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# Analysis of fractional hybrid differential equations with impulses in partially ordered Banach algebras*

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**Abstract.** In this paper, we investigate a class of fractional hybrid differential equations with impulses, which can be seen as nonlinear differential equations with a quadratic perturbation of second type and a linear perturbation in partially ordered Banach algebras. We deduce the existence and approximation of a mild solution for the initial value problems of this system by applying Dhage iteration principles and related hybrid fixed point theorems. Compared with previous works, we generalize the results to fractional order and extend some existing conclusions for the first time. Meantime, we take into consideration the effect of impulses. Our results indicate the influence of fractional order for nonlinear hybrid differential equations and improve some known results, which have wider applications as well. A numerical example is included to illustrate the effectiveness of the proposed results.

**Keywords:** impulsive fractional differential equation, hybrid fixed point theorem, Dhage iteration principle, quadratic.

## 1 Introduction

Numerous mathematical concepts have been presented to describe or characterize many phenomena in the real world. Differential equation, as one of the most valuable concepts, has been widely used in constructing mathematical models, which can be regarded as an essential tool toward the comprehension of nature. (See, for instance, [21, 23, 27] for applications tied to models from mathematical biology and physics.) On account of the significance of the theory and application of differential equation, it attracts scholars

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interest extensively to study the qualitative theories of systems constructed by differential equation. Especially, the theory of solvability is one of the fundamental and important topics in practice, such as the existence of the solutions for differential equation; we can refer to [18, 22, 30] for the recent developments.

The existence results about nonlinear systems utilize differential equations of integer order in the majority. As is known to us, fractional dynamic is widely applied in various fields of science, like physics, engineering, biology and modeling infectious diseases, neural networks, as well as anomalous diffusion [17, 26, 32]. For instance, scholars [25] simulated the outbreak of influenza A(H1N1) using a fractional-order SEIR model. Recently, it has a rapid development [1, 2, 19, 24, 25, 28].

One of the methods for solving the existence of solutions of nonlinear systems is relying on fixed point theorems. If the fixed point theory involves the mixed hypotheses of algebra, topology, and geometry, then it is called hybrid fixed point theory. Hybrid fixed point theorem can do well in solving existence of hybrid differential equations, which is also useful for the theory of nonlinear quadratic differential equations; see in literature [6, 20]. Recently, many researchers like Dhage [9, 11], Lakshmikantham [7], Bashiri [5] give several hybrid fixed point theorems to help perfecting the theories. Achievements have been made in this area.

In 2011, Zhao [31] studied fractional quadratic differential equations involving Riemann–Liouville differential operators of order $0 < \alpha \leqslant 1$

$$\begin{aligned} &D^{\alpha}\left[\frac{x(t)}{f(t,x(t))}\right] = g\bigl(t,x(t)\bigr), \quad t \in J, \\ &x(0) = 0. \end{aligned}$$

The existence theorem for fractional quadratic differential equations was proved under mixed Lipschitz and Caratheodory conditions.

Recently, Dhage [8] introduced a notion of partially condensing mappings in a partially ordered normed linear space and proved some hybrid fixed point theorems under certain mixed conditions of algebra, analysis, and topology. The author combined the compactness with some algebraic arguments to prove some hybrid fixed point theorems for the mappings in ordered spaces and apply the results to obtain the solutions of integral equations under some mixed compactness and monotonic conditions, which was the main contribution of the works; see Dhage [10, 12–15] for recent development.

In 2015, Dhage [16] considered the periodic boundary value problem of first-order nonlinear quadratic ordinary differential equations

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{x(t)}{f(t,x(t))}\right] + \lambda\left[\frac{x(t)}{f(t,x(t))}\right] = h\bigl(t,x(t)\bigr), \quad t \in J, \\ &x(0) = x(T), \end{aligned}$$

for $\lambda > 0$, $J = [0, T]$. They applied iteration principle embodied in hybrid fixed point theorem in partially ordered normed linear spaces to obtain the existence as well as approximations of the positive solutions for this system.

In 2020, Ardjouni [3] investigated the nonlinear third-order hybrid differential equation

$$\left(\frac{x(t)}{p(t)+\int_{t_0}^{t} g(s,x(s))\,\mathrm{d}t}\right)''' = h\big(t,x(t)\big), \quad t\in J,\ J=[0,T],$$
$$\left(\frac{x(t)}{p(t)+\int_{t_0}^{t} g(s,x(s))\,\mathrm{d}t}\right)^{(k)}\bigg|_{t=t_0} = a_k, \quad k=0,1,2.$$

By employing Dhage iteration principle in a partially ordered normed linear space they proved the existence and approximation of solutions of the initial value problems of this system.

The previous works were mostly based on differential equations of integer order. We intend to study a class of hybrid fractional differential equations with impulses in a partially ordered linear algebra, which contains linear perturbations and quadratic perturbations in the meantime. Our results not only rely on iteration principles and related hybrid fixed point theorems, which have been proposed, but also extend some known theories to fractional-order systems. Furthermore, the conclusions may be subject to the influence of impulses to some extent. These advantages inspire us to continue current works.

