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Exponentials of general multivector in 3D Clifford algebras

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Abstract. Closed form expressions to calculate the exponential of a general multivector (MV) in Clifford geometric algebras (GAs) $Cl_{p,q}$ are presented for $n = p+q = 3$. The obtained exponential formulas were applied to find exact GA trigonometric and hyperbolic functions of MV argument. We have verified that the presented exact formulas are in accord with series expansion of MV hyperbolic and trigonometric functions. The exponentials may be applied to solve GA differential equations, in signal and image processing, automatic control and robotics.

Keywords: Clifford (geometric) algebra, exponentials of Clifford numbers, computer-aided theory.

1 Introduction

In Clifford geometric algebra (GA) the exponential functions with the exponent represented by a simple blade are well known and used widely. In case of complex algebra (the complex number algebra is isomorphic to $Cl_{0,1}$ GA) the exponential can be expanded into a trigonometric function sum by de Moivre's theorem. In 2D vector space including Hamilton quaternions, the exponential is similar to de Moivre's formula multiplied by exponential of the scalar part [6, 13, 19, 21]. In 3D vector spaces, only special cases are known. Particularly, when the square of the blade is equal to ± 1, the exponential can be expanded in de Moivre-type sum of trigonometric or hyperbolic functions, respectively. However, general expansion in a symbolic form in case of 3D algebras $Cl_{3,0}$, $Cl_{1,2}$, $Cl_{2,1}$ and $Cl_{0,3}$, when the exponent is a general multivector (MV), is more difficult. The paper [6] considers general properties of functions of MV variable for Clifford algebras

$n = p + q \leqslant 3$, including the exponential function, for this purpose using the unique properties of a pseudoscalar I in $Cl_{3,0}$ and $Cl_{1,2}$ algebras. Namely, the pseudoscalar in these algebras commutes with all MV elements and $I^2 = -1$. This allows to introduce more general functions, in particular, the polar decomposition of all multivectors. A different approach to resolve the problem is to factor, if possible, the exponential into product of simpler exponentials, for example, in the polar form [15, 16, 18, 22]. General bivector exponentials in $Cl_{4,1}$ algebra were analyzed in [5]. In coordinate form the difficulty is connected with the appearance of both trigonometric and hyperbolic functions simultaneously in the expansion of exponentials as well as the mixing of scalar coefficients from different grades.

In this paper a different approach, which presents the exponential in coordinates and which is more akin to construction of de Moivre formula, was applied. Namely, to solve the problem, the GA exponential function is expanded into sum of basis elements (grades) using for this purpose the computer algebra (*Mathematica* package). Although in this way obtained final formulas are rather cumbersome, however, their analysis allows to identify the obstacles in constructing the GA coordinate-free formulas. The formulas presented in this paper can be also applied to general purpose programming languages such as Fortran, C^{++} or Python.

In Section 2 the notation is introduced. The final exponential formulas in the coordinates are presented in Sections 3–5 in a form of theorems. The particular cases that follow from general exponential formulas are given in Section 6. Relations of GA exponential to GA trigonometric and hyperbolic functions are presented in Section 7. Possible application of the exponential function in solving spinorial Pauli–Schrödinger equation is given in Section 8. In Section 9, we compare finite GA series of trigonometric functions with the exact formulas that follow from exponential. Finally, in Section 10, we discuss further development of the problem.

2 Notation

In the inverse degree lexicographic ordering used in this paper, the general MV in GA space is expanded in the orthonormal basis $\{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_{12}, \mathbf{e}_{13}, \mathbf{e}_{23}, \mathbf{e}_{123} \equiv I\}$, where $\mathbf{e}_i$ are basis vectors, $\mathbf{e}_{ij}$ are the bivectors and I is the pseudoscalar.[1] The number of subscripts indicates the grade. The scalar is a grade-0 element, the vectors $\mathbf{e}_i$ are the grade-1 elements, etc. In the orthonormalized basis the geometric products of basis vectors satisfy the anticommutation relation

$$\mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i = \pm 2\delta_{ij}.$$

For $Cl_{3,0}$ and $Cl_{0,3}$ algebras, the squares of basis vectors are, correspondingly, $\mathbf{e}_i^2 = +1$ and $\mathbf{e}_i^2 = -1$, where $i = 1, 2, 3$. For mixed signature algebras such as $Cl_{2,1}$ and $Cl_{1,2}$, we have $\mathbf{e}_1^2 = \mathbf{e}_2^2 = 1$, $\mathbf{e}_3^2 = -1$ and $\mathbf{e}_1^2 = 1$, $\mathbf{e}_2^2 = \mathbf{e}_3^2 = -1$, respectively. The general

[1] An increasing order of digits in basis elements is used, i.e., we write $\mathbf{e}_{13}$ instead of $\mathbf{e}_{31} = -\mathbf{e}_{13}$. This convention is reflected in opposite signs of some terms in formulas.

MV of real Clifford algebras $Cl_{p,q}$ for $n = p + q = 3$ can be expressed as

$$\begin{aligned}\mathsf{A} &= a_0 + a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 + a_{12}\mathbf{e}_{12} + a_{23}\mathbf{e}_{23} + a_{13}\mathbf{e}_{13} + a_{123}I\\ &\equiv a_0 + \mathbf{a} + \mathcal{A} + a_{123}I,\end{aligned}$$

where a_i, a_{ij} and a_{123} are the real coefficients, and $\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$ and $\mathcal{A} = a_{12}\mathbf{e}_{12} + a_{23}\mathbf{e}_{23} + a_{13}\mathbf{e}_{13}$ is, respectively, the vector and bivector. I is the pseudoscalar, $I = \mathbf{e}_{123}$. Similarly, the exponential B will be denoted as

$$\begin{aligned}\mathsf{B} = \mathrm{e}^{\mathsf{A}} &= b_0 + b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3 + b_{12}\mathbf{e}_{12} + b_{23}\mathbf{e}_{23} + ba_{13}\mathbf{e}_{13} + a_{123}I\\ &\equiv b_0 + \mathbf{b} + \mathcal{B} + b_{123}I.\end{aligned}$$

We start from the $Cl_{0,3}$ geometric algebra (GA), where the expanded exponential in the coordinate form has the simplest MV coefficients.

3 MV exponential in $Cl_{0,3}$ algebra

Theorem 1 [Exponential function of multivector in $Cl_{0,3}$]. *The exponential of MV*

$$\mathsf{A} = a_0 + a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 + a_{12}\mathbf{e}_{12} + a_{13}\mathbf{e}_{13} + a_{23}\mathbf{e}_{23} + a_{123}I$$

is another MV

$$\exp(\mathsf{A}) = b_0 + b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3 + b_{12}\mathbf{e}_{12} + b_{13}\mathbf{e}_{13} + b_{23}\mathbf{e}_{23} + b_{123}I,$$

where the real coefficients are

$$\begin{aligned}
b_0 &= \frac{1}{2}\mathrm{e}^{a_0}\big(\mathrm{e}^{a_{123}}\cos a_+ + \mathrm{e}^{-a_{123}}\cos a_-\big),\\
b_{123} &= \frac{1}{2}\mathrm{e}^{a_0}\big(\mathrm{e}^{a_{123}}\cos a_+ - \mathrm{e}^{-a_{123}}\cos a_-\big),\\
b_1 &= \frac{1}{2}\mathrm{e}^{a_0}\bigg(\mathrm{e}^{a_{123}}(a_1 - a_{23})\frac{\sin a_+}{a_+} + \mathrm{e}^{-a_{123}}(a_1 + a_{23})\frac{\sin a_-}{a_-}\bigg),\\
b_2 &= \frac{1}{2}\mathrm{e}^{a_0}\bigg(\mathrm{e}^{a_{123}}(a_2 + a_{13})\frac{\sin a_+}{a_+} + \mathrm{e}^{-a_{123}}(a_2 - a_{13})\frac{\sin a_-}{a_-}\bigg),\\
b_3 &= \frac{1}{2}\mathrm{e}^{a_0}\bigg(\mathrm{e}^{a_{123}}(a_3 - a_{12})\frac{\sin a_+}{a_+} + \mathrm{e}^{-a_{123}}(a_3 + a_{12})\frac{\sin a_-}{a_-}\bigg),\\
b_{12} &= \frac{1}{2}\mathrm{e}^{a_0}\bigg(-\mathrm{e}^{a_{123}}(a_3 - a_{12})\frac{\sin a_+}{a_+} + \mathrm{e}^{-a_{123}}(a_3 + a_{12})\frac{\sin a_-}{a_-}\bigg),\\
b_{13} &= \frac{1}{2}\mathrm{e}^{a_0}\bigg(\mathrm{e}^{a_{123}}(a_2 + a_{13})\frac{\sin a_+}{a_+} - \mathrm{e}^{-a_{123}}(a_2 - a_{13})\frac{\sin a_-}{a_-}\bigg),\\
b_{23} &= \frac{1}{2}\mathrm{e}^{a_0}\bigg(-\mathrm{e}^{a_{123}}(a_1 - a_{23})\frac{\sin a_+}{a_+} + \mathrm{e}^{-a_{123}}(a_1 + a_{23})\frac{\sin a_-}{a_-}\bigg),
\end{aligned} \tag{1}$$

and where

$$a_+ = \sqrt{(a_3 - a_{12})^2 + (a_2 + a_{13})^2 + (a_1 - a_{23})^2},$$
$$a_- = \sqrt{(a_3 + a_{12})^2 + (a_2 - a_{13})^2 + (a_1 + a_{23})^2}.$$

When either $a_+ = 0$ *or* $a_- = 0$ *or the both are equal to zero simultaneously, the formula yields special cases considered in Section* 3.1.

