Nonlinear Analysis: Modelling and Control, Vol. 26, No. 6, 1144–1165
https://doi.org/10.15388/namc.2021.26.24941

Hidden maximal monotonicity in evolutionary variational-hemivariational inequalities*

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Received: December 4, 2020 / **Revised:** April 12, 2021 / **Published online:** November 1, 2021

Abstract. In this paper, we propose a new methodology to study evolutionary variational-hemivariational inequalities based on the theory of evolution equations governed by maximal monotone operators. More precisely, the proposed approach, based on a hidden maximal monotonicity, is used to explore the well-posedness for a class of evolutionary variational-hemivariational inequalities involving history-dependent operators and related problems with periodic and antiperiodic boundary conditions. The applicability of our theoretical results is illustrated through applications to a fractional evolution inclusion and a dynamic semipermeability problem.

Keywords: evolutionary variational-hemivariational inequality, history-dependent operator, Clarke subdifferential, fractional evolution inclusion, semipermeability problem.

*This project has received funding from the NNSF of China grant Nos. 12001478, 12026255, and 12026256, the European Union's Horizon 2020 Research and Innovation Programme under the Marie Skłodowska-Curie grant agreement No. 823731 CONMECH, National Science Center of Poland under Preludium Project No. 2017/25/N/ST1/00611, and the Startup Project of Doctor Scientific Research of Yulin Normal University No. G2020ZK07. It is also supported by Natural Science Foundation of Guangxi grant Nos. 2021JJG110004 and 2020GXNSFBA297137 and the Ministry of Science and Higher Education of Republic of Poland under grants Nos. 4004/GGPJII/H2020/2018/0 and 440328/PnH2/2019. E. Vilches was partially supported by ANID-Chile under grant Fondecyt de Iniciación No. 11180098.

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1 Introduction

Variational and hemivariational inequalities serve as theoretical models for various problems arising in mechanics, physics, and engineering sciences. The representative literatures in the field include [1,4,10,11,13–15,17–19,21,23,24]. On the one hand, the theory of variational inequalities uses monotonicity and convexity as its main tools, including the properties of the subdifferential of a convex function and maximal monotone operators. On the other hand, the theory of hemivariational inequalities is based on the features of the subdifferential in the sense of Clarke defined for locally Lipschitz functions, which may be nonconvex.

Observantly, variational-hemivariational inequalities represent an intermediate class of inequalities in which both convex and nonconvex features are involved. Interest in their study is motivated by various problems in mechanics as discussed in [5, 8, 9, 12, 16, 21, 25]. It should be mentioned that the study of evolutionary variational-hemivariational inequalities has been performed typically through surjectivity results for pseudomonotone operators and fixed point theorems for nonlinear operators (see, e.g., [21] and the references therein). However, this paper aims to propose a new approach to study evolutionary variational-hemivariational inequalities based on the theory of evolution problems governed by maximal monotone operators. Indeed, the proposed method is quite different from the previous literature and is not based on surjectivity results for pseudomonotone operators.

Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be a separable Hilbert space, Y a Banach space, and $I = [0,T]$ for some $T > 0$ fixed. In this paper, we study and provide new applications to PDEs for the following class of evolutionary variational-hemivariational inequalities involving history-dependent operators:

$$\begin{aligned} \dot{x}(t) &\in f(t) - \mathcal{R}(x)(t) - \partial J\big(t, x(t)\big) - A\big(t, x(t)\big) \\ &\quad - \partial_c \varphi\big(t, \mathcal{S}(x)(t), x(t)\big), \quad \text{a.e. } t \in I, \\ x(0) &= x_0, \end{aligned} \tag{1}$$

where $f \in L^2(I;\mathcal{H})$, $J : I \times \mathcal{H} \to \mathbb{R}$ is a function, $A : I \times \mathcal{H} \to \mathcal{H}$ is a nonlinear operator, $\varphi : I \times Y \times \mathcal{H} \to \mathbb{R}$ is a given function, and $\mathcal{R}, \mathcal{S}$ are two history-dependent operators (see Definition 1 below) in which we refer to Section 4 for the precise hypothesis. Problem (1) was studied in [7] (see also [21, Chap. 7]) in the framework of evolution triple of spaces by using surjectivity results for pseudomonotone operators and a fixed point theorem for nonlinear operators. A key assumption to apply the surjectivity result is the so-called *relaxed monotonicity* for the subdifferential in the sense of Clarke (see Definition 2 below), which is a weaker notion than monotonicity, but which still permits to obtain the existence of solutions. We characterize this notion in terms of the convexity of an associated function (see Section 3). Then we consider the differential inclusion

$$\begin{aligned} \dot{x}(t) &\in f(t) - \partial J\big(t, (t)\big) - A\big(t, x(t)\big) - \partial_c \psi\big(t, x(t)\big), \quad \text{a.e. } t \in I, \\ x(0) &= x_0, \end{aligned} \tag{2}$$

where $\psi : I \times \mathcal{H} \to \mathbb{R}$ is a given function. We prove that the latter problem is, in fact, an evolution equation governed by a set-valued operator, which is a maximal monotone operator. Whereas the existence can be obtained through a recent result on the subject [22]. As a by-product, we obtain the existence for the periodic and antiperiodic version of (2). Moreover, we prove that every trajectory of the Cauchy problem (2) converges asymptotically to a periodic solution of (2).

