant \tau(t) \leqslant \tau$, $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathbb{R}^{n\times n}$ are three constant matrices, $\mathbf{J} = (J_1, J_2, \dots, J_n)^{\mathrm{T}}$ is a constant vector, which may be an external disturbance or the system bias.

The initial condition of system (1) is given by $\mathbf{x}(h) = \phi(h) \in C([-\tau, 0], \mathbb{R}^n)$, where $C([-\tau, 0], \mathbb{R}^n)$ represents the set of all n-dimensional continuous functions defined on the interval $[-\tau, 0]$.

In the case that system (1) reaches complete synchronization, we have response chaotic system described as follows:

$$\dot{\mathbf{y}}(t) = \mathbf{A}\mathbf{y}(t) + \mathbf{B}\mathbf{f}\big(\mathbf{y}(t)\big) + \mathbf{C}\mathbf{f}\big(\mathbf{y}\big(t - \tau(t)\big)\big) + \mathbf{J} + \mathbf{u}(t), \tag{2}$$

where $\mathbf{u}(t)$ is an appropriate control input designed in the following.

In order to obtain our main results, the following assumptions, definition and lemmas are necessary:

Assumption 1. (See [21].) For the nonlinear function $\mathbf{f}(t)$, there exist positive constant l such that for any $\mathbf{x}(t), \mathbf{y}(t) \in \mathbb{R}^n$,

$$\big\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})\big\| \leqslant l\|\mathbf{x} - \mathbf{y}\|.$$

Assumption 2. (See [21].) For the aperiodically intermittent control strategy, there exist two positive scalars $0 < \theta < \omega < +\infty$ such that, for $m \in Z^+$,

$$\inf_m (s_m - t_m) = \theta, \qquad \sup_m (t_{m+1} - t_m) = \omega,$$

where $t_m < s_m < t_{m+1} < s_{m+1}$.

Assumption 3. Time-delay function $\tau(t) : [0, +\infty) \to [0, +\infty)$ is real-valued continuous function and satisfies

$$\dot{\tau}(t) \leqslant \sigma < 1.$$

Definition 1. (See [21].) For the aperiodically intermittent control, define

$$\varPsi = \limsup_{m\to\infty} \frac{t_{m+1} - s_m}{t_{m+1} - t_m}.$$

Obviously, $0 \leqslant \varPsi < 1$ and $\varPsi \leqslant 1 - \theta/\omega$. When $\varPsi = 0$, the aperiodically intermittent control becomes the continuous control. In the following, without loss of generality, we always suppose that $0 < \varPsi < 1$.

Lemma 1. (*See* [18].) *If* $\mathbf{Y}$ *and* $\mathbf{Z}$ *are real matrices with appropriate dimensions, then there exists a positive constant* $\varsigma > 0$ *such that*

$$\mathbf{Y}^{\mathrm{T}}\mathbf{Z} + \mathbf{Z}^{\mathrm{T}}\mathbf{Y} \leqslant \varsigma\mathbf{Y}^{\mathrm{T}}\mathbf{Y} + \frac{1}{\varsigma}\mathbf{Z}^{\mathrm{T}}\mathbf{Z}.$$

Lemma 2. (*See* [18].) *Suppose that function* $Q(t)$ *is continuous and nonnegative for* $t \to [-\tau, +\infty)$ *and satisfies the following condition*:

$$\dot{Q}(t) \leqslant \begin{cases} -\gamma_1 Q(t) + \gamma_2(\sup_{t-\tau\leqslant s\leqslant t} Q(s)), & t_m \leqslant t \leqslant s_m, \\ \gamma_3 Q(t) + \gamma_4(\sup_{t-\tau\leqslant s\leqslant t} Q(s)), & s_m < t < t_{m+1}, \end{cases}$$

where γ_1, γ_2, γ_3, γ_4 are constants and $m \in Z^+$. Suppose that for the aperiodically intermittent control, there exists a constant $\Psi \in (0,1)$, where Ψ is defined in Definition 1. *If*

$$\gamma_1 > \gamma^* = \max\{\gamma_2, \gamma_4\} > 0, \qquad \rho = \gamma_1 + \gamma_3 > 0, \qquad \varpi = q - \rho\Psi > 0,$$

then

$$Q(t) \leqslant \Big(\sup_{-\tau\leqslant s\leqslant 0} Q(s)\Big)\exp\{-\varpi t\}, \quad t \geqslant 0,$$

where $q > 0$ is the unique positive solution of the equation $q - \gamma_1 + \gamma^* \exp\{q\tau\} = 0$.

3 Main results

Let $\mathbf{e}(t) = (e_1(t), e_2(t), \dots, e_n(t))^{\mathrm{T}} = \mathbf{y}(t) - \mathbf{x}(t)$ be synchronization errors. In this section, two control schemes will be designed to synchronize nonlinear system (1) to the desired state (2). The main results are stated in the following.

3.1 Intermittent control with constant control gains

To achieve the synchronization, firstly, a generalized intermittent control $\mathbf{u}(t)$ with adaptive constant control gain defined as follows:

$$\mathbf{u}(t) = \begin{cases} -d\,\mathbf{e}(t), & t_m \leqslant t \leqslant s_m, \\ 0, & s_m < t < t_{m+1}, \end{cases} \tag{3}$$

where $m \in Z^+$, $d > 0$ is a constant.

According to the control law (3), the error dynamical system is then governed as follows:

$$\dot{\mathbf{e}}(t) = \begin{cases} \mathbf{A}\mathbf{e}(t) + \mathbf{B}\big(\mathbf{f}(\mathbf{y}(t)) - \mathbf{f}(\mathbf{x}(t))\big) & \\ \quad + \mathbf{C}\big(\mathbf{f}\big(\mathbf{y}(t-\tau(t))\big) - \mathbf{f}\big(\mathbf{x}(t-\tau(t))\big)\big) - d\,\mathbf{e}(t), & t_m \leqslant t \leqslant s_m, \\ \mathbf{A}\mathbf{e}(t) + \mathbf{B}\big(\mathbf{f}(\mathbf{y}(t)) - \mathbf{f}(\mathbf{x}(t))\big) & \\ \quad + \mathbf{C}\big(\mathbf{f}\big(\mathbf{y}(t-\tau(t))\big) - \mathbf{f}\big(\mathbf{x}(t-\tau(t))\big)\big), & s_m < t < t_{m+1}. \end{cases} \tag{4}$$

It is easy to see that the synchronization of the delayed dynamical systems (1) and (2) are achieved if the zero solution of the error dynamical system (4) is stable.