Proof. The simplest way to prove the above formula $\exp(\mathsf{A})$ is to check explicitly its defining property

$$\frac{\partial \exp(\mathsf{A}t)}{\partial t}\bigg|_{t=1} = \mathsf{A}\exp(\mathsf{A}) = \exp(\mathsf{A})\mathsf{A}, \qquad (2)$$

where A is assumed to be independent of t. Since we have a single MV that always commutes with itself, the multiplications from left and right by A coincide. After differentiation with respect to scalar parameter t and then setting $t = 1$, we find that in this way, obtained result indeed is $\mathsf{A}\exp(\mathsf{A})$. To be sure, we also checked Eq. (2) by series expansions of $\exp(\mathsf{A}t)$ up to order 6 with symbolic coefficients and up to order 20 with random integers using for this purpose the *Mathematica* package [4]. □

3.1 Special cases of Theorem 1

Let $\operatorname{Det}(\mathsf{A})$ be the determinant of MV [2, 7, 17, 23]. The determinant of the sum of vector $\mathbf{a}$ and bivector $\mathcal{A}$ parts of A simplifies to

$$\begin{aligned}\operatorname{Det}(\mathbf{a} + \mathcal{A}) &= \big((a_3 - a_{12})^2 + (a_2 + a_{13})^2 + (a_1 - a_{23})^2\big)\\ &\quad\times \big((a_3 + a_{12})^2 + (a_2 - a_{13})^2 + (a_1 + a_{23})^2\big)\\ &= a_+^2 a_-^2\end{aligned}$$

from which follows that special cases will arise when $\operatorname{Det}(\mathbf{a}+\mathcal{A}) = 0$. Since the formulas for a_+ and a_- are expressed through square roots, it is interesting to find a MV to which the square roots are associated. In [3, 11] an algorithm to compute the square root of MV in 3D algebras is provided. It seems reasonable to conjecture that the special cases in exponential are related to isolated square roots of the center $a_S + a_I I$ of the considered algebra, where the scalars a_S and a_I are defined by

$$a_S = -(\mathbf{a} + \mathcal{A}) \cdot (\mathbf{a} + \mathcal{A}) = a_1^2 + a_2^2 + a_3^2 + a_{12}^2 + a_{13}^2 + a_{23}^2,$$
$$a_I = -(\mathbf{a} + \mathcal{A}) \wedge (\mathbf{a} + \mathcal{A})I = -2(a_3a_{12} - a_2a_{13} + a_1a_{23}).$$

In $Cl_{0,3}$ algebra the explicit formula for the center is $-(\mathbf{a} + \mathcal{A})(\mathbf{a} + \mathcal{A}) = a_S + a_I I$. In particular the square root of the center can be written as

$$\sqrt{a_S + a_I I} = a_R + a_P I, \qquad (3_1)$$

where

$$a_R + a_P I = \begin{cases} \pm\frac{a_S+\sqrt{a_S^2-a_I^2}+a_I I}{\sqrt{2}\sqrt{a_S+\sqrt{a_S^2-a_I^2}}}, \\ \pm\frac{a_S-\sqrt{a_S^2-a_I^2}+a_I I}{\sqrt{2}\sqrt{a_S-\sqrt{a_S^2-a_I^2}}} \end{cases} \qquad \text{if } a_S^2 > a_I^2. \tag{3_2}$$

From this follows that a_+ and a_- in Eq. (1) can be expressed as $a_+^2 = a_S + a_I = (a_R+a_P)^2$ and $a_-^2 = a_S - a_I = (a_R-a_P)^2$. Note that in (3) the both required conditions $a_S > 0$ and $a_S^2 > a_I^2$ are satisfied for all values of MV coefficients, except when the vector and bivector parts of MV are absent. From this we conclude that the condition $a_+ = 0$ (or $a_- = 0$) is equivalent to the determinant being zero, $\operatorname{Det}(a_S + a_I I) = (a_S + a_I)^2(a_S - a_I)^2 = a_+^4 a_-^4 = 0$.

The special cases in Theorem 1 occur when whichever of denominators, a_+ or a_-, in the coefficients turns to zero. Though at first glance, we could compute corresponding limits, for example, $\lim_{a_+\to 0} \sin a_+/a_+ = 1$, in fact, the formula in this case becomes simpler because the condition $a_+ = 0$ implies that $a_3 = a_{12}, a_2 = -a_{13}$ and $a_1 = a_{23}$. Therefore, the terms in vector and bivector components that include corresponding differences vanish altogether. Similarly, the case $a_- = 0$ implies three conditions $a_3 = -a_{12}, a_2 = a_{13}$ and $a_1 = -a_{23}$ that nullify the corresponding terms in vector and bivector components too. On the other hand, in scalar and pseudoscalar components, we can simply replace corresponding $\cos a_+$ and $\cos a_-$ by 1. Thus, the listed special cases actually represent the special cases already found in the analysis of algorithm of MV square root in [3]. After identification of a_0 and a_{123} with coefficients in [3], $a_0 \equiv s$ and $a_{123} \equiv S$, we find the following equivalence relations $a_+^2 = a_S + a_I = 0 \Leftrightarrow s = -S$, $a_- = a_S - a_I = 0 \Leftrightarrow s = S$ and $a_- = a_+ = 0 \Leftrightarrow s = S = 0$, respectively.

4 MV exponentials in $Cl_{3,0}$ and $Cl_{1,2}$ algebras

Theorem 2 [Exponential function in $Cl_{3,0}$ (upper) and $Cl_{1,2}$ (lower signs)]. *The exponential of MV*

$$\mathsf{A} = a_0 + a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 + a_{12}\mathbf{e}_{12} + a_{13}\mathbf{e}_{13} + a_{23}\mathbf{e}_{23} + a_{123}I$$

is another MV

$$\exp(\mathsf{A}) = \mathrm{e}^{a_0}\left(b_0 + \frac{b_1}{|c|}\mathbf{e}_1 + \frac{b_2}{|c|}\mathbf{e}_2 + \frac{b_3}{|c|}\mathbf{e}_3 + \frac{b_{12}}{|c|}\mathbf{e}_{12} + \frac{b_{13}}{|c|}\mathbf{e}_{13} + \frac{b_{23}}{|c|}\mathbf{e}_{23} + b_{123}I\right),$$

where real coefficients $b_{i\dots j}$ are

$$\begin{aligned} b_0 &= \cos a_{123} \cos a_- \cosh a_+ - \sin a_{123} \sin a_- \sinh a_+, \\ b_{123} &= \sin a_{123} \cos a_- \cosh a_+ + \cos a_{123} \sin a_- \sinh a_+, \\ b_1 &= \cosh a_+ \sin a_-\big((a_-a_1 - a_+a_{23}) \cos a_{123} - (a_+a_1 + a_-a_{23}) \sin a_{123}\big) \\ &\quad + \sinh a_+ \cos a_-\big((a_+a_1 + a_-a_{23}) \cos a_{123} + (a_-a_1 - a_+a_{23}) \sin a_{123}\big), \end{aligned} \tag{4_1}$$

$$\begin{aligned}
b_2 &= \pm\cosh a_+ \sin a_-\big((\pm a_- a_2 + a_+ a_{13})\cos a_{123} + (\mp a_+ a_2 + a_- a_{13})\sin a_{123}\big)\\
&\quad + \sinh a_+ \cos a_-\big((a_+ a_2 \mp a_- a_{13})\cos a_{123} + (a_- a_2 \pm a_+ a_{13})\sin a_{123}\big),\\
b_3 &= \cosh a_+ \sin a_-\big((a_- a_3 \mp a_+ a_{12})\cos a_{123} \mp (\pm a_+ a_3 + a_- a_{12})\sin a_{123}\big)\\
&\quad + \sinh a_+ \cos a_-\big((a_+ a_3 \pm a_- a_{12})\cos a_{123} + (a_- a_3 \mp a_+ a_{12})\sin a_{123}\big),\\
b_{12} &= \cosh a_+ \sin a_-\big((\pm a_+ a_3 + a_- a_{12})\cos a_{123} \pm (a_- a_3 \mp a_+ a_{12})\sin a_{123}\big)\\
&\quad + \sinh a_+ \cos a_-\big((\mp a_- a_3 + a_+ a_{12})\cos a_{123} + (\pm a_+ a_3 + a_- a_{12})\sin a_{123}\big),\\
b_{13} &= \mp\cosh a_+ \sin a_-\big((a_+ a_2 \mp a_- a_{13})\cos a_{123} + (a_- a_2 \pm a_+ a_{13})\sin a_{123}\big)\\
&\quad + \sinh a_+ \cos a_-\big((\pm a_- a_2 + a_+ a_{13})\cos a_{123} + (\mp a_+ a_2 + a_- a_{13})\sin a_{123}\big),\\
b_{23} &= \cosh a_+ \sin a_-\big((a_+ a_1 + a_- a_{23})\cos a_{123} + (a_- a_1 - a_+ a_{23})\sin a_{123}\big)\\
&\quad + \sinh a_+ \cos a_-\big((-a_- a_1 + a_+ a_{23})\cos a_{123} + (a_+ a_1 + a_- a_{23})\sin a_{123}\big)
\end{aligned} \tag{4_2}$$

