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Steady state non-Newtonian flow with strain rate dependent viscosity in domains with cylindrical outlets to infinity*

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Abstract. The paper deals with a stationary non-Newtonian flow of a viscous fluid in unbounded domains with cylindrical outlets to infinity. The viscosity is assumed to be smoothly dependent on the gradient of the velocity. Applying the generalized Banach fixed point theorem, we prove the existence, uniqueness and high order regularity of solutions stabilizing in the outlets to the prescribed quasi-Poiseuille flows. Varying the limit quasi-Poiseuille flows, we prove the stability of the solution.

Keywords: non-Newtonian flow, strain rate dependent viscosity, quasi-Poiseuille flows, domains with outlets to infinity.

1 Introduction

Asymptotic behaviour of solutions of elliptic and parabolic equations in domains with noncompact boundaries was considered in [12], where the first theorems on stabilization of solutions were proved. They were called Phrägmen–Lindelöf theorems. The stationary elasticity equations in unbounded domains are studied in [15], and the stabilization theorems were associated there with the Saint-Venant principle. For the stationary and nonstationary Stokes and Navier–Stokes equations with no-slip condition at the boundary of

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the outlets, these questions were studied in [1, 9–11, 21–24, 28], and for the viscoelastic flows, in [25]. For the non-Newtonian flows with viscosity depending on the gradient of the velocity, the existence, uniqueness and asymptotic behaviour in the outlets were studied in [13, 20]. Note that this non-Newtonian rheology governs the blood circulation in vessels (see [2, pp. 84–89, 196–200]).

A part of theoretical interest for partial differential equations, this set of questions is important for construction of asymptotic expansions of solutions in thin domains. Namely, matching of the asymptotic solutions via the boundary layer method leads exactly to the scaled partial differential equations in unbounded domains with cylindrical outlets (see, e.g., [16–19] for Newtonian flows and [14] for the power law fluids). In particular, results of the present paper are used for the construction of an asymptotic expansion of a non-Newtonian flow in a network of thin cylinders, modeling blood vessels.

In the present paper the results obtained in [20] will be extended and generalized. First, we reconstruct the pressure, while in [20], only the weak formulation of the problem without pressure was studied. Second, in order to reconstruct the pressure, we need to have more regularity for the solution, so we will prove the third-order regularity of the velocity and second-order regularity of the pressure in weighted spaces with exponential decay at infinity. Of course, we need more regularity (C^3) for the viscosity ν, depending on the shear rate y. However, we will rid of a restrictive condition of boundedness of $\nabla(\nu(y)y)$, which was assumed in [20]. Finally, we will focus on the questions of stability of solutions with respect to the quasi-Poiseuille flows to which they stabilize in the outlets. These new theorems are important for the construction of boundary layers of non-Newtonian flows.

The paper has the following structure. In Section 2, we give the definition of the domain with outlets. In Section 3, we cite and prove some auxiliary results: embedding inequalities in domains with cylindrical outlets and a lemma on the stabilization to a constant for functions with exponentially decaying gradient. In the same Section 3, we recall some results for the stationary Stokes equation and prove the weak Banach contraction principle. This theorem generalizes the classical Banach fixed point theorem. This result is well known in the mathematical community and is widely used. However, we could not find the proof in literature. Therefore, for the reader's convenience, we present a proof. This generalization is used in the proofs of the regularity of the solutions. The main problem for the stationary non-Newtonian flow in unbounded domains with outlets is formulated in Section 4. In Section 5 the quasi-Poiseuille flow for the stationary non-Newtonian equations in an infinite tube is studied. A Poiseuille flow is an exact solution to the equations of the fluid motion (Stokes, Navier–Stokes) in an infinite cylinder with the no-slip condition at the boundary, with a linear pressure with respect to the longitudinal variable, and with the velocity vector having only longitudinal component (called normal velocity) different from zero; this normal velocity depends only on the transversal variables. A quasi-Poiseille (or Hägen–Poiseuille) flow is an exact solution having the same structure and corresponding to some non-Newtonian rheology. Such flows for various rheologies were studied in [2, 6, 7, 26, 27]. Contrary to [20], where also the quasi-Poiseuille flow was studied, we focus on the regularity issues. Finally, Section 6 contains the main results of the paper: existence and uniqueness of a regular classical solution (velocity and pressure) and continuity of the solution with respect to the data of problem (stability). The proof of continuity of the solution in the norm $W^{2,2}$ for the velocity and L^2 norm for the gradient of the pressure needs the regularity "plus one" of the solution. It explains the difference of norms in Theorems 5 and 6.

2 Definitions of domains

Consider the domain $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, with J cylindrical outlets to infinity: $\Omega = \Omega_0 \cup (\bigcup_{j=1}^J \Omega_j)$, where Ω_0 is a bounded domain, $\Omega_0 \cap \Omega_j = \emptyset$ for $j \in \{1, \dots, J\}$, $\Omega_j \cap \Omega_l = \emptyset$ for $j \neq l$, $j, l \in \{1, \dots, J\}$, and the outlets to infinity Ω_j in some coordinate systems $x^{(j)} = (x_1^{(j)}, x_2^{(j)}, \dots, x_n^{(j)}) = (x_1^{(j)}, x^{(j)\prime})$, having the origins within the boundary of domain Ω_0, are given by the relations

$$\Omega_j = \{x^{(j)} \in \mathbb{R}^n\colon x^{(j)\prime} \in \sigma_j,\ x_1^{(j)} \geqslant 0\},$$

where σ_j are some bounded domains in $\mathbb{R}^{n-1}$, cross-sections of the cylinders (see Fig. 1). Assume that for any $k \in \{1, \dots, J\}$, there exists a $\delta_j > 0$ such that the cylinder $\{x^{(j)} \in \mathbb{R}^n\colon x^{(j)\prime} \in \sigma_j,\ -\delta_j < x_1^{(j)} < 0\} \subset \Omega_0$. Denote d_σ the maximal diameter of the cross-sections σ_j. We assume that the boundary $\partial\Omega$ is C^3-regular and that $\partial\Omega \cap \partial\Omega_0 \neq \emptyset$ has a positive measure. Evidently, there exists a positive real number $R > d_\sigma$ such that the ball $B_R = \{x \in \mathbb{R}^n\colon |x| < R\}$ contains Ω_0.

We introduce the following notation:

$$\Omega_{jk} = \{x \in \Omega_j\colon x_1^{(j)} < k\}, \qquad \omega_{jk} = \Omega_{jk+1} \setminus \Omega_{jk},$$
$$\widehat{\omega}_{jk} = \omega_{jk-1} \cup \omega_{jk} \cup \omega_{jk+1}, \qquad \Omega^{(k)} = \Omega_0 \cup \left(\bigcup_{j=0}^{J} \Omega_{jk}\right),$$

where $j = 1, \dots, J$, and $k \geqslant 0$ is an integer.

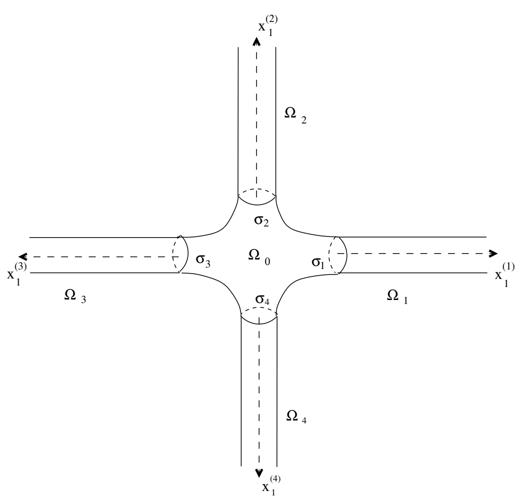

Figure 1. Domain Ω.

3 Auxiliary results

3.1 Embedding inequalities in domains with cylindrical outlets

Let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, be domain with J outlets to infinity. We define in Ω weighted function spaces. Denote $\boldsymbol{\beta} = (\beta, \dots, \beta)$, define a smooth function $\phi_{\boldsymbol{\beta}}(x)$,

$$\phi_{\boldsymbol{\beta}}(x) = \begin{cases} 0, & x \in \Omega_0, \\ \beta x_1^{(j)}, & x \in \Omega_j,\ x_1^{(j)} > 2,\ j = 1, \dots, J, \end{cases}$$

and set $E_{\boldsymbol{\beta}}(x) = \exp\{2\phi_{\boldsymbol{\beta}}(x)\}$.

Denote by $\mathcal{W}^{l,2}_{\boldsymbol{\beta}}(\Omega)$, $l \geqslant 0$, the space of functions obtained as the closure of $C_0^\infty(\Omega)$ in the norm

$$\|u\|_{\mathcal{W}^{l,2}_{\boldsymbol{\beta}}(\Omega)} = \left(\sum_{|\alpha|=0}^{l} \int_\Omega E_{\boldsymbol{\beta}}(x) \left| D^\alpha u(x) \right|^2 \mathrm{d}x \right)^{1/2}$$

and set $\mathcal{W}^{0,2}_{\boldsymbol{\beta}}(\Omega) = \mathcal{L}^2_{\boldsymbol{\beta}}(\Omega)$. Notice that for $\beta > 0$, the elements of the space $\mathcal{W}^{l,2}_{\boldsymbol{\beta}}(\Omega)$ exponentially vanish as $|x| \to +\infty$.

Lemma 1 [Poincaré's inequality]. *There exists a constant $C > 0$ independent of $\boldsymbol{\beta}$ such that for any function $u \in \mathcal{W}^{1,2}_{\boldsymbol{\beta}}(\Omega)$, the following inequality holds*:

$$\|u\|^2_{\mathcal{L}^2_{\boldsymbol{\beta}}(\Omega)} \leqslant C \|\nabla u\|^2_{\mathcal{L}^2_{\boldsymbol{\beta}}(\Omega)}.$$

Proof. For the proof, see, e.g., [20]. □

Lemma 2.

(i) *For any function $u \in \mathcal{W}^{2,2}_{\boldsymbol{\beta}}(\Omega)$, the inequality holds*:

$$\left\| E^{1/2}_{\boldsymbol{\beta}} u \right\|^4_{L^4(\Omega)} \leqslant C \|u\|^4_{\mathcal{W}^{1,2}_{\boldsymbol{\beta}}(\Omega)}. \tag{1}$$

(ii) *For any function $u \in \mathcal{W}^{2,2}_{\boldsymbol{\beta}}(\Omega)$, the inequality holds*:

$$\left\| E^{1/2}_{\boldsymbol{\beta}} u \right\|^2_{L^\infty(\Omega)} \leqslant C \|u\|^2_{\mathcal{W}^{2,2}_{\boldsymbol{\beta}}(\Omega)}. \tag{2}$$

Proof. (i) Let us represent the domain Ω as a union of bounded domains:

$$\Omega = \Omega_0 \cup \left(\bigcup_{j=1}^{J} \bigcup_{k=1}^{\infty} \omega_{jk} \right).$$

In every ω_{jk}, we have the inequalities

$$\|u\|^4_{L^4(\omega_{jk})} \leqslant c \|u\|^4_{W^{1,2}(\omega_{jk})} \tag{3}$$

with the constant c independent of k. Multiplying inequalities (3) by $\mathrm{e}^{4\beta k}$ and having in mind that $\mathrm{e}^{\beta k} \sim \mathrm{e}^{\beta x_1^{(j)}}$ in ω_{jk}, we obtain

$$\big\|\mathrm{e}^{\beta x_1^{(j)}} u\big\|^4_{L^4(\omega_{jk})} \leqslant c\big\|\mathrm{e}^{\beta x_1^{(j)}} u\big\|^4_{W^{1,2}(\omega_{jk})} \leqslant c\|u\|^2_{\mathcal{W}^{1,2}_{\boldsymbol{\beta}}(\Omega)}\big\|\mathrm{e}^{\beta x_1^{(j)}} u\big\|^2_{W^{1,2}(\omega_{jk})}.$$

Here the constants depend on β only. Summing these inequalities over k and j and adding the inequality $\|u\|^4_{L^4(\Omega_0)} \leqslant c\|u\|^4_{W^{1,2}(\Omega_0)}$, we obtain (1).

(ii) By same token, using the inequalities

$$\|u\|^2_{L^\infty(\omega_{jk})} \leqslant c\|u\|^2_{W^{2,2}(\omega_{jk})}, \qquad \|u\|^2_{L^\infty(\Omega_0)} \leqslant c\|u\|^2_{W^{2,2}(\Omega_0)},$$

we get (2). □

Lemma 3. *Let us define the half-cylinder $\Pi^+ = \{x = (z, x') \in \mathbb{R}^n\colon x' \in \sigma, z \in (0, +\infty)\}$, where σ is a bounded domain in $\mathbb{R}^{n-1}$ with Lipschitz boundary. Suppose that $p \in W^{1,2}_{\mathrm{loc}}(\Pi^+)$ and*

$$\int\limits_{\Pi^+} \big|\nabla p(x)\big|^2 \mathrm{e}^{2\beta z}\,\mathrm{d}x'\,\mathrm{d}z < +\infty, \quad \beta > 0.$$

Then there exists a constant p_0 such that the following estimate holds:

$$\int\limits_{\Pi^+} \big|p(z, x') - p_0\big|^2 \mathrm{e}^{2\beta z}\,\mathrm{d}x'\,\mathrm{d}z \leqslant \frac{c}{\beta^2}\int\limits_{\Pi^+} \big|\nabla p(x)\big|^2 \mathrm{e}^{2\beta z}\,\mathrm{d}x'\,\mathrm{d}z.$$

Proof. First, we prove that the mean value $\bar{p}(z) = \int_\sigma p(z, x')\,\mathrm{d}x'$ is a bounded function. Let $z > 0$. Since $\bar{p}(z) = \bar{p}(0) + \int_0^z \bar{p}'(r)\,\mathrm{d}r$ and since

$$\Bigg(\int\limits_0^z \bar{p}'(r)\,\mathrm{d}r\Bigg)^2 \leqslant \int\limits_0^z \big|\bar{p}'(r)\big|^2 \mathrm{e}^{2\beta r}\,\mathrm{d}r \int\limits_0^z \mathrm{e}^{-2\beta r}\,\mathrm{d}r,$$

we have

$$\begin{aligned}\big|\bar{p}(z)\big| &\leqslant \big|\bar{p}(0)\big| + c\Bigg(\int\limits_0^{+\infty} \big|\bar{p}'(z)\big|^2 \mathrm{e}^{2\beta z}\,\mathrm{d}z\Bigg)^{1/2} \\ &\leqslant \big|\bar{p}(0)\big| + c\Bigg(\int\limits_0^{+\infty}\int\limits_\sigma \big|\partial_z p(z, x')\big|^2 \mathrm{e}^{2\beta z}\,\mathrm{d}x'\,\mathrm{d}z\Bigg)^{1/2} \leqslant \mathrm{const}.\end{aligned}$$

It is easy to prove that there exists a constant p_0 such that $\lim_{z\to+\infty} \bar{p}(z) = p_0$. Indeed, since $\bar{p}(z)$ is bounded, there is a sequence $\{z_k\}$ such that $\lim_{k\to+\infty} \bar{p}(z_k) = p_0$ for some constant $p_0 < +\infty$. Consider

$$\big|\bar{p}(z) - \bar{p}(z_k)\big| = \Bigg|\int\limits_z^{z_k} \bar{p}'(r)\,\mathrm{d}r\Bigg| \leqslant c\Bigg(\int\limits_z^{+\infty}\int\limits_\sigma \big|\partial_z p(z, x')\big|^2 \mathrm{e}^{2\beta z}\,\mathrm{d}x'\,\mathrm{d}z\Bigg)^{1/2}.$$

Passing to the limit as $k \to +\infty$, we get

$$\left|\bar{p}(z) - p_0\right| \leqslant c\left(\int_z^{+\infty}\int_\sigma \left|\partial_z p(z,x')\right|^2 \mathrm{e}^{2\beta z}\,\mathrm{d}x'\,\mathrm{d}z\right)^{1/2} \to 0 \quad \text{as } z \to +\infty.$$

Now, by Poincaré's inequality,

$$\begin{aligned}
&\left|\frac{\mathrm{d}}{\mathrm{d}z}\int_\sigma \left|p(z,x') - \bar{p}(z)\right|^2 \mathrm{d}x'\right| \\
&\quad\leqslant 2\int_\sigma \left|p(z,x') - \bar{p}(z)\right|\left|\partial_z p(z,x')\right| \mathrm{d}x' + 2\int_\sigma \left|p(z,x') - \bar{p}(z)\right|\left|\bar{p}'(z)\right| \mathrm{d}x' \\
&\quad\leqslant 2\int_\sigma \left|p(z,x') - \bar{p}(z)\right|^2 \mathrm{d}x' + c\int_\sigma \left|\partial_z p(z,x')\right|^2 \mathrm{d}x' \\
&\quad\leqslant c\int_\sigma \left|\nabla p(z,x')\right|^2 \mathrm{d}x'.
\end{aligned}$$

Let $z < \xi < r < 2z$. Integrating the last inequality from ξ to r yields

$$\begin{aligned}
&\int_\sigma \left|p(\xi,x') - \bar{p}(\xi)\right|^2 \mathrm{d}x' \\
&\quad\leqslant \int_\sigma \left|p(r,x') - \bar{p}(r)\right|^2 \mathrm{d}x' + c\int_\xi^r\int_\sigma \left|\nabla p(z,x')\right|^2 \mathrm{d}x'\,\mathrm{d}z \\
&\quad\leqslant c\int_\sigma \left|\nabla_{x'} p(r,x')\right|^2 \mathrm{d}x' + c\int_\xi^r\int_\sigma \left|\nabla p(z,x')\right|^2 \mathrm{d}x'\,\mathrm{d}z.
\end{aligned}$$

Multiplying both sides of the last inequality by $\mathrm{e}^{2\beta r}$ and integrating with r from z to $2z$ yields

$$\begin{aligned}
&\frac{1}{2\beta}\mathrm{e}^{2\beta z}\left(\mathrm{e}^{2\beta z} - 1\right)\int_\sigma \left|p(\xi,x') - \bar{p}(\xi)\right|^2 \mathrm{d}x' \\
&\quad\leqslant c\int_z^{2z}\int_\sigma \left|\nabla_{x'} p(r,x')\right|^2 \mathrm{e}^{2\beta r}\,\mathrm{d}x'\,\mathrm{d}r + c\int_z^{2z} \mathrm{e}^{2\beta r}\int_\xi^r\int_\sigma \left|\nabla p(y,x')\right|^2 \mathrm{d}x'\,\mathrm{d}y\,\mathrm{d}r \\
&\quad\leqslant c\int_z^{2z}\int_\sigma \left|\nabla_{x'} p(r,x')\right|^2 \mathrm{e}^{2\beta r}\,\mathrm{d}x'\,\mathrm{d}r \\
&\qquad + \frac{c}{2\beta}\mathrm{e}^{2\beta z}\left(\mathrm{e}^{2\beta z} - 1\right)\int_z^{2z}\int_\sigma \left|\nabla_{x'} p(y,x')\right|^2 \mathrm{e}^{2\beta y}\,\mathrm{d}x'\,\mathrm{d}y.
\end{aligned}$$