Theorem 1. *Suppose that Assumptions* 1–2 *hold. Under controller* (3)*, if there exist positive constants ε_1, ε_2, a_1, a_2, l, d, q, Ψ such that*

(i) $\mathbf{A} + \mathbf{A}^{\mathrm{T}} + \varepsilon_1\mathbf{B}\mathbf{B}^{\mathrm{T}} + \varepsilon_2\mathbf{C}\mathbf{C}^{\mathrm{T}} + (l^2/\varepsilon_1 - d + a_1)\mathbf{I} < 0$,
(ii) $\mathbf{A} + \mathbf{A}^{\mathrm{T}} + \varepsilon_1\mathbf{B}\mathbf{B}^{\mathrm{T}} + \varepsilon_2\mathbf{C}\mathbf{C}^{\mathrm{T}} + (l^2/\varepsilon_1 - (a_2 - a_1))\mathbf{I} \leqslant 0$,
(iii) $l^2/\varepsilon_2 - a_1 < 0$,
(iv) $\eta = q - a_2\Psi > 0$, *where $q > 0$ is the unique positive solution of the equation $q - a_1 + (l^2/\varepsilon_2)\exp\{q\tau\} = 0$,*

then the delayed dynamical systems (1) *and* (2) *are globally asymptotically synchronized.*

Proof. Define the Lyapunov function as

$$V(t) = \frac{1}{2}\mathbf{e}^{\mathrm{T}}(t)\mathbf{e}(t).$$

Then its derivative with respect to time t along with solutions of (4) can be calculated as follows.

When $t_m \leqslant t \leqslant s_m$, $m \in Z^+$,

$$\begin{aligned}
\dot{V}(t) &= \mathbf{e}^{\mathrm{T}}(t)\big[\mathbf{A}\mathbf{e}(t) + \mathbf{B}\big(\mathbf{f}\big(\mathbf{y}(t)\big) - \mathbf{f}\big(\mathbf{x}(t)\big)\big) \\
&\quad + \mathbf{C}\big(\mathbf{f}\big(\mathbf{y}\big(t-\tau(t)\big)\big) - \mathbf{f}\big(\mathbf{x}\big(t-\tau(t)\big)\big)\big) - d\,\mathbf{e}(t)\big] \\
&\leqslant \frac{1}{2}\mathbf{e}^{\mathrm{T}}(t)\left[\mathbf{A} + \mathbf{A}^{\mathrm{T}} + \varepsilon_1\mathbf{B}\mathbf{B}^{\mathrm{T}} + \varepsilon_2\mathbf{C}\mathbf{C}^{\mathrm{T}} + \left(\frac{l^2}{\varepsilon_1} - d + a_1\right)\mathbf{I}\right]\mathbf{e}(t) \\
&\quad - \frac{a_1}{2}\mathbf{e}^{\mathrm{T}}(t)\mathbf{e}(t) + \frac{l^2}{\varepsilon_2}\Big(\sup_{t-\tau\leqslant s\leqslant t} V(s)\Big).
\end{aligned}$$

According to condition (i), it can be obtained that

$$\dot{V}(t) \leqslant -a_1 V(t) + \frac{l^2}{\varepsilon_2}\Big(\sup_{t-\tau\leqslant s\leqslant t} V(s)\Big).$$

Similarly, for $s_m < t < t_{m+1}$, using condition (ii), it can be derived that

$$\begin{aligned}
\dot{V}(t) &\leqslant \frac{1}{2}\mathbf{e}^{\mathrm{T}}(t)\left[\mathbf{A} + \mathbf{A}^{\mathrm{T}} + \varepsilon_1\mathbf{B}\mathbf{B}^{\mathrm{T}} + \varepsilon_2\mathbf{C}\mathbf{C}^{\mathrm{T}} + \frac{l^2}{\varepsilon_1}\mathbf{I}\right]\mathbf{e}(t) \\
&\quad + \frac{l^2}{2\varepsilon_2}\mathbf{e}^{\mathrm{T}}(t-\tau(t))\mathbf{e}\big(t-\tau(t)\big) \\
&\leqslant (a_2 - a_1)V(t) + \frac{l^2}{\varepsilon_2}\Big(\sup_{t-\tau\leqslant s\leqslant t} V(s)\Big).
\end{aligned}$$

Hence, it can be gotten that

$$\dot{V}(t) \leqslant \begin{cases} -a_1 V(t) + \frac{l^2}{\varepsilon_2}(\sup_{t-\tau\leqslant s\leqslant t} V(s)), & t_m \leqslant t \leqslant s_m, \\ (a_2 - a_1)V(t) + \frac{l^2}{\varepsilon_2}(\sup_{t-\tau\leqslant s\leqslant t} V(s)), & s_m < t < t_{m+1}. \end{cases}$$

By Lemma 2 and conditions (iii), (iv) it can be obtained that

$$V(t) \leqslant \Big(\sup_{t-\tau\leqslant s\leqslant t} V(s)\Big)\exp\{-\eta t\}.$$

Therefore, the asymptotical synchronization of the controlled system (2) is realized, and the proof of Theorem 1 is completed. □

3.2 Intermittent control with adaptive control gains

For the controlled system (2), the feedback control gains may be chosen according to Theorem 1 to derived synchronization. However, feedback gains may be given much larger than those needed in practice. A better way is to use adaptive approach to tune the feedback gains. In the section an adaptive method is introduced to determine the intermittent control gains, and the global synchronization will be investigated based on the aperiodically intermittent adaptive control gains. For the sake of simplicity, the control input $\mathbf{u}(t) = (u_1(t), u_2(t), \dots, u_n(t))^{\mathrm{T}}$ in the controlled system (2) is defined as follows:

$$u_i(t) = \begin{cases} -d_i(t)e_i(t) - \frac{\mu l^2}{2\varepsilon_2(1-\sigma)} \int_{t-\tau(t)}^{t} \mathbf{e}^{\mathrm{T}}(s)\mathbf{e}(s)\,\mathrm{d}s \frac{e_i(t)}{\|\mathbf{e}(t)\|^2}, \\ \qquad \|\mathbf{e}(t)\| \neq 0 \text{ and } t_m \leqslant t \leqslant s_m, \\ 0, \quad \|\mathbf{e}(t)\| = 0 \text{ or } s_m < t < t_{m+1}, \end{cases} \tag{5}$$

where

$$d_i(t) = \begin{cases} d_i(t_0), & t = t_0, \\ d_i(s_m), & t = t_{m+1}, \\ 0, & s_m < t < t_{m+1}, \end{cases} \tag{6}$$

and

$$\dot{d}_i(t) = \zeta_i e_i^2(t), \quad t_m \leqslant t \leqslant s_m, \tag{7}$$

here $m \in Z^+$, l, μ, ε_2, σ are the positive constant control strengths.