The contribution of this paper is threefold. First, we show that some evolutionary variational-hemivariational inequalities can be handled with the theory of evolution problems governed by maximal monotone operators. Second, we extend the results of [7] to the general functional setting. Finally, the applicability of our theoretical results is illustrated through applications to the study of a fractional evolution inclusion and a dynamic semipermeability problem.

The paper is organized as follows. After some preliminaries, in Section 3, we provide an impressive characterization of the relaxed monotonicity property, and then we prove the maximal monotonicity of the sum of operators that appear on the right-hand side of (2). Then, in Section 4, we establish the well-posedness for problems (1) and (2), respectively. Finally, in Section 5, we illustrate the applicability of our theoretical results to the study of a fractional evolution inclusion and a complicated dynamic semipermeability problem, respectively.

2 Notation and preliminaries

2.1 Elements of convex and variational analysis

Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be a separable Hilbert space. We denote by $\mathbb{B}$ the unit closed ball with center at the origin in $\mathcal{H}$. Given a set-valued map $A : \mathcal{H} \rightrightarrows \mathcal{H}$, we denote by $D(A)$ and $Gr(A)$, respectively, the domain and the graph of A defined by $D(A) := \{x \in \mathcal{H}\colon A(x) \neq \emptyset\}$ and $Gr(A) := \{(x, y) \in D(A) \times \mathcal{H}\colon y \in A(x)\}$. We say that an operator $A : \mathcal{H} \rightrightarrows \mathcal{H}$ is monotone if $\langle x^* - y^*, x - y\rangle \geqslant 0$ for all $x^* \in A(x)$, $y^* \in A(y)$. Moreover, an operator $A : \mathcal{H} \rightrightarrows \mathcal{H}$ is maximal monotone if it is monotone and its graph is maximal in the sense of the inclusion, i.e., $Gr(A)$ is not properly contained in the graph of any other monotone operator. We refer to [2] for more details on maximal monotone operators.

The Clarke subdifferential of a locally Lipschitz function $f : \mathcal{H} \to \mathbb{R}$ at $x \in \mathcal{H}$ is defined by $\partial f(x) = \{x^* \in \mathcal{H}\colon f^0(x; v) \geqslant \langle x^*, v\rangle$ for all $v \in \mathcal{H}\}$, where $f^0(x; v)$ stands for the generalized directional derivative of f at $x \in \mathcal{H}$ in the direction $v \in \mathcal{H}$ defined by $f^0(x; v) = \limsup_{y\to x, t\downarrow 0}(f(y + tv) - f(y))/t$. For a convex function $f : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$, the convex subdifferential of f at $x \in \mathcal{H}$ is given by $\partial_c f(x) = \{x^* \in \mathcal{H}\colon f(y) \geqslant f(x) + \langle x^*, y - x\rangle$ for all $y \in \mathcal{H}\}$. It is well known that for a proper, convex, and lower semicontinuous function, the convex subdifferential defines a maximal monotone operator. Moreover, for a convex and locally Lipschitz function, the convex subdifferential coincides to the Clarke subdifferential (see, e.g., [3]).

The following result is an important characterization of convexity. We refer to [3, Prop. 2.2.9] for its proof.

Proposition 1. *Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be a Hilbert space. Let $f : C \to \mathbb{R}$ be a locally Lipschitz function in an open convex set $C \subset \mathcal{H}$. Then f is convex on C if and only if the multi-function ∂f is monotone on C, that is, if and only if for all $x, y \in C$, $\langle x^* - y^*, x - y\rangle \geqslant 0$ for all $x^* \in \partial f(x)$ and $y^* \in \partial f(y)$.*

Proposition 2. *Let Y be a Banach space and consider $\varphi : I \times Y \times \mathcal{H} \to \mathbb{R}$ be a function such that for a.e. $t \in I$, the map $x \mapsto \varphi(t, y, x)$ is convex and lower semicontinuous on $\mathcal{H}$. Assume that for all $y_1, y_2 \in Y$ and $x_1, x_2 \in \mathcal{H}$, it holds*

$$\begin{aligned}&\varphi(t, y_1, x_2) - \varphi(t, y_1, x_1) + \varphi(t, y_2, x_1) - \varphi(t, y_2, x_2)\\&\quad \leqslant \beta_\varphi \|y_1 - y_2\|_Y \, \|x_1 - x_2\|_{\mathcal{H}}, \quad \text{a.e. } t \in I.\end{aligned}$$

Then, for all $\xi_1 \in \partial_c\varphi(t, y_1, x_1)$ and $\xi_2 \in \partial_c\varphi(t, y_2, x_2)$, the following inequality holds:

$$-\langle \xi_1 - \xi_2,\, x_1 - x_2\rangle \leqslant \beta_\varphi \|y_1 - y_2\|_Y \|x_1 - x_2\|_{\mathcal{H}}, \quad \text{a.e. } t \in I.$$