From this it follows that

$$\int_\sigma \left|p(\xi,x') - \bar{p}(\xi)\right|^2 \mathrm{d}x' \leqslant c\frac{\mathrm{e}^{-2\beta z}}{\mathrm{e}^{2\beta z}-1}\int_z^{2z}\int_\sigma \left|\nabla_{x'}p(r,x')\right|^2 \mathrm{e}^{2\beta r}\,\mathrm{d}x'\,\mathrm{d}r + c\int_z^{2z}\int_\sigma \left|\nabla_{x'}p(y,x')\right|^2 \mathrm{e}^{2\beta y}\,\mathrm{d}x'\,\mathrm{d}y \to 0 \quad \text{as } \xi \to +\infty.$$

Here we used that $\xi \in [z,2z]$, and so, $\xi \to +\infty$ implies $z \to +\infty$. By the triangle inequality we get $\lim_{z\to+\infty}\int_\sigma |p(z,x')-p_0|^2\,\mathrm{d}x' = 0$. In order to finish the proof of the lemma, we need an auxiliary inequality

$$\int_0^{+\infty} \left|f(t)\right|^2 \mathrm{e}^{2\beta t}\,\mathrm{d}t \leqslant \frac{1}{\beta^2}\int_0^{+\infty} \left|f'(t)\right|^2 \mathrm{e}^{2\beta t}\,\mathrm{d}t \tag{4}$$

(for the proof see [18, Cor. 7.1]), which holds for any function f such that $\int_0^{+\infty}|f'(t)|^2 \times \mathrm{e}^{2\beta t}\,\mathrm{d}t < +\infty$, $\lim_{t\to+\infty} f(t) = 0$. Applying (4) to $f(z) = (\int_\sigma |p(z,x')-p_0|^2\,\mathrm{d}x')^{1/2}$, we obtain

$$\begin{aligned}
&\int_0^{+\infty} \mathrm{e}^{2\beta z}\int_\sigma \left|p(z,x')-p_0\right|^2 \mathrm{d}x'\,\mathrm{d}z \\
&\quad\leqslant \frac{1}{\beta^2}\int_0^{+\infty} \mathrm{e}^{2\beta z}\left|\partial_z\left(\int_\sigma \left|p(z,x')-p_0\right|^2 \mathrm{d}x'\right)^{1/2}\right|^2 \mathrm{d}z \\
&\quad\leqslant \frac{1}{\beta^2}\int_0^{+\infty} \mathrm{e}^{2\beta z}\left|\frac{1}{(\int_\sigma |p(z,x')-p_0|^2\,\mathrm{d}x')^{1/2}}\int_\sigma \left|p(z,x')-p_0\right|\left|\partial_z p(z,x')\right|\mathrm{d}x'\right|^2 \mathrm{d}z \\
&\quad\leqslant \frac{1}{\beta^2}\int_0^{+\infty} \mathrm{e}^{2\beta z}\int_\sigma \left|\partial_z p(z,x')\right|^2 \mathrm{d}x'\,\mathrm{d}z.
\end{aligned}$$

□

Remark 1. From the last lemma it follows that if $\int_\Omega E_{\boldsymbol\beta}(x)|\nabla p(x)|^2\,\mathrm{d}x < +\infty$, then there exist constants p_j, $j = 1,2,\dots,J$, such that

$$\int_{\Omega_0} \left|p(x)\right|^2 \mathrm{d}x + \sum_{j=1}^J \int_{\Omega_j} \exp\bigl(2\beta x_1^{(j)}\bigr)\left|p(x)-p_j\right|^2 \mathrm{d}x \leqslant c\int_\Omega E_{\boldsymbol\beta}(x)\left|\nabla p(x)\right|^2 \mathrm{d}x.$$

3.2 Stokes problem

Consider now the Stokes problem in the domain Ω with J outlets to infinity:

$$\begin{aligned}
&-\nu\Delta\mathbf{v} + \nabla p = \mathbf{f}, \quad x \in \Omega, \\
&\operatorname{div}\mathbf{v} = \mathbf{0}, \quad \mathbf{x} \in \Omega, \qquad \mathbf{v}|_{\partial\Omega} = \mathbf{0}.
\end{aligned} \tag{5}$$

Let $H(\Omega)$ be the space of divergence-free functions of $\mathring{W}^{1,2}(\Omega)$. It is well known (see [3, 4, 8]) that there exists a unique weak solution $\mathbf{v} \in H(\Omega)$ to (5), which satisfies the integral identity $\nu \int_\Omega \nabla\mathbf{v} \cdot \nabla\boldsymbol{\eta}\,\mathrm{d}x = \int_\Omega \mathbf{f} \cdot \boldsymbol{\eta}\,\mathrm{d}x$ for all $\boldsymbol{\eta} \in H(\Omega)$ and the estimate $\|\nabla\mathbf{v}\|^2_{L^2(\Omega)} \leqslant c\|\mathbf{f}\|^2_{L^2(\Omega)}$.

The following Agmon–Duglis–Nirenberg (ADN)-type theorem is proved in [24] (see Theorem III.3.2).

Theorem 1. *Let l be an integer, $l \geqslant 0$. Let $\partial\Omega \in C^{l+2}$. There exists a positive β_* such that for all $\beta \in (0, \beta_*]$ and $\mathbf{f} \in \mathcal{W}^{l,2}_{\boldsymbol{\beta}}(\Omega)$, the weak solution $\mathbf{v}$ belongs to the space $\mathcal{W}^{l+2,\,2}_{\boldsymbol{\beta}}(\Omega)$, and there exists a pressure function p with $\nabla p \in \mathcal{W}^{l,2}_{\boldsymbol{\beta}}(\Omega)$ such that the pair $(\mathbf{v}(x), p(x))$ satisfies equations* (5) *almost everywhere in Ω. The following estimate holds*:

$$\|\mathbf{v}\|_{\mathcal{W}^{\mathbf{l+2,2}}_{\boldsymbol{\beta}}(\Omega)} + \|\nabla\mathbf{p}\|_{\mathcal{W}^{\mathbf{l,2}}_{\boldsymbol{\beta}}(\Omega)} \leqslant \mathbf{c}\,\|\mathbf{f}\|_{\mathcal{W}^{\mathbf{l,2}}_{\boldsymbol{\beta}}(\Omega)}.$$

Moreover, the local estimate

$$\|\mathbf{v}\|_{\mathcal{W}^{l+2,2}_{\boldsymbol{\beta}}(\Omega(K))} + \|\nabla p\|_{\mathcal{W}^{l,2}_{\boldsymbol{\beta}}(\Omega(K))} \leqslant c\big(\|\mathbf{f}\|_{\mathcal{W}^{l,2}_{\boldsymbol{\beta}}(\Omega(2K))} + \|\nabla\mathbf{v}\|_{\mathcal{L}^2_{\boldsymbol{\beta}}(\Omega(2K))}\big) \tag{6}$$

holds with the constant c independent of K.

3.3 Weak Banach contraction principle

Theorem 2. *Let X and Y be reflexive Banach spaces, $X \subset Y$, $\|x\|_Y \leqslant \|x\|_X$ for all $x \in X$. Suppose that $M \subset X$ is a closed, bounded set, $M \neq \emptyset$, and the mapping $T : M \mapsto M$ satisfies the inequality*

$$\|Tx - Ty\|_Y \leqslant k\|x - y\|_Y \quad \textit{with } k < 1\ \forall x, y \in M. \tag{7}$$

Then T admits exactly one fixed point $x_* \in M$: $Tx_* = x_*$.

Proof. Let us define a sequence $\{x_n\}$ by the recurrent formulas

$$x_{n+1} = Tx_n, \quad x_0 \in M. \tag{8}$$

Since T maps the bounded set M to itself, there exists a positive constant c_0 such that $\|x_n\|_X \leqslant c_0$ and $\|Tx_n\|_X \leqslant c_0$. Since the space X is reflexive, there exists a subsequence $\{x_{n_k}\}$ such that

$$x_{n_k} \overset{X}{\rightharpoonup} x_*, \quad Tx_{n_k} \overset{X}{\rightharpoonup} y_*, \quad x_*, y_* \in M. \tag{9}$$

For simplicity, we will not distinguish in notation the subsequence $\{x_{n_k}\}$ and the sequence $\{x_n\}$. From (7) it follows that

$$\|Tx_n - Tx_{n+1}\|_Y \leqslant k^n\|x_0 - x_1\|_Y \to 0 \quad \text{as } n \to +\infty.$$

Therefore, $\{Tx_n\}$ is strongly convergent in Y and $Tx_n \overset{Y}{\to} y_*$. From (8) we obtain

$$x_n = Tx_{n-1} \overset{Y}{\to} y_* \overset{(9)}{=} x_*. \tag{10}$$

Thus, $\|Tx_n - Tx_*\|_Y \leqslant k\|x_n - x_*\|_Y \to 0$ as $n \to +\infty$, and hence,

$$Tx_* = y_*. \tag{11}$$

Relations (10) and (11) yield $Tx_* = x_*$. The uniqueness of the fixed point is obvious. □

4 Formulation of the problem

Let $n = 2, 3$, ν_0, λ be positive constants. Let ν be a bounded C^3-smooth function $\mathbb{R}^{n(n+1)/2} \to \mathbb{R}$ such that for all $y \in \mathbb{R}^{n(n+1)/2}$,

$$\bigl|\nu(y)\bigr| \leqslant A, \qquad \bigl|\nabla\nu(y)\bigr| \leqslant A, \qquad \bigl|\nabla^2\bigl(\nu(y)\bigr)\bigr| \leqslant A, \qquad \bigl|\nabla^3\bigl(\nu(y)\bigr)\bigr| \leqslant A, \tag{12}$$

where A is a positive constant independent of y.

Consider the steady state boundary value problem for the non-Newtonian fluid motion equations in the domain Ω

$$\begin{aligned} &-\operatorname{div}\bigl(\bigl(\nu_0 + \lambda\nu\bigl(\dot\gamma(\mathbf{v})\bigr)\bigr)D(\mathbf{v})\bigr) + \nabla p = \mathbf{f}, \quad x \in \Omega, \\ &\operatorname{div}\mathbf{v} = 0, \quad x \in \Omega, \qquad \mathbf{v}|_{\partial\Omega} = 0, \end{aligned} \tag{13}$$

where $D(\mathbf{v})$ is the strain rate matrix with the elements $d_{ij} = (\partial v_i/\partial x_j + \partial v_j/\partial x_i)/2$, $\dot\gamma(\mathbf{v}) = (d_{12}, d_{13}, d_{23}, d_{11}, d_{22}, d_{33})$ if $n = 3$, and $\dot\gamma(\mathbf{v}) = (d_{12}, d_{11}, d_{22})$ if $n = 2$, $\mathbf{f} \in \mathcal{W}^{1,2}_\beta(\Omega)$, $\beta > 0$.

We look for the solution $\mathbf{v}$ having prescribed fluxes F_j over the cross sections σ_j of outlets to infinity:

$$\int_{\sigma_j} \mathbf{v}\cdot\mathbf{n}\,\mathrm{d}S = F_j, \quad j = 1, 2, \dots, J, \tag{14}$$

where

$$\sum_{j=1}^{J} F_j = 0. \tag{15}$$

Here and below an integral over σ_j is understood as an integral over any orthogonal cross-section of Ω_j. Note that this integral for a divergence-free vector function is independent of the position of this cross-section.

Since $\operatorname{div}\mathbf{v} = 0$, equations (13) can be written in the form

$$\begin{aligned} &-\frac{\nu_0}{2}\Delta\mathbf{v} + \nabla p = \lambda\operatorname{div}\bigl(\nu\bigl(\dot\gamma(\mathbf{v})\bigr)D(\mathbf{v})\bigr) + \mathbf{f}, \quad x \in \Omega, \\ &\operatorname{div}\mathbf{v} = 0, \quad x \in \Omega, \qquad \mathbf{v}|_{\partial\Omega} = 0. \end{aligned}$$

5 Non-Newtonian quasi-Poiseuille flow

5.1 Existence of non-Newtonian Poiseuille flow with prescribed pressure slope

The non-Newtonian Poiseuille flow with the strain rate dependent viscosity was studied in the book [2] and recently in [20]. We will need below some extended versions of theorems proved there.

Let us recall the definition of a quasi-Poiseuille flow for equations (13). Let σ be a bounded domain with Lipschitz boundary in $\mathbb{R}^{n-1}$. Consider in the infinite cylinder $\Pi = \mathbb{R} \times \sigma$ the Dirichlet boundary value problem

$$\begin{aligned} &-\operatorname{div}\left(\left(\nu_0 + \lambda\nu\left(\dot{\gamma}(\mathbf{u})\right)\right) D(\mathbf{u})\right) + \nabla p = 0, \quad x \in \Pi, \\ &\operatorname{div} \mathbf{u} = 0, \quad x \in \Pi, \qquad \mathbf{u}|_{\partial\Pi} = 0, \end{aligned}$$

where $\dot{\gamma}(\mathbf{v}) = (d_{12}, d_{13}, d_{23}, 0, 0, 0)$ if $n = 3$, and $\dot{\gamma}(\mathbf{v}) = (d_{12}, 0, 0)$ if $n = 2$ (below we will see that for the quasi-Poiseuille flow, $d_{ii} = 0$).

Define a quasi-Poiseuille flow as a couple $(\mathbf{V}_{P_\alpha}, \mathcal{P}_{P_\alpha})$ such that $\mathbf{V}_{P_\alpha}(x) = (v_{P_\alpha}(x'), 0, \dots, 0)^{\mathrm{T}}$, and $\mathcal{P}_{P_\alpha}(x) = -\alpha x_1 + \beta$, $\alpha, \beta \in \mathbb{R}$, $x' = (x_2, \dots, x_n)$, where v_{P_α} is the solution of the following problem:

$$\begin{aligned} &-\frac{1}{2}\operatorname{div}_{x'}\left(\left(\nu_0 + \lambda\nu\left(\dot{\gamma}_P(v_{P_\alpha})\right)\right)\nabla_{x'} v_{P_\alpha}\right) = \alpha, \quad x' \in \sigma, \\ &v_{P_\alpha}|_{\partial\sigma} = 0. \end{aligned} \tag{16}$$

Here $\dot{\gamma}_P(v_{P_\alpha}) = (\nabla_{x'} v_{P_\alpha}/2, 0, 0)$ if $n = 2$, $\dot{\gamma}_P(v_{P_\alpha}) = (\nabla_{x'} v_{P_\alpha}/2, 0, 0, 0)$ if $n = 3$, and α is the given pressure slope.

Theorem 3. *Let $\partial\sigma \in C^3$. For any $\alpha_0 > 0$, there exists $\lambda_0 = \lambda_0(\alpha_0)$ such that for all $\lambda \in (0, \lambda_0]$ and any $|\alpha| \leqslant \alpha_0$, problem* (16) *admits a unique*[1] *solution $v_{P_\alpha} \in \mathring{W}^{1,2}(\sigma) \cap W^{3,2}(\sigma)$. The solution v_{P_α} satisfies the estimate*

$$\|v_{P_\alpha}\|_{W^{3,2}(\sigma)} \leqslant c|\alpha|, \tag{17}$$

where the constant c depends only on σ.

Proof. Let $\mathcal{L}$ be an operator $\mathring{W}^{1,2}(\sigma) \cap W^{3,2}(\sigma) \to \mathring{W}^{1,2}(\sigma) \cap W^{3,2}(\sigma)$ such that for any $v \in \mathring{W}^{1,2}(\sigma) \cap W^{3,2}(\sigma)$, $V = \mathcal{L}v$ is a solution of the Poisson problem

$$-\frac{\nu_0}{2}\Delta V = h(v) + \alpha, \quad x \in \sigma, \qquad V|_{\partial\sigma} = 0, \tag{18}$$

where

$$\begin{aligned} h(v) &= \frac{1}{2}\lambda \operatorname{div}_{x'}\left(\nu\left(\dot{\gamma}_P(v)\right)\nabla_{x'} v\right) \\ &= \frac{1}{2}\lambda\left[\nu\left(\dot{\gamma}_P(v)\right)\Delta_{x'} v + \left(\nabla_y \nu\left(\dot{\gamma}_P(v)\right)\right)^{\mathrm{T}}\left(\nabla_{x'}\left(\dot{\gamma}_P(v)\right)\right)^{\mathrm{T}} \cdot \nabla_{x'} v\right]. \end{aligned}$$

Using the embedding $W^{3,2}(\sigma) \hookrightarrow W^{1,\infty}(\sigma)$ and conditions (12), we obtain

$$\begin{aligned} \left\|h(v)\right\|^2_{L^2(\sigma)} &\leqslant c\lambda^2\left(\|v\|^2_{W^{2,2}(\sigma)} + \|v\|^2_{W^{3,2}(\sigma)}\|v\|^2_{W^{2,2}(\sigma)}\right) \\ &\leqslant c\lambda^2\left(\|v\|^2_{W^{3,2}(\sigma)} + \|v\|^4_{W^{3,2}(\sigma)}\right). \end{aligned} \tag{19}$$

[1]Here and below the uniqueness takes place only in some ball, where the contraction principle is applied.

Analogously,

$$\left|\nabla_{x'} h(v)\right| \leqslant c\lambda\left(\left|\nabla^3_{x'} v\right| + \left|\nabla^3_{x'} v\right|\left|\nabla_{x'} v\right| + \left|\nabla^2_{x'} v\right|^2 + \left|\nabla^2_{x'} v\right|^2\left|\nabla_{x'} v\right|\right),$$

and using, in addition, the embedding $W^{3,2}(\sigma) \hookrightarrow W^{2,4}(\sigma)$, we derive

$$\left\|\nabla_{x'} h(v)\right\|^2_{L^2(\sigma)} \leqslant c\lambda^2\left(\|v\|^2_{W^{3,2}(\sigma)} + \|v\|^4_{W^{3,2}(\sigma)} + \|v\|^6_{W^{3,2}(\sigma)}\right). \tag{20}$$

Define in $W^{3,2}(\sigma) \cap \mathring{W}^{1,2}(\sigma)$ a closed bounded set $\mathcal{B}_{R_0} = \{u \in W^{3,2}(\sigma) \cap \mathring{W}^{1,2}(\sigma)\colon \|u\|_{W^{3,2}(\sigma)} \leqslant R_0\}$. Assume that $v \in \mathcal{B}_{R_0}$. Then (19) and (20) yield the estimate

$$\|h + \alpha\|^2_{W^{1,2}(\sigma)} \leqslant c\lambda^2\left(R_0^2 + R_0^4 + R_0^6\right) + c|\alpha|^2. \tag{21}$$

Then for the solution of the Poisson equation (18), the following estimate holds:

$$\|V\|^2_{W^{3,2}(\sigma)} \leqslant c_1\lambda^2\left(R_0^2 + R_0^4 + R_0^6\right) + c_2|\alpha|^2. \tag{22}$$

Set $M_0^2 = c_2|\alpha_0|^2$ and $R_0^2 = 2M_0^2$ and suppose that

$$\lambda^2 \leqslant \frac{1}{c_1(2 + 4M_0^2 + 8M_0^4)} = \lambda_*^2.$$

Then from (22) it follows that $\|\mathcal{L}v\|^2_{W^{3,2}(\sigma)} \leqslant R_0^2$. The last inequality implies that the operator $\mathcal{L}$ maps the closed bounded set $\mathcal{B}_{R_0} \subset W^{3,2}(\sigma) \cap \mathring{W}^{1,2}(\sigma)$ onto itself.