According to the control law (5)–(7), the error dynamical system is then governed as follows:

$$\dot{\mathbf{e}}(t) = \begin{cases} \mathbf{A}\mathbf{e}(t) + \mathbf{B}(\mathbf{f}(\mathbf{y}(t)) - \mathbf{f}(\mathbf{x}(t))) \\ \quad + \mathbf{C}(\mathbf{f}(\mathbf{y}(t-\tau(t))) - \mathbf{f}(\mathbf{x}(t-\tau(t)))) \\ \quad - \mathbf{d}(t)\mathbf{e}(t) - \frac{\mu l^2}{2\varepsilon_2(1-\sigma)} \int_{t-\tau(t)}^{t} \mathbf{e}^{\mathrm{T}}(s)\mathbf{e}(s)\,\mathrm{d}s \frac{\mathbf{e}(t)}{\|\mathbf{e}(t)\|^2}, & t_m \leqslant t \leqslant s_m, \\ \mathbf{A}\mathbf{e}(t) + \mathbf{B}(\mathbf{f}(\mathbf{y}(t)) - \mathbf{f}(\mathbf{x}(t))) \\ \quad + \mathbf{C}(\mathbf{f}(\mathbf{y}(t-\tau(t))) - \mathbf{f}(\mathbf{x}(t-\tau(t)))), & s_m < t < t_{m+1}, \end{cases} \tag{8}$$

where $\mathbf{d}(t) = \operatorname{diag}\{d_1(t), d_2(t), \dots, d_n(t)\}$ is a diagonal matrix.

It is easy to see that the synchronization of the delayed dynamical systems (1) and (2) is achieved if the zero solution of the error dynamical system (8) is stable.

Theorem 2. *Suppose that Assumptions* 1–3 *hold. Under controller* (5) *satisfying* (6) *and* (7)*, systems* (1) *and* (2) *are globally asymptotically synchronized.*

Proof. Let λ is the largest eigenvalue of the matrix

$$\mathbf{A} + \mathbf{A}^{\mathrm{T}} + \varepsilon_1 \mathbf{B}\mathbf{B}^{\mathrm{T}} + \varepsilon_2 \mathbf{C}\mathbf{C}^{\mathrm{T}} + \left(\frac{l^2}{\varepsilon_1} + \frac{l^2}{\varepsilon_2(1-\sigma)}\right)\mathbf{I}.$$

Choose a positive number μ such that $\alpha = \mu\theta - \lambda(\omega - \theta) > 0$. Construct a piecewise function described by

$$W(t) = \begin{cases} \frac{1}{2}\exp\{-\mu(t - t_m)\}\sum_{i=1}^{n}\frac{1}{\zeta_i}(d_i^* - d_i(t))^2, \\ \quad t_m \leqslant t \leqslant s_m, \\ \frac{1}{2}\exp\{\lambda(t - s_m) - \mu(s_m - t_m)\}\sum_{i=1}^{n}\frac{1}{\zeta_i}(d_i^* - d_i(s_m))^2, \\ \quad s_m < t < t_{m+1} \end{cases} \tag{9}$$

for $m \in Z^+$ in which d_i^* is a positive constant to be determined later. It follows from (9) that $W(t)$ is continuous except for $t = t_{m+1}$ with $m \in Z^+$ and

$$W_+(t_{m+1}) = \exp\bigl\{\mu(s_m - t_m) - \lambda(t_{m+1} - s_m)\bigr\}W_-(t_{m+1}), \tag{10}$$

where $W_+(t_{m+1})$ and $W_-(t_{m+1})$ denote the right limit and the left limit of $W(t)$ at time t_{m+1}, respectively.

Define the Lyapunov function as

$$V(t) = U(t) + W(t),$$

where

$$U(t) = \frac{1}{2}\mathbf{e}^{\mathrm{T}}(t)\mathbf{e}(t) + \frac{l^2}{2\varepsilon_2(1-\sigma)}\int\limits_{t-\tau(t)}^{t}\mathbf{e}^{\mathrm{T}}(s)\mathbf{e}(s)\,\mathrm{d}s.$$

Evidently, $U(t)$ is continuous for all $t \geqslant t_0$, and $V(t)$ is continuous except for $t = t_{m+1}$ with $m \in Z^+$ and it is right continuous at $t = t_{m+1}$.

Then its derivative with respect to time t along with solutions of (8) can be calculated as follows.

When $t_m \leqslant t \leqslant s_m$, $m \in Z^+$,

$$\begin{aligned}
\dot{V}(t) \leqslant{}& \mathbf{e}^{\mathrm{T}}(t)\Biggl[\mathbf{A}\mathbf{e}(t) + \mathbf{B}\bigl(\mathbf{f}\bigl(\mathbf{y}(t)\bigr) - \mathbf{f}\bigl(\mathbf{x}(t)\bigr)\bigr) \\
&+ \mathbf{C}\bigl(\mathbf{f}\bigl(\mathbf{y}\bigl(t-\tau(t)\bigr)\bigr) - \mathbf{f}\bigl(\mathbf{x}\bigl(t-\tau(t)\bigr)\bigr)\bigr) \\
&- \mathbf{d}(t)\mathbf{e}(t) - \frac{\mu l^2}{2\varepsilon_2(1-\sigma)}\int\limits_{t-\tau(t)}^{t}\mathbf{e}^{\mathrm{T}}(s)\mathbf{e}(s)\,\mathrm{d}s\frac{\mathbf{e}(t)}{\|\mathbf{e}(t)\|^2}\Biggr] \\
&- \frac{l^2}{2\varepsilon_2}\mathbf{e}^{\mathrm{T}}\bigl(t-\tau(t)\bigr)\mathbf{e}\bigl(t-\tau(t)\bigr) + \frac{l^2}{2\varepsilon_2(1-\sigma)}\mathbf{e}^{\mathrm{T}}(t)\mathbf{e}(t) \\
&- \frac{\mu}{2}\exp\bigl\{-\mu(t-t_m)\bigr\}\sum_{i=1}^{n}\frac{1}{\zeta_i}\bigl(d_i^* - d_i(t)\bigr)^2 \\
&- \exp\bigl\{-\mu(t-t_m)\bigr\}\sum_{i=1}^{n}\bigl(d_i^* - d_i(t)\bigr)e_i^2(t)
\end{aligned}$$