In this paper, we discuss the following hybrid fractional (in the sense of Caputo) differential equation with impulses, which is given by

$$\begin{aligned} &{}^cD^{\alpha}\left[\frac{x(t)-f(t,x(t))}{g(t,x(t))}\right] = h\big(t,x(t)\big), \quad t\in J',\\ &\Delta x(t_k) = I_k\big(x(t_k)\big), \quad k=1,\dots,m,\\ &x(0)=x_0, \end{aligned} \tag{1}$$

where ${}^cD^{\alpha}$ denotes the Caputo fractional derivative of order $\alpha$, $0<\alpha\leqslant 1$, $f,h\in C(J\times\mathbb{R},\mathbb{R}^+)$, $g\in C(J\times\mathbb{R},\mathbb{R}^+\setminus\{0\})$, $I_k(x(t_k))\in C(\mathbb{R},\mathbb{R})$, $k=1,\dots,m$, $J=[0,1]$, $0=t_0<t_1<\dots<t_k<\dots<t_m<t_{m+1}=1$, $J'=J\setminus\{t_1,\dots,t_m\}$, $J_0=[0,t_1]$, $J_1=(t_1,t_2],\dots,J_m=(t_m,1]$, $\Delta x(t_k)=x(t_k^+)-x(t_k^-)$, $x(t_k^-)=x(t_k)$, $k=1,\dots,m$.

The plan of this paper is as follows. In Section 2, we recall some necessary notions, definitions, and lemmas that used in the rest. In Section 3, we apply hybrid fixed point theorem to solve the initial value problems of (1) and give rigorous proof in the meantime. In Section 4, an example is given to illustrate our conclusions.

## 2 Preliminaries

In this section, we recall some basic and essential definitions of fractional calculus. Meanwhile, we introduce some definitions, lemmas, as well as relative theories in linear partially ordered spaces for better obtaining our main results.

**Definition 1.** (See [26].) The Riemann–Liouville fractional integral of order $\alpha > 0$ of a function $f : [0, +\infty) \to \mathbb{R}$ is given by

$$I_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\,\mathrm{d}s, \quad t > 0,$$

where $\Gamma(\alpha)$ is the Gamma function, provided the right side is pointwise defined on $[0, +\infty)$.

**Definition 2.** (See [26].) The Caputo fractional derivative of order $\alpha > 0$ for a function $f \in C^n[0, 1]$ is defined by

$${}^cD^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)\,\mathrm{d}s,$$

$n - 1 < \alpha < n$, $n = [\alpha] + 1$.

**Lemma 1.** (*See* [26].) *Let* $\alpha > 0$. *Then the solution of the fractional differential equation*

$${}^cD^{\alpha} f(t) = 0$$

*is*

$$f(t) = C_0 + C_1 t + C_2 t^2 + \cdots + C_{n-1} t^{n-1},$$

$C_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n$, $n = [\alpha] + 1$.

Unless otherwise specified, we assume that $(S, \preccurlyeq, \|\cdot\|)$ denotes a partially ordered Banach space with an order relation $\preccurlyeq$ and the norm $\|\cdot\|$. Two elements $x$ and $y$ in $S$ are said to be comparable if either the relation $x \preccurlyeq y$ or $y \preccurlyeq x$ holds. A nonempty subset $C$ of $S$ is called a chain or totally ordered if all the elements of $C$ are comparable. The set $S$ is said to be regular if each sequence $\{x_n\}$ is nondecreasing (resp. nonincreasing) in $S$ such that $x_n \to x^*$ as $n \to \infty$, then $x_n \preccurlyeq x^*$ (resp. $x_n \succcurlyeq x^*$) for all $n \in \mathbb{N}$. The details can be found in [8] and the reference therein.

**Definition 3.** (See [16].) A mapping $F : S \to S$ is called isotone or nondecreasing if it preserves the order relation $\preccurlyeq$, that is, if $x \preccurlyeq y$ implies $Fx \preccurlyeq Fy$ for all $x, y \in S$.

**Definition 4.** (See [16].) A mapping $F : S \to S$ is called partially continuous at a point $a \in S$ if for $\epsilon > 0$, there exists a $\delta > 0$ such that $\|Fx - Fa\| < \epsilon$ whenever $x$ is comparable to $a$ and $\|x - a\| < \delta$. $F$ is called partially continuous on $S$ if it is partially continuous at every point of $S$. It is clear that if $F$ is partially continuous on $S$, then it is continuous on every chain $C$ contained in $S$.

**Definition 5.** (See [16].) A mapping $F : S \to S$ is called partially bounded if $F(C)$ is bounded for every chain $C$ in $S$. $F$ is called uniformly partially bounded if $F(C)$ is bounded for every chain $C$ in $S$ by a unique constant. $F$ is called bounded if $F(S)$ is a bounded subset of $S$.

**Definition 6.** (See [4].) The set $S$ is said to be quasi-equicontinuous in $[0, c]$ if for any $\varepsilon > 0$, there exists a $\delta > 0$ such that if $u \in S$, $k \in \mathbb{Z}^+$, $t', t'' \in (t_{k-1}, t_k] \bigcap [0, c]$, and $|t' - t''| < \delta$, then $|u(t') - u(t'')| < \varepsilon$.