Let us show that $\mathcal{L}$ is a contraction in $\mathring{W}^{1,2}(\sigma)$. Multiplying equations (18) by an arbitrary $\eta \in \mathring{W}^{1,2}(\sigma)$ and integrating by parts, we get

$$\frac{\nu_0}{2}\int\limits_\sigma \nabla_{x'}(\mathcal{L}v)\cdot\nabla_{x'}\eta\,\mathrm{d}x' = -\frac{1}{2}\lambda\int\limits_\sigma \nu\left(\dot\gamma_P(v)\right)\nabla_{x'}v\cdot\nabla_{x'}\eta\,\mathrm{d}x' + \alpha\int\limits_\sigma \eta\,\mathrm{d}x'.$$

Thus, for any $v_1, v_2 \in \mathcal{B}_{R_0}$, the following equality holds:

$$\begin{aligned}
&\frac{\nu_0}{2}\int\limits_\sigma \nabla_{x'}(\mathcal{L}v_1 - \mathcal{L}v_2)\cdot\nabla_{x'}\eta\,\mathrm{d}x' \\
&\quad= -\frac{1}{2}\lambda\int\limits_\sigma \nu\left(\dot\gamma_P(v_1)\right)\left(\nabla_{x'}v_1 - \nabla_{x'}v_2\right)\cdot\nabla_{x'}\eta\,\mathrm{d}x' \\
&\qquad - \frac{1}{2}\lambda\int\limits_\sigma \left(\nu\left(\dot\gamma_P(v_1)\right) - \nu\left(\dot\gamma_P(v_2)\right)\right)\nabla_{x'}v_2\cdot\nabla_{x'}\eta\,\mathrm{d}x' \\
&\quad= J_1 + J_2.
\end{aligned} \tag{23}$$

Using Young's inequality, we obtain

$$|J_1| \leqslant \frac{\nu_0}{8}\int\limits_\sigma |\nabla_{x'}\eta|^2\,\mathrm{d}x + C_3\frac{\lambda^2}{\nu_0}\int\limits_\sigma |\nabla_{x'}v_1 - \nabla_{x'}v_2|^2\,\mathrm{d}x'.$$

Using (12),

$$\left|\nu\big(\dot\gamma_P(v_1)\big)-\nu\big(\dot\gamma_P(v_2)\big)\right|^2 \leqslant \sup_y |\nabla_y \nu(y)|^2 |\nabla_{x'}v_1-\nabla_{x'}v_2|^2 \\ \leqslant c_2|\nabla_{x'}v_1-\nabla_{x'}v_2|^2,$$

we have

$$\begin{aligned}|J_2| &\leqslant \frac{\nu_0}{8}\int_\sigma |\nabla_{x'}\eta|^2\,\mathrm{d}x' + \frac{c_2\lambda^2}{\nu_0}\int_\sigma |\nabla_{x'}v_1-\nabla_{x'}v_2|^2|\nabla_{x'}v_2|^2\,\mathrm{d}x' \\ &\leqslant \frac{\nu_0}{8}\int_\sigma |\nabla_{x'}\eta|^2\,\mathrm{d}x' + \frac{c_2\lambda^2}{\nu_0}\sup_{x'\in\sigma}|\nabla_{x'}v_2|^2\int_\sigma |\nabla_{x'}v_1-\nabla_{x'}v_2|^2\,\mathrm{d}x' \\ &\leqslant \frac{\nu_0}{8}\int_\sigma |\nabla_{x'}\eta|^2\,\mathrm{d}x' + \frac{c_3\lambda^2}{\nu_0}\|v_2\|^2_{W^{3,2}(\sigma)}\int_\sigma |\nabla_{x'}v_1-\nabla_{x'}v_2|^2\,\mathrm{d}x' \\ &\leqslant \frac{\nu_0}{8}\int_\sigma |\nabla_{x'}\eta|^2\,\mathrm{d}x' + \frac{c_3\lambda^2}{\nu_0}R_0^2\int_\sigma |\nabla_{x'}v_1-\nabla_{x'}v_2|^2\,\mathrm{d}x'.\end{aligned}$$

Therefore, taking in (23) $\eta=\mathcal{L}v_1-\mathcal{L}v_2$, we derive the inequality

$$\frac{\nu_0}{2}\big\|\nabla_{x'}(\mathcal{L}v_1-\mathcal{L}v_2)\big\|^2_{L^2(\sigma)} \leqslant \frac{\nu_0}{4}\big\|\nabla_{x'}(\mathcal{L}v_1-\mathcal{L}v_2)\big\|^2_{L^2(\sigma)} \\ + \frac{\lambda^2}{\nu_0}\big[c_3R_0^2+C_3\big]\big\|\nabla_{x'}(v_1-v_2)\big\|^2_{L^2(\sigma)},$$

and it follows that

$$\big\|\nabla_{x'}(\mathcal{L}v_1-\mathcal{L}v_2)\big\|^2_{L^2(\sigma)} \leqslant \lambda^2\frac{4(C_3+c_3R_0^2)}{\nu_0^2}\big\|\nabla_{x'}(v_1-v_2)\big\|^2_{L^2(\sigma)}.$$

Let

$$\lambda_0^2=\min\left\{\lambda_*^2,\frac{\nu_0^2}{4(C_3+c_3R_0^2)}\right\}.$$

Then for any $\lambda\in(0,\lambda_0)$, operator $\mathcal{L}$ is a contraction in $\mathring{W}^{1,2}(\sigma)$ with the contraction factor

$$\mu=\lambda^2\frac{4(C_3+c_3R_0^2)}{\nu_0^2}<1,$$

and by Theorem 2 there exists a unique fixed point v_{P_α} of the operator $\mathcal{L}$, which is a solution of problem (16).

From estimates (20), (21) applied to the fixed point v_{P_α} it follows that

$$\|h+\alpha\|^2_{W^{1,2}(\sigma)} \leqslant c\lambda^2\big(1+R_0^2+R_0^4\big)\|v_{P_\alpha}\|^2_{W^{3,2}(\sigma)}+c|\alpha|^2,$$

and thus, by (22)

$$\|v_{P_\alpha}\|^2_{W^{3,2}(\sigma)} \leqslant c_1\lambda^2\big(1+R_0^2+R_0^4\big)\|v_{P_\alpha}\|^2_{W^{3,2}(\sigma)}+c_2|\alpha|^2.$$

If $\lambda<\lambda_0$, the last estimate implies (17). □

5.2 Operator relating the pressure slope and the flux

Define $F(\alpha) = \int_\sigma v_{P_\alpha}(x')\,\mathrm{d}x'$ the flux corresponding to the pressure slope $-\alpha$. Note that in the case of the steady Newtonian flow (the steady form of Navier–Stokes or Stokes equations), $F(\alpha)$ is proportional to α. This case corresponds to the value $\lambda = 0$, and so, $F(\alpha) = \kappa\alpha$, where $\kappa = \int_\sigma \tilde{v}_P(x')\,\mathrm{d}x'$, and $\tilde{v}_P$ is a solution of the Poisson equation

$$-\frac{\nu_0}{2}\Delta_{x'}\tilde{v}_P = 1, \quad x' \in \sigma, \qquad \tilde{v}_P = 0, \quad x' \in \partial\sigma.$$

We consider as well the operator (function) corrector of the non-Newtonian flux with respect to the Newtonian one: $G(\alpha) = F(\alpha) - \kappa\alpha$, and prove that for sufficiently small $\lambda > 0$, $G(\alpha)$ is a contraction.

The next lemma is an extension of Lemma 2.3 and Corollary 2.4 [20].

Lemma 4. *For any $\alpha_0 > 0$, there exists a number $\lambda_1 = \lambda_1(\alpha_0)$ such that for any $\lambda \in (0, \lambda_1]$ and every $|\alpha| \leqslant \alpha_0$, the solution v_{P_α} of problem* (16) *is a Lipschitz-continuous function with respect to α in the norm $\|\nabla_{x'}\cdot\|_{L^2(\sigma)}$. Moreover $F(\alpha)$ is a Lipschitz-continuous function with respect to α.*

Proof. Let $v_{P_{\alpha_1}} \in \mathring{W}^{1,2}(\sigma) \cap W^{3,2}(\sigma)$ and $v_{P_{\alpha_2}} \in \mathring{W}^{1,2}(\sigma) \cap W^{3,2}(\sigma)$ be two solutions of problem (16) corresponding to $\alpha = \alpha_1$ and $\alpha = \alpha_2$, respectively. By Theorem 3 these solutions exist if $\lambda \in (0, \lambda_0(\alpha_0))$. Moreover, the following estimates hold:

$$\|v_{P_{\alpha_i}}\|_{W^{3,2}(\sigma)} \leqslant c|\alpha_0|, \quad i = 1, 2.$$

Using the integral identities

$$\frac{\nu_0}{2}\int_\sigma \nabla_{x'}v_{P_{\alpha_i}} \cdot \nabla_{x'}\eta\,\mathrm{d}x' = -\frac{1}{2}\lambda\int_\sigma \nu\big(\dot{\gamma}_P(v_{P_{\alpha_i}})\big)\nabla_{x'}v_{P_{\alpha_i}} \cdot \nabla_{x'}\eta\,\mathrm{d}x' + \alpha_i\int_\sigma \eta\,\mathrm{d}x', \quad i = 1, 2, \ \forall\eta \in \mathring{W}^{1,2}(\sigma), \tag{24}$$

subtracting (24) with $i = 2$ from (24) with $i = 1$, taking $\eta = v_{P_{\alpha_1}} - v_{P_{\alpha_2}}$ and using arguments similar to those at the end of the proof of Theorem 3, we get

$$\begin{aligned}
&\|\nabla_{x'}v_{P_{\alpha_1}} - \nabla_{x'}v_{P_{\alpha_2}}\|^2_{L^2(\sigma)} \\
&\quad\leqslant \frac{\lambda}{\nu_0}\int_\sigma \big|\nu\big(\dot{\gamma}_P(v_{P_{\alpha_1}})\big) - \nu\big(\dot{\gamma}_P(v_{P_{\alpha_2}})\big)\big||\nabla_{x'}v_{P_{\alpha_1}}||\nabla_{x'}v_{P_{\alpha_1}} - \nabla_{x'}v_{P_{\alpha_2}}|\,\mathrm{d}x' \\
&\qquad+ \frac{\lambda}{\nu_0}\int_\sigma \big|\nu\big(\dot{\gamma}_P(v_{P_{\alpha_2}})\big)\big||\nabla_{x'}v_{P_{\alpha_1}} - \nabla_{x'}v_{P_{\alpha_2}}|^2\,\mathrm{d}x' \\
&\qquad+ |\alpha_1 - \alpha_2|\int_\sigma |v_{P_{\alpha_1}} - v_{P_{\alpha_2}}|\,\mathrm{d}x' \\
&\quad\leqslant c_4\lambda|\alpha_1|\|\nabla_{x'}v_{P_{\alpha_1}} - \nabla_{x'}v_{P_{\alpha_2}}\|^2_{L^2(\sigma)} + |\alpha_1 - \alpha_2|\sqrt{|\sigma|}\|v_{P_{\alpha_1}} - v_{P_{\alpha_2}}\|_{L^2(\sigma)},
\end{aligned}$$

where $|\sigma| = \mathrm{mes}(\sigma)$. If $\lambda < \min\{\lambda_0(\alpha_0),\ 1/(c_4\alpha_0)\} \equiv \lambda_1$, then from the last inequality it follows that

$$\left\|\nabla_{x'}(v_{P_{\alpha_1}} - v_{P_{\alpha_2}})\right\|_{L^2(\sigma)} \leqslant c_5\sqrt{|\sigma|}|\alpha_1 - \alpha_2|. \tag{25}$$

Further,

$$\begin{aligned}\left|F(\alpha_1) - F(\alpha_2)\right| &\leqslant \int\limits_{\sigma} \left|v_{P_{\alpha_1}}(x') - v_{P_{\alpha_2}}(x')\right| \mathrm{d}x' \leqslant \sqrt{|\sigma|}\|v_{P_{\alpha_1}} - v_{P_{\alpha_2}}\|_{L^2(\sigma)} \\ &\leqslant c_6\sqrt{|\sigma|}\|\nabla_{x'}(v_{P_{\alpha_1}} - v_{P_{\alpha_2}})\|_{L^2(\sigma)} \leqslant c_7\sqrt{|\sigma|}|\alpha_1 - \alpha_2|,\end{aligned}$$

and this estimate completes the proof. □

Lemma 5. *For any $\alpha_0 > 0$, there exists a number $\lambda_2 = \lambda_2(\alpha_0) \leqslant \lambda_1(\alpha_0)$ such that for all $\lambda \in (0, \lambda_2]$, the operator $\kappa^{-1}G(\alpha)$ is a contraction on the interval $[-\alpha_0, \alpha_0]$.*

Proof. Denote $\tilde{v}_{\alpha_1}(x') = \alpha_1\tilde{v}_P(x')$, $\tilde{v}_{\alpha_2}(x') = \alpha_2\tilde{v}_P(x')$, where $|\alpha_i| \leqslant \alpha_0$. Then $\tilde{v}_{\alpha_1} - v_{P_{\alpha_1}}$ and $\tilde{v}_{\alpha_2} - v_{P_{\alpha_2}}$ satisfy the following problems for $m = 1$ and $m = 2$:

$$\begin{aligned}&-\frac{\nu_0}{2}\Delta_{x'}(\tilde{v}_{\alpha_m} - v_{P_{\alpha_m}}) = \frac{\lambda}{2}\operatorname{div}_{x'}\left(\nu\left(\dot{\gamma}_P(v_{P_{\alpha_m}})\right)\nabla_{x'}v_{P_{\alpha_m}}\right), \quad x' \in \sigma, \\ &(\tilde{v}_{\alpha_m} - v_{P_{\alpha_m}})|_{\partial\sigma} = 0.\end{aligned}$$

Subtracting one problem from another, we get for $w = (\tilde{v}_{\alpha_1} - v_{P_{\alpha_1}}) - (\tilde{v}_{\alpha_2} - v_{P_{\alpha_2}})$ the following relations:

$$\begin{aligned}-\frac{\nu_0}{2}\Delta_{x'}w = \frac{\lambda}{2}&\operatorname{div}_{x'}\left(\nu\left(\dot{\gamma}_P(v_{P\alpha_1})\right)\nabla_{x'}v_{P\alpha_1}\right. \\ &\left.- \nu\left(\dot{\gamma}_P(v_{P\alpha_2})\right)\nabla_{x'}v_{P\alpha_2}\right), \quad x' \in \sigma, \\ w|_{\partial\sigma} = 0.&\end{aligned}$$

Applying a standard a priori estimate for the solution of the Poisson equation with Dirichlet conditions, we obtain

$$\|\nabla_{x'}w\|_{L^2(\sigma)} \leqslant c\frac{\lambda}{\nu_0}\left\|\nu\left(\dot{\gamma}_P(v_{P\alpha_1})\right)\nabla_{x'}v_{P\alpha_1} - \nu\left(\dot{\gamma}_P(v_{P\alpha_2})\right)\nabla_{x'}v_{P\alpha_2}\right\|_{L^2(\sigma)},$$

and by using similar arguments as before we obtain from inequalities (17), (25) for $\lambda < \lambda_1(\alpha_0)$

$$\|\nabla_{x'}w\|_{L^2(\sigma)} \leqslant c_6\lambda(\alpha_0 + 1)\left\|\nabla_{x'}(v_{P_{\alpha_1}} - v_{P_{\alpha_2}})\right\|_{L^2(\sigma)} \leqslant c_7\lambda|\alpha_1 - \alpha_2|.$$

So, finally,

$$\left|\int\limits_{\sigma} w\,\mathrm{d}x'\right| \leqslant \sqrt{|\sigma|}\|w\|_{L^2(\sigma)} \leqslant c\sqrt{|\sigma|}\|\nabla_{x'}w\|_{L^2(\sigma)} \leqslant c_8\lambda|\alpha_1 - \alpha_2|.$$

Since

$$\int_\sigma w \,\mathrm{d}x' = \bigl(\kappa\alpha_1 - F(\alpha_1)\bigr) - \bigl(\kappa\alpha_2 - F(\alpha_2)\bigr) = G(\alpha_2) - G(\alpha_1),$$

we have

$$\kappa^{-1}\bigl|G(\alpha_2) - G(\alpha_1)\bigr| \leqslant c_8\kappa^{-1}\lambda|\alpha_1 - \alpha_2|.$$

So, if $\lambda < \lambda_2(\alpha_0) = \min\{\lambda_1(\alpha_0), \kappa/c_8\}$, then G/κ is a contraction with the factor $q = c_8\kappa^{-1}\lambda < 1$. □

Remark 2. The same proof shows that if the constant κ^{-1} is replaced by another constant $K^{-1} > 0$, then for any $\alpha_0 > 0$, there exists a number $\lambda_2' = \lambda_2'(\alpha_0) \leqslant \lambda_1(\alpha_0)$ such that for all $\lambda \in (0, \lambda_2']$, the operator $K^{-1}G(\alpha)$ is a contraction on the interval $[-\alpha_0, \alpha_0]$.

Lemma 6. $\mathrm{sgn}(F(\alpha)) = \mathrm{sgn}(\alpha)$.

Proof. Indeed,

$$\begin{aligned}
\alpha F(\alpha) &= \int_\sigma \alpha v_{P_\alpha} \,\mathrm{d}x' \\
&= -\int_\sigma \frac{1}{2}\,\mathrm{div}_{x'}\,\bigl(\bigl(\nu_0 + \lambda\nu\bigl(\dot\gamma_P(v_{P_\alpha})\bigr)\bigr)\nabla_{x'}v_{P_\alpha}\bigr)v_{P_\alpha}\,\mathrm{d}x' \\
&= \int_\sigma \frac{1}{2}\bigl(\nu_0 + \lambda\nu\bigl(\dot\gamma_P(v_{P_\alpha})\bigr)\bigr)\nabla_{x'}v_{P_\alpha}\cdot\nabla_{x'}v_{P_\alpha}\,\mathrm{d}x' \geqslant 0.
\end{aligned}$$ □

Lemma 7. *For any $F_0 > 0$, there exists $\lambda_3 = \lambda_3(F_0)$ such that for all $\lambda \in (0, \lambda_3]$ and every $F \in (-F_0, F_0)$, there is a unique pair (v_{P_α}, α) satisfying* (16) *and such that $F(\alpha) = \int_\sigma v_{P_\alpha}(x')\,\mathrm{d}x' = F$. Moreover, the following estimates hold*:

$$\|v_{P_\alpha}\|_{W^{1,2}(\sigma)} \leqslant C|F|, \qquad |\alpha| \leqslant c|F|. \tag{26}$$

Proof. F is a Lipschitz continuous function. So, for any fixed F_0, we can find a number $\alpha_0 = \alpha_0(F_0)$ such that for all $\lambda \in (0, \min\{\lambda_1(\alpha_0), \lambda_2(\alpha_0)\})$, Lemma 4 holds, and for $\alpha \in [-\alpha_0, \alpha_0]$, we have $|F(\alpha)| \leqslant F_0$. So,

$$\bigl|\kappa^{-1}F(\alpha)\bigr| = \bigl|\kappa^{-1}F(\alpha) - \alpha + \alpha\bigr| = \bigl|\kappa^{-1}G(\alpha) + \alpha\bigr| \geqslant \bigl||\alpha| - \bigl|\kappa^{-1}G(\alpha)\bigr|\bigr|.$$

Since by Lemma 4, $F(\alpha)$ is Lipschitz continuous and, by Lemma 5, $\kappa^{-1}G(\alpha)$ is a contraction, we conclude that there exist constants $0 < a_1 < a_2$ such that

$$a_1|\alpha| < \bigl|\kappa^{-1}F(\alpha)\bigr| < a_2|\alpha|.$$

Therefore, for every $F \in (-F_0, F_0)$, there exists at least one $\alpha \in [-\alpha_0(F_0), \alpha_0(F_0)]$ such that $F(\alpha) = F$.