$$\begin{aligned}
&\leqslant \frac{1}{2}\mathbf{e}^{\mathrm{T}}(t)\biggl[\mathbf{A}+\mathbf{A}^{\mathrm{T}}+\varepsilon_1\mathbf{B}\mathbf{B}^{\mathrm{T}}+\varepsilon_2\mathbf{C}\mathbf{C}^{\mathrm{T}}\\
&\quad+\biggl(\frac{l^2}{\varepsilon_1}+\frac{l^2}{\varepsilon_2(1-\sigma)}+\mu-2\hat{d}\exp\{-\mu\omega\}\biggr)\mathbf{I}\biggr]\mathbf{e}^{\mathrm{T}}(t)\\
&\quad-\frac{\mu l^2}{2\varepsilon_2(1-\sigma)}\int\limits_{t-\tau(t)}^{t}\mathbf{e}^{\mathrm{T}}(s)\mathbf{e}(s)\,\mathrm{d}s-\frac{1}{2}\mu\mathbf{e}^{\mathrm{T}}(t)\mathbf{e}(t)-\mu W(t),
\end{aligned}$$

where $\hat{d}=\min\{d_i^*, i\in\mathscr{I}\}$. It is easy to see that some suitable $\hat{d}$ can be chosen such that

$$\mathbf{A}+\mathbf{A}^{\mathrm{T}}+\varepsilon_1\mathbf{B}\mathbf{B}^{\mathrm{T}}+\varepsilon_2\mathbf{C}\mathbf{C}^{\mathrm{T}}+\biggl(\frac{l^2}{\varepsilon_1}+\frac{l^2}{\varepsilon_2(1-\sigma)}+\mu-2\hat{d}\exp\{-\mu\omega\}\biggr)\mathbf{I}\leqslant 0,$$

which shows that

$$\dot{V}(t)\leqslant -\mu V(t),\quad t_m\leqslant t\leqslant s_m.$$

So,

$$V(t)\leqslant \exp\bigl\{-\mu(t-t_m)\bigr\}V_+(t_m),\quad t_m\leqslant t\leqslant s_m. \tag{11}$$

Similarly, for $s_m<t<t_{m+1}$,

$$\begin{aligned}
\dot{V}(t)&\leqslant \mathbf{e}^{\mathrm{T}}(t)\bigl[\mathbf{A}\mathbf{e}(t)+\mathbf{B}\bigl(\mathbf{f}\bigl(\mathbf{y}(t)\bigr)-\mathbf{f}\bigl(\mathbf{x}(t)\bigr)\bigr)\\
&\quad+\mathbf{C}\bigl(\mathbf{f}\bigl(\mathbf{y}\bigl(t-\tau(t)\bigr)\bigr)-\mathbf{f}\bigl(\mathbf{x}\bigl(t-\tau(t)\bigr)\bigr)\bigr)\bigr]\\
&\quad-\frac{l^2}{2\varepsilon_2}\mathbf{e}^{\mathrm{T}}\bigl(t-\tau(t)\bigr)\mathbf{e}\bigl(t-\tau(t)\bigr)+\frac{l^2}{2\varepsilon_2(1-\sigma)}\mathbf{e}^{\mathrm{T}}(t)\mathbf{e}(t)+\lambda W(t)\\
&\leqslant \frac{1}{2}\mathbf{e}^{\mathrm{T}}(t)\biggl[\mathbf{A}+\mathbf{A}^{\mathrm{T}}+\varepsilon_1\mathbf{B}\mathbf{B}^{\mathrm{T}}+\varepsilon_2\mathbf{C}\mathbf{C}^{\mathrm{T}}+\frac{l^2}{\varepsilon_1}\mathbf{I}+\frac{l^2}{\varepsilon_2(1-\sigma)}\mathbf{I}\biggr]\mathbf{e}(t)+\lambda W(t)\\
&\leqslant \lambda V(t).
\end{aligned}$$

Hence, for $s_m<t<t_{m+1}$,

$$\dot{V}(t)\leqslant \lambda V(t),$$

which leads to

$$V(t)\leqslant \exp\bigl\{\lambda(t-s_m)\bigr\}V(s_m). \tag{12}$$

In the following, it can be proved that

$$\lim_{t\to+\infty}U(t)=0.$$

By virtue of (10), (11) and (12) it can be derived that

$$\begin{aligned}
V_-(t_{m+1})&\leqslant \exp\bigl\{\lambda(t_{m+1}-s_m)\bigr\}V(s_m)\\
&\leqslant \exp\bigl\{\lambda(t_{m+1}-s_m)\bigr\}\exp\bigl\{-\mu(s_m-t_m)\bigr\}V_+(t_m)\\
&= \exp\bigl\{\lambda(t_{m+1}-s_m)\bigr\}\exp\bigl\{-\mu(s_m-t_m)\bigr\}U(t_m)+W_-(t_m)\\
&\leqslant \exp\{-\alpha\}U(t_m)+W_-(t_m)\\
&= \exp\{-\alpha\}V_-(t_m)+\bigl(1-\exp\{-\alpha\}\bigr)W_-(t_m),
\end{aligned}$$

which implies that

$$\begin{aligned} V_-(t_{m+1}) - V_-(t_m) &\leqslant \big(\exp\{-\alpha\} - 1\big)V_-(t_m) + \big(1 - \exp\{-\alpha\}\big)W_-(t_m) \\ &= \big(\exp\{-\alpha\} - 1\big)U(t_m), \end{aligned}$$

and then

$$V_-(t_{m+1}) - V(t_0) \leqslant \big(\exp\{-\alpha\} - 1\big)\sum_{i=0}^{m} U(t_i).$$