**Lemma 2 [Compactness criterion].** (*See* [4].) *The set* $S \subset PC([0, c], \mathbb{R}^n)$ *(*$c < +\infty$*) is relatively compact if and only if*

(i) $S$ *is bounded,*
(ii) $S$ *is quasi-equicontinuous in* $[0, c]$.

**Definition 7.** (See [16].) A mapping $F : S \to S$ is called partially compact if $F(C)$ is a relatively compact subset of $S$ for all totally ordered sets or chains $C$ in $S$.

**Definition 8.** (See [13,14].) The order relation $\preccurlyeq$ and the metric $d$ on a nonempty set $S$ are said to be $D$-compatible if $\{x_n\}$ is a monotone sequence, that is, monotone nondecreasing or monotone nondecreasing sequence in $S$ and if a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to $x^*$ implies that the original sequence $\{x_n\}$ converges to $x^*$. Similarly, given a partially ordered normed linear space $(S, \preccurlyeq, \|\cdot\|)$, the order relation $\preccurlyeq$ and the norm $\|\cdot\|$ are said to be $D$-compatible if $\preccurlyeq$ and the metric $d$ defined through the norm $\|\cdot\|$ are $D$-compatible. A subset $E$ of $S$ is called a Janhavi set if the order relation $\preccurlyeq$ and the metric $d$ or the norm $\|\cdot\|$ are $D$-compatible in it. In particular, if $E = S$, then $S$ is called a Janhavi metric space or Janhavi Banach space.

**Definition 9.** (See [16].) A upper semicontinuous and nondecreasing function $\varphi$: $\mathbb{R}^+ \to \mathbb{R}^+$ is called a $D$-function, provided $\varphi(0) = 0$. Let $(S, \preccurlyeq, \|\cdot\|)$ be a partially ordered normed linear space. A mapping $T : S \to S$ is called partially nonlinear $D$-Lipschitz if there exists a $D$-function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\|Tx - Ty\| \leqslant \varphi(\|x - y\|)$ for all comparable elements $x, y \in S$. If $\varphi(x) = kx$, $k > 0$, then $T$ is called a partially Lipschitz with a Lipschitz constant $k$.

We recall that a nonempty closed and convex subset $\mathcal{K}$ of the Banach algebra $S$ is called a cone if

(i) $\mathcal{K} + \mathcal{K} \subseteq \mathcal{K}$,
(ii) $\lambda\mathcal{K} \subseteq \mathcal{K}$ for $\lambda \in \mathbb{R}$, $\lambda > 0$, and
(iii) $\{-\mathcal{K}\} \cap \mathcal{K} = \{\theta\}$, where $\theta$ is a zero element of $S$.

The cone $\mathcal{K}$ in $S$ is called positive if

(i) $\mathcal{K} \circ \mathcal{K} \subseteq \mathcal{K}$, where "$\circ$" is a multiplicative composition in $S$; see Dhage [10] and the reference therein.

Now we assume that $\mathcal{K}$ denotes a positive cone.

**Lemma 3.** (*See* [16].) *If* $x_1, x_2, y_1, y_2 \in \mathcal{K}$ *are such that* $x_1 \preccurlyeq y_1$ *and* $x_2 \preccurlyeq y_2$*, then* $x_1 x_2 \preccurlyeq y_1 y_2$.

**Lemma 4.** (*See* [14].) *Every ordered Banach space* $(S, \mathcal{K})$ *is regular.*

**Lemma 5.** (*See* [14].) *Every partially compact subset* $E$ *of an ordered Banach space* $(S, \mathcal{K})$ *is a Janhavi set.*

**Lemma 6.** (*See* [9].) *Let $(S, \preccurlyeq, \|\cdot\|)$ be a regular partially ordered complete normed linear algebra, and let every compact chain $C$ in $S$ be Janhavi set. Let $\mathcal{A}, \mathcal{B} : S \to \mathcal{K}$ and $\mathcal{C} : S \to S$ be three nondecreasing operators such that*:

(i) *$\mathcal{A}$ and $\mathcal{C}$ are partially bounded and partially nonlinear $D$-Lipschitz with $D$-functions $\psi$ and $\varphi$, respectively*;

(ii) *$\mathcal{B}$ is partially continuous and compact*;

(iii) *$L\psi(r) + \varphi(r) < r$, $r > 0$, where $L = \sup\{\|\mathcal{B}(C)\|$: $C$ is a chain in $S\}$; and*

(iv) *there exists an element $x_0 \in S$ such that $x_0 \preccurlyeq \mathcal{A}x_0 \cdot \mathcal{B}x_0 + \mathcal{C}x_0$ or $x_0 \succeq \mathcal{A}x_0 \cdot \mathcal{B}x_0 + \mathcal{C}x_0$.*

*Then the operator equation $\mathcal{A}x \cdot \mathcal{B}x + \mathcal{C}x = x$ has a solution $x^* \in S$, and the sequence $\{x_n\}$ of successive iterations defined by $x_{n+1} = \mathcal{A}x_n \cdot \mathcal{B}x_n + \mathcal{C}x_n$, $n = 0, 1, \dots,$ converges monotonically to $x^*$.*