Proof. Let $\xi_1 \in \partial_c\varphi(t, y_1, x_1)$ and $\xi_2 \in \partial_c\varphi(t, y_2, x_2)$. Then, according to the definition of the convex subdifferential, for all $x \in \mathcal{H}$, we have

$$\begin{aligned}\langle \xi_1,\, x - x_1\rangle + \varphi(t, y_1, x_1) &\leqslant \varphi(t, y_1, x),\\ \langle \xi_2,\, x - x_2\rangle + \varphi(t, y_2, x_2) &\leqslant \varphi(t, y_2, x).\end{aligned}$$

Hence, taking $x = x_2$ and $x = x_1$ in the inequalities above, respectively, and summing the resulting inequalities, we get

$$\begin{aligned}&-\langle \xi_1 - \xi_2,\, x_1 - x_2\rangle\\&\quad \leqslant \varphi(t, y_1, x_2) - \varphi(t, y_1, x_1) + \varphi(t, y_2, x_1) - \varphi(t, y_2, x_2).\end{aligned}$$

Hence, we get the desired inequality. □

Definition 1. Let X, Y be normed spaces. An operator $\mathcal{F} : L^2(I; X) \to L^2(I; Y)$ is called a *history-dependent operator* if there exists $L > 0$ such that for all $v_1, v_2 \in L^2(I; X)$,

$$\big\|\mathcal{F}(v_1)(t) - \mathcal{F}(v_2)(t)\big\|_Y \leqslant L \int_0^t \big\|v_1(s) - v_2(s)\big\|_X \,\mathrm{d}s, \quad \text{a.e. } t \in I.$$

The following result is an essential fixed point property for history-dependent operators (see, e.g., [21, p. 118]).

Theorem 1. *Let X be a Banach space and $\mathcal{F} : L^2(I; X) \to L^2(I; X)$ be a history-dependent operator. Then $\mathcal{F}$ has a unique fixed point.*

We end this subsection with a technical lemma related to differential inequalities.

Lemma 1. *Let $x_1, x_2 : I \to \mathcal{H}$ be two absolutely continuous functions such that*

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big\|x_1(t) - x_2(t)\big\|^2 \leqslant \alpha(t)\big\|x_1(t) - x_2(t)\big\|, \quad \text{a.e. } t \in I,$$

where $\alpha : I \to \mathbb{R}$ is a nonnegative function. Then it holds

$$\frac{\mathrm{d}}{\mathrm{d}t}\big\|x_1(t)-x_2(t)\big\| \leqslant \alpha(t), \quad \text{a.e. } t\in I.$$

Proof. Let us consider the sets $\Omega_1 := \{t \in I\colon x_1(t) \neq x_2(t)\}$ and $\Omega_2 := \{t \in I\colon x_1(t) = x_2(t)\}$. On the one hand, for a.e. $t \in \Omega_1$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big\|x_1(t)-x_2(t)\big\|^2 = \big\|x_1(t)-x_2(t)\big\|\frac{\mathrm{d}}{\mathrm{d}t}\big\|x_1(t)-x_2(t)\big\|,$$

which implies the desired inequality. On the other hand, for any $t \in \Omega_2$, we can see that the map $t \mapsto \|x_1(t) - x_2(t)\|$ attains a minimum. Thus, for a.e. $t \in \Omega_2$, we have $(\mathrm{d}/\mathrm{d}t)\|x_1(t) - x_2(t)\| = 0$, which implies the desired inequality. The proof is then complete. □

2.2 Elements of PDEs

Let Ω be a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary, and let $s \in (0, 1)$ be such that $N > 2s$. We adopt the symbols $\mathcal{S} := (\mathbb{R}^N \setminus \Omega) \times (\mathbb{R}^N \setminus \Omega)$, $\mathcal{P} := \mathbb{R}^{2N} \setminus \mathcal{S}$, and $2^*_s := 2N/(N - 2s)$ to denote the fractional critical exponent. Also, we denote by $u|_\Omega$ the function u restricted to the domain Ω. In what follows, we assume that function $K : \mathbb{R}^N \setminus \{0\} \to (0, +\infty)$ satisfies the conditions:

(H_K) $K : \mathbb{R}^N \setminus \{0\} \to (0, +\infty)$ is such that

- (i) the function $x \mapsto \min\{|x|^2, 1\}K(x)$ belongs to $L^1(\mathbb{R}^N)$;
- (ii) there exists a constant $m_K > 0$ such that $K(x) \geqslant m_K|x|^{-(N+2s)}$ for all $x \in \mathbb{R}^N \setminus \{0\}$;
- (iii) for each $x \in \mathbb{R}^N \setminus \{0\}$, we have $K(x) = K(-x)$.

Consider the function space $X := \{u\colon \mathbb{R}^N \to \mathbb{R}\colon u|_\Omega \in L^2(\Omega)$ and $(u(x)-u(y))^2 \times K(x - y) \in L^2(\mathcal{P})\}$. It is clear (see, e.g., [20]) that X is a Banach space endowed with the norm $\|u\|_X := \|u\|_{L^2(\Omega)} + (\int_{\mathcal{P}} |u(x) - u(y)|^2 K(x - y)\,\mathrm{d}y\,\mathrm{d}x)^{1/2}$, $u \in X$. We also introduce a subspace of X given by $X_0 := \{u \in X\colon u = 0, \text{ a.e. } x \in \mathbb{R}^N \setminus \Omega\}$.