It shows that

$$\sum_{i=0}^{\infty} U(t_i) \leqslant \frac{V(t_0)}{1 - \exp\{-\alpha\}},$$

therefore, by the theory of series,

$$\lim_{i\to+\infty} U(t_i) = 0.$$

In addition, for $t_m < t < t_{m+1}$, in view of the nonnegativity of $d_i(t)$ and $\mu l_2/(1-\sigma)$, it is easy to estimate that

$$\begin{aligned} \dot{U}(t) &\leqslant \mathbf{e}^{\mathrm{T}}(t)\big[\mathbf{A}\mathbf{e}(t) + \mathbf{B}\big(\mathbf{f}\big(\mathbf{y}(t)\big) - \mathbf{f}\big(\mathbf{x}(t)\big)\big) \\ &\quad + \mathbf{C}\big(\mathbf{f}\big(\mathbf{y}(t-\tau(t))\big) - \mathbf{f}\big(\mathbf{x}\big(t-\tau(t)\big)\big)\big)\big] \\ &\quad - \frac{l^2}{2\varepsilon_2}\mathbf{e}^{\mathrm{T}}\big(t-\tau(t)\big)\mathbf{e}\big(t-\tau(t)\big) + \frac{l^2}{2\varepsilon_2(1-\sigma)}\mathbf{e}^{\mathrm{T}}(t)\mathbf{e}(t) \\ &\leqslant \frac{1}{2}\mathbf{e}^{\mathrm{T}}(t)\left[\mathbf{A}+\mathbf{A}^{\mathrm{T}}+\varepsilon_1\mathbf{B}\mathbf{B}^{\mathrm{T}}+\varepsilon_2\mathbf{C}\mathbf{C}^{\mathrm{T}} + \left(\frac{l^2}{\varepsilon_1}+\frac{l^2}{\varepsilon_2(1-\sigma)}\right)\mathbf{I}\right]\mathbf{e}(t) \\ &\leqslant \lambda U(t). \end{aligned}$$

In view of this, it can be gotten that

$$U(t) \leqslant \exp\big\{\lambda(t-t_m)\big\}U(t_m) \leqslant \exp\{\lambda\omega\}U(t_m), \quad t_m \leqslant t \leqslant t_{m+1}.$$

Evidently, $m \to \infty$ when $t \to \infty$, by the above results we obtain

$$\lim_{t\to+\infty} U(t) = 0.$$

Therefore, the asymptotical synchronization of the controlled system (2) is realized, and the proof of Theorem 2 is completed. □

If for all $m \in Z^+$, $t_{m+1} - t_m = T$ and $s_m - t_m = \delta T$, where $T > 0$ and $0 < \delta < 1$, the aperiodically intermittent adaptive control (5) satisfying (6) and (7) becomes the following periodically intermittent adaptive control:

$$u_i(t) = \begin{cases} -d_i(t)e_i(t) - \frac{\mu l^2}{2\varepsilon_2(1-\sigma)}\int_{t-\tau(t)}^{t}\mathbf{e}^{\mathrm{T}}(s)\mathbf{e}(s)\,\mathrm{d}s\,\frac{e_i(t)}{\|\mathbf{e}(t)\|^2}, \\ \qquad \|\mathbf{e}(t)\| \neq 0 \text{ and } t_0 + mT \leqslant t \leqslant t_0 + (m+\delta)T, \\ 0, \quad \|\mathbf{e}(t)\| = 0 \text{ or } t_0 + (m+\delta)T < t < t_0 + (m+1)T, \end{cases} \tag{13}$$

where

$$d_i(t) = \begin{cases} d_i(t_0), & t = t_0, \\ d_i(t_0 + (m+\delta)T), & t = t_0 + (m+1)T, \\ 0, & t_0 + (m+\delta)T < t < t_0 + (m+1)T, \end{cases} \tag{14}$$

and

$$\dot{d}_i(t) = \zeta_i e_i^2(t), \quad t_0 + mT \leqslant t \leqslant t_0 + (m+\delta)T, \tag{15}$$

here $m \in Z^+$, l, μ, ε_2, σ are the positive constant control strengths.

Then based on Theorem 2, the following Corollary 1 is immediately obtained.

Corollary 1. *Suppose that Assumptions* 1–3 *hold. Under controller* (13) *satisfying* (14) *and* (15)*, system* (1) *and the controlled delayed system* (2) *are synchronized.*

If for all $m \in Z^+$, $s_m = t_{m+1}$, the aperiodically intermittent adaptive control becomes the following general continuous control:

$$u_i(t) = \begin{cases} -d_i(t)e_i(t) - \frac{\mu l^2}{2\varepsilon_2(1-\sigma)} \int_{t-\tau(t)}^{t} \mathbf{e}^{\mathrm{T}}(s)\mathbf{e}(s)\,\mathrm{d}s \frac{e_i(t)}{\|\mathbf{e}(t)\|^2}, \\ \dot{d}_i(t) = \zeta_i e_i^2(t). \end{cases} \tag{16}$$

Then, based on Theorem 2, the following Corollary 2 is immediately gotten.

Corollary 2. *Suppose that Assumptions* 1–3 *hold. Under controller* (16)*, system* (1) *and the controlled delayed dynamical system* (2) *are synchronized.*

Suppose $\tau(t) = 0$, we can rewrite nonlinear system model (1) as follows:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{f}\big(\mathbf{x}(t)\big) + \mathbf{J}, \quad i \in \mathscr{I}. \tag{17}$$

Correspondingly, the response chaotic system $\mathbf{y}(t)$ is represented by

$$\dot{\mathbf{y}}(t) = \mathbf{A}\mathbf{y}(t) + \mathbf{B}\mathbf{f}\big(\mathbf{y}(t)\big) + \mathbf{U}(t) + \mathbf{J}, \quad i \in \mathscr{I}, \tag{18}$$

where $\mathbf{u}(t) = (u_1(t), \dots, u_n(t))^{\mathrm{T}}$ is the aperiodically intermittent adaptive control gain defined as follows:

$$u_i(t) = \begin{cases} -d_i(t)e_i(t), & t_m \leqslant t \leqslant s_m, \\ 0, & s_m < t < t_{m+1}, \end{cases} \tag{19}$$

where

$$d_i(t) = \begin{cases} d_i(t_0), & t = t_0, \\ d_i(s_m), & t = t_{m+1}, \\ 0, & s_m < t < t_{m+1}, \end{cases} \tag{20}$$

and

$$\dot{d}_i(t) = \zeta_i e_i^2(t), \quad t_m \leqslant t \leqslant s_m. \tag{21}$$

Based on Theorem 2, we can get the following Corollary 3.