We consider the space $PC(J, \mathbb{R})$ defined on $J$ with norm $\|\cdot\|$, which is given by

$$PC(J,\mathbb{R}) := \bigl\{x : J \to \mathbb{R} \bigm| x \in C(J_k),\ x\bigl(t_k^+\bigr), x\bigl(t_k^-\bigr) \text{ exist}, \\ x\bigl(t_k^-\bigr) = x(t_k),\ k = 0, 1, \dots, m\bigr\},$$

$\|x\| = \sup_{t\in J} |x(t)|$. Clearly, $PC(J, \mathbb{R})$ is a Banach space with the supremum norm. We define the order cone $K$ in $PC(J, \mathbb{R})$ by

$$K = \bigl\{x \in PC(J,\mathbb{R}) \bigm| x(t) \geqslant 0 \text{ for all } t \in J\bigr\},$$

which is a positive cone in $PC(J, \mathbb{R})$, and define order relation $\preccurlyeq$ in $PC(J, \mathbb{R})$ by

$$x \preccurlyeq y \quad \Longleftrightarrow \quad x(t) \leqslant y(t), \quad t \in J.$$

Clearly, $(PC(J, \mathbb{R}), K)$ is a regular ordered Banach space with respect to the above norm and order relation in $PC(J, \mathbb{R})$, and every compact chain $C$ in $PC(J, \mathbb{R})$ is Janhavi set in view of the Lemmas 4 and 5. For the sake of convenience, we denote $f(0, 0) := f_0$, $g(0, 0) := g_0$ in the sequel.

**Lemma 7.** (*See* [29].) *Let $x(t) \in C^n[0, 1]$ be a solution of the following impulsive hybrid fractional differential equation*:

$$\begin{aligned} &{}^{c}D^{\alpha}\left(\frac{x(t) - f(t, x(t))}{g(t, x(t))}\right) = h(t), \quad t \in J', \\ &\Delta x(t_k) = I_k\bigl(x(t_k)\bigr), \quad k = 1, 2, \dots, m, \\ &x(0) = x_0. \end{aligned} \tag{2}$$

*Then it satisfies the following impulsive hybrid fractional integral equation*:

$$x(t) = g\bigl(t, x(t)\bigr)\left(\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}h(s)\,\mathrm{d}s + \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}h(s)\,\mathrm{d}s \right. \\ \left. + \sum_{i=1}^{k}\frac{I_i(x(t_i))}{g(t_i, x(t_i))} + \frac{x_0 - f_0}{g_0}\right) + f\bigl(t, x(t)\bigr), \quad t \in J_k,\ k = 0, 1, \dots, m.$$

**Definition 10.** Let $x(t) \in PC(J, \mathbb{R})$ satisfy

$$x(t) = g\big(t, x(t)\big)\left(\frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} h(s)\,\mathrm{d}s + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} h(s)\,\mathrm{d}s \right.$$
$$\left. + \sum_{i=1}^{k} \frac{I_i(x(t_i))}{g(t_i, x(t_i))} + \frac{x_0 - f_0}{g_0}\right) + f\big(t, x(t)\big), \quad t \in J.$$

Then the function $x$ is called a mild solution of IVPs (2).

**Definition 11.** Let $u \in PC(J, \mathbb{R})$ satisfy

$$u(t) \leqslant g\big(t, u(t)\big)\left(\frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} h(s)\,\mathrm{d}s + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} h(s)\,\mathrm{d}s \right.$$
$$\left. + \sum_{i=1}^{k} \frac{I_i(u(t_i))}{g(t_i, u(t_i))} + \frac{x_0 - f_0}{g_0}\right) + f\big(t, u(t)\big), \quad t \in J.$$

Then the function $u$ is called a mild lower solution of IVPs (2).

## 3 Main results

In this section, we attempt to obtain the existence and approximation for a mild solution of IVPs for impulsive hybrid fractional differential equation (1).

In order to obtain our main results, hypotheses (H1)–(H5) are given by:

(H1) There exist positive constants $M$ and $N$ such that

$$0 < f(t, x) \leqslant M, \qquad 0 < g(t, x) \leqslant N,$$

and $(x_0 - f_0)/g_0 \geqslant 0$, where $t \in J$, $x \in \mathbb{R}$.

(H2) There exist $D$-functions $\varphi$ and $\psi$ such that

$$0 \leqslant f(t, x) - f(t, y) \leqslant \varphi(x - y), \qquad 0 \leqslant g(t, x) - g(t, y) \leqslant \psi(x - y),$$

where $t \in J$, $x, y \in \mathbb{R}$, $x \geqslant y$.

(H3) $h(t, x)$ is nondecreasing in $x$ for each $t \in J$, and there exists a constant $K$ such that

$$0 < h(t, x) \leqslant K,$$

where $t \in J$, $x \in \mathbb{R}$.

(H4) $I_k(x)/g(t, x)$ is nondecreasing in $x$ for each $t \in J$, and there exists a constant $H$ such that

$$0 \leqslant \frac{I_k(x)}{g(t, x)} \leqslant H,$$

where $t \in J$, $x \in \mathbb{R}$, $k = 1, \dots, m$.