In order to prove the uniqueness of α, we argue by contradiction: suppose that there are two such number α_1 and α_2, i.e., $F(\alpha_1) = F(\alpha_2) = F$. Then, by definition, $|\kappa^{-1}G(\alpha_1) - \kappa^{-1}G(\alpha_2)| = |\alpha_1 - \alpha_2|$. But this contradicts the fact that $\kappa^{-1}G(\alpha)$ is a contraction, and so $\alpha_1 = \alpha_2$. □

5.3 Continuity of the non-Newtonian Poiseuille flow

Theorem 4.

(i) *For any α_0, there exists $\lambda_4 = \lambda_4(\alpha_0)$ such that for all $\lambda \in (0, \lambda_4]$ and every $\alpha_1, \alpha_2 \in (-\alpha_0, \alpha_0)$, there holds the estimate*

$$\|v_{P_{\alpha_1}} - v_{P_{\alpha_2}}\|_{W^{2,2}(\sigma)} \leqslant c|\alpha_1 - \alpha_2|. \tag{27}$$

(ii) *For any F_0, there exists $\lambda_5 = \lambda_5(F_0)$ such that for all $\lambda \in (0, \lambda_5]$ and every $F_1, F_2 \in (-F_0, F_0)$, there holds the estimate*

$$\|v_{P_{\alpha_1}} - v_{P_{\alpha_2}}\|_{W^{2,2}(\sigma)} \leqslant c|F_1 - F_2|, \tag{28}$$

where $F_i = \int_\sigma v_{P_{\alpha_i}}(x')\,\mathrm{d}x'$, $i = 1, 2$.

Proof. (i) Let $F_0 > 0$, $\alpha_0 > 0$ such that $\alpha_0 \leqslant cF_0$ with c from (26), and let $\lambda_1 = \lambda_1(\alpha_0)$ be the number defined in Theorem 3, $\lambda_3 = \lambda_3(F_0)$ be the number defined in Lemma 7. Then due to these theorem and lemma, for $\lambda \in (0, \min\{\lambda_1, \lambda_3\}]$ and every $\alpha_1, \alpha_2 \in (-\alpha_0, \alpha_0)$, there exist solutions $(v_{P_{\alpha_1}}, \alpha_1)$ and $(v_{P_{\alpha_2}}, \alpha_2)$ of problem (16) such that $v_{P_{\alpha_i}} \in W^{3,2}(\sigma)$, and the following estimates

$$\|v_{P_{\alpha_i}}\|_{W^{3,2}(\sigma)} \leqslant C|\alpha_i|, \quad i = 1, 2,$$

hold. Moreover, $|\alpha_i| \leqslant cF_0$. The difference $v = v_{P_{\alpha_1}} - v_{P_{\alpha_2}}$ satisfies the equations

$$\begin{aligned} &-\frac{\nu_0}{2}\Delta(v_{P_{\alpha_1}} - v_{P_{\alpha_2}}) = h(v_{P_{\alpha_1}}) - h(v_{P_{\alpha_2}}) + (\alpha_1 - \alpha_2), \quad x \in \sigma, \\ &v|_{\partial\sigma} = 0, \end{aligned} \tag{29}$$

where

$$\begin{aligned} h(u) &= \frac{1}{2}\lambda \operatorname{div}_{x'}\big(\nu\big(\dot\gamma_P(u)\big)\nabla_{x'}u\big) \\ &= \frac{1}{2}\lambda\big[\nu\big(\dot\gamma_P(u)\big)\Delta_{x'}u + \big(\nabla_y\nu\big(\dot\gamma_P(u)\big)\big)^{\mathrm{T}}\big(\nabla_{x'}(\nabla_{x'}u)\big)^{\mathrm{T}} \cdot \nabla_{x'}u\big]. \end{aligned}$$

It is easy to calculate that

$$\begin{aligned} &\big|h(v_{P_{\alpha_1}}) - h(v_{P_{\alpha_2}})\big| \\ &\quad\leqslant c\lambda\big(\big|\nabla^2(v_{P_{\alpha_1}} - v_{P_{\alpha_2}})\big| + \big|\nabla^2 v_{P_{\alpha_2}}\big|\big|\nabla(v_{P_{\alpha_1}} - v_{P_{\alpha_2}})\big| \\ &\qquad + \big|\nabla^2 v_{P_{\alpha_1}}\big||\nabla v_{P_{\alpha_1}}|\big|\nabla(v_{P_{\alpha_1}} - v_{P_{\alpha_2}})\big| + |\nabla v_{P_{\alpha_1}}|\big|\nabla^2(v_{P_{\alpha_1}} - v_{P_{\alpha_2}})\big|\big). \end{aligned}$$

By using Sobolev embedding theorems we get the inequality

$$\begin{aligned} &\big\|h(v_{P_{\alpha_1}}) - h(v_{P_{\alpha_2}})\big\|_{L^2(\sigma)} \\ &\quad\leqslant c\lambda\big(\big\|\nabla^2(v_{P_{\alpha_1}} - v_{P_{\alpha_2}})\big\|_{L^2(\sigma)} + \|v_{P_{\alpha_2}}\|_{W^{3,2}(\sigma)}\|v_{P_{\alpha_1}} - v_{P_{\alpha_2}}\|_{W^{2,2}(\sigma)} \\ &\qquad + \|v_{P_{\alpha_1}}\|^2_{W^{3,2}(\sigma)}\|v_{P_{\alpha_1}} - v_{P_{\alpha_2}}\|_{W^{2,2}(\sigma)} + \|v_{P_{\alpha_1}}\|_{W^{3,2}(\sigma)}\|v_{P_{\alpha_1}} - v_{P_{\alpha_2}}\|_{W^{2,2}(\sigma)}\big) \\ &\quad\leqslant c\lambda\big(F_0 + F_0^2\big)\|v_{P_{\alpha_1}} - v_{P_{\alpha_2}}\|_{W^{2,2}(\sigma)}. \end{aligned}$$

Therefore, the classical estimate for the Poisson equation (29) yields

$$\begin{aligned}\bigl\|(v_{P_{\alpha_1}} - v_{P_{\alpha_2}})\bigr\|_{W^{2,2}(\sigma)} &\leqslant c_* \lambda\bigl(F_0 + F_0^2\bigr)\|v_{P_{\alpha_1}} - v_{P_{\alpha_2}}\|_{W^{2,2}(\sigma)} \\ &\quad + c|\alpha_1 - \alpha_2|. \end{aligned} \tag{30}$$

If $\lambda < 1/(c_*(F_0 + F_0^2))$, (30) implies

$$\|v_{P_{\alpha_1}} - v_{P_{\alpha_2}}\|_{W^{2,2}(\sigma)} \leqslant c|\alpha_1 - \alpha_2|. \tag{31}$$

Thus, inequality (27) is proved.

(ii) Let $\lambda_2 = \lambda_3(F_0)$ be the number defined in Lemma 7. By the definition of the function $G(\alpha) = F(\alpha) - \kappa\alpha$ we have

$$G(\alpha_1) - G(\alpha_2) = \bigl(F(\alpha_1) - F(\alpha_2)\bigr) - \kappa(\alpha_1 - \alpha_2).$$

Thus,

$$|\alpha_1 - \alpha_2| \leqslant \kappa^{-1}\bigl|F(\alpha_1) - F(\alpha_2)\bigr| + \bigl|\kappa^{-1}G(\alpha_1) - \kappa^{-1}G(\alpha_2)\bigr|.$$

Since for sufficiently small λ, the operator $\kappa^{-1}G(\alpha)$ is a contraction (see Lemma 5), the last estimate yields

$$|\alpha_1 - \alpha_2| \leqslant \kappa^{-1}\bigl|F(\alpha_1) - F(\alpha_2)\bigr| + \gamma|\alpha_1 - \alpha_2|$$

with $\gamma < 1$, and thus,

$$|\alpha_1 - \alpha_2| \leqslant c\bigl|F(\alpha_1) - F(\alpha_2)\bigr| = c|F_1 - F_2|. \tag{32}$$

From (31) and (32) follows (28). □

6 The non-Newtonian flow equations in domain with cylindrical outlets to infinity

6.1 Existence and uniqueness of a solution

Consider the domain $\Omega \subset \mathbb{R}^n$ with J cylindrical outlets to infinity. We assume that the boundary $\partial\Omega$ is C^4-regular. Consider in Ω problem (13)–(15). Denote $\mathsf{F} = \sqrt{\sum_{j=1}^J F_j^2}$. Let F_0 be a nonnegative number. By Lemma 7 there exists a number λ_{00} depending on F_0 such that for every $\lambda \in (0, \lambda_{00})$ and for any set of fluxes $(F_1, \dots, F_J)$ such that $\mathsf{F} \leqslant F_0$, there exist J pressure slopes α_j and corresponding J quasi-Poiseuille flows $\mathbf{V}_{P_{\alpha_j}}(x) = (v_{P_{\alpha_j}}(x'), 0, \dots, 0)^{\mathrm{T}} \in W^{3,2}(\sigma_j)$, defined in cylinders $\Pi_j = \{x^{(j)} \in \mathbb{R}^n, x^{(j)\prime} \in \sigma_j, x_1^{(j)} \in \mathbb{R}\}$, $j = 1, \dots, J$, such that $F(\alpha_j) = F_j$.

We define cut-off functions χ_j associated to each outlet Ω_j as $C^{(3)}$-smooth functions vanishing everywhere in Ω except for the outlet Ω_j, where they depend on the local longitudinal variable $x_1^{(j)}$ only, are equal to zero if $x_1^{(j)} < 1$, and equal to one if $x_1^{(j)} > 2$, and put

$$\mathbf{V}_\chi = \sum_{j=1}^J \chi_j \mathbf{V}_{P\alpha_j}, \quad P_\chi = -\sum_{j=1}^J \chi_j \alpha_j x_1^{(j)}.$$

It is easy to see that for $h(x) = \operatorname{div} \mathbf{V}_\chi(x) = \sum_{j=1}^J \chi_j'(x_1^{(j)})\, v_{P_{\alpha_j}}(x^{(j)\prime})$,

$$\operatorname{supp} h \subset \overline{\Omega^{(2)} \setminus \Omega^{(1)}}.$$

Moreover, from the condition $\sum_{j=1}^J F_j = 0$ it follows that $\int_{\Omega^{(2)}} h(x)\,\mathrm{d}x = 0$. Finally, estimates (17) and (26) yield

$$\|h\|_{W^{2,2}(\Omega^{(3)})} \leqslant c \sum_{j=1}^J \|v_{P_{\alpha_j}}\|_{W^{2,2}(\sigma_j)} \leqslant c\mathsf{F}.$$

Since $h \in W^{2,2}(\Omega^{(3)})$, by results in [5], there exits a vector field $\mathbf{W} \in \mathring{W}^{1,2}(\Omega^{(3)}) \cap W^{3,2}(\Omega^{(3)})$ such that $\operatorname{div} \mathbf{W}(x) = -h(x)$ and

$$\|\mathbf{W}\|_{W^{3,2}(\Omega^{(3)})} \leqslant c\,\|h\|_{W^{2,2}(\Omega^{(3)})} \leqslant c\mathsf{F}. \tag{33}$$

Moreover, since $\operatorname{supp} h \subset \overline{\Omega}^{(2)}$, $\mathbf{W}$ can be constructed such that $\operatorname{supp} \mathbf{W} \subset \overline{\Omega}^{(3)}$.

Extend the functions $\mathbf{W}$ and $\mathbf{V}_\chi$ by zero into the whole Ω and set

$$\widehat{\mathbf{V}}_\chi(x) = \mathbf{W}(x) + \mathbf{V}_\chi(x). \tag{34}$$

Then

$$\operatorname{div} \widehat{\mathbf{V}}_\chi(x) = 0, \qquad \widehat{\mathbf{V}}_\chi(x)|_{\partial\Omega} = 0,$$
$$\int_{\sigma_j} \widehat{\mathbf{V}}_\chi(x) \cdot \mathbf{n}(x)\,\mathrm{d}s = F_j, \quad j = 1, \dots, J,$$

and for $x \in \Omega_j \setminus \Omega_{j3}$, the vector-field $\widehat{\mathbf{V}}_\chi(x)$ coincides with the velocity part $\mathbf{V}_{P_{\alpha_j}}(x^{(j)\prime})$ of the corresponding Poiseuille flow. Note that the vector field $\mathbf{W}$ has zero flux.

By denoting in (13)

$$\mathbf{v} = \mathbf{u} + \widehat{\mathbf{V}}_\chi, \qquad p = q + \mathcal{P}_\chi, \tag{35}$$

where $\mathcal{P}_\chi = \sum_{j=1}^J \chi_j \alpha_j x_1^j$, we obtain the following problem:

$$\begin{aligned} &-\operatorname{div}\big[\big(\nu_0 + \lambda\nu\big(\dot\gamma(\mathbf{u} + \widehat{\mathbf{V}}_\chi)\big)\big) D(\mathbf{u} + \widehat{\mathbf{V}}_\chi)\big] + \nabla(q + \mathcal{P}_\chi) = \mathbf{f} \quad \text{in } \Omega,\\ &\operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega, \qquad \mathbf{u} = 0 \quad \text{on } \partial\Omega,\\ &\int_{\sigma_j} \mathbf{u} \cdot \mathbf{n}\,\mathrm{d}S = 0, \quad j = 1, \dots, J. \end{aligned} \tag{36}$$

Theorem 5. *Assume that $\partial\Omega \in C^4$. Then for any $f_0 > 0$ and $\mathsf{F}_0 > 0$, there exist numbers $\Lambda_0 = \Lambda_0(F_0, f_0) > 0$ and $\beta_* > 0$ such that for all $\lambda \in (0, \Lambda_0]$, $\beta \in (0, \beta_*]$ and for any $\mathbf{f} \in \mathcal{W}_\beta^{1,2}(\Omega)$ satisfying $\|\mathbf{f}\|_{\mathcal{W}_{\boldsymbol\beta}^{1,2}(\Omega)} \leqslant f_0$ and any set $(F_1, \dots, F_J)$ with $\mathsf{F}^2 = \sum_{j=1}^J F_j^2 \leqslant F_0^2$, problem* (13), (14), (15) *possesses a unique solution $(\mathbf{v}, p)$*[2] *admitting representation* (35) *with $\mathbf{u} \in \mathcal{W}_{\boldsymbol\beta}^{3,2}(\Omega)$, $\nabla q \in \mathcal{W}_{\boldsymbol\beta}^{1,2}(\Omega)$, $\int_{\Omega^{(3)}} q(x)\,\mathrm{d}x = 0$.*

[2]The uniqueness takes place only in some ball, where the contraction principle is applied, and we have in mind the uniqueness only for solutions admitting representation (35).

The following estimate holds:

$$\|\mathbf{u}\|^2_{\mathcal{W}^{3,2}_{\boldsymbol{\beta}}(\Omega)} + \|\nabla q\|^2_{\mathcal{W}^{1,2}_{\boldsymbol{\beta}}(\Omega)} \leqslant c\bigl(\|\mathbf{f}\|^2_{\mathcal{W}^{1,2}_{\boldsymbol{\beta}}(\Omega)} + \mathsf{F}^2\bigr). \tag{37}$$

Moreover, there exist constants $q_1, q_2, \dots, q_J$ *such that*

$$\begin{aligned}&\int_{\Omega_0} |q(x)|^2\,\mathrm{d}x + \sum_{j=1}^{J} \int_{\Omega_j} \exp\bigl(2\beta x_1^{(j)}\bigr)\bigl|q(x) - q_j\bigr|^2\,\mathrm{d}x \\ &\quad\leqslant c\int_{\Omega} E_{\boldsymbol{\beta}}(x)\bigl|\nabla q(x)\bigr|^2\,\mathrm{d}x \leqslant c\bigl(\|\mathbf{f}\|^2_{\mathcal{W}^{1,2}_{\boldsymbol{\beta}}(\Omega)} + \mathsf{F}^2\bigr).\end{aligned} \tag{38}$$

Proof. Define $\mathcal{K}$ as the operator $\mathcal{W}^{3,2}_{\boldsymbol{\beta}}(\Omega) \cap H(\Omega) \to \mathcal{W}^{3,2}_{\boldsymbol{\beta}}(\Omega) \cap H(\Omega)$ such that for any $\mathbf{U} \in \mathcal{W}^{3,2}_{\boldsymbol{\beta}}(\Omega) \cap H(\Omega)$, $(\mathcal{K}\mathbf{U}, q)$ is a solution of the problem

$$\begin{aligned}&-\frac{\nu_0}{2}\Delta\mathcal{K}\mathbf{U} + \nabla q = \mathbf{H}(\mathbf{U} + \widehat{\mathbf{V}}_\chi) + \mathbf{f}, \\ &\operatorname{div}\mathcal{K}\mathbf{U} = 0, \qquad \mathcal{K}\mathbf{U}|_{\partial\Omega} = 0,\end{aligned} \tag{39}$$

where

$$\mathbf{H}(\mathbf{U} + \widehat{\mathbf{V}}_\chi) = \frac{\nu_0}{2}\Delta\widehat{\mathbf{V}}_\chi - \nabla\mathcal{P}_\chi + \lambda\operatorname{div}\bigl[\nu\bigl(\dot\gamma(\mathbf{U} + \widehat{\mathbf{V}}_\chi)\bigr)D(\mathbf{U} + \widehat{\mathbf{V}}_\chi)\bigr].$$

After subtracting and adding the expression $\lambda\operatorname{div}(\nu(\dot\gamma(\widehat{\mathbf{V}}_\chi))D(\widehat{\mathbf{V}}_\chi))$, we write $\mathbf{H}$ in the form

$$\mathbf{H}(\mathbf{U} + \widehat{\mathbf{V}}_\chi) = \mathbf{g} + \lambda\operatorname{div}\bigl[\nu\bigl(\dot\gamma(\mathbf{U} + \widehat{\mathbf{V}}_\chi)\bigr)D(\mathbf{U} + \widehat{\mathbf{V}}_\chi) - \nu\bigl(\dot\gamma(\widehat{\mathbf{V}}_\chi)\bigr)D(\widehat{\mathbf{V}}_\chi)\bigr],$$

where $\mathbf{g} = \nu_0/2\Delta\widehat{\mathbf{V}}_\chi + \lambda\operatorname{div}(\nu(\dot\gamma(\widehat{\mathbf{V}}_\chi))D(\widehat{\mathbf{V}}_\chi)) - \nabla\mathcal{P}_\chi$. The function $\mathbf{g}$ has compact support, $\operatorname{supp}\mathbf{g} \subset \overline{\Omega^{(3)}}$, and

$$\|\mathbf{g}\|^2_{W^{1,2}(\Omega^{(3)})} \leqslant c|\mathsf{F}|^2 = c\sum_{j=1}^{J} F_j^2. \tag{40}$$