with

$$|c| = \sqrt{a_S^2 + a_I^2} = a_+^2 + a_-^2, \tag{5_1}$$

where

$$\begin{aligned}
a_S &= a_1^2 \pm a_2^2 \pm a_3^2 \mp a_{12}^2 \mp a_{13}^2 - a_{23}^2,\\
a_I &= 2(a_3 a_{12} - a_2 a_{13} + a_1 a_{23}),
\end{aligned} \tag{5_2}$$

and

$$\begin{aligned}
a_+ &= \begin{cases} \frac{1}{\sqrt{2}}\sqrt{a_S + |c|}, & a_I \neq 0,\\ \sqrt{a_S}, & a_I = 0 \textit{ and } a_S > 0,\\ 0, & a_I = 0 \textit{ and } a_S < 0,\end{cases}\\
a_- &= \begin{cases} \frac{1}{\sqrt{2}}\frac{a_I}{\sqrt{a_S + |c|}}, & a_I \neq 0,\\ 0, & a_I = 0 \textit{ and } a_S > 0,\\ \sqrt{-a_S}, & a_I = 0 \textit{ and } a_S < 0.\end{cases}
\end{aligned} \tag{5_3}$$

When both $a_+ = 0$ and $a_- = 0$ or, alternatively, both $a_S = 0$ and $a_I = 0$, the formulas are associated with special cases considered below in Section 4.1.

Proof. It is enough to check the defining property (2). The validity was also checked by expanding in Taylor series up to order 6 with symbolic coefficients and up to order 20 using random integers. □

Since both $Cl_{1,2}$ and $Cl_{3,0}$ algebras are represented by $\mathbb{C}(2)$ matrices, they are mutually isomorphic. Therefore, the same formula may be used for $Cl_{3,0}$ and $Cl_{1,2}$ algebras without modification if one takes into account one-to-one equivalence. For example, either $\mathbf{e}_2 \leftrightarrow \mathbf{e}_{13}$, $\mathbf{e}_3 \leftrightarrow \mathbf{e}_{12}$ or, alternatively,

$$\begin{aligned}
&\mathbf{e}_1 \leftrightarrow \mathbf{e}_{12}, && \mathbf{e}_2 \leftrightarrow \mathbf{e}_{13}, && \mathbf{e}_3 \leftrightarrow \mathbf{e}_1,\\
&\mathbf{e}_{12} \leftrightarrow \mathbf{e}_{23}, && \mathbf{e}_{13} \leftrightarrow \mathbf{e}_2, && \mathbf{e}_{23} \leftrightarrow \mathbf{e}_3.
\end{aligned}$$

Those not explicitly listed being the same.

4.1 Special cases of Theorem 2

The determinant of sum of vector and bivector parts of MV A in this case is

$$\begin{aligned}\operatorname{Det}(\mathbf{a}+\mathcal{A}) &= \bigl(4(a_3a_{12}-a_2a_{13}+a_1a_{23})^2 \\ &\quad + \bigl(a_1^2 \pm a_2^2 \pm a_3^2 \mp a_{12}^2 \mp a_{13}^2 - a_{23}^2\bigr)^2\bigr)^2 \\ &= \bigl(a_S^2+a_I^2\bigr)^2 = a_+^2 + a_-^2,\end{aligned}$$

where upper signs are for $Cl_{3,0}$, and lower for $Cl_{1,2}$ algebra. Equation (5) shows that special cases occur again when $\operatorname{Det}(\mathbf{a}+\mathcal{A}) = 0$. The isolated square roots of $c = a_S + a_I I$ of $Cl_{3,0}$ algebra are given by (both signs for both algebras)

$$\sqrt{c} = \sqrt{a_S + a_I I} = \pm\frac{a_S + \sqrt{a_S^2+a_I^2} + a_I I}{\sqrt{2}\sqrt{a_S + \sqrt{a_S^2+a_I^2}}} = \pm(a_+ + a_- I),$$

where the root $\sqrt{a_S^2+a_I^2}$ is a norm: $|c| = \sqrt{c\tilde{\bar{c}}} = \sqrt{a_S^2+a_I^2} = a_+^2 + a_-^2$. The coefficients a_S and a_I represent coefficients at scalar and pseudoscalar of geometric product $\mathbf{a}+\mathcal{A}$ by itself. In particular, for $Cl_{3,0}$ algebra, the explicit form is $(\mathbf{a}+\mathcal{A})(\mathbf{a}+\mathcal{A}) = a_S + a_I I$, where a_S and a_I are expressed through inner and outer products, $a_S = (\mathbf{a}+\mathcal{A})\cdot(\mathbf{a}+\mathcal{A}) = a_1^2 \pm a_2^2 \pm a_3^2 \mp a_{12}^2 \mp a_{13}^2 - a_{23}^2$ (upper signs for $Cl_{3,0}$ and lower for $Cl_{1,2}$ algebra) and $a_I = -(\mathbf{a}+\mathcal{A}) \wedge (\mathbf{a}+\mathcal{A}) I = 2(a_3a_{12} - a_2a_{13} + a_1a_{23})$.

The denominator in (4) vanishes when $|c| = \sqrt{a_S^2+a_I^2} = a_+^2 + a_-^2 = 0$. It is easy to see that in this case, all vector and bivector coefficients become zero $b_1 = b_2 = b_3 = b_{12} = b_{13} = b_{23} = 0$. Then, in the expressions for b_0 and b_{123}, we have to take $\cosh a_+ = \cos a_- = 1$ and $\sinh a_+ = \sin a_- = 0$. After identification with coefficients of [3], $a_0 \equiv s$ and $a_{123} \equiv S$, these conditions again are analogues of the only possible special case when $s = S = 0$ in the square root of MV for $Cl_{3,0}$ [3].

5 MV exponential in $Cl_{2,1}$ algebra

Theorem 3 [Exponential function in $Cl_{2,1}$]. *Exponential of MV*

$$\mathsf{A} = a_0 + a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 + a_{12}\mathbf{e}_{12} + a_{13}\mathbf{e}_{13} + a_{23}\mathbf{e}_{23} + a_{123}I$$

is another MV

$$\exp(\mathsf{A}) = b_0 + b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3 + b_{12}\mathbf{e}_{12} + b_{13}\mathbf{e}_{13} + b_{23}\mathbf{e}_{23} + b_{123}I,$$

where

$$\begin{aligned} b_0 &= \frac{1}{2}\mathrm{e}^{a_0}\bigl(\mathrm{e}^{a_{123}}\operatorname{co}(a_+^2) + \mathrm{e}^{-a_{123}}\operatorname{co}(a_-^2)\bigr), \\ b_{123} &= \frac{1}{2}\mathrm{e}^{a_0}\bigl(\mathrm{e}^{a_{123}}\operatorname{co}(a_+^2) - \mathrm{e}^{-a_{123}}\operatorname{co}(a_-^2)\bigr), \end{aligned} \tag{6_1}$$

$$
\begin{aligned}
b_1 &= \frac{1}{2}\mathrm{e}^{a_0}\big(\mathrm{e}^{a_{123}}(a_1+a_{23})\,\mathrm{si}\big(a_+^2\big)+\mathrm{e}^{-a_{123}}(a_1-a_{23})\,\mathrm{si}\big(a_-^2\big)\big),\\
b_2 &= \frac{1}{2}\mathrm{e}^{a_0}\big(\mathrm{e}^{a_{123}}(a_2-a_{13})\,\mathrm{si}\big(a_+^2\big)+\mathrm{e}^{-a_{123}}(a_2+a_{13})\,\mathrm{si}\big(a_-^2\big)\big),\\
b_3 &= \frac{1}{2}\mathrm{e}^{a_0}\big(\mathrm{e}^{a_{123}}(a_3-a_{12})\,\mathrm{si}\big(a_+^2\big)+\mathrm{e}^{-a_{123}}(a_3+a_{12})\,\mathrm{si}\big(a_-^2\big)\big),\\
b_{12} &= \frac{1}{2}\mathrm{e}^{a_0}\big(-\mathrm{e}^{a_{123}}(a_3-a_{12})\,\mathrm{si}\big(a_+^2\big)+\mathrm{e}^{-a_{123}}(a_3+a_{12})\,\mathrm{si}\big(a_-^2\big)\big),\\
b_{13} &= \frac{1}{2}\mathrm{e}^{a_0}\big(-\mathrm{e}^{a_{123}}(a_2-a_{13})\,\mathrm{si}\big(a_+^2\big)+\mathrm{e}^{-a_{123}}(a_2+a_{13})\,\mathrm{si}\big(a_-^2\big)\big),\\
b_{23} &= \frac{1}{2}\mathrm{e}^{a_0}\big(\mathrm{e}^{a_{123}}(a_1+a_{23})\,\mathrm{si}\big(a_+^2\big)-\mathrm{e}^{-a_{123}}(a_1-a_{23})\,\mathrm{si}\big(a_-^2\big)\big)
\end{aligned}
\tag{6_2}
$$