Also, we recall the following lemma (see [20]), which will be used in Section 5.

Lemma 2. *Let $s \in (0, 1)$ and Ω be a bounded, open subset of $\mathbb{R}^N$ with Lipschitz boundary and $N > 2s$. Then we have*

(i) *X_0 is a Hilbert space with the inner product*

$$\langle u, v\rangle_{X_0} := \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \big[u(x)-u(y)\big]\big[v(x)-v(y)\big]K(x-y)\,\mathrm{d}x\,\mathrm{d}y, \quad u,v\in X_0.$$

(ii) *If $p \in [1, 2^*_s]$, then there exists a positive constant $c(p)$ such that for all $u \in X_0$, $\|u\|_{L^p(\mathbb{R}^N)} \leqslant c(p)\|u\|_{X_0}$.*

(iii) *The embedding from X_0 to $L^p(\mathbb{R}^N)$ is compact if $p \in [1, 2^*_s)$.*

3 Technical assumptions and hidden maximal monotonicity

For the sake of readability, furthermore, we collect the hypotheses used along with the paper.

Hypotheses on the operator $A : I \times \mathcal{H} \to \mathcal{H}$

$(\mathcal{H}_A)$ $A : I \times \mathcal{H} \to \mathcal{H}$ is a nonlinear operator satisfying:

- (i) the operator $t \mapsto A(t, x)$ is measurable on I for all $x \in \mathcal{H}$;
- (ii) for a.e. $t \in I$, the map $x \mapsto A(t, x)$ is hemicontinuous, that is, for all $x, y, w \in \mathcal{H}$, $\lim_{\eta\downarrow 0}\langle w, A(t,\, x + \eta y)\rangle = \langle w, A(t, x)\rangle$;
- (iii) there exist $\alpha, \beta \in L^2_+(I)$ such that $\|A(t, x)\| \leqslant \alpha(t) + \beta(t)\|x\|_{\mathcal{H}}$ for all $x \in \mathcal{H}$ and a.e. $t \in I$;
- (iv) there exists $m_A \geqslant 0$ such that $\langle A(t, x) - A(t, y),\, x - y\rangle \geqslant m_A\|x - y\|^2_{\mathcal{H}}$ for all $x, y \in \mathcal{H}$ and a.e. $t \in I$.

Hypotheses on the function $J : I \times \mathcal{H} \to \mathbb{R}$

$(\mathcal{H}_J)$ The function $J : I \times \mathcal{H} \to \mathbb{R}$ satisfies:

- (i) for all $x \in \mathcal{H}$, $t \mapsto J(t, x)$ is measurable on I;
- (ii) for a.e. $t \in I$, $x \mapsto J(t, x)$ is locally Lipschitz continuous;
- (iii) there exist $\gamma, \delta \in L^2_+(I)$ such that for a.e. $t \in I$ and all $x \in \mathcal{H}$, $|\partial J(t, x)| := \inf\{\|x^*\|_{\mathcal{H}}\colon x^* \in \partial J(t, x)\} \leqslant \gamma(t) + \delta(t)\|x\|_{\mathcal{H}}$;
- (iv) there exists $m_J \geqslant 0$ such that $\langle x^* - y^*,\, x - y\rangle \geqslant -m_J\|x - y\|^2_{\mathcal{H}}$ for all $x^* \in \partial J(t, x)$ and $y^* \in \partial J(t, y)$ and a.e. $t \in I$.

Here ∂J denotes the Clarke subdifferential of the map $x \mapsto J(t, x)$ for a fixed $t \in I$.

Hypotheses on the function $\psi : I \times \mathcal{H} \to \mathbb{R}$

$(\mathcal{H}_\psi)$ The function $\psi : I \times \mathcal{H} \to \mathbb{R}$ satisfies:

- (i) for all $x \in \mathcal{H}$, the map $t \mapsto \psi(t, x)$ is measurable on I;
- (ii) for a.e. $t \in I$, the map $x \mapsto \psi(t, x)$ is convex and l.s.c. on $\mathcal{H}$;
- (iii) there exist $c_{0\psi} \in L^2_+(I)$ and $c_{1\psi} > 0$ such that for $x \in \mathcal{H}$ and a.e. $t \in I$, $\sup_{x^* \in \partial_c\psi(t,x)}\|x^*\|_{\mathcal{H}} \leqslant c_{0\psi}(t) + c_{1\psi}\|x\|_{\mathcal{H}}$. Here $\partial_c\psi(t, x)$ denotes the convex subdifferential of the map $x \mapsto \psi(t, x)$.

Hypotheses on the function $\varphi : I \times \mathcal{H} \times Y \to \mathbb{R}$

$(\mathcal{H}_\varphi)$ The function $\varphi : I \times Y \times \mathcal{H} \to \mathbb{R}$ satisfies:

- (i) for all $x \in \mathcal{H}$, $y \in Y$, the map $t \mapsto \varphi(t, y, x)$ is measurable on I;
- (ii) for a.e. $t \in I$, for $x \in X$, the map $y \mapsto \varphi(t, y, x)$ is continuous;
- (iii) for a.e. $t \in I$, for $y \in Y$, the map $x \mapsto \varphi(t, y, x)$ is convex and l.s.c. on $\mathcal{H}$.