Corollary 3. *Suppose that Assumptions* 1 *and* 2 *hold. Under controller* (19) *satisfying* (20) *and* (21)*, system* (17) *and the controlled delayed system* (18) *are synchronized.*

Remark 1. Due to the finite information transmission and processing speeds among the units, time delays are usually encountered in dynamical networks and may result in undesirable dynamic behaviors such as oscillation behavior and network instability. Hence, time delays should be taken into account in realistic modeling of dynamical networks. Otherwise, because of the technical reasons, time delays is not considered in [9, 10, 12, 14, 28]. In this paper, by establishing piecewise auxiliary function, piecewise analysis technique and constructing novel generalized adaptive intermittent control, the synchronization of delayed dynamical systems has been realized. When $\tau(t) = 0$, Corollary 3 in the paper is equivalent to Theorem 4 in [9]. This is to say, results in [9] are the special case of our results. So, the model we choose in the paper is more close to the reality.

Remark 2. In this paper a generalized adaptive intermittent control strategy, which contains the traditional periodically intermittent control and the aperiodic case, is introduced. Especially, when $t_{m+1} - t_m \equiv T$, $s_{m+} - t_m \equiv \delta T$, where T, δ are positive constants, $0 < \delta < 1$, the generalized adaptive intermittent control becomes the adaptive periodic one which have been studied in [10, 23]. When $s_m = t_{m+1}$, the adaptive intermittent control is reduced to the continuous-time adaptive control, which have been studied in [1, 4, 15, 25]. This is to say, our results in the paper are less conservative and more practically applicable.

Remark 3. Evidently, it follows from adaptive intermittent strategy (5)–(7) that the adaptive gains $d_i(t)$ for $i = 1, 2, 3$ are increasing in each work time according to the update law (7) and identically equal to zero in the rest time. When the synchronization is realized, the values of $d_i(t)$ are converge to some positive constants in each work time. The adaptive gains will be shown in the numerical examples in the four section.

Remark 4. In [24] the authors utilize the method of adaptive intermittent control and the theory of Lyapunov stability to realize the synchronization of chaotic systems with time-varying delay. The synchronization of chaotic system is obtained by constructing a conventional Lyapunov function in [24]. In the paper, by using a piecevise function described by

$$W(t) = \begin{cases} \frac{1}{2}\exp\{-\mu(t - t_m)\}\sum_{i=1}^{n}\frac{1}{\zeta_i}(d_i^* - d_i(t))^2, & t_m \leqslant t \leqslant s_m, \\ \frac{1}{2}\exp\{\lambda(t - s_m) - \mu(s_m - t_m)\}\sum_{i=1}^{n}\frac{1}{\zeta_i}(d_i^* - d_i(s_m))^2, & s_m < t < t_{m+1}, \end{cases}$$

the synchronization of delayed chaotic system is gotten.

4 Numerical simulations

In this section, two numerical examples are given to present the effectiveness of our results achieved in this paper.

Example 1. Consider the following 3-D oscillator model with variable delay as follows:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{f}\big(\mathbf{x}(t)\big) + \mathbf{C}\mathbf{f}\big(\mathbf{x}\big(t-\tau(t)\big)\big) + \mathbf{J}, \tag{22}$$

where $\mathbf{x}(t) = (x_1(t), x_2(t), x_3(t))^{\mathrm{T}} \in \mathbb{R}^3$, $\mathbf{f}(\mathbf{x}) = (\tanh(x_1), \tanh(x_2), \tanh(x_3))$, $\tau(t) = 0.1\mathrm{e}^t/(1+e^t)$, $\mathbf{J} = 0$ and

$$\mathbf{A} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 1.25 & -0.32 & -0.32 \\ -0.32 & 1.1 & -0.44 \\ -0.32 & 0.44 & 1 \end{pmatrix},$$
$$\mathbf{C} = \begin{pmatrix} -1.5 & -0.1 & -0.1 \\ -0.1 & -1.5 & -0.1 \\ -0.1 & -1.5 & -1.1 \end{pmatrix}.$$

In the following, we consider the synchronization between (22) and the following response system:

$$\dot{\mathbf{y}}(t) = \mathbf{A}\mathbf{y}(t) + \mathbf{B}\mathbf{f}\big(\mathbf{y}(t)\big) + \mathbf{C}\mathbf{f}\big(\mathbf{y}\big(t-\tau(t)\big)\big) + \mathbf{J} + \mathbf{U}(t), \tag{23}$$

where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{f}$, $\mathbf{J}$ and $\tau(t)$ are defined in system (22), the controller $\mathbf{U}(t)$ is an intermittent protocol defined in (3).

The dynamic property of (22) with the initial values $(x_1(h), x_2(h), x_3(h))^{\mathrm{T}} = (0.2, 0.6, -0.2)^{\mathrm{T}}$ with $h \in [-0.1, 0]$ can be emerged, which is revealed in Fig. 1, and the state is chaotic attractor in this case. Furthermore, we choose $\varepsilon_1 = \varepsilon_2 = 1$. Hereinafter, we will choose suitable control parameters such that (22) and (23) achieves the global synchronization.

By computation it can be obtained that $\tau = 0.1$, $l = 1$. Moreover, the different dynamic properties of (23) with the initial values $(y_1(h), y_2(h), y_3(h))^{\mathrm{T}} = (-0.2, -0.6, 0.2)^{\mathrm{T}}$ with $h \in [-1, 0]$ are presented in Figs. 2–7. The intermittent control exists on time span

$$[0,3] \cup [3.2, 6.4] \cup [6.5, 9.5] \cup [9.8, 12.8] \cup [13, 16] \cup [16.2, 19.2] \\ \cup [19.5, 22.6] \cup [22.8, 25.8] \cup \cdots.$$

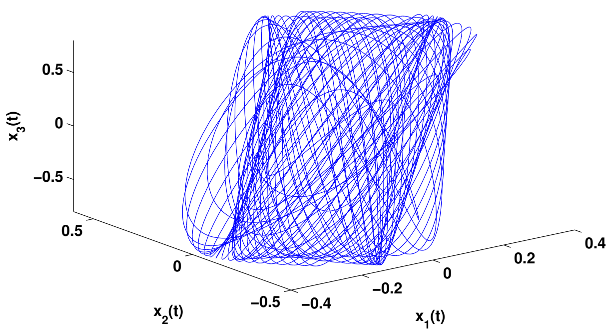

Figure 1. Phase trajectory of chaotic system (22).