with

$$
\begin{aligned}
a_+^2 &= -(a_3-a_{12})^2+(a_2-a_{13})^2+(a_1+a_{23})^2,\\
a_-^2 &= -(a_3+a_{12})^2+(a_2+a_{13})^2+(a_1-a_{23})^2,
\end{aligned}
$$

and

$$
\mathrm{si}\big(a_\pm^2\big)=\begin{cases}\frac{\sin\sqrt{a_\pm^2}}{\sqrt{a_\pm^2}}, & a_\pm^2>0,\\ \frac{\sinh\sqrt{-a_\pm^2}}{\sqrt{-a_\pm^2}}, & a_\pm^2<0,\end{cases}\qquad \mathrm{co}\big(a_\pm^2\big)=\begin{cases}\cos\sqrt{a_\pm^2}, & a_\pm^2>0,\\ \cosh\sqrt{-a_\pm^2}, & a_\pm^2<0.\end{cases}
$$

When either $a_+^2=0$ or $a_-^2=0$ or both are zeroes, the formula yields special cases considered in Section 5.1.

Proof. The same as for $Cl_{0,3}$ and $Cl_{3,0}$ algebras; see Eq. (2). □

5.1 Special cases of Theorem 3

Determinant of the sum of vector and bivector in A yields

$$
\begin{aligned}
\mathrm{Det}(\mathbf{a}+\mathcal{A}) &= \big(-(a_3-a_{12})^2+(a_2-a_{13})^2+(a_1+a_{23})^2\big)\\
&\quad\times\big(-(a_3+a_{12})^2+(a_2+a_{13})^2+(a_1-a_{23})^2\big)\\
&= a_+^2a_-^2.
\end{aligned}
$$

The special cases occur when $\mathrm{Det}(\mathbf{a}+\mathcal{A})=0$. As for previous algebras, they are related to the isolated roots of the element of the center a_S+a_II of $Cl_{2,1}$. In particular, for the root of the center, we find

$$
\sqrt{a_S+a_II}=a_R+a_PI,
$$

$$
a_R+a_PI=\begin{cases}\pm\frac{a_S+\sqrt{a_S^2-a_I^2}+a_II}{\sqrt{2}\sqrt{a_S+\sqrt{a_S^2-a_I^2}}} & \text{if } a_S+\sqrt{a_S^2-a_I^2}>0 \text{ and } a_S^2>a_I^2,\\ \pm\frac{a_S-\sqrt{a_S^2-a_I^2}+a_II}{\sqrt{2}\sqrt{a_S-\sqrt{a_S^2-a_I^2}}} & \text{if } a_S-\sqrt{a_S^2-a_I^2}>0 \text{ and } a_S^2>a_I^2.\end{cases}
$$

So, in $Cl_{2,1}$ algebra, we have up to four roots. The real coefficients a_S and a_I are equal to coefficients (which are elements of the algebra center) of geometric product $\mathbf{a}+\mathcal{A}$ by itself. In particular, for $Cl_{2,1}$ algebra, the explicit form is $(\mathbf{a}+\mathcal{A})(\mathbf{a}+\mathcal{A}) = a_S + a_I I$, where $a_S = (\mathbf{a}+\mathcal{A})\cdot(\mathbf{a}+\mathcal{A}) = a_1^2 + a_2^2 - a_3^2 - a_{12}^2 + a_{13}^2 + a_{23}^2$ and $a_I = (\mathbf{a}+\mathcal{A})\wedge(\mathbf{a}+\mathcal{A})I = 2(a_3 a_{12} - a_2 a_{13} + a_1 a_{23})$.

In (6), a_+ and a_- then again can be expressed as

$$a_+^2 = a_S + a_I = (a_R + a_P)^2 = -(a_3 - a_{12})^2 + (a_2 - a_{13})^2 + (a_1 + a_{23})^2$$
$$a_-^2 = a_S - a_I = (a_R - a_P)^2 = -(a_3 + a_{12})^2 + (a_2 + a_{13})^2 + (a_1 - a_{23})^2.$$

After comparison with $Cl_{0,3}$ algebra case, we see that the explicit expressions now have different signs and, in general, can acquire positive and negative values. Since these expressions are present inside the square root of exponential, we formally have to introduce functions $\mathrm{si}(a_\pm^2)$ and $\mathrm{co}(a_\pm^2)$ (see Eq. (6)) in order to ensure real arguments for both functions.

When denominator a_+ or a_- in Eq. (6) acquires zero value, we have a special case. This corresponds to the condition $\mathrm{Det}(a_S + a_I I) = (a_S - a_I)^2(a_S + a_I)^2 = 0$. Therefore, conditions $a_+^2 = a_S + a_I = 0$ and $a_-^2 = a_S - a_I = 0$ define special cases. This requires to modify some of the terms in Eq. (6), i.e., these terms have to be replaced by limits $\lim_{a_\pm \to 0} \mathrm{si}(a_\pm^2) = 1$. Note that now the coefficients in vector and bivector components that include a_+ or a_-, in general, do not necessary vanish, unless the both a_+^2 and a_-^2 are equal to zero simultaneously. This is a different situation compared to $Cl_{0,3}$ algebra for which the corresponding terms in the component expressions always vanish.

Once more, we note that after identification of the coefficients with those in [3], $a_0 \equiv s$ and $a_{123} \equiv S$, the mentioned special cases correspond to special cases of square root of MV when $a_+^2 = a_S + a_I = 0 \Leftrightarrow s = -S$, $a_- = a_S - a_I = 0 \Leftrightarrow s = S$ and $a_- = a_+ = 0 \Leftrightarrow s = S = 0$, respectively.

6 Particular cases: Pure bivector, vector and (pseudo)scalar

Equating appropriate coefficients (b_i or b_{ij}) to zero from formulas (1), (4) and (6), one can derive the exponentials of blades and compare them with those in the literature, mainly for $Cl_{3,0}$ and $Cl_{0,3}$ algebras. For mixed signature algebras, to authors knowledge, such formulas are presented below for the first time.

6.1 Exponential of bivector

In this case the exponential of a pure bivector $\mathcal{A} = a_{12}\mathbf{e}_{12} + a_{13}\mathbf{e}_{13} + a_{23}\mathbf{e}_{23}$ can be expressed in a coordinate-free form. The general formulas (1), (4) and (6) then reduce to

$$\mathrm{e}^{\mathcal{A}} = \begin{cases} \cos|\mathcal{A}| + \frac{\mathcal{A}}{|\mathcal{A}|}\sin|\mathcal{A}| & \text{for } Cl_{3,0}, Cl_{0,3},\ \mathcal{A}^2 < 0, \\ \cosh|\mathcal{A}| + \frac{\mathcal{A}}{|\mathcal{A}|}\sinh|\mathcal{A}| & \text{for } Cl_{1,2}, Cl_{2,1},\ \mathcal{A}^2 > 0, \\ \cos|\mathcal{A}| + \frac{\mathcal{A}}{|\mathcal{A}|}\sin|\mathcal{A}| & \text{for } Cl_{1,2}, Cl_{2,1},\ \mathcal{A}^2 < 0, \end{cases}$$

where

$$|\mathcal{A}| = \begin{cases} \sqrt{\mathcal{A}^2} & \text{if } \mathcal{A}^2 > 0, \\ \sqrt{-\mathcal{A}^2} & \text{if } \mathcal{A}^2 < 0, \end{cases} \quad \text{and} \quad \mathcal{A}^2 = \begin{cases} -a_{12}^2 - a_{13}^2 - a_{23}^2 & \text{for } Cl_{3,0}, Cl_{0,3}, \\ -a_{12}^2 + a_{13}^2 + a_{23}^2 & \text{for } Cl_{2,1}, \\ +a_{12}^2 + a_{13}^2 - a_{23}^2 & \text{for } Cl_{1,2}. \end{cases}$$