(iv) there exist $c_{0\varphi} \in L^2_+(I)$ and $c_{1\varphi}$, $c_{2\varphi} \geqslant 0$ such that for all $y \in Y$, $x \in \mathcal{H}$ and a.e. $t \in I$, $\sup_{x^* \in \partial_c \varphi(t,y,x)} \|x^*\|_{\mathcal{H}} \leqslant c_{0\varphi}(t) + c_{1\varphi}\|y\|_Y + c_{2\varphi}\|x\|_{\mathcal{H}}$. Here $\partial_c \varphi(t, y, x)$ is the convex subdifferential of the map $x \mapsto \varphi(t, y, x)$.

(v) There exists $\beta_\varphi \geqslant 0$ such that for all $y_1, y_2 \in Y$ and $x_1, x_2 \in \mathcal{H}$ and a.e. $t \in I$, $\varphi(t, y_1, x_2) - \varphi(t, y_1, x_1) + \varphi(t, y_2, x_1) - \varphi(t, y_2, x_2) \leqslant \beta_\varphi \|y_1 - y_2\|_Y \|x_1 - x_2\|_{\mathcal{H}}$.

Hypotheses on the operators $\mathcal{R}$ and $\mathcal{S}$

$(\mathcal{H}_{RS})$ The operators $\mathcal{R} : L^2(I; \mathcal{H}) \to L^2(I; \mathcal{H})$ and $\mathcal{S} : L^2(I; \mathcal{H}) \to L^2(I; Y)$ satisfy:

(i) The operator $\mathcal{R}$ is a history-dependent, i.e., there exists $c_{\mathcal{R}} > 0$ such that $\|\mathcal{R}v_1(t) - \mathcal{R}v_2(t)\|_{\mathcal{H}} \leqslant c_{\mathcal{R}} \int_0^t \|v_1(s) - v_2(s)\|_{\mathcal{H}}\,\mathrm{d}s$ for all $v_1, v_2 \in L^2(I; \mathcal{H})$ and a.e. $t \in I$.

(ii) The operator $\mathcal{S}$ is a history-dependent, i.e., there exists $c_{\mathcal{S}} > 0$ such that $\|\mathcal{S}v_1(t) - \mathcal{S}v_2(t)\|_Y \leqslant c_{\mathcal{S}} \int_0^t \|v_1(s) - v_2(s)\|_{\mathcal{H}}\,\mathrm{d}s$ for all $v_1, v_2 \in L^2(I; \mathcal{H})$ and a.e. $t \in I$.

Next, we characterize the so-called *relaxed monotonicity condition* for a locally Lipschitz function $f : \mathcal{H} \to \mathbb{R}$ sum of a quadratic term. With this result in hand, we prove the maximal monotonicity of the sum of the Clarke subdifferential of f plus an appropriate strongly monotone operator A, which can be understood as a hidden maximal monotonicity property.

Definition 2. We say that a locally Lipschitz function $f : \mathcal{H} \to \mathbb{R}$ satisfies the *m-relaxed monotonicity condition* if there exists $m \geqslant 0$ such that

$$\langle x_1^* - x_2^*,\, x_1 - x_2 \rangle \geqslant -m\|x_1 - x_2\|^2_{\mathcal{H}}, \quad x_1^* \in \partial f(x_1),\ x_2^* \in \partial f(x_2). \tag{3}$$

Condition (3) has been used extensively in the literature, we refer to [6, 13] for more details. The following result gives a characterization of m-relaxed monotonicity in terms of an associated convex function.

Proposition 3. *Let $f : \mathcal{H} \to \mathbb{R}$ be a locally Lipschitz function and $m \geqslant 0$. Then f satisfies the m-relaxed monotonicity condition if and only if the map $x \mapsto f(x) + (m/2)\|x\|^2_{\mathcal{H}}$ is convex on $\mathcal{H}$.*

Proof. Assume that f satisfies the m-relaxed monotonicity condition. Then, due to calculus rules for the Clarke subdifferential, $\partial(f + m\|\cdot\|^2_{\mathcal{H}}/2)(x) = \partial f(x) + mx$ for all $x \in \mathcal{H}$. Thus, by the m-relaxed monotonicity condition, the map $x \rightrightarrows \partial(f + m\|\cdot\|^2/2)(x)$ is, clearly, monotone. Therefore, by Proposition 1, the function $x \mapsto f(x) + m\|x\|^2_{\mathcal{H}}/2$ is convex on $\mathcal{H}$. Reciprocally, assume that the map $x \mapsto f(x) + m\|x\|^2/2$ is convex.

On the other side, suppose that the set-valued map $x \mapsto \partial(f + m\|\cdot\|^2_{\mathcal{H}}/2)(x)$ is monotone. Hence, f satisfies the m-relaxed monotonicity condition, which ends the proof. □

The following result shows that the relaxed monotonicity added with an appropriate strongly monotone operator generates a maximal monotone operator.