Note that

$$\begin{aligned}\nabla\nu\bigl(\dot\gamma(\mathbf{U} + \widehat{\mathbf{V}}_\chi)\bigr) &= \bigl(\nabla_y\nu(y)|_{y=\dot\gamma(\mathbf{U}+\widehat{\mathbf{V}}_\chi)}\bigr)^{\mathrm{T}}\nabla\dot\gamma(\mathbf{U} + \widehat{\mathbf{V}}_\chi), \\ \nabla\nu\bigl(\dot\gamma(\widehat{\mathbf{V}}_\chi)\bigr) &= \bigl(\nabla_y\nu(y)|_{y=\dot\gamma(\widehat{\mathbf{V}}_\chi)}\bigr)^{\mathrm{T}}\nabla\dot\gamma(\widehat{\mathbf{V}}_\chi),\end{aligned}$$

where $\nabla\dot\gamma$ is the Jacobian matrix of $\dot\gamma$, and

$$\begin{aligned}&\bigl\|\lambda\bigl(\nabla_y\nu(y)|_{y=\dot\gamma(\mathbf{U}+\widehat{\mathbf{V}}_\chi)}\bigr)^{\mathrm{T}}\nabla\dot\gamma(\mathbf{U} + \widehat{\mathbf{V}}_\chi) - \lambda\bigl(\nabla_y\nu(y)|_{y=\dot\gamma(\mathbf{U}+\widehat{\mathbf{V}}_\chi)}\bigr)^{\mathrm{T}}\nabla\dot\gamma(\widehat{\mathbf{V}}_\chi)\bigr\|_{\mathcal{L}^2_{\boldsymbol{\beta}}(\Omega)} \\ &\quad\leqslant c\lambda\sup_y\bigl\|\nabla_y\nu(y)\bigr\|\nabla^2\mathbf{U}\|_{\mathcal{L}^2_{\boldsymbol{\beta}}(\Omega)}, \\ &\bigl|\lambda\bigl(\nabla_y\nu(y)|_{y=\dot\gamma(\mathbf{U}+\widehat{\mathbf{V}}_\chi)}\bigr)^{\mathrm{T}}\nabla\dot\gamma(\widehat{\mathbf{V}}_\chi) - \lambda\bigl(\nabla_y\nu(y)|_{y=\dot\gamma(\widehat{\mathbf{V}}_\chi)}\bigr)^{\mathrm{T}}\nabla\dot\gamma(\widehat{\mathbf{V}}_\chi)\bigr| \\ &\quad\leqslant c\lambda\sup_y\bigl|\nabla_y^2\nu(y)\bigr|\bigl|\nabla\dot\gamma(\widehat{\mathbf{V}}_\chi)\bigr||\nabla\mathbf{U}|.\end{aligned}$$

Since

$$\begin{aligned}
&\lambda \operatorname{div}\bigl[\nu\bigl(\dot\gamma(\mathbf{U}+\widehat{\mathbf{V}}_\chi)\bigr)D(\mathbf{U}+\widehat{\mathbf{V}}_\chi)-\nu\bigl(\dot\gamma(\widehat{\mathbf{V}}_\chi)\bigr)D(\widehat{\mathbf{V}}_\chi)\bigr]\\
&\quad=\lambda \operatorname{div}\bigl[\nu\bigl(\dot\gamma(\mathbf{U}+\widehat{\mathbf{V}}_\chi)\bigr)D(\mathbf{U})+\bigl(\nu\bigl(\dot\gamma(\mathbf{U}+\widehat{\mathbf{V}}_\chi)\bigr)-\nu\bigl(\dot\gamma(\widehat{\mathbf{V}}_\chi)\bigr)\bigr)D(\mathbf{V}_\chi)\bigr]\\
&\quad=\lambda\nabla\nu^{\mathrm{T}}\bigl(\dot\gamma(\mathbf{U}+\widehat{\mathbf{V}}_\chi)\bigr)\cdot D(\mathbf{U})+\lambda\nu\bigl(\dot\gamma(\mathbf{U}+\widehat{\mathbf{V}}_\chi)\bigr)\operatorname{div}D(\mathbf{U})\\
&\qquad+\lambda\bigl(\nabla^{\mathrm{T}}\nu\bigl(\dot\gamma(\mathbf{U}+\widehat{\mathbf{V}}_\chi)\bigr)-\nabla^{\mathrm{T}}\nu\bigl(\dot\gamma(\widehat{\mathbf{V}}_\chi)\bigr)\bigr)\cdot D(\widehat{\mathbf{V}}_\chi)\\
&\qquad+\lambda\bigl(\nu\bigl(\dot\gamma(\mathbf{U}+\widehat{\mathbf{V}}_\chi)\bigr)-\nu\bigl(\dot\gamma(\widehat{\mathbf{V}}_\chi)\bigr)\bigr)\operatorname{div}D(\widehat{\mathbf{V}}_\chi),
\end{aligned}$$

by using (12) we obtain the estimate

$$\begin{aligned}
&\lambda\bigl|\operatorname{div}\bigl[\nu\bigl(\dot\gamma(\mathbf{U}+\widehat{\mathbf{V}}_\chi)\bigr)D(\mathbf{U}+\widehat{\mathbf{V}}_\chi)-\nu\bigl(\dot\gamma(\widehat{\mathbf{V}}_\chi)\bigr)D(\widehat{\mathbf{V}}_\chi)\bigr]\bigr|\\
&\quad\leqslant c\lambda\sup_y\bigl|\nabla_y\nu(y)\bigr||\nabla\mathbf{U}|\bigl(\bigl|\nabla^2\mathbf{U}\bigr|+\bigl|\nabla^2\widehat{\mathbf{V}}_\chi\bigr|\bigr)+c\lambda\sup_y\bigl|\nu(y)\bigr|\bigl|\nabla^2\mathbf{U}\bigr|\\
&\qquad+c\lambda\sup_y\bigl|\nabla_y\nu(y)\bigr|\bigl|\nabla^2\mathbf{U}\bigr||\nabla\widehat{\mathbf{V}}_\chi|\\
&\qquad+c\lambda\sup_y\bigl|\nabla_y^2\nu(y)\bigr||\nabla\mathbf{U}||\nabla\widehat{\mathbf{V}}_\chi|\bigl(\bigl|\nabla^2\widehat{\mathbf{V}}_\chi\bigr|+\bigl|\nabla^2\mathbf{U}\bigr|\bigr)\\
&\quad\leqslant c\lambda A\bigl(|\nabla\mathbf{U}|\bigl|\nabla^2\mathbf{U}\bigr|+\bigl|\nabla^2\mathbf{U}\bigr|+|\nabla\mathbf{U}|\bigl|\nabla^2\widehat{\mathbf{V}}_\chi\bigr|\\
&\qquad+\bigl|\nabla^2\mathbf{U}\bigr||\nabla\widehat{\mathbf{V}}_\chi|+|\nabla\mathbf{U}||\nabla\widehat{\mathbf{V}}_\chi|\bigl(\bigl|\nabla^2\widehat{\mathbf{V}}_\chi\bigr|+\bigl|\nabla^2\mathbf{U}\bigr|\bigr)\bigr).
\end{aligned}$$

Using the embedding inequalities (1), (2) and estimates (26), (33), we obtain

$$\bigl\||\nabla\mathbf{U}|\bigl|\nabla^2\mathbf{U}\bigr|\bigr\|^2_{\mathcal{L}^2_{\boldsymbol\beta}(\Omega)}\leqslant\sup_{x\in\Omega}\bigl|E_{\boldsymbol\beta}^{1/2}\nabla\mathbf{U}\bigr|^2\bigl\|\nabla^2\mathbf{U}\bigr\|^2_{\mathcal{L}^2_{\boldsymbol\beta}(\Omega)}\leqslant c\|\mathbf{U}\|^4_{\mathcal{W}^{3,2}_{\boldsymbol\beta}(\Omega)};$$

$$\begin{aligned}
&\bigl\||\nabla\widehat{\mathbf{V}}_\chi|\bigl|\nabla^2\mathbf{U}\bigr|\bigr\|^2_{\mathcal{L}^2_{\boldsymbol\beta}(\Omega)}\\
&\quad\leqslant\int\limits_{\Omega^{(3)}}\bigl|\nabla\mathbf{W}\bigr|^2\bigl|\nabla^2\mathbf{U}\bigr|^2\,\mathrm{d}x+\sum_{j=1}^J\sum_{k=2}^\infty\int\limits_{\omega_{jk}}E_{\boldsymbol\beta}|\nabla(\chi_j\mathbf{V}_{P_{\alpha_j}})|^2|\nabla^2\mathbf{U}|^2\,\mathrm{d}x\\
&\quad\leqslant\sup_{x\in\Omega^{(3)}}\bigl|\nabla\mathbf{W}(x)\bigr|^2\int\limits_{\Omega^{(3)}}\bigl|\nabla^2\mathbf{U}\bigr|^2\,\mathrm{d}x\\
&\qquad+c\sum_{j=1}^J\sum_{k=2}^\infty\sup_{x\in\sigma_j}\bigl(|v_{P_{\alpha_j}}|^2+|\nabla_{x^{(j)\prime}}v_{P_{\alpha_j}}|^2\bigr)\int\limits_{\omega_{jk}}E_{\boldsymbol\beta}\bigl|\nabla^2\mathbf{U}\bigr|^2\,\mathrm{d}x\\
&\quad\leqslant c\|\mathbf{W}\|^2_{W^{3,2}(\Omega^{(3)})}\int\limits_{\Omega^{(3)}}\bigl|\nabla^2\mathbf{U}\bigr|^2\,\mathrm{d}x\\
&\qquad+c\sum_{j=1}^J\|v_{P_{\alpha_j}}\|^2_{W^{3,2}(\sigma_j)}\sum_{j=1}^J\sum_{k=2}^\infty\int\limits_{\omega_{jk}}E_{\boldsymbol\beta}\bigl|\nabla^2\mathbf{U}\bigr|^2\,\mathrm{d}x\\
&\quad\leqslant c\mathsf{F}^2\Biggl(\int\limits_{\Omega^{(3)}}\bigl|\nabla^2\mathbf{U}\bigr|^2\,\mathrm{d}x+\sum_{j=1}^J\sum_{k=2}^\infty\int\limits_{\omega_{jk}}E_{\boldsymbol\beta}\bigl|\nabla^2\mathbf{U}\bigr|^2\,\mathrm{d}x\Biggr)\leqslant c\,\mathsf{F}^2\|\mathbf{U}\|^2_{\mathcal{W}^{3,2}_{\boldsymbol\beta}(\Omega)}.
\end{aligned}$$

Let us estimate the integrals containing the terms $|\nabla\mathbf{U}|^2|\nabla\widehat{\mathbf{V}}_\chi|^2|\widehat{\nabla}^2\mathbf{V}_\chi|^2$ and $|\nabla^2\widehat{\mathbf{V}}_\chi|^2|\nabla\mathbf{U}|^2$. Inequalities (1), (2) and (3) yield

$$\begin{aligned}
&\big\||\nabla^2\widehat{\mathbf{V}}_\chi||\nabla\mathbf{U}|\big\|^2_{\mathcal{L}^2_{\boldsymbol{\beta}}(\Omega)}\\
&\quad\leqslant \int\limits_{\Omega^{(3)}} \big|\nabla^2\mathbf{W}\big|^2|\nabla\mathbf{U}|^2\,\mathrm{d}x + \sum_{j=1}^{J}\sum_{k=2}^{\infty}\int\limits_{\omega_{jk}} E_{\boldsymbol{\beta}}\big|\nabla^2(\chi_j\mathbf{V}_{P_{\alpha_j}})\big|^2|\nabla\mathbf{U}|^2\,\mathrm{d}x\\
&\quad\leqslant \sup_{x\in\Omega^{(3)}}\big|\nabla\mathbf{U}(x)\big|^2\int\limits_{\Omega^{(3)}}\big|\nabla^2\mathbf{W}\big|^2\,\mathrm{d}x\\
&\qquad+\sum_{j=1}^{J}\sum_{k=2}^{\infty}\bigg(\int\limits_{\omega_{jk}}\big|\nabla^2(\chi_j v_{P_{\alpha_j}})\big|^4\,\mathrm{d}x\bigg)^{1/2}\bigg(\int\limits_{\omega_{jk}}E^2_{\boldsymbol{\beta}}|\nabla\mathbf{U}|^4\,\mathrm{d}x\bigg)^{1/2}\\
&\quad\leqslant c\|\mathbf{U}\|^2_{\mathcal{W}^{3,2}_{\boldsymbol{\beta}}(\Omega)}\|\mathbf{W}\|^2_{W^{3,2}(\Omega^{(3)})}+c\sum_{j=1}^{J}\|v_{P_{\alpha_j}}\|^2_{W^{3,2}(\sigma_j)}\sum_{j=1}^{J}\sum_{k=2}^{\infty}\|\mathbf{U}\|^2_{\mathcal{W}^{3,2}_{\boldsymbol{\beta}}(\omega_{jk})}\\
&\quad\leqslant c\,\mathsf{F}^2\|\mathbf{U}\|^2_{\mathcal{W}^{3,2}_{\boldsymbol{\beta}}(\Omega)}.
\end{aligned}$$

Similar considerations give us also the estimates

$$\begin{aligned}
&\big\||\nabla^2\widehat{\mathbf{V}}_\chi||\nabla\widehat{\mathbf{V}}_\chi||\nabla\mathbf{U}|\big\|^2_{\mathcal{L}^2_{\boldsymbol{\beta}}(\Omega)}\\
&\quad\leqslant\int\limits_{\Omega^{(3)}}\big(\big|\nabla^2\mathbf{W}\big|^2|\nabla\mathbf{W}|^2+\big|\nabla^2\mathbf{W}\big|^2|\nabla\widehat{\mathbf{V}}|^2+\big|\nabla^2\widehat{\mathbf{V}}\big|^2|\nabla\mathbf{W}|^2\big)|\nabla\mathbf{U}|^2\,\mathrm{d}x\\
&\qquad+\sum_{j=1}^{J}\sum_{k=2}^{\infty}\int\limits_{\omega_{jk}}E_{\boldsymbol{\beta}}\big|\nabla^2(\chi_j\mathbf{V}_{P_{\alpha_j}})\big|^2\big|\nabla(\chi_j\mathbf{V}_{P_{\alpha_j}})\big|^2|\nabla\mathbf{U}|^2\,\mathrm{d}x\\
&\quad\leqslant c\,\mathsf{F}^4\|\mathbf{U}\|^2_{\mathcal{W}^{3,2}_{\boldsymbol{\beta}}(\Omega)},
\end{aligned}$$

$$\big\||\nabla\widehat{\mathbf{V}}_\chi|\,|\nabla\mathbf{U}|\,|\nabla^2\mathbf{U}|\big\|^2_{\mathcal{L}^2_{\boldsymbol{\beta}}(\Omega)}\leqslant c\,\mathsf{F}^2\|\mathbf{U}\|^4_{\mathcal{W}^{3,2}_{\boldsymbol{\beta}}(\Omega)}.$$

Collecting the above inequalities and adding (40), we derive

$$\begin{aligned}
\big\|\mathbf{H}(\mathbf{U}+\widehat{\mathbf{V}}_\chi)\big\|^2_{\mathcal{L}^2_{\boldsymbol{\beta}}(\Omega)}\leqslant c\big(&\mathsf{F}^2+\lambda^2\big(\|\mathbf{U}\|^4_{\mathcal{W}^{3,2}_{\boldsymbol{\beta}}(\Omega)}+\big\|\mathbf{U}\|^2_{\mathcal{W}^{3,2}_{\boldsymbol{\beta}}(\Omega)}+\mathsf{F}^2\big\|\mathbf{U}\|^2_{\mathcal{W}^{3,2}_{\boldsymbol{\beta}}(\Omega)}\\
&+\mathsf{F}^4\big\|\mathbf{U}\|^2_{\mathcal{W}^{3,2}_{\boldsymbol{\beta}}(\Omega)}+\mathsf{F}^2\big\|\mathbf{U}\|^4_{\mathcal{W}^{3,2}_{\boldsymbol{\beta}}(\Omega)}\big)\big).
\end{aligned}\tag{41}$$

Similarly,

$$\begin{aligned}
&\big|\nabla\operatorname{div}\big[\nu\big(\dot\gamma(\mathbf{U}+\widehat{\mathbf{V}}_\chi)\big)D(\mathbf{U}+\widehat{\mathbf{V}}_\chi)-\nu\big(\dot\gamma(\widehat{\mathbf{V}}_\chi)\big)D(\widehat{\mathbf{V}}_\chi)\big]\big|\\
&\quad\leqslant c\big(\big(\big|\nabla^3\mathbf{U}\big|+\big|\nabla^2\mathbf{U}\big|^2\big)\big(1+|\nabla\mathbf{U}|+|\nabla\mathbf{U}||\nabla\widehat{\mathbf{V}}_\chi|\big)\\
&\qquad+\big|\nabla^3\mathbf{U}\big||\nabla\widehat{\mathbf{V}}_\chi|+\big|\nabla^2\mathbf{U}\big|\big|\nabla^2\widehat{\mathbf{V}}_\chi\big|+\big|\nabla^2\mathbf{U}\big||\nabla\mathbf{U}|\big|\nabla^2\widehat{\mathbf{V}}_\chi\big|
\end{aligned}$$

$$+ \left|\nabla^2\mathbf{U}\right|^2|\nabla\widehat{\mathbf{V}}_\chi| + \left|\nabla^2\mathbf{U}\right|\left|\nabla^2\widehat{\mathbf{V}}_\chi\right||\nabla\widehat{\mathbf{V}}_\chi| + |\nabla\mathbf{U}|\big(\left|\nabla^3\widehat{\mathbf{V}}_\chi\right| \\ + \left|\nabla^3\widehat{\mathbf{V}}_\chi\right||\nabla\widehat{\mathbf{V}}_\chi| + \left|\nabla^2\widehat{\mathbf{V}}_\chi\right|^2 + \left|\nabla^2\widehat{\mathbf{V}}_\chi\right|^2|\nabla\widehat{\mathbf{V}}_\chi|\big)\big).$$