6.2 Exponential of vector

In the case of pure vector $\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$, the magnitude is $|\mathbf{a}| = \sqrt{\pm\mathbf{a}^2}$, where the root must be a positive real number. Then the general formulas reduce to

$$\mathrm{e}^{\mathbf{a}} = \begin{cases} \cosh|\mathbf{a}| + \frac{\mathbf{a}}{|\mathbf{a}|}\sinh|\mathbf{a}| & \text{for } Cl_{3,0},\ \mathbf{a}^2 > 0, \\ \cos|\mathbf{a}| + \frac{\mathbf{a}}{|\mathbf{a}|}\sin|\mathbf{a}| & \text{for } Cl_{0,3},\ \mathbf{a}^2 < 0, \\ \cosh|\mathbf{a}| + \frac{\mathbf{a}}{|\mathbf{a}|}\sinh|\mathbf{a}| & \text{for } Cl_{1,2}, Cl_{2,1},\ \mathbf{a}^2 > 0, \\ \cos|\mathbf{a}| + \frac{\mathbf{a}}{|\mathbf{a}|}\sin|\mathbf{a}| & \text{for } Cl_{1,2}, Cl_{2,1},\ \mathbf{a}^2 < 0, \end{cases}$$

where

$$|\mathbf{a}| = \begin{cases} \sqrt{\mathbf{a}^2}, & \mathbf{a}^2 > 0, \\ \sqrt{-\mathbf{a}^2}, & \mathbf{a}^2 < 0, \end{cases}$$

and

$$\mathbf{a}^2 = \begin{cases} \pm(a_1^2 + a_2^2 + a_3^2) & \text{for } Cl_{3,0}\ (+\text{ sign}),\ Cl_{0,3}\ (-\text{ sign}), \\ a_1^2 + a_2^2 - a_3^2 & \text{for } Cl_{2,1}, \\ a_1^2 - a_2^2 - a_3^, & \text{for } Cl_{1,2}. \end{cases}$$

Thus, $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$ for both $Cl_{3,0}$ and $Cl_{0,3}$.

6.3 Exponent of scalar + pseudoscalar

When $\mathsf{A} = a_0 + a_{123}I$, the type of the function depends on sign of I^2, minus sign for $Cl_{3,0}$ and $Cl_{1,2}$, and plus sign for $Cl_{0,3}$ and $Cl_{2,1}$,

$$\mathrm{e}^{a_0 + a_{123}I} = \begin{cases} \mathrm{e}^{a_0}(\cos a_{123} + I\sin a_{123}) & \text{for } Cl_{3,0}, Cl_{1,2}, \\ \mathrm{e}^{a_0}(\cosh a_{123} + I\sinh a_{123}) & \text{for } Cl_{0,3}, Cl_{2,1}. \end{cases}$$

All listed in this section formulas are well known [21], and they readily follow from general formulas (1), (4) and (6). One also can check that the identity $\mathrm{e}^{\mathsf{A}}\mathrm{e}^{-\mathsf{A}} = \mathrm{e}^{-\mathsf{A}}\mathrm{e}^{\mathsf{A}} = 1$ holds, i.e., the inverse of exponential can be obtained by changing the sign of the exponent.

7 Relations of the exponential to GA trigonometric and hyperbolic functions

The geometric product is noncommutative. However, any two GA functions of the same argument, for example, $f(\mathsf{A})$ and $g(\mathsf{A})$, that can be expanded in the Taylor series, commute: $f(\mathsf{A})g(\mathsf{A}) = g(\mathsf{A})f(\mathsf{A})$. Indeed, for any chosen finite series expansion, we have a product of two polynomials of a single variable A. Since the MV always commutes with itself, it follows that a well-behaved functions of the same MV argument commute too.

As known, the elementary trigonometric and hyperbolic functions in GA are defined by exactly the same series expansions as their commutative counterparts [6,19,21]. For an arbitrary MV, the GA hyperbolic functions can be defined similarly as for ordinary functions. However, the GA trigonometric functions, in general, exist for specific real GAs only. The latter are characterized by a commutative pseudoscalar and property $I^2 = -1$ [6] and therefore can be defined for Clifford algebras $Cl_{3,0}$ and $Cl_{1,2}$ only. In order to define them for the algebras $Cl_{0,3}$ and $Cl_{2,1}$, we have to introduce imaginary unit, i.e., in these algebras, trigonometric functions exist only when they are complexified.

As known, scalar trigonometric and hyperbolic functions are linked up through the imaginary unit $\mathrm{i} = \sqrt{-1}$, for example, $\cos(\mathrm{i}x) = \cosh(x)$ for all $x \in \mathbb{R}$. For MV functions, similar relations also exist if apart from i the pseudoscalar I is included:

$$\cosh(I\mathsf{A}) = \cos(\mathrm{i}I\mathsf{A}), \quad \sinh(I\mathsf{A}) = -\mathrm{i}\sin(\mathrm{i}I\mathsf{A}).$$

Also, trigonometric and hyperbolic functions of MV A can be expressed through the exponentials if one remembers that $I^2 = -1$ for $Cl_{3,0}$ and $Cl_{1,2}$, and $I^2 = +1$ for $Cl_{0,3}$ and $Cl_{2,1}$,

$$\begin{aligned} \sin\mathsf{A} &= \begin{cases} \frac{I}{2}(\mathrm{e}^{-I\mathsf{A}} - \mathrm{e}^{I\mathsf{A}}) & \text{for } Cl_{3,0}, Cl_{1,2}, \\ \frac{\mathrm{i}}{2}(\mathrm{e}^{-\mathrm{i}I\mathsf{A}} - \mathrm{e}^{\mathrm{i}I\mathsf{A}}) & \text{for } Cl_{0,3}, Cl_{2,1}, \end{cases} \\ \cos\mathsf{A} &= \begin{cases} \frac{1}{2}(\mathrm{e}^{-I\mathsf{A}} + \mathrm{e}^{I\mathsf{A}}) & \text{for } Cl_{3,0}, Cl_{1,2}, \\ \frac{1}{2}(\mathrm{e}^{-\mathrm{i}I\mathsf{A}} + \mathrm{e}^{\mathrm{i}I\mathsf{A}}) & \text{for } Cl_{0,3}, Cl_{2,1}, \end{cases} \end{aligned} \tag{7}$$

where $I\mathsf{A}$ is the dual to multivector A. As suggested at the beginning of this section, the hyperbolic GA functions do not require imaginary unit, thus we have

$$\sinh\mathsf{A} = \frac{1}{2}\big(\mathrm{e}^{\mathsf{A}} - \mathrm{e}^{-\mathsf{A}}\big), \qquad \cosh\mathsf{A} = \frac{1}{2}\big(\mathrm{e}^{\mathsf{A}} + \mathrm{e}^{-\mathsf{A}}\big). \tag{8}$$

From the above formulas follows various relations between GA trigonometric and hyperbolic functions that are analogues of the well-known scalar relations. As an example, a few of them are given below:

$$\begin{aligned} &\cos^2\mathsf{A} + \sin^2\mathsf{A} = 1, \quad \cosh^2\mathsf{A} - \sinh^2\mathsf{A} = 1, \\ &\sin(2\mathsf{A}) = 2\sin\mathsf{A}\cos\mathsf{A} = 2\cos\mathsf{A}\sin\mathsf{A}, \\ &\cos(2\mathsf{A}) = \cos^2\mathsf{A} - \sin^2\mathsf{A}. \end{aligned}$$

Also, it should be noted that GA sine and cosine functions, as well as hyperbolic GA sine and cosine functions, commute: $\sin\mathsf{A}\cos\mathsf{A} = \cos\mathsf{A}\sin\mathsf{A}$ and $\sinh\mathsf{A}\cosh\mathsf{A} = \cosh\mathsf{A}\sinh\mathsf{A}$.

Apart from relations between the exact hyperbolic sine-cosine functions and the exponential given in Eq. (8) we can write an exact formula for hyperbolic tangent as well (see the beginning of this section),

$$\tanh \mathsf{A} = \sinh \mathsf{A} \cosh^{-1} \mathsf{A} = \cosh^{-1} \mathsf{A} \sinh \mathsf{A}, \tag{9}$$

and likewise for $\coth \mathsf{A}$ functions. After substitution of exponential formulas (8) into the right-hand side of (9), we obtain general $\tanh \mathsf{A}$. However, as a first step in deriving exact formula for $\tanh \mathsf{A}$, at first, one must compute the exact inverse of hyperbolic cosine. How to compute the inverse MV in case of general Clifford algebras is described in [2, 14, 23]. For this purpose, the adjugate and determinant of MV may be needed,

$$\mathsf{A}^{-1} = \frac{\mathrm{Adj}(\mathsf{A})}{\mathrm{Det}(\mathsf{A})}, \qquad \mathrm{Adj}(\mathsf{A})\,\mathsf{A} = \mathsf{A}\,\mathrm{Adj}(\mathsf{A}) = \mathrm{Det}(\mathsf{A}). \tag{10}$$

Here Det is the determinant of MV, which in 3D can be computed with the help of involutions [2, 7, 17]

$$\mathrm{Det}(\mathsf{A}) = \mathsf{A}\widetilde{\mathsf{A}}\widehat{\mathsf{A}}\widehat{\widetilde{\mathsf{A}}}, \tag{11}$$

where $\widetilde{\mathsf{A}}$ denotes reverse MV, and $\widehat{\mathsf{A}}$ is grade inverse of MV A. Although the computation of inverse of general 3D MV is straightforward, the resulting symbolic expression is too large to be presented here. For this purpose, numerical calculations are more suited. Also, in Section 9, we shall profit from numerical calculations by *Mathematica*.