Lemma 3. *Let $J : \mathcal{H} \to \mathbb{R}$ be a locally Lipschitz function and $A : \mathcal{H} \to \mathcal{H}$ be a nonlinear operator such that*:

(i) *The map $x \mapsto A(x)$ is hemicontinuous, that is,* $\lim_{\eta\downarrow 0}\langle w, A(x+\eta y)\rangle = \langle w, A(x)\rangle$ *for all $x, y, w \in \mathcal{H}$.*
(ii) *There exists $m_A \geqslant 0$ such that $\langle A(x) - A(y), x - y\rangle \geqslant m_A\|x - y\|^2_{\mathcal{H}}$ for all $x, y \in \mathcal{H}$.*
(iii) *There exists $m_J \geqslant 0$ such that $\langle x^* - y^*, x - y\rangle \geqslant -m_J\|x - y\|^2_{\mathcal{H}}$ for all $x^* \in \partial J(x)$ and $y^* \in \partial J(y)$.*

Then, if $m_A \geqslant m_J$, the operator $x \mapsto \partial J(x) + A(x)$ is maximal monotone.

Proof. Let us consider $\tilde{J}(x) := J(x)+m_J\|x\|^2_{\mathcal{H}}/2$ and $\tilde{A}(x) := A(x)-m_J x$. According to [3, Prop. 2.3.3], $\partial\tilde{J}(x) = \partial J(x) + m_J x$ for all $x \in \mathcal{H}$. Hence, by virtue of (iii), the operator $x \mapsto \partial\tilde{J}(x)$ is monotone. Therefore, due to Proposition 1, the function $x \mapsto \tilde{J}(x)$ is convex, which implies that $x \mapsto \partial\tilde{J}(x)$ is maximal monotone. On the other hand, due to (i) and (ii), the operator $x \mapsto \tilde{A}(x)$ is monotone and hemicontinuous. Therefore, as a result of [2, Prop. 20.27], the map $x \mapsto \tilde{A}(x)$ is maximal monotone. Finally, the maximal monotonicity of $x \mapsto \partial J(x) + A(x) = \partial\tilde{J}(x) + \tilde{A}(x)$ follows from [2, Cor. 25.4]. □

4 Well-posedness results

In this section, we explore several well-posedness results for evolutionary variational-hemivariational inequalities.

4.1 Cauchy problems

In this subsection, we prove the existence of solutions for the following Cauchy problem:

$$\begin{aligned} &\dot{x}(t) \in f(t) - \partial J\bigl(t, x(t)\bigr) - A\bigl(t, x(t)\bigr) - \partial_c\psi\bigl(t, x(t)\bigr), \quad \text{a.e. } t \in I, \\ &x(0) = x_0. \end{aligned} \tag{4}$$

The following result provides the well-posedness for (4).

Theorem 2. *Assume that $(\mathcal{H}_A)$, $(\mathcal{H}_J)$, and $(\mathcal{H}_\psi)$ hold. If $m_A \geqslant m_J$, then for each $f \in L^2(I;\mathcal{H})$ and $x_0 \in \mathcal{H}$, problem* (4) *has a unique solution $x(\cdot, f, x_0) \in W^{1,2}(I;\mathcal{H})$. Moreover, the solution operator $(f, x_0) \mapsto x(\cdot, f, x_0)$ is Lipschitz continuous from $L^2(I;\mathcal{H}) \times \mathcal{H}$ into $C(I;\mathcal{H})$.*

Proof. We will employ [22, Thm. 1] to obtain the desired conclusion. So, the proof is divided into three steps.

Step 1. For a.e. $t \in I$, the operator $x \mapsto \partial J(t, x) + A(t, x) + \partial_c\psi(t, x)$ is maximal monotone.

Proof of Step 1. It follows directly from Lemma 3.

Step 2. For all $x \in \mathcal{H}$, the operator $t \mapsto \partial J(t,x) + A(t,x) + \partial\psi(t,x)$ is measurable.

Proof of Step 2. The measurability can be obtained directly from the separability of $\mathcal{H}$ and hypotheses $(\mathcal{H}_A)$(i), $(\mathcal{H}_J)$(i), and $(\mathcal{H}_\psi)$(i).

Step 3. There exist $\tilde{\alpha}$, $\tilde{\beta} \in L^2_+(I)$ such that for all $x \in \mathcal{H}$,

$$\begin{aligned}&\bigl|\partial J(t,x) + A(t,x) + \partial_c\psi(t,x)\bigr| \\ &\quad := \inf_{\xi \in \partial J(t,x)+A(t,x)+\partial_c\psi(t,x)} \|\xi\|_{\mathcal{H}} \leqslant \tilde{\alpha}(t) + \tilde{\beta}(t)\|x\|_{\mathcal{H}}, \quad \text{a.e. } t \in I.\end{aligned}$$

Proof of Step 3. Indeed, conditions $(\mathcal{H}_J)$(iii), $(\mathcal{H}_A)$(iii), and $(\mathcal{H}_\psi)$(iii) indicate that

$$\bigl|\partial J(t,x) + A(t,x) + \partial\psi(t,x)\bigr| \leqslant \tilde{\alpha}(t) + \tilde{\beta}(t)\|x\|_{\mathcal{H}},$$

where $\tilde{\alpha} := \alpha + \gamma + c_{0\psi}$ and $\tilde{\beta} := \beta + \delta + c_{1\psi}$, which proves Step 3.