(H5) IVPs (1) has a mild lower solution $u \in PC(J, \mathbb{R})$.

**Theorem 1.** *Under assumptions* (H1)–(H5) *and further, we assume that*

$$\overline{L}\psi(r) + \varphi(r) < r, \quad r > 0,$$

*where we denote*

$$\overline{L} = \frac{(m+1)K}{\Gamma(\alpha+1)} + \frac{x_0 - f_0}{g_0} + (m+1)H. \tag{3}$$

*Then IVPs* (1) *has a mild solution $x^*$, and the sequence $\{x_n\}$ of successive approximations defined by*

$$\begin{aligned} x_1(t) &= u(t), \\ x_{n+1}(t) &= g\big(t, x_n(t)\big)\Bigg(\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}h\big(s, x_n(s)\big)\,\mathrm{d}s \\ &\quad + \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}h\big(s, x_n(s)\big)\,\mathrm{d}s + \sum_{i=1}^{k}\frac{I_i(x_n(t_i))}{g(t_i, x_n(t_i))} \\ &\quad + \frac{x_0 - f_0}{g_0}\Bigg) + f\big(t, x_n(t)\big), \quad t \in J_k,\ k = 0, 1, \dots, m, \end{aligned} \tag{4}$$

*converges monotonically to $x^*$.*

*Proof.* At first, we define $S = PC(J, \mathbb{R})$. Then by Lemma 5 we have that every compact chain $C$ in $S$ possesses the compatibility property with respect to the norm $\|\cdot\|$, and the order relation $\preccurlyeq$ so that every compact chain $C$ is a Janhavi set in $S$.

By Lemma 7 a mild solution of impulsive hybrid fractional differential equation (1) is a solution for impulsive hybrid integral equation

$$\begin{aligned} x(t) &= g\big(t, x(t)\big)\Bigg(\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}h\big(s, x(s)\big)\,\mathrm{d}s \\ &\quad + \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}h\big(s, x(s)\big)\,\mathrm{d}s + \sum_{i=1}^{k}\frac{I_i(x(t_i))}{g(t_i, x(t_i))} + \frac{x_0 - f_0}{g_0}\Bigg) \\ &\quad + f\big(t, x(t)\big), \quad t \in J_k,\ k = 0, 1, \dots, m. \end{aligned} \tag{5}$$

Define operators $\mathcal{A}, \mathcal{B} : S \to K$ and $\mathcal{C} : S \to S$ by

$$\mathcal{A}x(t) = g\big(t, x(t)\big), \qquad \mathcal{C}x(t) = f\big(t, x(t)\big),$$

$$\begin{aligned} \mathcal{B}x(t) &= \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}h\big(s, x(s)\big)\,\mathrm{d}s + \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}h\big(s, x(s)\big)\,\mathrm{d}s \\ &\quad + \sum_{i=1}^{k}\frac{I_i(x(t_i))}{g(t_i, x(t_i))} + \frac{x_0 - f_0}{g_0}, \quad t \in J_k,\ k = 0, 1, \dots, m. \end{aligned}$$

In this way, the impulsive hybrid fractional integral equation (5) can be written as the operator equation

$$\mathcal{A}x(t)\cdot\mathcal{B}x(t)+\mathcal{C}x(t)=x(t),\quad x\in S,\ t\in J_k,\ k=0,1,\dots,m.$$

For the sake of showing our main results, we prove that $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ satisfy the conditions of Lemma 6. This is achieved in the following steps.

*Step 1.* $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are nondecreasing operators on $S$.

Let $x,y\in S$ be such that $x\succeq y$. Then by hypothesis (H2) we obtain

$$\begin{aligned}\mathcal{A}x(t)&=g\bigl(t,x(t)\bigr)\geqslant g\bigl(t,y(t)\bigr)=\mathcal{A}y(t),\\ \mathcal{C}x(t)&=f\bigl(t,x(t)\bigr)\geqslant f\bigl(t,y(t)\bigr)=\mathcal{C}y(t),\end{aligned}$$

for all $t\in J_k$, $k=0,1,\dots,m$. By the definition of the order relation in $S$ we get $\mathcal{A}x\succeq\mathcal{A}y,\mathcal{C}x\succeq\mathcal{C}y$. This shows that $\mathcal{A}$ and $\mathcal{C}$ are nondecreasing operators. Similarly, using hypotheses (H3), (H4), we can get

$$\begin{aligned}\mathcal{B}x(t)&=\frac{1}{\Gamma(\alpha)}\int\limits_{t_k}^{t}(t-s)^{\alpha-1}h\bigl(s,x(s)\bigr)\,\mathrm{d}s+\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\int\limits_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}h\bigl(s,x(s)\bigr)\,\mathrm{d}s\\ &\quad+\sum_{i=1}^{k}\frac{I_i(x(t_i))}{g(t_i,x(t_i))}+\frac{x_0-f_0}{g_0}\\ &\geqslant\frac{1}{\Gamma(\alpha)}\int\limits_{t_k}^{t}(t-s)^{\alpha-1}h\bigl(s,y(s)\bigr)\,\mathrm{d}s+\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\int\limits_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}h\bigl(s,y(s)\bigr)\,\mathrm{d}s\\ &\quad+\sum_{i=1}^{k}\frac{I_i\bigl(y(t_i)\bigr)}{g(t_i,y(t_i))}+\frac{y_0-f_0}{g_0}\\ &=By(t),\quad t\in J_k,\ k=0,1,\dots,m.\end{aligned}$$

It is shown that the operator $\mathcal{B}$ is also nondecreasing. Thus, $\mathcal{A},\mathcal{B}$, and $\mathcal{C}$ are nondecreasing positive operators.