8 Applications

8.1 Time-dependent GA equation with a simple Hamiltonian

The spinor evolution under the action of magnetic field is considered. The field (vector) is assumed to consist of two parts, constant parallel to $\mathbf{e}_3$ and rotating in $\mathbf{e}_{12}$ plane with angular frequency ω,

$$\mathbf{B}(t) = B_0\mathbf{e}_3 + B_1\big(\mathbf{e}_1 \cos(\omega t) + \sigma \mathbf{e}_2 \sin(\omega t)\big).$$

The sign number σ determines the rotation sense. When $\sigma = -1$, the field of amplitude B_1 is rotating clockwise, and when $\sigma = 1$, anticlockwise.

The time-dependent Pauli–Schrödinger equation in the presence of homogeneous $\mathbf{B}(t)$ field for a spinor ψ, which is the MV of $Cl_{3,0}$ algebra, is

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = \frac{1}{2}\gamma I \mathbf{B}(t)\psi, \tag{12}$$

where γ is the gyromagnetic ratio. This GA equation can be solved by rotating frame method (in physics it is called the rotating wave approximation) if the following rotor

$S = \exp(-\sigma \mathbf{e}_{12}\omega t/2)$ is applied to Eq. (12). Multiplying from left by reverse of S and then differentiating with respect to time, we find

$$\begin{aligned}\frac{\mathrm{d}(\widetilde{S}\psi)}{\mathrm{d}t} &= \frac{\mathrm{d}\widetilde{S}}{\mathrm{d}t}\psi + \widetilde{S}\frac{\mathrm{d}\psi}{\mathrm{d}t} = \frac{1}{2}\big(\sigma\mathbf{e}_{12}\omega\widetilde{S}\psi + \widetilde{S}\gamma I\mathbf{B}(t)\psi\big)\\ &= \frac{1}{2}\big(\sigma\mathbf{e}_{12}\omega + \gamma I\widetilde{S}\mathbf{B}(t)S\big)(\widetilde{S}\psi).\end{aligned}$$

When $\sigma = \pm 1$, the product

$$\begin{aligned}\widetilde{S}\mathbf{B}(t)S = B_0\mathbf{e}_3 &+ B_1\cos(\sigma\omega t)(\mathbf{e}_1\cos\omega t + \sigma\mathbf{e}_2\sin\omega t)\\ &+ B_1\sin(\sigma\omega t)(\mathbf{e}_2\cos\omega t + \sigma\mathbf{e}_1\sin\omega t)\end{aligned}$$

reduces to time-independent field $\widetilde{S}\mathbf{B}(t)S = B_1\mathbf{e}_1 + B_0\mathbf{e}_3$. Therefore, the GA differential equation becomes

$$\frac{\mathrm{d}(\widetilde{S}\psi)}{\mathrm{d}t} = \frac{1}{2}(\sigma\mathbf{e}_{12}\omega + \mathbf{e}_{23}\omega_1 + \mathbf{e}_{12}\omega_0)(\widetilde{S}\psi), \tag{13}$$

where $\omega_0 = \gamma B_0$, $\omega_1 = \gamma B_1$. Since Eq. (13) has a constant MV coefficient, its solution is the exponential function

$$(\widetilde{S}\psi) = \exp\bigg(\frac{1}{2}(\sigma\mathbf{e}_{12}\omega + \mathbf{e}_{23}\omega_1 + \mathbf{e}_{12}\omega_0)\bigg)(\widetilde{S}\psi)_0.$$

At $t = 0$ the initial MV is $(\widetilde{S}\psi)_0 = \psi(0)$. Multiplying from left by S and expanding the second exponential according to Section 6.1, finally, we have

$$\psi = \mathrm{e}^{-\sigma\mathbf{e}_{12}\omega t/2}\bigg(\cos\frac{\alpha t}{2} + \alpha^{-1}\big(\mathbf{e}_{23}\omega_1 + \mathbf{e}_{12}(\omega_0 + \sigma\omega)\big)\sin\frac{\alpha t}{2}\bigg)\psi(0), \tag{14}$$

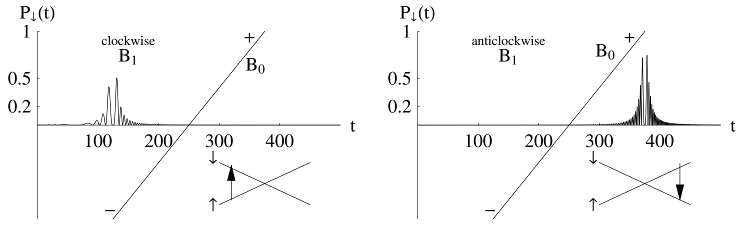
where $\alpha = ((\sigma\omega + \omega_0)^2 + \omega_1^2)^{1/2}/2$.

Equation (14) describes the evolution of the total spinor, which is a mixture of up and down spinor states $\psi = \psi_\uparrow + \psi_\downarrow$ and normalized, $\psi\widetilde{\psi} = 1$. In GA the up and down spinor eigenstates are, respectively, described by basis scalar 1 and basis bivector $\mathbf{e}_{13}$ [12]. We shall assume that the spinor initially is in the up eigenstate, $\psi(0) = \psi_\uparrow = 1$. Then the evolution of the state $\psi_\downarrow$ is given by projecting ψ onto the down eigenstate $\mathbf{e}_{13}$ [12]. The result is

$$\psi_\downarrow = -\langle\mathbf{e}_{13}\psi\rangle + \langle\mathbf{e}_{13}\psi\mathbf{e}_{12}\rangle\mathbf{e}_{12} = \alpha^{-1}\bigg(\sin\frac{\alpha t}{2}\bigg(\mathbf{e}_{12}\cos\frac{\alpha t}{2} - \sin\frac{\sigma\alpha t}{2}\bigg)\bigg).$$

The probability to detect the down spin at the moment t then is

$$P_\downarrow(t) = \psi_\downarrow\widetilde{\psi}_\downarrow = \bigg(\frac{\omega_1\sin(\frac{1}{2}t\sqrt{(\sigma\omega + \omega_0)^2 + \omega_1^2})}{\sqrt{(\sigma\omega + \omega_0)^2 + \omega_1^2}}\bigg)^2.$$



(a) (b)

Figure 1. The probability $P_{\downarrow}(t)$ to find the spin in the down direction when the magnetic field $\mathbf{B}_0 = B_0\mathbf{e}_3$ linearly increases from $B_0 = -2$ to $B_0 = 2$ in the time interval $T = (t_{\text{fin}} - t_{\text{ini}}) = 500$. The exciting field $\mathbf{B}_1$ that flips the spin from $\uparrow$ to $\downarrow$ direction, as shown in the insets by long vertical arrows, is rotating in $\mathbf{e}_{12}$-plane: clockwise in (a) and anticlockwise in (b). The insets show the up and down spin eigenenergies as a function of magnetic field strength. In (a) the absorption and in (b) the stimulated emission take place that are the main processes that determine performance of a laser, which is the acronym of "light absorption [and] stimulated emission [of] radiation". Other parameters in the calculation: $\omega = 1, \omega_1 = 0.05$.

At resonance, when $\sigma\omega + \omega_0 = 0$ (for clockwise rotation, $-\omega + \omega_0 = 0$, and for anticlockwise rotation, $\omega - \omega_0 = 0$) the probability oscillates, $P_{\downarrow}(t) = \sin^2(\omega_1 t/2) = \sin^2(\gamma B_1 t/2)$ with the frequency that depends on exciting field amplitude B_1. In quantum mechanics, such rotating field induced oscillations between up and down states are called Rabi oscillations. If magnetic field $\omega_0 = \gamma B_0$ changes very slowly (adiabatically) in the interval $T >> 2\pi/\omega$, then in the vicinity of resonance the probability peaks related with Rabi oscillations will appear; see Fig. 1. The moment of the appearance depends on the rotation sense via sign number σ. The observed asymmetry between (a) and (b) panels in Fig. 1 is the manifestation of selection rules for quantum transition under action by rotating magnetic field.