Therefore, by virtue of Steps 1–3 and [22, Thm. 1], the Cauchy problem (4) has a unique solution $x(\cdot, f, x_0) \in W^{1,2}(I;\mathcal{H})$.

Furthermore, let $x_0^1, x_0^2 \in \mathcal{H}$ and $f_1, f_2 \in L^2(I;\mathcal{H})$ and set $x_1 := x(\cdot, f_1, x_0^1)$ and $x_2 := x(\cdot, f_2, x_0^2)$. Then, due to the monotonicity of the set-valued map $x \mapsto \partial J(t,x) + A(t,x) + \partial_c\psi(t,x)$, for a.e. $t \in I$, it follows that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\bigl\|x_1(t) - x_2(t)\bigr\|^2_{\mathcal{H}} &= \bigl\langle x_1(t) - x_2(t),\, \dot{x}_1(t) - \dot{x}_2(t)\bigr\rangle \\ &\leqslant \bigl\langle x_1(t) - x_2(t),\, f_1(t) - f_2(t)\bigr\rangle, \quad \text{a.e. } t \in I.\end{aligned}$$

Therefore, by virtue of Lemma 1, for a.e. $t \in I$, it holds $(\mathrm{d}/\mathrm{d}t)\|x_1(t) - x_2(t)\|_{\mathcal{H}} \leqslant \|f_1(t) - f_2(t)\|_{\mathcal{H}}$, which implies that

$$\bigl\|x_1(t) - x_2(t)\bigr\|_{\mathcal{H}} \leqslant \bigl\|x_0^1 - x_0^2\bigr\|_{\mathcal{H}} + \int_0^t \bigl\|f_1(s) - f_2(s)\bigr\|_{\mathcal{H}}\,\mathrm{d}s, \quad t \in I.$$

We conclude that the solution operator $(f, x_0) \mapsto x(\cdot, f, x_0)$ is Lipschitz from $L^2(I;\mathcal{H}) \times \mathcal{H}$ into $C(I;\mathcal{H})$ by using Hölder inequality. □

Remark 1. From the proof of Theorem 2 we can see that hypotheses $(\mathcal{H}_J)$(iii), $(\mathcal{H}_A)$(iii), and $(\mathcal{H}_\psi)$(iii) are only used to ensure that

$$\begin{aligned}\bigl|\partial J(t,x) + A(t,x) + \partial_c\psi(t,x)\bigr| &:= \inf_{\xi \in \partial J(t,x)+A(t,x)+\partial_c\psi(t,x)} \|\xi\|_{\mathcal{H}} \\ &\leqslant \tilde{\alpha}(t) + \tilde{\beta}(t)\|x\|_{\mathcal{H}}\end{aligned}$$

for some functions $\tilde{\alpha}$, $\tilde{\beta}$ in $L^2_+(I)$. So, Theorem 2 holds too if we interchange the infimum by a supremum in $(\mathcal{H}_J)$(iv) and the supremum by a infimum in $(\mathcal{H}_\psi)$(iii).

Remark 2. According to [22, Thm. 1] and Lemma 3, Theorem 2 still holds if we consider a set-valued operator $A : I \times \mathcal{H} \rightrightarrows \mathcal{H}$ such that the map $x \mapsto A(t,x)$ is maximal monotone with a full domain.

4.2 Cauchy problems with history-dependent operators

In this subsection, we focus our attention on the study of the existence of solutions for the following Cauchy problem involving history-dependent operators:

$$\begin{aligned} &\dot{x}(t) \in f(t) - \mathcal{R}(x)(t) - \partial J\bigl(t, x(t)\bigr) - A\bigl(t, x(t)\bigr) \\ &\qquad\qquad - \partial_c \varphi\bigl(t, \mathcal{S}(x)(t), x(t)\bigr), \quad \text{a.e. } t \in I, \\ &x(0) = x_0, \end{aligned} \tag{5}$$

where $\mathcal{R}$ and $\mathcal{S}$ are two history-dependent operators, i.e., $(\mathcal{H}_{\mathcal{RS}})$ is satisfied.

Theorem 3. *Assume that* $(\mathcal{H}_A)$*,* $(\mathcal{H}_J)$*,* $(\mathcal{H}_\varphi)$*, and* $(\mathcal{H}_{RS})$ *hold. If* $m_A \geqslant m_J$*, then for each* $f \in L^2(I; \mathcal{H})$ *and* $x_0 \in \mathcal{H}$*, problem* (5) *has a unique solution* $x(\cdot, f, x_0) \in W^{1,2}(I; \mathcal{H})$*. Moreover, the solution operator* $(f, x_0) \mapsto x(\cdot, f, x_0)$ *is Lipschitz continuous from* $L^2(I; \mathcal{H}) \times \mathcal{H}$ *into* $C(I; \mathcal{H})$*.*

Proof. Fix $v \in L^2(I; \mathcal{H})$ and let us consider the intermediate problem:

$$\begin{aligned} &\dot{x}(t) \in f(t) - \mathcal{R}(v)(t) - \partial J\bigl(t, x(t)\bigr) - A\bigl(t, x(t)\bigr) \\ &\qquad\qquad - \partial_c \varphi\bigl(t, \mathcal{S}(v)(t), x(t)\bigr), \quad \text{a.e. } t \in I, \\ &x(0) = x_0. \end{aligned} \tag{6}$$

Our aim is to prove that (6) has a unique fixed point in $W^{1,2}(I; \mathcal{H})$, which is clearly a solution of (5). The proof is divided into several steps.