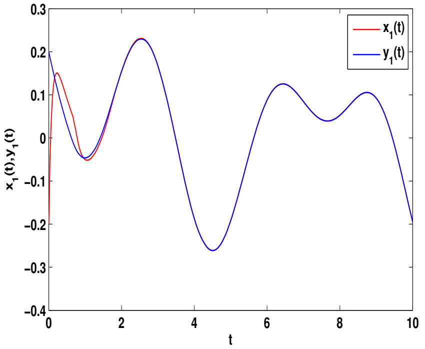

Figure 2. Synchronization evaluation between $x_1(t)$ and $y_1(t)$ in systems (22) and (23) under the intermittent control (3).

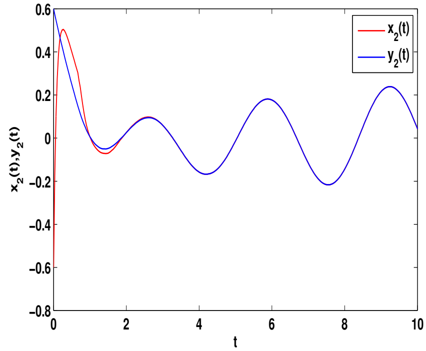

Figure 3. Synchronization evaluation between $x_2(t)$ and $y_2(t)$ in systems (22) and (23) under the intermittent control (3).

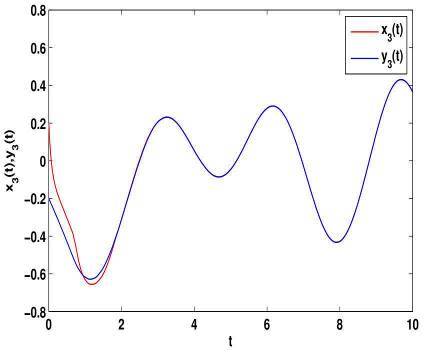

Figure 4. Synchronization evaluation between $x_3(t)$ and $y_3(t)$ in systems (22) and (23) under the intermittent control (3).

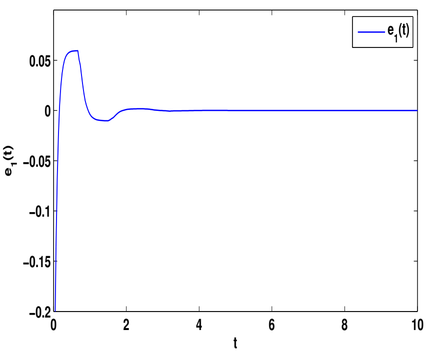

Figure 5. Synchronization error $e_1(t)$ between systems (22) and (23) under the intermittent control (3).

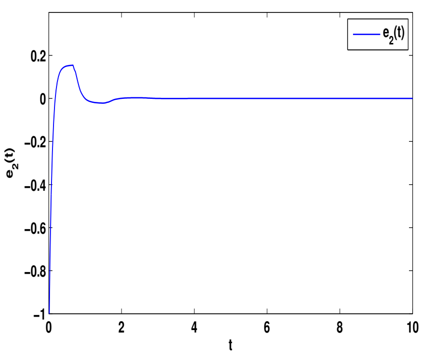

Figure 6. Synchronization error $e_2(t)$ between systems (22) and (23) under the intermittent control (3).

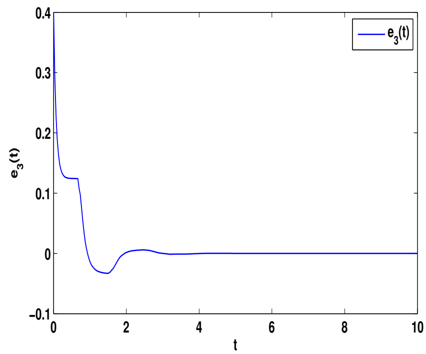

Figure 7. Synchronization error $e_3(t)$ between systems (22) and (23) under the intermittent control (3).

So, $\theta = 3$, $\omega = 3.3$. By simple computing it is easy to verify that $\Psi = 1/11$. Then let $a_1 = 4$, and it can be obtained that $q = 3.3042$ is the unique positive solution of the equation $q - a_1 + (l^2/\varepsilon_2)\exp\{q\tau\} = 0$. Let $a_2 = 11$, $d = 12$, then we can derive that conditions (i)–(iv) of Theorem 1 are satisfied. From Theorem 1 systems (22) and (23) can be global asymptotical synchronized. Figures 2–4 show the synchronization of dynamics, and Figs. 5–7 show the errors of dynamics.

Example 2. Consider the 3-D neural networks model with variable delay as follows:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{f}\big(\mathbf{x}(t)\big) + \mathbf{C}\mathbf{f}\big(\mathbf{x}\big(t - \tau(t)\big)\big) + \mathbf{J}, \tag{24}$$

where $\mathbf{x}(t) = (x_1(t), x_2(t), x_3(t))^{\mathrm{T}} \in \mathbb{R}^3$, $\mathbf{f}(\mathbf{x}) = (\tanh(x_1), \tanh(x_2), \tanh(x_3))$, $\tau(t) = e^t/(1 + e^t)$, $\mathbf{J} = 0$ and

$$\mathbf{A} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 1.25 & -0.32 & -0.32 \\ -0.32 & 1.1 & -0.44 \\ -0.32 & 0.44 & 1 \end{pmatrix},$$

$$\mathbf{C} = \begin{pmatrix} -1.5 & -0.1 & -0.1 \\ -0.1 & -1.5 & -0.1 \\ -0.1 & -1.5 & -1.1 \end{pmatrix}.$$

In the following, we investigate the synchronization of system (24) and the response system (25) described by

$$\dot{\mathbf{y}}(t) = \mathbf{A}\mathbf{y}(t) + \mathbf{B}\mathbf{f}\big(\mathbf{y}(t)\big) + \mathbf{C}\mathbf{f}\big(\mathbf{y}\big(t - \tau(t)\big)\big) + \mathbf{J} + \mathbf{U}(t), \tag{25}$$

where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{f}$, $\mathbf{J}$ and $\tau(t)$ are defined in system (24), the controller $\mathbf{U}(t)$ is an intermittent adaptive protocol defined in (5)–(7).