8.2 Relations to even $Cl^{+}_{1,3}$ and $Cl^{+}_{3,1}$ geometric algebras

If isomorphism rules between 4-dimensional even subalgebra of $Cl_{1,3}$ and full $Cl_{3,0}$ algebra are made of, namely,

$$Cl^{+}_{1,3} \leftrightarrow Cl_{3,0}, \qquad \mathbf{e}_{23} \leftrightarrow \mathbf{e}_{12}, \qquad \mathbf{e}_{24} \leftrightarrow \mathbf{e}_{13}, \qquad \mathbf{e}_{34} \leftrightarrow \mathbf{e}_{23},$$
$$\mathbf{e}_{1234} \leftrightarrow \mathbf{e}_{123}, \qquad \mathbf{e}_{12} \leftrightarrow \mathbf{e}_{1}, \qquad \mathbf{e}_{13} \leftrightarrow \mathbf{e}_{2}, \qquad \mathbf{e}_{14} \leftrightarrow \mathbf{e}_{3}$$

or, alternatively,

$$Cl^{+}_{1,3} \leftrightarrow Cl_{3,0}, \qquad \mathbf{e}_{23} \leftrightarrow \mathbf{e}_{23}, \qquad \mathbf{e}_{24} \leftrightarrow \mathbf{e}_{13}, \qquad \mathbf{e}_{34} \leftrightarrow \mathbf{e}_{12},$$
$$\mathbf{e}_{1234} \leftrightarrow \mathbf{e}_{123}, \qquad \mathbf{e}_{12} \leftrightarrow \mathbf{e}_{3}, \qquad \mathbf{e}_{13} \leftrightarrow \mathbf{e}_{2}, \qquad \mathbf{e}_{14} \leftrightarrow \mathbf{e}_{1},$$

it is easy to obtain explicit formulas for physically important cases of exponentials of general even MVs that represents spinors in $Cl_{1,3}$ algebra.

In case of $Cl_{3,1}$ the following rules may be used for this purpose:

$$Cl^{+}_{3,1} \leftrightarrow Cl_{3,0}, \qquad \mathbf{e}_{12} \leftrightarrow \mathbf{e}_{12}, \qquad \mathbf{e}_{13} \leftrightarrow \mathbf{e}_{13}, \qquad \mathbf{e}_{23} \leftrightarrow \mathbf{e}_{23},$$
$$\mathbf{e}_{1234} \leftrightarrow \mathbf{e}_{123}, \qquad \mathbf{e}_{14} \leftrightarrow \mathbf{e}_{1}, \qquad \mathbf{e}_{24} \leftrightarrow \mathbf{e}_{2}, \qquad \mathbf{e}_{34} \leftrightarrow \mathbf{e}_{3}$$

or, alternatively,

$$Cl^{+}_{3,1} \leftrightarrow Cl_{3,0}, \qquad \mathbf{e}_{12} \leftrightarrow \mathbf{e}_{23}, \qquad \mathbf{e}_{13} \leftrightarrow \mathbf{e}_{13}, \qquad \mathbf{e}_{23} \leftrightarrow \mathbf{e}_{12},$$
$$\mathbf{e}_{1234} \leftrightarrow \mathbf{e}_{123}, \qquad \mathbf{e}_{14} \leftrightarrow \mathbf{e}_{3}, \qquad \mathbf{e}_{24} \leftrightarrow \mathbf{e}_{2}, \qquad \mathbf{e}_{34} \leftrightarrow \mathbf{e}_{1}.$$

9 Numerical comparison between exact formulas and their series expansion

In this section a comparison between exact MV formulas obtained in Section 7 and finite series expansion is made. The numerical form of MVs is used for this purpose. The knowledge of exact formulas allows to investigate the rate of convergence of finite GA trigonometric and hyperbolic series in $Cl_{3,0}$ algebra. The following MV

$$\mathsf{A}' = \frac{1}{N}\bigl(4 + \mathbf{e}_1 + 3\mathbf{e}_2 - 5\mathbf{e}_3 + 10\mathbf{e}_{12} + 9\mathbf{e}_{13} - 9\mathbf{e}_{23} - 4I\bigr), \quad I = \mathbf{e}_{123}, \tag{15}$$

is used for this purpose where the integer numbers were generated randomly. The normalization factor N helps to make trigonometric series convergent. Up to 8 significant figures are given in numerical evaluation of symbolic (exact) formulas from Section 7. Of course, obtained exact formulas can be used to compute trigonometric functions of any MV even if respective Taylor series does not converge, for example, at large coefficients and $N = 1$. Our primary intention here is however to compare answers provided by exact formula and series expansion.

9.1 GA hyperbolic functions

The trigonometric function series can be made to converge if in (15), we chose large enough N but not too large. We have found that the optimal factor must be larger than the determinant norm of MV in Eq. (11). The norm is defined as the determinant of A raised to fractional power $1/k$, where $k = 2^{\lceil n/2\rceil}$, i.e., $|\mathsf{A}| = (\mathrm{Det}(\mathsf{A}))^{1/k}$. This norm can be interpreted as a number of MVs A in a MV product needed to define $\mathrm{Det}(\mathsf{A})$. In our case, $\mathrm{Det}(\mathsf{A})$ in Eq. (11) consists of geometric product of four MVs, therefore, for 3D algebras ($n = 3$), we have $k = 2^{\lceil 3/2\rceil} = 2^2 = 4$, and the determinant norm is $|\mathsf{A}| = \sqrt[4]{\mathrm{Det}(\mathsf{A})}$. For the chosen MV A', we find $\mathrm{Det}(\mathsf{A}') = 71129$ and $|\mathsf{A}'| = \sqrt[4]{71129} \approx 16.33$. Since the strict analysis of convergence[2] of multivector series is outside the scope of this article, we will divide the chosen MV by the nearest larger integer $17 > 16.33$. Due to multiplicative property of the determinant $\mathrm{Det}(\mathsf{AA}) = \mathrm{Det}(\mathsf{A})\,\mathrm{Det}(\mathsf{A})$, division by any scalar that is larger than the determinant norm factor $1/|\mathsf{A}|$ ensures that determinants of series terms make a decreasing sequence, i.e., $|\,\mathrm{Det}(A)| > |\,\mathrm{Det}(\mathsf{AA})| > \cdots > |\,\mathrm{Det}(\mathsf{AA}\cdots\mathsf{A})|$, and, therefore, we may anticipate that MV series will tend to converge or at least will yield meaningful answer. For GA series, we have profited by the standard exponential,

[2]If, instead, for example, we divide the MV by the largest coefficient in the considered MV, then we immediately would find that $\tanh \mathsf{A}$ series fails to converge.

trigonometric and hyperbolic series [1]. In particular, for $\tanh \mathsf{A}$, we have used $\tanh \mathsf{A} = \mathsf{A} - (1/3)\mathsf{A}^3 + (2/15)\mathsf{A}^5 - (17/315)\mathsf{A}^7 + (62/2835)\mathsf{A}^9 + \cdots$.

To illustrate, let us compute hyperbolic functions $\sinh \mathsf{A}$, $\cosh \mathsf{A}$, $\cosh^{-1} \mathsf{A}$ and[3] $\tanh \mathsf{A}$ of normalized MV argument A'',

$$\mathsf{A}'' = \frac{1}{17}(4 + \mathbf{e}_1 + 3\mathbf{e}_2 - 5\mathbf{e}_3 + 10\mathbf{e}_{12} + 9\mathbf{e}_{13} - 9\mathbf{e}_{23} - 4I). \tag{16}$$

Substituting A'' into exact symbolic formulas Eqs. (8) and (9) (where inverse MV is computed using (10) and (11)) and then evaluating exact expressions numerically up to 8 significant figures (the last digit is exact), we obtain

$$\begin{aligned}
\sinh \mathsf{A}'' &= 0.0806082 - 0.0230640\mathbf{e}_1 + 0.0787983\mathbf{e}_2 - 0.1724390\mathbf{e}_3 \\
&\quad + 0.5504206\mathbf{e}_{12} + 0.4830460\mathbf{e}_{13} - 0.4666026\mathbf{e}_{23} - 0.2082492I, \\
\cosh \mathsf{A}'' &= 0.6039792 - 0.1111834\mathbf{e}_1 - 0.0900922\mathbf{e}_2 + 0.0825265\mathbf{e}_3 \\
&\quad + 0.1730832\mathbf{e}_{12} + 0.1354867\mathbf{e}_{13} - 0.1084358\mathbf{e}_{23} - 0.2939648I, \\
\tanh \mathsf{A}'' &= 0.6231177 + 0.3099294\mathbf{e}_1 + 0.4271905\mathbf{e}_2 - 0.5723737\mathbf{e}_3 \\
&\quad + 0.4466951\mathbf{e}_{12} + 0.4439088\mathbf{e}_{13} - 0.4997530\mathbf{e}_{23} + 0.0547345I.
\end{aligned}$$