The $\mathcal{L}^2_{\boldsymbol{\beta}}$-norm of this expression is evaluated according to the following scheme: in each product of gradients, the first-order terms $|\nabla\mathbf{U}|$ and $|\nabla\widehat{\mathbf{V}}_\chi|$ are evaluated by $\sup_{x\in\Omega}|\nabla\mathbf{U}(x)|$ and $\sup_{x\in\Omega}|\nabla\widehat{\mathbf{V}}_\chi(x)|$, the second-order terms $|\nabla^2\mathbf{U}|$ and $|\nabla^2\widehat{\mathbf{V}}_\chi|$ are evaluated in the $\mathcal{L}^4_{\boldsymbol{\beta}}$-norm, finally, the third-order terms $|\nabla^3\mathbf{U}|$ and $|\nabla^3\widehat{\mathbf{V}}_\chi|$ are evaluated in the $\mathcal{L}^2_{\boldsymbol{\beta}}$-norm. Then we apply the embedding inequalities of Lemma 2. So, for the gradient of $\mathbf{H}$, we obtain the estimate

$$\begin{aligned}\left\|\nabla\mathbf{H}(\mathbf{U}+\widehat{\mathbf{V}}_\chi)\right\|^2_{\mathcal{L}^2_{\boldsymbol{\beta}}(\Omega)} \leqslant c\big(\mathsf{F}^2 + \lambda^2\big(&\|\mathbf{U}\|^6_{\mathcal{W}^{3,2}_{\boldsymbol{\beta}}(\Omega)} + \|\mathbf{U}\|^4_{\mathcal{W}^{3,2}_{\boldsymbol{\beta}}(\Omega)} + \|\mathbf{U}\|^2_{\mathcal{W}^{3,2}_{\boldsymbol{\beta}}(\Omega)} \\ &+ \mathsf{F}^2\|\mathbf{U}\|^6_{\mathcal{W}^{3,2}_{\boldsymbol{\beta}}(\Omega)} + \mathsf{F}^2\|\mathbf{U}\|^4_{\mathcal{W}^{3,2}_{\boldsymbol{\beta}}(\Omega)} + \mathsf{F}^2\|\mathbf{U}\|^2_{\mathcal{W}^{3,2}_{\boldsymbol{\beta}}(\Omega)} \\ &+ \mathsf{F}^4\|\mathbf{U}\|^2_{\mathcal{W}^{3,2}_{\boldsymbol{\beta}}(\Omega)} + \mathsf{F}^6\|\mathbf{U}\|^2_{\mathcal{W}^{3,2}_{\boldsymbol{\beta}}(\Omega)}\big)\big).\end{aligned} \tag{42}$$

From (41) and (42) it follows that the right-hand side $\mathbf{R} = \mathbf{f} + \mathbf{H}(\mathbf{U} + \widehat{\mathbf{V}}_\chi)$ of system (39) satisfies the estimate

$$\begin{aligned}\|\mathbf{R}\|^2_{\mathcal{W}^{1,2}_{\boldsymbol{\beta}}(\Omega)} \leqslant c_1\big(\|\mathbf{f}\|^2_{\mathcal{W}^{1,2}_{\boldsymbol{\beta}}(\Omega)} + \mathsf{F}^2\big) + \lambda^2 c_2\big(&\|\mathbf{U}\|^6_{\mathcal{W}^{3,2}_{\boldsymbol{\beta}}(\Omega)} + \|\mathbf{U}\|^4_{\mathcal{W}^{3,2}_{\boldsymbol{\beta}}(\Omega)} \\ &+ \|\mathbf{U}\|^2_{\mathcal{W}^{3,2}_{\boldsymbol{\beta}}(\Omega)} + \mathsf{F}^2\|\mathbf{U}\|^6_{\mathcal{W}^{3,2}_{\boldsymbol{\beta}}(\Omega)} + \mathsf{F}^2\|\mathbf{U}\|^4_{\mathcal{W}^{3,2}_{\boldsymbol{\beta}}(\Omega)} \\ &+ \mathsf{F}^2\|\mathbf{U}\|^2_{\mathcal{W}^{3,2}_{\boldsymbol{\beta}}(\Omega)} + \mathsf{F}^4\|\mathbf{U}\|^2_{\mathcal{W}^{3,2}_{\boldsymbol{\beta}}(\Omega)} + \mathsf{F}^6\|\mathbf{U}\|^2_{\mathcal{W}^{3,2}_{\boldsymbol{\beta}}(\Omega)}\big).\end{aligned}$$

Then, by Theorem 1, for sufficiently small $\beta > 0$, the solution $(\mathcal{K}\mathbf{U}, q)$ of the Stokes problem (39) is subject to

$$\begin{aligned}&\|\mathcal{K}\mathbf{U}\|^2_{\mathcal{W}^{3,2}_{\boldsymbol{\beta}}(\Omega)} + \|\nabla q\|^2_{\mathcal{W}^{1,2}_{\boldsymbol{\beta}}(\Omega)} \\ &\quad\leqslant c_3\|\mathbf{R}\|^2_{\mathcal{W}^{1,2}_{\boldsymbol{\beta}}(\Omega)} \\ &\quad\leqslant c_4\big(\|\mathbf{f}\|^2_{\mathcal{W}^{1,2}_{\boldsymbol{\beta}}(\Omega)} + \mathsf{F}^2\big) + \lambda^2 c_5\big(\|\mathbf{U}\|^6_{\mathcal{W}^{3,2}_{\boldsymbol{\beta}}(\Omega)} + \|\mathbf{U}\|^4_{\mathcal{W}^{3,2}_{\boldsymbol{\beta}}(\Omega)} \\ &\qquad+ \|\mathbf{U}\|^2_{\mathcal{W}^{3,2}_{\boldsymbol{\beta}}(\Omega)} + \mathsf{F}^2\|\mathbf{U}\|^4_{\mathcal{W}^{3,2}_{\boldsymbol{\beta}}(\Omega)} + \mathsf{F}^2\|\mathbf{U}\|^6_{\mathcal{W}^{3,2}_{\boldsymbol{\beta}}(\Omega)} \\ &\qquad+ \mathsf{F}^2\|\mathbf{U}\|^2_{\mathcal{W}^{3,2}_{\boldsymbol{\beta}}(\Omega)} + \mathsf{F}^4\|\mathbf{U}\|^2_{\mathcal{W}^{3,2}_{\boldsymbol{\beta}}(\Omega)} + \mathsf{F}^6\|\mathbf{U}\|^2_{\mathcal{W}^{3,2}_{\boldsymbol{\beta}}(\Omega)}\big).\end{aligned} \tag{43}$$

Assume that

$$\|\mathbf{U}\|^2_{\mathcal{W}^{3,2}_{\boldsymbol{\beta}}(\Omega)} \leqslant 2c_4\big(\|\mathbf{f}\|^2_{\mathcal{W}^{1,2}_{\boldsymbol{\beta}}(\Omega)} + F_0^2\big) \equiv M^2.$$

Then from (43) it follows

$$\begin{aligned}&\|\mathcal{K}\mathbf{u}\|^2_{\mathcal{W}^{3,2}_{\boldsymbol{\beta}}(\Omega)} + \|\nabla q\|^2_{\mathcal{W}^{1,2}_{\boldsymbol{\beta}}(\Omega)} \\ &\quad\leqslant \frac{1}{2}M^2 + c_5\lambda^2 M^2\big(M^4 + M^2 + 1 + F_0^2M^2 + F_0^2M^4 + F_0^2 + F_0^4 + F_0^6\big),\end{aligned}$$

and if λ satisfies

$$\lambda^2 \leqslant \frac{1}{2c_5\left(M^4 + M^2 + 1 + F_0^2 M^2 + F_0^2 M^4 + F_0^2 + F_0^4 + F_0^6\right)} = \Lambda_*^2,$$

we obtain the estimate

$$\|\mathcal{K}\mathbf{U}\|^2_{\mathcal{W}^{3,2}_{\boldsymbol{\beta}}(\Omega)} + \|\nabla q\|^2_{\mathcal{W}^{1,2}_{\boldsymbol{\beta}}(\Omega)} \leqslant M^2. \tag{44}$$

Thus, by (44), the operator $\mathcal{K}$ maps the ball $\mathcal{B}_M = \{\mathbf{U} \in \mathcal{W}^{3,2}_{\boldsymbol{\beta}}(\Omega)\colon\ \|\mathbf{v}\|_{\mathcal{W}^{3,2}_{\boldsymbol{\beta}}(\Omega)} \leqslant M\}$ into itself.

Let us show that $\mathcal{K}$ is a contraction in the space $H(\Omega)$. Multiplying equations (39) by arbitrary $\boldsymbol{\eta} \in H(\Omega)$ and integrating by parts, we get

$$\begin{aligned}
&\frac{\nu_0}{2}\int\limits_\Omega \nabla(\mathcal{K}\mathbf{U})\cdot\nabla\boldsymbol{\eta}\,\mathrm{d}x \\
&\quad = -\lambda\int\limits_\Omega \nu\bigl(\dot\gamma(\mathbf{U}+\widehat{\mathbf{V}}_\chi)\bigr)D(\mathbf{U}+\widehat{\mathbf{V}}_\chi)\cdot\nabla\boldsymbol{\eta}\,\mathrm{d}x \\
&\qquad + \lambda\int\limits_\Omega \nu(\dot\gamma(\widehat{\mathbf{V}}_\chi)\bigr)D(\widehat{\mathbf{V}}_\chi)\cdot\nabla\boldsymbol{\eta}\,\mathrm{d}x + \int\limits_\Omega \bigl(\mathbf{g}+\mathbf{f}\bigr)\cdot\boldsymbol{\eta}\,\mathrm{d}x.
\end{aligned} \tag{45}$$

From (45) it follows that for any $\mathbf{U}_1, \mathbf{U}_2 \in \mathcal{B}_M$, the following equality holds:

$$\begin{aligned}
&\frac{\nu_0}{2}\int\limits_\Omega \nabla(\mathcal{K}\mathbf{U}_1 - \mathcal{K}\mathbf{U}_2)\cdot\nabla\boldsymbol{\eta}\,\mathrm{d}x \\
&\quad = -\lambda\int\limits_\Omega \nu\bigl(\dot\gamma(\mathbf{U}_1+\widehat{\mathbf{V}}_\chi)\bigr)\bigl(D(\mathbf{U}_1)-D(\mathbf{U}_2)\bigr)\cdot\nabla\boldsymbol{\eta}\,\mathrm{d}x \\
&\qquad - \lambda\int\limits_\Omega \bigl(\nu(\dot\gamma(\mathbf{U}_1+\widehat{\mathbf{V}}_\chi)\bigr) - \nu\bigl(\dot\gamma(\mathbf{U}_2+\widehat{\mathbf{V}}_\chi)\bigr)\bigr)D(\mathbf{U}_2+\widehat{\mathbf{V}}_\chi)\cdot\nabla\boldsymbol{\eta}\,\mathrm{d}x \\
&\quad = J_1 + J_2.
\end{aligned} \tag{46}$$

Applying Young's inequality, we have

$$|J_1| \leqslant \frac{\nu_0}{8}\int\limits_\Omega |\nabla\boldsymbol{\eta}|^2\,\mathrm{d}x + \frac{2c\lambda^2 A^2}{\nu_0}\int\limits_\Omega |\nabla\mathbf{U}_1 - \nabla\mathbf{U}_2|^2\,\mathrm{d}x.$$

Since, by (12),

$$\begin{aligned}
&\bigl|\nu(\dot\gamma(\mathbf{U}_1+\widehat{\mathbf{V}}_\chi)\bigr) - \nu\bigl(\dot\gamma(\mathbf{U}_2+\widehat{\mathbf{V}}_\chi)\bigr)\bigr|^2 \\
&\quad \leqslant \sup_y \bigl|\nabla_y\nu(y)\bigr|^2\bigl|D(\mathbf{U}_1)-D(\mathbf{U}_2)\bigr|^2 \leqslant cA^2|\nabla\mathbf{U}_1 - \nabla\mathbf{U}_2|^2,
\end{aligned}$$

we get

$$\begin{aligned}
|J_2| &\leqslant \frac{\nu_0}{8}\int\limits_{\Omega}|\nabla\boldsymbol{\eta}|^2\,\mathrm{d}x + \frac{2c\lambda^2A^2}{\nu_0}\int\limits_{\Omega}|\nabla\mathbf{U}_1-\nabla\mathbf{U}_2|^2\big|\nabla(\mathbf{U}_2+\widehat{\mathbf{V}}_\chi)\big|^2\,\mathrm{d}x\\
&\leqslant \frac{\nu_0}{8}\int\limits_{\Omega}|\nabla\boldsymbol{\eta}|^2\,\mathrm{d}x + \frac{2c\lambda^2A^2}{\nu_0}\sup_{x\in\Omega}\big|\nabla(\mathbf{U}_2+\widehat{\mathbf{V}}_\chi)\big|^2\int\limits_{\Omega}|\nabla\mathbf{U}_1-\nabla\mathbf{U}_2|^2\,\mathrm{d}x\\
&\leqslant \frac{\nu_0}{8}\int\limits_{\Omega}|\nabla\boldsymbol{\eta}|^2\,\mathrm{d}x + \frac{2c\lambda^2A^2}{\nu_0}\big(\|\mathbf{U}_2\|^2_{\mathcal{W}^{3,2}_{\boldsymbol{\beta}}(\Omega)}+F_0^2\big)\int\limits_{\Omega}|\nabla\mathbf{U}_1-\nabla\mathbf{U}_2|^2\,\mathrm{d}x\\
&\leqslant \frac{\nu_0}{8}\int\limits_{\Omega}|\nabla\boldsymbol{\eta}|^2\,\mathrm{d}x + \frac{2c\lambda^2A^2}{\nu_0}\big(M^2+F_0^2\big)\int\limits_{\Omega}|\nabla\mathbf{U}_1-\nabla\mathbf{U}_2|^2\,\mathrm{d}x.
\end{aligned}$$

Taking in (46) $\boldsymbol{\eta}=\mathcal{K}\mathbf{U}_1-\mathcal{K}\mathbf{U}_2$, we derive the inequality

$$\begin{aligned}
&\frac{\nu_0}{2}\big\|\nabla(\mathcal{K}\mathbf{U}_1-\mathcal{K}\mathbf{U}_2)\big\|^2_{L^2(\Omega)}\\
&\quad\leqslant \frac{\nu_0}{4}\Big\|\nabla(\mathcal{K}\mathbf{U}_1-\mathcal{K}\mathbf{U}_2)\big\|^2_{L^2(\Omega)} + \lambda^2\frac{c_6A^2\big[1+M^2+F_0^2\big]}{\nu_0}\Big\|\nabla(\mathbf{U}_1-\mathbf{U}_2)\|^2_{L^2(\Omega)}.
\end{aligned}$$

Therefore,

$$\big\|\nabla(\mathcal{K}\mathbf{U}_1-\mathcal{K}\mathbf{U}_2)\big\|^2_{L^2(\Omega)} \leqslant \lambda^2\frac{4c_6A^2\big[1+M^2+F_0^2\big]}{\nu_0^2}\big\|\nabla(\mathbf{U}_1-\mathbf{U}_2)\big\|^2_{L^2(\Omega)}.$$

Let

$$\Lambda_0^2=\min\left\{\Lambda_*^2,\ \frac{\nu_0^2}{4c_6A^2[1+M^2+F_0^2]}\right\}.$$

Then for any $\lambda\in(0,\Lambda_0)$, the operator $\mathcal{K}$ is a contraction with the contraction factor

$$\mu=\lambda^2\frac{4c_6A^2\big[1+M^2+F_0^2\big]}{\nu_0^2}<1,$$

and, by Theorem 2, there exists a unique fixed point $\mathbf{U}$ of the operator $\mathcal{K}$, which is a solution (together with the corresponding pressure function q) of problem (36). Estimate (37) for the fixed point $\mathbf{u}$ and the pressure q follows from the fact that $\mathbf{u}\in\mathcal{B}_M$ (see inequality (44)).

The existence of the constants $q_1,\dots,q_J$ and estimate (38) follows from Lemma 3 and Remark 1. $\square$

6.2 Continuity of the solution with respect to data of the problem

Assume that we have two sets of fluxes $(F_1^{(1)},\dots,F_J^{(1)})$ and $(F_1^{(2)},\dots,F_J^{(2)})$ satisfying condition (14) and two functions $\mathbf{f}^{(1)},\mathbf{f}^{(2)}\in\mathcal{W}^{1,2}_{\boldsymbol{\beta}}(\Omega)$. Let $\widehat{\mathbf{V}}_\chi^{(1)}$ and $\widehat{\mathbf{V}}_\chi^{(2)}$ be flux carriers corresponding to fluxes $(F_1^{(1)},\dots,F_J^{(1)})$ and $(F_1^{(2)},\dots,F_J^{(2)})$, respectively (see formula (34)). Denote by $(\mathbf{u}^{(1)},q^{(1)})$ and $(\mathbf{u}^{(2)},q^{(2)})$ the solutions of problem (36) corresponding to flux carriers $\widehat{\mathbf{V}}_\chi^{(1)}$, $\widehat{\mathbf{V}}_\chi^{(2)}$ and right-hand sides $\mathbf{f}^{(1)},\mathbf{f}^{(2)}$. Assume that

$$\mathsf{F}^{(i)} \leqslant F_0, \quad \|\mathbf{f}^{(i)}\|_{\mathcal{W}^{1,2}_{\boldsymbol{\beta}}(\Omega)} \leqslant f_0, \quad i=1,2. \tag{47}$$

Denote

$$Q=\sum_{j=1}^{J}\bigl|F_j^{(1)}-F_j^{(2)}\bigr|^2+\bigl\|\mathbf{f}^{(1)}-\mathbf{f}^{(2)}\bigr\|^2_{\mathcal{W}^{1,2}_{\boldsymbol{\beta}}(\Omega)}.$$

Theorem 6. *There exists $\Lambda=\Lambda_1(\mathsf{F}_0,f_0)$ such that for all $\lambda\in(0,\Lambda_1]$ and sufficiently small Q, for arbitrary $\mathbf{f}^{(i)}$ and $(F_1^{(i)},\dots,F_J^{(i)})$, $i=1,2$, satisfying* (47)*, the following estimate holds*:

$$\bigl\|\mathbf{u}^{(1)}-\mathbf{u}^{(2)}\bigr\|^2_{W^{2,2}(\Omega)}+\bigl\|\nabla\bigl(q^{(1)}-q^{(2)}\bigr)\bigr\|^2_{L^2(\Omega)}\leqslant cQ|\ln Q|. \tag{48}$$

Moreover, if $q^{(1)}(x)$ and $q^{(2)}(x)$ are normalized by the condition $\int_{\Omega^{(3)}} q^{(1)}(x)\,\mathrm{d}x=\int_{\Omega^{(3)}} q^{(2)}(x)\,\mathrm{d}x$, then the limit constants $q_1^{(1)},\dots,q_J^{(1)}$ at infinity of $q^{(1)}(x)$ and the corresponding constants $q_1^{(2)},\dots,q_J^{(2)}$ of $q^{(2)}(x)$ satisfy the estimate

$$\sum_{j=1}^{J}\bigl|q_j^{(1)}-q_j^{(2)}\bigr|^2\leqslant cQ(\ln Q)^2.$$

Proof. Due to condition (47) and inequality (37), $\mathbf{u}^{(i)}\in\mathcal{B}_M\subset\mathcal{W}^{3,2}_{\boldsymbol{\beta}}(\Omega)$, $\nabla q^{(i)}\in b_{M_1}\subset\mathcal{W}^{1,2}_{\boldsymbol{\beta}}(\Omega)$, $q_j^{(i)}\in[-M_2,M_2]$, where $\mathcal{B}_M$ and b_{M_1} are balls of the radius M and M_1, respectively, M, M_1, M_2 are positive numbers defined by F_0, f_0 of condition (47).