The dynamic property of (24) with the initial values $(x_1(h), x_2(h), x_3(h))^{\mathrm{T}} = (0.8, -0.6, 0.2)^{\mathrm{T}}$ with $h \in [-1, 0]$ can be emerged, which is revealed in Fig. 8, and the state is chaotic attractor in this case. Moreover, the different dynamic properties of (25) with the initial values $(y_1(h), y_2(h), y_3(h))^{\mathrm{T}} = (0.1, -0.45, 0.45)^{\mathrm{T}}$ with $h \in [-1, 0]$ are presented in Figs. 9–12. The intermittent control exists on time span

$$\begin{aligned} &[0,3] \cup [4,7] \cup [9,13] \cup [14,17] \cup [19,22] \cup [23,27] \\ &\quad \cup [28,32] \cup [33,36] \cup [37,40] \cup [42,46] \cup [47,50] \\ &\quad \cup [52,56] \cup [19,22] \cup [57,60] \cup [61,64] \cup [66,70] \cup \cdots. \end{aligned}$$

So, $\theta = 3$, $\omega = 5$. From Theorem 2 the networks (24) and (25) can be global asymptotical synchronized under the adaptive intermittent control rules (5)–(7). Figure 9 shows the error of dynamics, and Figs. 10–12 show the synchronization of dynamics. Time evolution of adaptive aperiodic intermittent control gain $d_i(t)$ with $d_i(0) = 1.1$ and $\zeta_i = 0$ for $i = 1, 2, 3$ are given in Figs. 13–15.

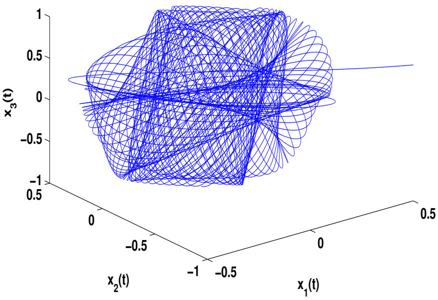

Figure 8. Phase trajectory of chaotic system (24).

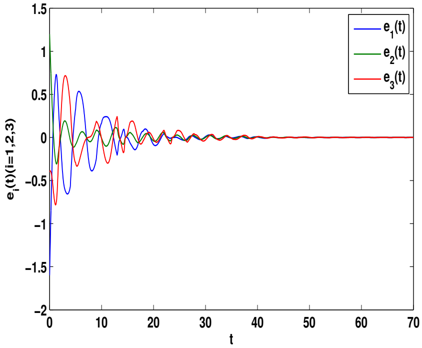

Figure 9. Synchronization error between systems (24) and (25) under the adaptive intermittent control (5)–(7).

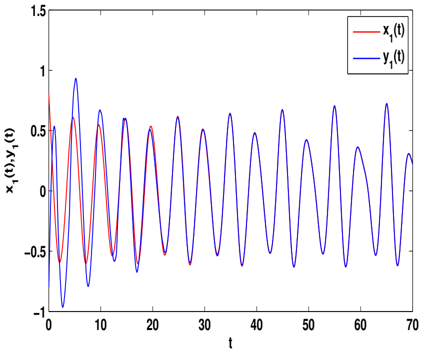

Figure 10. Synchronization evaluation between $x_1(t)$ and $y_1(t)$ in systems (24) and (25) under the control (5)–(7).

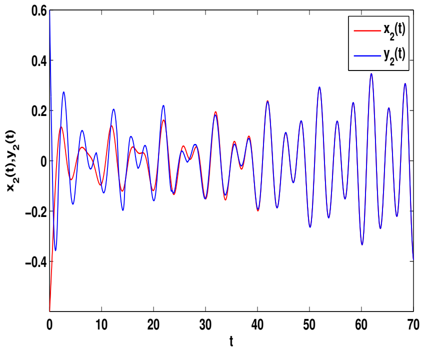

Figure 11. Synchronization evaluation between $x_2(t)$ and $y_2(t)$ in systems (23) and (24) under the control (5)–(7).

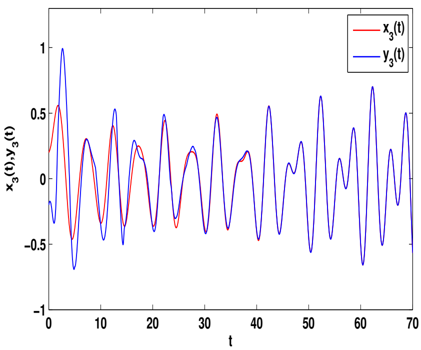

Figure 12. Synchronization evaluation between $x_3(t)$ and $y_3(t)$ in systems (24) and (25) under the control (5)–(7).

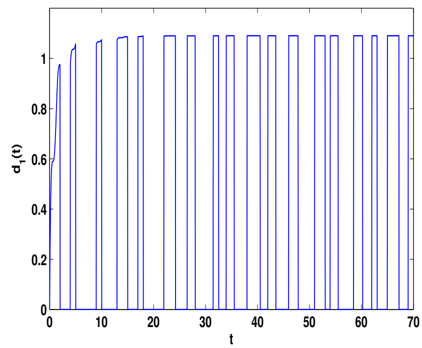

Figure 13. Synchronization control gain $d_1(t)$ of the adaptive intermittent control (5)–(7).

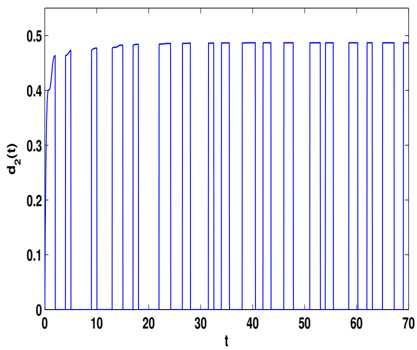

Figure 14. Synchronization control gain $d_2(t)$ of the adaptive intermittent control (5)–(7).

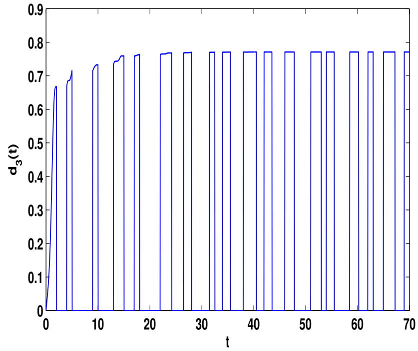

Figure 15. Synchronization control gain $d_3(t)$ of the adaptive intermittent control (5)–(7).

5 Conclusion

The paper deal with synchronization of delayed chaotic systems by means of a novel generalized intermittent and its adaptive strategy. This is to say, this article solves the open question mentioned in [9, 10, 28]. And it is noted that the control protocols in the paper are more general and practical than the traditional periodic intermittent control. Some novel global synchronization criteria have been derived based on the method of piecewise auxiliary function and piecewise analysis technique via the designed control protocols designed in the paper. Finally, two numerical examples are provided to demonstrate the feasibility of the proposed theoretical results.

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