For comparison, we provide answers obtained by finite series expansions

$$\begin{aligned}
\sinh_6 \mathsf{A}'' &= 0.0806569 - 0.0229633\mathbf{e}_1 + 0.0788338\mathbf{e}_2 - 0.1724240\mathbf{e}_3 \\
&\quad + 0.5500202\mathbf{e}_{12} + 0.4827078\mathbf{e}_{13} - 0.4662941\mathbf{e}_{23} - 0.2076350I, \\
\cosh_6 \mathsf{A}'' &= 0.6040721 - 0.1111303\mathbf{e}_1 - 0.0900394\mathbf{e}_2 + 0.0824681\mathbf{e}_3 \\
&\quad + 0.1730517\mathbf{e}_{12} + 0.1354672\mathbf{e}_{13} - 0.1084281\mathbf{e}_{23} - 0.2939312I, \\
\tanh_6 \mathsf{A}'' &= 0.7629316 + 0.3616722\mathbf{e}_1 + 0.5029447\mathbf{e}_2 - 0.6765545\mathbf{e}_3 \\
&\quad + 0.5446755\mathbf{e}_{12} + 0.5387139\mathbf{e}_{13} - 0.6033886\mathbf{e}_{23} - 0.1176009I, \\
\tanh_{40} \mathsf{A}'' &= 0.6231595 + 0.3099902\mathbf{e}_1 + 0.4272145\mathbf{e}_2 - 0.5723697\mathbf{e}_3 \\
&\quad + 0.4464672\mathbf{e}_{12} + 0.4437168\mathbf{e}_{13} - 0.4995786\mathbf{e}_{23} + 0.0550762I.
\end{aligned}$$

The subscripts at hyperbolic functions indicate the number of terms that has been included in the summation of finite series to get the result. It can be seen that $\tanh \mathsf{A}$ converges much slower than $\cosh \mathsf{A}$ and $\sinh \mathsf{A}$. The latters are directly related to exponential. For $\tanh \mathsf{A}$, we have had to include 50 terms to get six exact figures. If instead in (15) we would take different factor $N \leqslant \sqrt[4]{71129}$ and then try to compute $\tanh \mathsf{A}$ by standard (textbook) series expansion, then we would immediately find that the series fails to converge, whereas exact formula that follows from exponential yields meaningful answer. One can also easily check that all MV functions of the same argument commute pairwise up to assumed precision.

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[3] The function $\cosh^{-1}(\mathsf{A})$ approximately can be calculated from series $\cosh^{-1}(\mathsf{A}) = \sum_{n=0}^{\infty} \mathrm{E}_n \mathsf{A}^n/n!$, where E_n are the Euler coefficients, and the condition $\cosh^{-1}(\mathsf{A})\cosh(\mathsf{A}) = 1$. In fact the latter condition gives the Euler numbers. In case of inverse trigonometric function, we have $\cos^{-1}(\mathsf{A}) = \sum_{n=0}^{\infty} \mathrm{E}_n \mathsf{A}^{2n}/(2n!)$ and $\cos^{-1}(\mathsf{A})\cos(\mathsf{A}) = 1$. Similar relations exist for hyperbolic and trigonometric tangent functions but now, instead of Euler numbers, there appear Bernoulli numbers B_n [1].

9.2 GA trigonometric functions

Here we restrict ourselves to $Cl_{3,0}$ algebra for which $I^2 = -1$. The exact formulas in the exponential form for $\sin \mathsf{A}$ and $\cos \mathsf{A}$ in Eqs. (7) have been used. The numerical MV given by Eq. (16) was inserted to find the following exact GA functions presented below with 8 significant figures,

$$\begin{aligned}
\sin \mathsf{A}'' &= 0.4142215 + 0.1561775\mathbf{e}_1 + 0.2887099\mathbf{e}_2 - 0.4312306\mathbf{e}_3 \\
&\quad + 0.6127064\mathbf{e}_{12} + 0.5664210\mathbf{e}_{13} - 0.5864014\mathbf{e}_{23} - 0.2430952I, \\
\cos \mathsf{A}'' &= 1.3837580 + 0.1075001\mathbf{e}_1 + 0.0726490\mathbf{e}_2 - 0.0516785\mathbf{e}_3 \\
&\quad - 0.2436586\mathbf{e}_{12} - 0.1984718\mathbf{e}_{13} + 0.1707105\mathbf{e}_{23} + 0.4152926I, \\
\tan \mathsf{A}'' &= 0.0520468 - 0.0321336\mathbf{e}_1 + 0.0568865\mathbf{e}_2 - 0.1373908\mathbf{e}_3 \\
&\quad + 0.4876809\mathbf{e}_{12} + 0.4261388\mathbf{e}_{13} - 0.4091069\mathbf{e}_{23} - 0.1473168I.
\end{aligned}$$

On the other hand, using series expansion of $\sin \mathsf{A}$, $\cos \mathsf{A}$ and $\tan \mathsf{A}$, we find

$$\begin{aligned}
\sin_6 \mathsf{A}'' &= 0.4141938 + 0.1560854\mathbf{e}_1 + 0.2886852\mathbf{e}_2 - 0.4312611\mathbf{e}_3 \\
&\quad + 0.6131181\mathbf{e}_{12} + 0.5667705\mathbf{e}_{13} - 0.5867229\mathbf{e}_{23} - 0.2437297I, \\
\cos_6 \mathsf{A}'' &= 1.3838520 + 0.1075543\mathbf{e}_1 + 0.0727025\mathbf{e}_2 - 0.0517373\mathbf{e}_3 \\
&\quad - 0.2436926\mathbf{e}_{12} - 0.1984933\mathbf{e}_{13} + 0.1707199\mathbf{e}_{23} + 0.4153298I, \\
\tan_6 \mathsf{A}'' &= 0.0958579 + 0.0035747\mathbf{e}_1 + 0.0832419\mathbf{e}_2 - 0.1588803\mathbf{e}_3 \\
&\quad + 0.4184797\mathbf{e}_{12} + 0.3705885\mathbf{e}_{13} - 0.3625310\mathbf{e}_{23} - 0.0454115I, \\
\tan_{40} \mathsf{A}'' &= 0.0522097 - 0.0320415\mathbf{e}_1 + 0.0569781\mathbf{e}_2 - 0.1374922\mathbf{e}_3 \\
&\quad + 0.4876273\mathbf{e}_{12} + 0.4261060\mathbf{e}_{13} - 0.4090946\mathbf{e}_{23} - 0.1472580I.
\end{aligned}$$

The subscripts at trigonometric functions show the number of terms that has been used in series expansion to get the result.

10 Discussion and conclusions

Since the obtained exponentials are expressed in coordinates, the final formulas appear rather complicated. In geometric Clifford algebra the formulas in coordinate-free form may be desirable. The main problem is with vectors and bivectors the components of which, as seen from Eqs. (1), (4) and (6), are entangled mutually. To avoid the entanglement, a better strategy[4] would be to avoid MV expansion in components at all as done in [6].

Let us take $Cl_{3,0}$ and introduce the following complex quantity [9]

$$\begin{aligned}
H &= \bigl((a_1 - \mathrm{i}a_{23})^2 + (a_2 - \mathrm{i}a_{31})^2 + (a_3 - \mathrm{i}a_{12})^2\bigr)^{1/2}, \\
H^* &= \bigl((a_1 + \mathrm{i}a_{23})^2 + (a_2 + \mathrm{i}a_{31})^2 + (a_3 + \mathrm{i}a_{12})^2\bigr)^{1/2}
\end{aligned}$$

[4] It should be noted that at present the existing symbolic packages can do calculations in a concrete orthogonal frame (basis) rather than with simple blades directly.

so that HH^* is a real number. Introduction of the imaginary unit makes the formulas more compact and permits trigonometric-hyperbolic expansion of e^{A}. In the expanded form the vector and bivector coefficients in HH^* represent the sum of 81 terms that consist of various products of a_i and a_{ij}. However, the function H can be written very compactly if coordinate-free form is used [9],

$$H = \mathbf{a}^2 + \mathcal{A}^2 + 2\mathrm{i}I\,\mathbf{a} \wedge \mathcal{A}.$$

The same motive is seen in the coefficients a_+ and a_- that appear in Theorem 1. Furthermore, the coefficients may be given a similar shape:

$$a_+ = \sqrt{\mathbf{a}^2 - \mathcal{A}^2 + 2I\mathbf{a} \wedge \mathcal{A}}, \qquad a_- = \sqrt{\mathbf{a}^2 - \mathcal{A}^2 - 2I\mathbf{a} \wedge \mathcal{A}}.$$

So, there appears a chance to construct a MV exponential functions having a compact and coordinate-free forms, which will be more useful and efficient in various practical GA applications.

In conclusion, we have been able to expand the GA exponential function of a general argument into MV in the coordinate form for all four 3D Clifford geometric algebras. The expansion has been applied to get exact expressions for trigonometric and hyperbolic GA functions and to investigate the convergence of respective series. It was found that both trigonometric and hyperbolic GA sine-cosine series convergence is satisfactory if GA series is limited to more than 6 terms. However, the convergence of tangent series is slower, about 40 significant figures are needed to reach similar precision. We think that such an expansion of the exponential will be useful in solving GA differential equations [8,10,24], in signal and image processing, in automatic control and robotics [20].

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