The difference $\mathbf{u}=\mathbf{u}^{(1)}-\mathbf{u}^{(2)}\in\mathcal{W}^{3,2}_{\boldsymbol{\beta}}(\Omega)$ satisfies the equations

$$\begin{aligned}
&-\nu_0\Delta\mathbf{u}+\nabla q\\
&\quad=\mathbf{g}+\mathbf{f}+\lambda\operatorname{div}\bigl(\nu\bigl(\dot\gamma\bigl(\mathbf{u}^{(1)}+\widehat{\mathbf{V}}_\chi^{(1)}\bigr)\bigr)D\bigl(\mathbf{u}^{(1)}+\widehat{\mathbf{V}}_\chi^{(1)}\bigr)\\
&\qquad-\nu\bigl(\dot\gamma\bigl(\widehat{\mathbf{V}}_\chi^{(1)}\bigr)\bigr)D\bigl(\widehat{\mathbf{V}}_\chi^{(1)}\bigr)-\nu\bigl(\dot\gamma\bigl(\mathbf{u}^{(2)}+\widehat{\mathbf{V}}_\chi^{(2)}\bigr)\bigr)D\bigl(\mathbf{u}^{(2)}+\widehat{\mathbf{V}}_\chi^{(2)}\bigr)\\
&\qquad+\nu\bigl(\dot\gamma\bigl(\widehat{\mathbf{V}}_\chi^{(2)}\bigr)\bigr)D\bigl(\widehat{\mathbf{V}}_\chi^{(2)}\bigr)\bigr),\\
&\operatorname{div}\mathbf{u}=0,\qquad \mathbf{u}|_{\partial\Omega}=0,
\end{aligned} \tag{49}$$

where $q=q^{(1)}-q^{(2)}$, $\mathbf{g}^{(i)}=\nu_0\Delta\widehat{\mathbf{V}}_\chi^{(i)}+\lambda\operatorname{div}(\nu(\dot\gamma(\widehat{\mathbf{V}}_\chi^{(i)}))D(\widehat{\mathbf{V}}_\chi^{(i)}))-\nabla\mathcal{P}_\chi^{(i)}$, $i=1,2$, $\mathbf{g}=\mathbf{g}^{(1)}-\mathbf{g}^{(2)}$, $\mathbf{f}=\mathbf{f}^{(1)}-\mathbf{f}^{(2)}$.

From (28), (33) and (40) it follows that $\operatorname{supp}(\mathbf{g}^{(1)}-\mathbf{g}^{(2)})\subset\overline{\Omega}^{(3)}$ and

$$\bigl\|\mathbf{g}^{(1)}-\mathbf{g}^{(2)}\bigr\|^2_{L^2(\Omega^{(3)})}\leqslant c\sum_{j=1}^{J}\bigl|F_j^{(1)}-F_j^{(2)}\bigr|^2. \tag{50}$$

Since $\mathbf{u}\in\mathcal{W}^{2,2}_{\boldsymbol{\beta}}(\Omega)$, $\nabla q\in\mathcal{L}^2_{\boldsymbol{\beta}}(\Omega)$, there exists an integer k_Q such that

$$\|\mathbf{u}\|^2_{\mathcal{W}^{2,2}_{\boldsymbol{\beta}/4}(\Omega\setminus\Omega^{(k_Q)})}+\|\nabla q\|^2_{\mathcal{L}^2_{\boldsymbol{\beta}/4}(\Omega\setminus\Omega^{(k_Q)})}\leqslant Q. \tag{51}$$

Obviously, for every sufficiently large K,

$$\|\mathbf{u}\|^2_{\mathcal{W}^{2,2}_{\boldsymbol{\beta}/4}(\Omega\setminus\Omega^{(K)})} + \|\nabla q\|^2_{\mathcal{L}^2_{\boldsymbol{\beta}/4}(\Omega\setminus\Omega^{(K)})} \leqslant c_7 \mathrm{e}^{-3K\beta/2}$$

with the constant c_7 defined by F_0 and f_0. In particular, condition (51) holds if $c_7\mathrm{e}^{-3k_Q\beta/2} = Q$, i.e.,

$$k_Q = \left|\frac{2\ln(Q/c_7)}{3\beta}\right| = O\bigl(|\ln Q|\bigr). \tag{52}$$

We assume without loss of generality that $k_Q > 1$.

Let us estimate the norm $\|\nabla\mathbf{u}\|^2_{L^2(\Omega\cap\Omega^{(2K_Q)})}$. Consider the function

$$\boldsymbol{\varphi}(x) = \zeta_{k_Q}(x)\mathbf{u}(x) + \boldsymbol{\Phi}(x),$$

where $\zeta_{k_Q}(x) = \zeta_{k_Q}(x_1^{(j)})$ for $x \in \Omega_j$,

$$\zeta_{k_Q}(t) = \begin{cases} 1, & t \in [0, k_Q], \\ \cos^2(\frac{\pi}{2}\frac{t-k_Q}{k_Q}), & t \in [k_Q, 2k_Q], \\ 0, & t \in [2k_Q, +\infty), \end{cases}$$

and

$$\begin{aligned} \operatorname{div}\boldsymbol{\Phi} &= -\nabla\zeta_{k_Q}(x)\cdot\mathbf{u}(x) \quad \text{in } \Omega^{(2k_Q)}, \\ \boldsymbol{\Phi} &= 0 \quad \text{on } \partial\Omega^{(2k_Q)} = 0. \end{aligned} \tag{53}$$

Let us construct $\boldsymbol{\Phi}$.

Notice that

$$\operatorname{supp}\bigl(\nabla\zeta_{k_Q}(x)\cdot\mathbf{u}(x)\bigr) \subset \bigcup_{j=1}^{J}\bigl\{x \in \Omega_j\colon 0 < x_1^{(j)} < 2k_Q\bigr\}$$

and

$$\int\limits_{\sigma_j} \mathbf{u}\cdot\mathbf{n}\,\mathrm{d}S = 0 \quad j = 1,\dots,J.$$

Therefore, for any ω_{jk}, $k = 0,\dots,2k_Q-1$, $j = 1,\dots,J$, we have

$$\int\limits_{\sigma_j} \nabla\zeta_{k_Q}(x)\cdot\mathbf{u}(x)\,\mathrm{d}x^{(j)'} = 0,$$

and there exist functions $\boldsymbol{\Phi}_{jk} \in \mathring{W}^{1,2}(\omega_{jk})$ satisfying the equation

$$\operatorname{div}\boldsymbol{\Phi}_{jk}(x) = -\nabla\zeta_{k_Q}(x)\cdot\mathbf{u}(x) \quad \text{in } \omega_{jk},$$

and the estimate

$$\|\nabla\boldsymbol{\Phi}_{jk}\|_{L^2(\omega_{jk})} \leqslant c\|\nabla\zeta_{k_Q}\cdot\mathbf{u}\|_{L^2(\omega_{jk})} \leqslant ck_Q^{-1}\|\mathbf{u}\|_{L^2(\omega_{jk})}$$

with the constant c independent of k and j. Extend $\boldsymbol{\Phi}_{jk}$ by zero to Ω. Putting $\boldsymbol{\Phi}(x) = \sum_{j=1}^{J}\sum_{k=0}^{2k_Q-1}\boldsymbol{\Phi}_{jk}(x)$, we obtain the function belonging to $\mathring{W}^{1,2}(\bigcup_{j=1}^{J}\Omega_{j\,2k_Q})$, which satisfies equation (53) and obeys the estimate

$$\|\nabla\boldsymbol{\Phi}_{jk}\|_{L^2(\bigcup_{j=1}^{J}\Omega_{j\,2k_Q})} \leqslant ck_Q^{-1}\|\mathbf{u}\|_{L^2(\bigcup_{j=1}^{J}\Omega_{j\,2k_Q})} \tag{54}$$

with the constant c independent of k_Q.

Since, by construction, $\boldsymbol{\varphi}$ is solenoidal and $\operatorname{supp}\boldsymbol{\varphi}\subset\Omega^{(2k_Q)}$, multiplying (49) by $\boldsymbol{\varphi}$ and integrating by parts, we obtain

$$\begin{aligned}
\nu_0\int\limits_{\Omega^{(k_Q)}}|\nabla\mathbf{u}|^2\,\mathrm{d}x &= -\nu_0\int\limits_{\Omega^{(2k_Q)}\setminus\Omega^{(k_Q)}}\zeta_{k_Q}|\nabla\mathbf{u}|^2\,\mathrm{d}x \\
&\quad -\nu_0\int\limits_{\Omega^{(2k_Q)}}\nabla\mathbf{u}\nabla\zeta_{k_Q}\cdot\mathbf{u}\,\mathrm{d}x+\nu_0\int\limits_{\Omega^{(2k_Q)}}\nabla\mathbf{u}\cdot\nabla\boldsymbol{\Phi}\,\mathrm{d}x \\
&\quad +\nu_0\int\limits_{\Omega^{(2k_Q)}}(\mathbf{g}+\mathbf{f})\cdot\boldsymbol{\varphi}\,\mathrm{d}x+\lambda\int\limits_{\Omega^{(2k_Q)}}\mathbf{M}\big(\mathbf{u}^{(1)},\mathbf{u}^{(2)}\big)\cdot\boldsymbol{\varphi}\,\mathrm{d}x \\
&\quad -\lambda\int\limits_{\Omega^{(2k_Q)}}\mathbf{N}\big(\widehat{\mathbf{V}}_\chi^{(1)},\widehat{\mathbf{V}}_\chi^{(2)}\big)\cdot\boldsymbol{\varphi}\,\mathrm{d}x \\
&= \sum_{l=1}^{6}K_l,
\end{aligned} \tag{55}$$

where

$$\begin{aligned}
\mathbf{M}\big(\mathbf{u}^{(1)},\mathbf{u}^{(2)}\big) &= \operatorname{div}\big(\nu\big(\dot\gamma\big(\mathbf{u}^{(1)}+\widehat{\mathbf{V}}_\chi^{(1)}\big)\big)D\big(\mathbf{u}^{(1)}+\widehat{\mathbf{V}}_\chi^{(1)}\big) \\
&\quad -\nu\big(\dot\gamma\big(\mathbf{u}^{(2)}+\widehat{\mathbf{V}}_\chi^{(2)}\big)\big)D\big(\mathbf{u}^{(2)}+\widehat{\mathbf{V}}_\chi^{(2)}\big)\big), \\
\mathbf{N}\big(\widehat{\mathbf{V}}_\chi^{(1)},\widehat{\mathbf{V}}_\chi^{(2)}\big) &= \operatorname{div}\big(\nu\big(\dot\gamma\big(\widehat{\mathbf{V}}_\chi^{(1)}\big)\big)D\big(\widehat{\mathbf{V}}_\chi^{(1)}\big)-\nu\big(\dot\gamma\big(\widehat{\mathbf{V}}_\chi^{(2)}\big)\big)D\big(\widehat{\mathbf{V}}_\chi^{(2)}\big)\big).
\end{aligned}$$

Let us estimate the right-hand side of (55). First of all, we notice that

$$\begin{aligned}
\int\limits_{\Omega^{(2k_Q)}}|\nabla\boldsymbol{\varphi}|^2\,\mathrm{d}x &\leqslant c\int\limits_{\Omega^{(2k_Q)}}|\nabla\zeta_{k_Q}|^2|\mathbf{u}|^2\,\mathrm{d}x \\
&\quad + c\int\limits_{\Omega^{(2k_Q)}}|\zeta_{k_Q}|^2|\nabla\mathbf{u}|^2\,\mathrm{d}x+\int\limits_{\Omega^{(2k_Q)}}|\nabla\boldsymbol{\Phi}|^2\,\mathrm{d}x \\
&\leqslant c\int\limits_{\Omega^{(2k_Q)}}|\nabla\mathbf{u}|^2\,\mathrm{d}x,
\end{aligned}$$

where, in order to estimate the last term, we applied inequality (54). Moreover, by (50), (51) and (54),

$$|K_1| \leqslant c \int\limits_{\Omega\setminus\Omega^{(k_Q)}} |\nabla \mathbf{u}|^2 \,\mathrm{d}x \leqslant c\,Q, \tag{56}$$

$$\begin{aligned}|K_2| &\leqslant \biggl| \int\limits_{\Omega^{(2k_Q)}} \nabla \mathbf{u} \nabla \zeta_{k_Q} \cdot \mathbf{u} \,\mathrm{d}x \biggr| \\ &\leqslant \frac{c}{k_Q} \int\limits_{\Omega^{(2k_Q)}} |\nabla \mathbf{u}||\mathbf{u}| \,\mathrm{d}x \leqslant \frac{c}{k_Q} \int\limits_{\Omega^{(k_Q)}} |\nabla \mathbf{u}|^2 \,\mathrm{d}x + c \int\limits_{\Omega\setminus\Omega^{(k_Q)}} |\nabla \mathbf{u}|^2 \,\mathrm{d}x \\ &\leqslant \frac{c}{k_Q} \int\limits_{\Omega^{(k_Q)}} |\nabla \mathbf{u}|^2 \,\mathrm{d}x + c\,Q, \end{aligned} \tag{57}$$

$$\begin{aligned}|K_3| &\leqslant \nu_0 \biggl(\int\limits_{\Omega^{(2k_Q)}} |\nabla \mathbf{u}|^2 \,\mathrm{d}x \biggr)^{1/2} \biggl(\int\limits_{\Omega^{(2k_Q)}} |\nabla \boldsymbol{\Phi}|^2 \,\mathrm{d}x \biggr)^{1/2} \\ &\leqslant \frac{c}{k_Q} \int\limits_{\Omega^{(k_Q)}} |\nabla \mathbf{u}|^2 \,\mathrm{d}x + c\,Q, \end{aligned} \tag{58}$$

$$\begin{aligned}|K_4| &\leqslant c(\varepsilon) \biggl(\|\mathbf{g}\|^2_{L^2(\Omega^{(3)})} + \int\limits_{\Omega^{(2k_Q)}} |\mathbf{f}|^2 \,\mathrm{d}x \biggr) + \varepsilon \int\limits_{\Omega^{(2k_Q)}} |\nabla \boldsymbol{\varphi}|^2 \,\mathrm{d}x \\ &\leqslant c(\varepsilon) Q + \varepsilon \int\limits_{\Omega^{(k_Q)}} |\nabla \mathbf{u}|^2 \,\mathrm{d}x + \varepsilon \int\limits_{\Omega\setminus\Omega^{(k_Q)}} |\nabla \mathbf{u}|^2 \,\mathrm{d}x \\ &\leqslant cQ + \varepsilon \int\limits_{\Omega^{(k_Q)}} |\nabla \mathbf{u}|^2 \,\mathrm{d}x. \end{aligned} \tag{59}$$

Substituting estimates (56)–(59) into (55), for sufficiently small ε and sufficiently large k_Q, we obtain

$$\begin{aligned}\frac{\nu_0}{2} \int\limits_{\Omega^{(k_Q)}} |\nabla \mathbf{u}|^2 \,\mathrm{d}x &\leqslant cQ + \lambda \int\limits_{\Omega^{(2k_Q)}} \mathbf{M}\bigl(\mathbf{u}^{(1)}, \mathbf{u}^{(2)}\bigr) \cdot \boldsymbol{\varphi} \,\mathrm{d}x \\ &\quad + \lambda \int\limits_{\Omega^{(2k_Q)}} \mathbf{N}\bigl(\widehat{\mathbf{V}}^{(1)}_\chi, \widehat{\mathbf{V}}^{(2)}_\chi\bigr) \cdot \boldsymbol{\varphi} \,\mathrm{d}x. \end{aligned} \tag{60}$$

Consider the term K_6 containing $\mathbf{N}(\widehat{\mathbf{V}}^{(1)}_\chi, \widehat{\mathbf{V}}^{(2)}_\chi)$. We have

$$\begin{aligned}\mathbf{N}\bigl(\widehat{\mathbf{V}}^{(1)}_\chi, \widehat{\mathbf{V}}^{(2)}_\chi\bigr) &= \operatorname{div}\bigl(\nu\bigl(\dot\gamma\bigl(\widehat{\mathbf{V}}^{(1)}_\chi\bigr)\bigr) D\bigl(\widehat{\mathbf{V}}^{(1)}_\chi\bigr) - \nu\bigl(\dot\gamma\bigl(\widehat{\mathbf{V}}^{(2)}_\chi\bigr)\bigr) D\bigl(\widehat{\mathbf{V}}^{(2)}_\chi\bigr)\bigr) \\ &= \operatorname{div}\bigl(\nu\bigl(\dot\gamma\bigl(\widehat{\mathbf{V}}^{(1)}_\chi\bigr)\bigr)\bigl(D\bigl(\widehat{\mathbf{V}}^{(1)}_\chi\bigr) - D\bigl(\widehat{\mathbf{V}}^{(2)}_\chi\bigr)\bigr) \\ &\quad + \bigl(\nu\bigl(\dot\gamma\bigl(\widehat{\mathbf{V}}^{(1)}_\chi\bigr)\bigr) - \nu\bigl(\dot\gamma\bigl(\widehat{\mathbf{V}}^{(2)}_\chi\bigr)\bigr)\bigr) D\bigl(\widehat{\mathbf{V}}^{(2)}_\chi\bigr)\bigr). \end{aligned}$$

Therefore,

$$
\begin{aligned}
\left|\mathbf{N}\left(\widehat{\mathbf{V}}_\chi^{(1)},\widehat{\mathbf{V}}_\chi^{(2)}\right)\right| \leqslant c\bigl(&\left|\nabla^2\widehat{\mathbf{V}}_\chi^{(1)}\right|\left|\nabla\left(\widehat{\mathbf{V}}_\chi^{(1)}-\widehat{\mathbf{V}}_\chi^{(2)}\right)\right| + \left|\nabla^2\left(\widehat{\mathbf{V}}_\chi^{(1)}-\widehat{\mathbf{V}}_\chi^{(2)}\right)\right| \\
&+ \left|\nabla^2\left(\widehat{\mathbf{V}}_\chi^{(1)}-\widehat{\mathbf{V}}_\chi^{(2)}\right)\right|\left|\nabla\widehat{\mathbf{V}}_\chi^{(2)}\right| + \left|\nabla\left(\widehat{\mathbf{V}}_\chi^{(1)}-\widehat{\mathbf{V}}_\chi^{(2)}\right)\right|\left|\nabla^2\widehat{\mathbf{V}}_\chi^{(2)}\right| \\
&+ \left|\nabla\left(\widehat{\mathbf{V}}_\chi^{(1)}-\widehat{\mathbf{V}}_\chi^{(2)}\right)\right|\left|\nabla\widehat{\mathbf{V}}_\chi^{(2)}\right|\left|\nabla^2\widehat{\mathbf{V}}_\chi^{(1)}\right|\bigr).
\end{aligned} \tag{61}
$$