*Step 2.* We note that $\mathcal{A},\mathcal{C}$ are partially bounded and partially $D$-Lipschitz operators on $S$.

Let $x\in S$ be arbitrary. Then by (H1)

$$\bigl|\mathcal{A}x(t)\bigr|=\bigl|g\bigl(t,x(t)\bigr)\bigr|\leqslant N,\qquad\bigl|\mathcal{C}x(t)\bigr|=\bigl|f\bigl(t,x(t)\bigr)\bigr|\leqslant M$$

for all $t\in J_k$, $k=0,1,\dots,m$. Taking supremum over $t$, we obtain $\|\mathcal{A}x\|\leqslant N$, $\|\mathcal{C}x\|\leqslant M$. So, $\mathcal{A}$ and $\mathcal{C}$ are bounded. This further implies that $\mathcal{A}$ and $\mathcal{C}$ are partially bounded on $S$.

Let $x, y \in S$ and $x \succeq y$. Then by (H2) we have

$$\begin{aligned}\big|\mathcal{A}x(t) - \mathcal{A}y(t)\big| &= \big|g\big(t, x(t)\big) - g\big(t, y(t)\big)\big| \leqslant \psi\big(\big|x(t) - y(t)\big|\big) \leqslant \psi\big(\|x - y\|\big),\\ \big|\mathcal{C}x(t) - \mathcal{C}y(t)\big| &= \big|f\big(t, x(t)\big) - f\big(t, y(t)\big)\big| \leqslant \varphi\big(\big|x(t) - y(t)\big|\big) \leqslant \varphi\big(\|x - y\|\big).\end{aligned}$$

Taking the supremum over $t$, we obtain

$$\|\mathcal{A}x - \mathcal{A}y\| \leqslant \psi\big(\|x - y\|\big), \qquad \|\mathcal{C}x - \mathcal{C}y\| \leqslant \varphi\big(\|x - y\|\big).$$

Therefore, $\mathcal{A}$ and $\mathcal{C}$ are partially nonlinear $D$-Lipschitz on $S$. Assumption (i) of Lemma 6 is satisfied.

*Step 3*. We show that $\mathcal{B}$ is a partially continuous operator on $S$.

At first, let $\{x_n\}$ be a sequence in a chain $C$ of $S$, which satisfies $x_n \to x$, $n \to \infty$. Then by Lebesgue dominated convergence theorem

$$\begin{aligned}&\lim_{n\to\infty} \mathcal{B}x_n(t)\\ &\quad= \frac{1}{\Gamma(\alpha)} \lim_{n\to\infty} \int_{t_k}^{t} (t-s)^{\alpha-1} h\big(s, x_n(s)\big)\,\mathrm{d}s\\ &\qquad+ \frac{1}{\Gamma(\alpha)} \lim_{n\to\infty} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} h\big(s, x_n(s)\big)\,\mathrm{d}s + \lim_{n\to\infty} \sum_{i=1}^{k} \frac{I_i(x_n(t_i))}{g(t_i, x_n(t_i))}\\ &\qquad+ \frac{u_0 - f_0}{g_0}\\ &= \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} h\big(s, x(s)\big)\,\mathrm{d}s + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} h\big(s, x(s)\big)\,\mathrm{d}s\\ &\qquad+ \sum_{i=1}^{k} \frac{I_i(x(t_i))}{g(t_i, x(t_i))} + \frac{x_0 - f_0}{g_0}\\ &= \mathcal{B}x(t), \quad t \in J_k,\ k = 0, 1, \dots, m.\end{aligned}$$

This indicates that $\mathcal{B}x_n$ converges monotonically to $\mathcal{B}x$ pointwise on $J_k$, $k = 0, 1, \dots, m$.

Next, we prove that $\mathcal{B}x_n$ is a quasi-equicontinuous sequence of function in $S$. Let $t', t'' \in J_k$ be arbitrary, $k = 0, 1, \dots, m$. Then by hypothesis (H3)

$$\begin{aligned}&\big|\mathcal{B}x_n(t'') - \mathcal{B}x_n(t')\big|\\ &\quad\leqslant \left|\frac{1}{\Gamma(\alpha)} \int_{t_k}^{t''} (t'' - s)^{\alpha-1} h\big(s, x_n(s)\big)\,\mathrm{d}s - \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t'} (t'' - s)^{\alpha-1} h\big(s, x_n(s)\big)\,\mathrm{d}s\right|\end{aligned}$$