Arguing as in the proof of Theorems 3 and 5 and using (17), (1), (47), we get

$$
\begin{aligned}
&\int\limits_{\Omega^{(2k_Q)}} \left|\nabla^2\widehat{\mathbf{V}}_\chi^{(1)}\right|^2\left|\nabla\left(\widehat{\mathbf{V}}_\chi^{(1)}-\widehat{\mathbf{V}}_\chi^{(2)}\right)\right|^2 \mathrm{d}x \\
&\quad\leqslant c\int\limits_{\Omega^{(3)}} \left|\nabla^2\mathbf{W}^{(1)}\right|^2\left|\nabla\left(\widehat{\mathbf{V}}_\chi^{(1)}-\widehat{\mathbf{V}}_\chi^{(2)}\right)\right|^2 \mathrm{d}x \\
&\qquad + c\sum_{j=1}^{J}\sum_{l=0}^{2k_Q}\int\limits_{\omega_l} \left|\nabla^2\mathbf{V}_\chi^{(1)}\right|^2\left|\nabla\left(\widehat{\mathbf{V}}_\chi^{(1)}-\widehat{\mathbf{V}}_\chi^{(2)}\right)\right|^2 \mathrm{d}x \\
&\quad\leqslant c\left\|\mathbf{W}^{(1)}\right\|^2_{W^{3,2}\left(\Omega^{(3)}\right)}\left\|\widehat{\mathbf{V}}_\chi^{(1)}-\widehat{\mathbf{V}}_\chi^{(2)}\right\|^2_{W^{2,2}(\Omega^{(3)})} \\
&\qquad + c\int\limits_{\Omega^{(3)}} \left|\nabla^2\mathbf{V}_\chi^{(1)}\right|^2\left|\nabla\left(\widehat{\mathbf{V}}_\chi^{(1)}-\widehat{\mathbf{V}}_\chi^{(2)}\right)\right|^2 \mathrm{d}x \\
&\qquad + c\sum_{j=1}^{J}\sum_{l=0}^{2k_Q}\int\limits_{l}^{l+1}\int\limits_{\sigma_j} \left|\nabla^2\mathbf{V}_\chi^{(1)}\right|^2\left|\nabla\left(\widehat{\mathbf{V}}_\chi^{(1)}-\widehat{\mathbf{V}}_\chi^{(2)}\right)\right|^2 \mathrm{d}x_1^{(j)}\,\mathrm{d}x^{(j)\prime} \\
&\quad\leqslant c\sum_{j=1}^{J}\left|F_j^{(1)}-F_j^{(2)}\right|^2 \\
&\qquad + c\sum_{j=1}^{J}\sum_{l=0}^{2k_Q}\int\limits_{l}^{l+1}\int\limits_{\sigma_j} \left\|\widehat{\mathbf{V}}^{(1)}\right\|^2_{W^{3,2}(\sigma_j)}\left\|\widehat{\mathbf{V}}_\chi^{(1)}-\widehat{\mathbf{V}}_\chi^{(2)}\right\|^2_{W^{2,2}(\sigma_j)} \mathrm{d}x_1^{(j)} \\
&\quad\leqslant c\sum_{j=1}^{J}\left|F_j^{(1)}-F_j^{(2)}\right|^2 + c\sum_{j=1}^{J}\left|F_j^{(1)}-F_j^{(2)}\right|^2\sum_{j=1}^{J}\sum_{l=0}^{2k_Q}\int\limits_{l}^{l+1}\mathrm{d}x_1^{(j)} \\
&\quad\leqslant c\bigl((1+2Jk_Q)\bigr)\sum_{j=1}^{J}\left|F_j^{(1)}-F_j^{(2)}\right|^2 \leqslant cQ|\ln Q|.
\end{aligned}
$$

Estimating analogously the other terms in (61) and using (52), we derive

$$
\int\limits_{\Omega^{(2k_Q)}} \left|\mathbf{N}\left(\widehat{\mathbf{V}}_\chi^{(1)},\widehat{\mathbf{V}}_\chi^{(2)}\right)\right|^2 \mathrm{d}x \leqslant cQ|\ln Q|. \tag{62}
$$

Thus, similarly to (59), we get

$$\begin{aligned}
|K_6| &\leqslant c(\varepsilon) \int\limits_{\Omega^{(2k_Q)}} \bigl|\mathbf{N}\bigl(\widehat{\mathbf{V}}^{(1)}_{\chi}, \widehat{\mathbf{V}}^{(2)}_{\chi}\bigr)\bigr|^2 \,\mathrm{d}x + \varepsilon \int\limits_{\Omega^{(2k_Q)}} |\boldsymbol{\varphi}|^2 \,\mathrm{d}x \\
&\leqslant cQ|\ln Q| + \varepsilon \int\limits_{\Omega^{(2k_Q)}} |\nabla \mathbf{u}|^2 \,\mathrm{d}x \\
&\leqslant cQ|\ln Q| + \varepsilon \int\limits_{\Omega^{(k_Q)}} |\nabla \mathbf{u}|^2 \,\mathrm{d}x + \varepsilon \int\limits_{\Omega \setminus \Omega^{(k_Q)}} |\nabla \mathbf{u}|^2 \,\mathrm{d}x \\
&\leqslant cQ|\ln Q| + \varepsilon \int\limits_{\Omega^{(k_Q)}} |\nabla \mathbf{u}|^2 \,\mathrm{d}x.
\end{aligned} \tag{63}$$

Similarly, we have

$$\begin{aligned}
&\bigl|\mathbf{M}\bigl(\mathbf{u}^{(1)}, \mathbf{u}^{(2)}\bigr)\bigr| \\
&\quad\leqslant c\bigl(\bigl|\nabla^2\bigl(\mathbf{u}^{(1)} + \widehat{\mathbf{V}}^{(1)}_{\chi}\bigr)\bigr|\bigl|\nabla\bigl((\mathbf{u}^{(1)} + \widehat{\mathbf{V}}^{(1)}_{\chi}) - \bigl(\mathbf{u}^{(2)} + \widehat{\mathbf{V}}^{(2)}_{\chi}\bigr)\bigr)\bigr| \\
&\qquad + \bigl|\nabla^2\bigl(\bigl(\mathbf{u}^{(1)} + \widehat{\mathbf{V}}^{(1)}_{\chi}\bigr) - \bigl(\mathbf{u}^{(2)} + \widehat{\mathbf{V}}^{(2)}_{\chi}\bigr)\bigr)\bigr| \\
&\qquad + \bigl|\nabla^2\bigl(\bigl(\mathbf{u}^{(1)} + \widehat{\mathbf{V}}^{(1)}_{\chi}\bigr) - \bigl(\mathbf{u}^{(2)} + \widehat{\mathbf{V}}^{(2)}_{\chi}\bigr)\bigr)\bigr|\bigl|\nabla\bigl(\mathbf{u}^{(2)} + \widehat{\mathbf{V}}^{(2)}\bigr)\bigr| \\
&\qquad + \bigl|\nabla\bigl(\mathbf{u}^{(1)} + \widehat{\mathbf{V}}^{(1)}_{\chi}\bigr) - \nabla\bigl(\mathbf{u}^{(2)} + \widehat{\mathbf{V}}^{(2)}_{\chi}\bigr)\bigr|\bigl|\nabla^2\bigl(\mathbf{u}^{(2)} + \widehat{\mathbf{V}}^{(2)}_{\chi}\bigr)\bigr| \\
&\qquad + \bigl|\nabla\bigl(\mathbf{u}^{(1)} + \widehat{\mathbf{V}}^{(1)}_{\chi}\bigr) - \nabla\bigl(\mathbf{u}^{(2)} + \widehat{\mathbf{V}}^{(2)}_{\chi}\bigr)\bigr|\bigl|\nabla\bigl(\mathbf{u}^{(2)} + \widehat{\mathbf{V}}^{(2)}_{\chi}\bigr)\bigr|\bigl|\nabla^2\bigl(\mathbf{u}^{(1)} + \widehat{\mathbf{V}}^{(1)}_{\chi}\bigr)\bigr|\bigr) \\
&\quad= \sum_{l=1}^{5} M_l.
\end{aligned}$$

Arguing as in the proof of Theorem 5, we obtain

$$\begin{aligned}
\int\limits_{\Omega^{(2k_Q)}} M_1^2 \,\mathrm{d}x &\leqslant c\Biggl(\int\limits_{\Omega^{(2k_Q)}} \bigl|\nabla^2\mathbf{u}^{(1)}\bigr|^2|\nabla\mathbf{u}|^2 \,\mathrm{d}x + \int\limits_{\Omega^{(2k_Q)}} \bigl|\nabla^2\widehat{\mathbf{V}}^{(1)}_{\chi}\bigr|^2|\nabla\mathbf{u}|^2 \,\mathrm{d}x \\
&\quad + \int\limits_{\Omega^{(2k_Q)}} \bigl|\nabla^2\mathbf{u}^{(1)}\bigr|^2\bigl|\nabla\bigl(\widehat{\mathbf{V}}^{(1)}_{\chi} - \widehat{\mathbf{V}}^{(2)}_{\chi}\bigr)\bigr|^2 \,\mathrm{d}x \\
&\quad + \int\limits_{\Omega^{(2k_Q)}} \bigl|\nabla^2\widehat{\mathbf{V}}^{(1)}_{\chi}\bigr|^2\bigl|\nabla\bigl(\widehat{\mathbf{V}}^{(1)}_{\chi} - \widehat{\mathbf{V}}^{(2)}_{\chi}\bigr)\bigr|^2 \,\mathrm{d}x\Biggr) \\
&\leqslant c\Bigl(\bigl\|\mathbf{u}^{(1)}\bigr\|^2_{W^{3,2}(\Omega^{(2k_Q)})}\|\mathbf{u}\|^2_{W^{2,2}(\Omega^{(2k_Q)})} \\
&\quad + \Bigl(\bigl\|\mathbf{W}^{(1)}\bigr\|^2_{W^{3,2}(\Omega^{(3)})} + \sum_{j=1}^{J}\bigl\|\mathbf{V}^{(1)}_{\chi}\bigr\|^2_{W^{3,2}(\sigma_j)}\Bigr)\|\mathbf{u}\|^2_{W^{2,2}(\Omega^{(2k_Q)})}
\end{aligned}$$

$$
\begin{aligned}
&+\Bigg(\big\|\mathbf{W}^{(1)}-\mathbf{W}^{(2)}\big\|^2_{W^{2,2}(\Omega^{(3)})}+\sum_{j=1}^{J}\big\|\mathbf{V}^{(1)}_{\chi}-\mathbf{V}^{(2)}_{\chi}\big\|^2_{W^{2,2}(\Omega^{(2k_Q)})}\Bigg)\\
&\times\big\|\mathbf{u}^{(1)}\big\|^2_{W^{3,2}(\Omega^{(2k_Q)})}\\
&+c|\ln Q|\Bigg(\big\|\mathbf{W}^{(1)}\big\|^2_{W^{3,2}(\Omega^{(3)})}+\sum_{j=1}^{J}\big\|\mathbf{V}^{(1)}_{\chi}\big\|^2_{W^{3,2}(\sigma_j)}\Bigg)\\
&\times\sum_{j=1}^{J}\big\|\widehat{\mathbf{V}}^{(1)}_{\chi}-\widehat{\mathbf{V}}^{(2)}_{\chi}\big\|^2_{W^{2,2}(\sigma_j)}\Bigg)\\
\leqslant{}& c\big(\|\mathbf{u}\|^2_{W^{2,2}(\Omega^{(2k_Q)})}+Q|\ln Q|\big).
\end{aligned}
$$

Estimating the integral containing $|\nabla^2\widehat{\mathbf{V}}^{(1)}_{\chi}|^2|\nabla(\widehat{\mathbf{V}}^{(1)}_{\chi}-\widehat{\mathbf{V}}^{(2)}_{\chi})|^2$, we have used the same argument as in the proof of (62) (see Lemma 2).

The other terms M_i, $i=2,\dots,5$, can be estimated similarly (for M_5, we estimate $|\nabla\mathbf{u}^{(2)}|$ in L^∞-norm via $W^{3,2}(\Omega)$-norm of $\mathbf{u}^{(2)}$ and $|\nabla\widehat{V}^{(2)}_{\chi}|$ via $W^{3,2}(\sigma_j)$-norm of $\widehat{V}^{(2)}_{\chi}$), and we derive the following estimate for the norm of $\mathbf{M}(\mathbf{u}^{(1)},\mathbf{u}^{(2)})$:

$$
\int\limits_{\Omega^{(2k_Q)}}\big|\mathbf{M}(\mathbf{u}^{(1)},\mathbf{u}^{(2)})\big|^2\,\mathrm{d}x\leqslant c\big(\|\mathbf{u}\|^2_{W^{2,2}(\Omega^{(k_Q)})}+Q|\ln Q|\big). \tag{64}
$$

This gives the estimate for K_5:

$$
\begin{aligned}
|K_5|&\leqslant c(\varepsilon)\int\limits_{\Omega^{(2k_Q)}}\big|\mathbf{M}(\mathbf{u}^{(1)},\mathbf{u}^{(2)})\big|^2\,\mathrm{d}x+\varepsilon\int\limits_{\Omega^{(2k_Q)}}|\boldsymbol{\varphi}|^2\,\mathrm{d}x\\
&\leqslant cQ|\ln Q|+\varepsilon\int\limits_{\Omega^{(k_Q)}}|\nabla\mathbf{u}|^2\,\mathrm{d}x+c\|\mathbf{u}\|^2_{W^{2,2}(\Omega^{(k_Q)})}.
\end{aligned}\tag{65}
$$

Substituting (63), (65) into (60) and choosing ε sufficiently small, we obtain the estimate

$$
\int\limits_{\Omega^{(k_Q)}}|\nabla\mathbf{u}|^2\,\mathrm{d}x\leqslant c\lambda\|\mathbf{u}\|^2_{W^{2,2}(\Omega^{(k_Q)})}+cQ|\ln Q|. \tag{66}
$$

Now we apply the local ADN estimate (6) to the solution $(\mathbf{u},q)$ of problem (49),

$$
\begin{aligned}
&\|\mathbf{u}\|^2_{W^{2,2}(\Omega^{(k_Q)})}+\|\nabla q\|^2_{L^2(\Omega^{(k_Q)})}\\
&\quad\leqslant c\big(\|\mathbf{g}\|^2_{L^2(\Omega^{(3)})}+\|\mathbf{f}\|^2_{L^2(\Omega^{(2k_Q)})}+\lambda\|\mathbf{N}\|^2_{L^2(\Omega^{(2k_Q)})}\\
&\qquad+\lambda\|\mathbf{M}\|^2_{L^2(\Omega^{(2k_Q)})}+\|\nabla\mathbf{u}\|^2_{L^2(\Omega^{(2k_Q)})}\big).
\end{aligned}\tag{67}
$$

Applying estimates (50), (66), (64) and (62) to the right-hand side of (67), we derive

$$\begin{aligned}
&\|\mathbf{u}\|^2_{W^{2,2}(\Omega^{(k_Q)})} + \|\nabla q\|^2_{L^2(\Omega^{(k_Q)})} \\
&\quad \leqslant c\big(Q|\ln Q| + \lambda\|\mathbf{u}\|^2_{W^{2,2}(\Omega^{(k_Q)})} + \|\nabla\mathbf{u}\|^2_{L^2(\Omega\setminus\Omega^{(k_Q)})}\big) \\
&\quad \leqslant c\big(Q|\ln Q| + \lambda\|\mathbf{u}\|^2_{W^{2,2}(\Omega^{(k_Q)})}\big).
\end{aligned} \tag{68}$$

Thus, if $c\lambda < 1/2$, from (68) follows estimate (48).

Let us estimate the differences $q_{j0} = q_j^{(1)} - q_j^{(2)}$, $j = 1, \dots, J$. Let the pressure $q(x) = q^{(1)}(x) - q^{(2)}(x)$ be normalized by the condition $\int_{\Omega^{(3)}} q(x)\,\mathrm{d}x = 0$. Then q satisfies the inequality $\|q\|^2_{L^2(\Omega^{(3)})} \leqslant c\|\nabla q\|^2_{L^2(\Omega^{(3)})}$. Denote $\bar q_j(x_1^{(j)}) = \int_{\sigma_j} q(x_1^{(j)}, x^{(j)\prime})\,\mathrm{d}x^{(j)\prime}$. Since $\int_{\Omega_j} |q(x) - q_{j0}|^2\,\mathrm{d}x < +\infty$ (see Lemma 3), we can assume, without loss of generality, that k_Q is chosen such that $\int_{\Omega_j\setminus\Omega^{(k_Q)}} |q(x) - q_{j0}|^2\,\mathrm{d}x \leqslant Q$. Then

$$\begin{aligned}
&\left|\int_{k_Q}^{k_Q+1} \big(\bar q_j\big(x_1^{(j)}\big) - q_{j0}\operatorname{mes}\sigma_j\big)\,\mathrm{d}x_1^{(j)}\right| \\
&\quad = \left|\int_{(k_Q,\,k_Q+1)\times\sigma_j} \big(q(x) - q_{j0}\big)\,\mathrm{d}x\right| \leqslant \sqrt{\operatorname{mes}\sigma_j}\,\big\|q(x) - q_{j0}\big\|_{L^2(\Omega_j\setminus\Omega^{(k_Q)})} \\
&\quad \leqslant \sqrt{\operatorname{mes}\sigma_j Q}.
\end{aligned}$$

So,

$$\begin{aligned}
\operatorname{mes}\sigma_j|q_{j0}| &\leqslant \left|\int_{k_Q}^{k_Q+1} \big(\bar q_j(x_1^{(j)}) - q_{j0}\operatorname{mes}\sigma_j\big)\,\mathrm{d}x_1^{(j)}\right| + \left|\int_{k_Q}^{k_Q+1} \bar q_j\big(x_1^{(j)}\big)\,\mathrm{d}x_1^{(j)}\right| \\
&\leqslant \sqrt{\operatorname{mes}\sigma_j}\sqrt{Q} + \left|\int_{k_Q}^{k_Q+1} \left(\bar q_j(x_1^{(j)} - k_Q) + \int_{x_1^{(j)}-k_Q}^{x_1^{(j)}} \bar q_j'(t)\,\mathrm{d}t\right)\mathrm{d}x_1^{(j)}\right| \\
&\leqslant \sqrt{\operatorname{mes}\sigma_j}\sqrt{Q} + \left|\int_0^1 \bar q_j\big(x_1^{(j)}\big)\,\mathrm{d}x_1^{(j)}\right| + \int_0^{k_Q+1} \big|\bar q_j'\big(x_1^{(j)}\big)\big|\,\mathrm{d}x_1^{(j)} \\
&\leqslant \sqrt{\operatorname{mes}\sigma_j}\sqrt{Q} + \|q\|_{L^2(\Omega^{(3)})}\sqrt{\operatorname{mes}\sigma_j} + \|\nabla q\|_{L^2(\Omega)}\sqrt{\operatorname{mes}\sigma_j}\sqrt{k_Q+1} \\
&\leqslant \sqrt{\operatorname{mes}\sigma_j}\sqrt{Q} + C\|\nabla q\|_{L^2(\Omega^{(3)})} + C\|\nabla q\|_{L^2(\Omega)}\sqrt{|\ln Q|} \\
&\leqslant C\sqrt{Q}|\ln Q|,
\end{aligned}$$

and thus, $|q_{j0}|^2 \leqslant CQ(\ln Q)^2$. □

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