$$\leqslant \left|\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t'}(t'-s)^{\alpha-1}h\bigl(s,x_n(s)\bigr)\,\mathrm{d}s-\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t'}(t''-s)^{\alpha-1}h\bigl(s,x_n(s)\bigr)\,\mathrm{d}s\right|$$
$$+\left|\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t'}(t''-s)^{\alpha-1}h\bigl(s,x_n(s)\bigr)\,\mathrm{d}s-\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t''}(t''-s)^{\alpha-1}h\bigl(s,x_n(s)\bigr)\,\mathrm{d}s\right|$$
$$\leqslant \frac{K}{\Gamma(\alpha)}\left(\left|\int_{t_k}^{t'}(t'-s)^{\alpha-1}-(t''-s)^{\alpha-1}\,\mathrm{d}s\right|+\left|\int_{t'}^{t''}(t''-s)^{\alpha-1}\,\mathrm{d}s\right|\right)$$
$$=\frac{K}{\Gamma(\alpha)}\left(\frac{|(t''-s)^{\alpha}-(t'-s)^{\alpha}|}{\alpha}\bigg|_{t_k}^{t'}+\frac{|(t'-t'')^{\alpha}|}{\alpha}\right).$$

Note that $0<\alpha\leqslant 1$, then we have

$$\bigl|(t''-s)^{\alpha}-(t'-s)^{\alpha}\bigr|\leqslant\bigl|(t''-t')^{\alpha}\bigr|.$$

So

$$\bigl|\mathcal{B}x_n(t'')-\mathcal{B}x_n(t')\bigr|\leqslant\frac{2|(t'-t'')^{\alpha}|}{\alpha}\to 0$$

as $t'\to t''$ for all $n\in\mathbb{N}$, which shows that the convergence $\mathcal{B}x_n\to\mathcal{B}x$ is uniform, and hence, $\mathcal{B}$ is partially continuous on $S$.

*Step 4*. $\mathcal{B}$ is a partially compact operator on $S$.

Let $C$ be an arbitrary chain in $S$. We need to prove that $\mathcal{B}(C)$ is a uniformly bounded set in $S$. Let $y\in\mathcal{B}(C)$ be any element. Then there is an element $x\in C$ such that $y=\mathcal{B}x$. Now by hypotheses (H3), (H4) we have

$$\bigl|y(t)\bigr|=\bigl|\mathcal{B}x(t)\bigr|$$
$$\leqslant\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}\bigl|h\bigl(s,x(s)\bigr)\bigr|\,\mathrm{d}s+\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\bigl|h\bigl(s,x(s)\bigr)\bigr|\,\mathrm{d}s$$
$$+\sum_{i=1}^{k}\frac{|I_i\bigl(x(t_i)\bigr)|}{|g(t_i,x(t_i))|}+\frac{|x_0-f_0|}{|g_0|}$$
$$\leqslant\frac{(m+1)K}{\Gamma(\alpha+1)}+(m+1)H+\frac{x_0-f_0}{g_0}:=\overline{L},$$

where $t\in J_k$, $k=0,1,\dots,m$. Taking supremum over $t$, we obtain $\|y\|=\|\mathcal{B}x\|\leqslant\overline{L}$ for all $y\in\mathcal{B}(C)$. Hence, $\mathcal{B}(C)$ is a uniformly bounded subset of $S$. Moreover, $\|\mathcal{B}(C)\|\leqslant\overline{L}$ for all chains in $S$. Hence, $\mathcal{B}$ is uniformly partially bounded on $S$.

Next, we prove that $\mathcal{B}(C)$ is a quasi-equicontinuous set in $S$. Let $t', t'' \in J_k$ be arbitrary, $k = 0, 1, \dots, m$. Likewise, we discuss above in Step 3:

$$\begin{aligned}
&\left|\mathcal{B}x(t') - \mathcal{B}x(t'')\right| \\
&\quad = \left|\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t'}(t'' - s)^{\alpha-1}h\bigl(s, v(s)\bigr)\,\mathrm{d}s - \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t'}(t'' - s)^{\alpha-1}h\bigl(s, v(s)\bigr)\,\mathrm{d}s\right| \\
&\quad \to 0
\end{aligned}$$

when $t' \to t''$. Hence, $\mathcal{B}(C)$ is a quasi-equicontinuous subset of $S$. Now, $\mathcal{B}(C)$ is a uniformly bounded and quasi-equicontinuous set of functions in $S$. By applying Lemma 2 it is compact. Consequently, $\mathcal{B}$ is a partially compact operator on $S$ into itself.

Therefore, assumption (ii) of Lemma 6 is satisfied.

*Step 5.* There is a $u \in S$ such that satisfies $u \leqslant \mathcal{A}u \cdot \mathcal{B}u + \mathcal{C}u$. IVPs (1) has a mild lower solution $u$. Then by application of Lemma 7 and Definition 11 we have

$$\begin{aligned}
u(t) \leqslant{}& g\bigl(t, u(t)\bigr)\left(\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t - s)^{\alpha-1}h\bigl(s, u(s)\bigr)\,\mathrm{d}s\right. \\
&\left. + \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i - s)^{\alpha-1}h\bigl(s, u(s)\bigr)\,\mathrm{d}s + \sum_{i=1}^{k}\frac{I_i(u(t_i))}{g(t_i, u(t_i))} + \frac{x_0 - f_0}{g_0}\right) \\
&+ f\bigl(t, u(t)\bigr), \quad t \in J_k,\ k = 0, 1, \dots, m.
\end{aligned}$$

Then from the definitions of $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ we can deduce that

$$u(t) \leqslant \mathcal{A}u(t) \cdot \mathcal{B}u(t) + \mathcal{C}u(t), \quad t \in J_k,\ k = 0, 1, \dots, n.$$

Hence, $u \leqslant \mathcal{A}u \cdot \mathcal{B}u + \mathcal{C}u$, which meets (iii) in Lemma 6.

*Step 6.* $\psi$ and $\varphi$ satisfy $L\psi(r) + \varphi(r) < r$, $r > 0$.

It is clear that $\psi$ and $\varphi$ are $D$-functions of operators $\mathcal{A}$ and $\mathcal{C}$ from condition (H2), $L$ denotes $\sup\{\|\mathcal{B}(C)\|\colon C \text{ is a chain in } S\}$ from the proof given in Step 4. Then, combining with condition (3), we have

$$L\psi(r) + \varphi(r) < \overline{L}\psi(r) + \varphi(r) < r.$$

Assumption (iv) of Lemma 6 is satisfied.

In a word, all the conditions of Lemma 6 are satisfied. Thus, we can conclude that the operator equation $\mathcal{A}x \cdot \mathcal{B}x + \mathcal{C}x = x$ has a solution $x^*$. Consequently, IVPs (1) has a mild solution $x^*$ defined on $J_k$, $k = 0, 1, \dots, m$. Furthermore, the sequence $\{x_n\}$ of successive approximations defined by (4) converges monotonically to $x^*$. This completes the proof. □

## 4 Example

Given the interval $J = [0, 1]$ and the points $t_1 = 1/4$, $t_2 = 1/2$, $t_3 = 3/4$, we consider the IVPs of the following impulsive hybrid fractional differential equation:

$$\begin{aligned}
&{}^{c}D^{1/2}\left[\frac{x(t) - f(t, x(t))}{g(t, x(t))}\right] = h\big(t, x(t)\big), \quad t \in J := [0, 1] \setminus \{t_1, t_2, t_3\},\\
&\Delta x(t_k) = I_k\big(x(t_k)\big), \quad k = 1, 2, 3,\\
&x(0) = \frac{1}{15},
\end{aligned} \tag{6}$$

where functions $f$, $g$, $h$ are defined as follows:

$$f(t, x) = \frac{1}{15\pi}(\arctan x + \pi), \qquad h(t, x) = \frac{1}{40}(\tanh x + 2),$$

$$g(t, x) = \begin{cases} \frac{1}{2}, & x \leqslant 0,\\ \frac{1+x}{2}, & 0 < x < 1,\\ 1, & x \geqslant 1, \end{cases} \qquad I_k(x) = \begin{cases} 0, & x \leqslant 0,\\ \frac{1}{(k+1)^2}\frac{x}{1+x}, & x > 0,\ k = 1, 2, 3, \end{cases}$$

for arbitrary $x \in \mathbb{R}$, $t \in J$. Through calculation, we have:

(i) $0 < f(t, x) \leqslant 1/10$, $0 < g(t, x) \leqslant 1$.
(ii) $0 < f(t, x) - f(t, y) \leqslant (x - y)/(15\pi)$, $0 < g(t, x) - g(t, y) \leqslant (x - y)/2$ when $x \geqslant y$.
(iii) $h(t, x)$ is nondecreasing, and $0 < h(t, x) \leqslant 3/40$.
(iv) $I_k(x)/g(t, x)$ is nondecreasing, and $0 \leqslant I_k(x)/g(t, x) \leqslant 1/4$, $k = 1, 2, 3$.
(v) It is clear that $u(t) = 0$, $t \in J$, is a mild lower solution of IVP (6) by Definition 11.

Take $\varphi(r) = r/(15\pi)$, $\psi(r) = r/2$, $N = 1$, $M = 1/10$, $K = 3/40$, $H = 1/4$ in Theorem 1. Then we calculate that $\overline{L} = 1.338$, moreover, $\overline{L}\psi(r) + \varphi(r) < r$, $r > 0$. Therefore, all the conditions of Theorem 1 are satisfied. Then we can conclude that system (6) has a mild solution $x^*$, and $\{x_n\}$ of successive approximation is defined by

$$\begin{aligned}
x_{n+1}(t) = g\big(t, x_n(t)\big)\Bigg(&\frac{1}{40\Gamma(\alpha)}\Bigg(\int_{t_k}^{t}(t - s)^{\alpha-1}\big(\tanh x_n(s) + 2\big)\,\mathrm{d}s\\
&+ \sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i - s)^{\alpha-1}\big(\tanh x_n(s) + 2\big)\,\mathrm{d}s\Bigg) + \sum_{i=1}^{k}\frac{I_i(x_n(t_i))}{g(t_i, x_n(t_i))}\Bigg)\\
&+ \frac{1}{15\pi}\big(\arctan x_n(t) + \pi\big), \quad t \in J_k,\ k = 0, 1, \dots, m,
\end{aligned}$$

where $x_1 = u = 0$, converges monotonically to $x^*$.

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