Step 1. For $v \in L^2(I; \mathcal{H})$, problem (6) has a unique solution $x(v) \in W^{1,2}(I; \mathcal{H})$.

Proof of Step 1. It follows directly from Theorem 2.

We now denote by $\mathcal{F} : L^2(I; \mathcal{H}) \to W^{1,2}(I; \mathcal{H})$ the operator, which associates to any $v \in L^2(I; \mathcal{H})$ for the unique solution $x(v) \in W^{1,2}(I; \mathcal{H})$ of (6).

Step 2. The operator $\mathcal{F}$ is history-dependent. More precisely, for all $v_1, v_2 \in L^2(I; \mathcal{H})$, we have

$$\bigl\|\mathcal{F}(v_1)(t) - \mathcal{F}(v_2)(t)\bigr\|_{\mathcal{H}} \leqslant (c_{\mathcal{R}} + \beta_\varphi c_{\mathcal{S}}) T \int_0^t \bigl\|v_1(s) - v_2(s)\bigr\|_{\mathcal{H}} \,\mathrm{d}s, \quad \text{a.e. } t \in I.$$

Proof of Step 2. Denote $x_1 := \mathcal{F}(v_1)$ and $x_2 := \mathcal{F}(v_2)$. Let us consider ξ_1 and ξ_2 such that $\xi_i(t) \in \partial_c \varphi(t, \mathcal{S}(v_i)(t), x_i(t))$ for all $t \in I$ and $i = 1, 2$, and $\dot{x}_i(t) + \xi_i(t) \in f(t) - \mathcal{R}(v_i)(t) - \partial J(t, x_i(t)) - A(t, x_i(t))$ for all $t \in I$ and $i = 1, 2$. Define $h(t) := \|x_1(t) - x_2(t)\|_{\mathcal{H}}^2/2$. Then h is absolutely continuous, and it holds

$$\begin{aligned} \dot{h}(t) &= \bigl\langle \dot{x}_1(t) - \dot{x}_2(t), x_1(t) - x_2(t) \bigr\rangle \\ &\leqslant c_{\mathcal{R}} \int_0^t \bigl\|v_1(s) - v_2(s)\bigr\|_{\mathcal{H}} \,\mathrm{d}s \,\bigl\|x_1(t) - x_2(t)\bigr\|_{\mathcal{H}} \\ &\quad + \beta_\varphi c_{\mathcal{S}} \int_0^t \bigl\|v_1(s) - v_2(s)\bigr\|_{\mathcal{H}} \,\mathrm{d}s \,\bigl\|x_1(t) - x_2(t)\bigr\|_{\mathcal{H}}, \quad \text{a.e. } t \in I, \end{aligned}$$

where $m := m_A - m_J \geqslant 0$, and we have used the monotonicity of the set-valued map $x \mapsto \partial J(t,x) + A(t,x)$, Proposition 2, and hypotheses $(\mathcal{H}_{RS})$. Therefore, by virtue of Lemma 1, for a.e. $t \in I$, we conclude $(\mathrm{d}/\mathrm{d}t)\|x_1(t) - x_2(t)\|_{\mathcal{H}} \leqslant (c_{\mathcal{R}} + \beta_\varphi c_{\mathcal{S}}) \times \int_0^t \|v_1(s) - v_2(s)\|_{\mathcal{H}}\,\mathrm{d}s$, which implies that

$$\|x_1(t) - x_2(t)\|_{\mathcal{H}} \leqslant (c_{\mathcal{R}} + \beta_\varphi c_{\mathcal{S}})T \int_0^t \|v_1(s) - v_2(s)\|_{\mathcal{H}}\,\mathrm{d}s, \quad \text{a.e. } t \in I,$$

which proves Step 2.

Step 3. Problem (5) has a unique solution $x^* \in W^{1,2}(I;\mathcal{H})$.

Proof of Step 3. Since $\mathcal{F}$ is a history-dependent operator, employing Theorem 1 implies that the operator $\mathcal{F} : L^2(I;\mathcal{H}) \to W^{1,2}(I;\mathcal{H})$ has a unique fixed point x^*, which clearly solves (5).

To end the proof, let $x_0^1, x_0^2 \in \mathcal{H}$ and $f_1, f_2 \in L^2(I;\mathcal{H})$ and consider $x_1 := x(\cdot, f_1, x_0^1)$ and $x_2 := x(\cdot, f_2, x_0^2)$. Then, by virtue of the monotonicity of the operator $x \mapsto \partial J(t,x) + A(t,x)$ for a.e. $t \in I$, it follows that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|x_1(t) - x_2(t)\|_{\mathcal{H}}^2 \\
&\quad = \langle x_1(t) - x_2(t),\, \dot{x}_1(t) - \dot{x}_2(t)\rangle \\
&\quad \leqslant \|x_1(t) - x_2(t)\|_{\mathcal{H}}\|f_1(t) - f_2(t)\|_{\mathcal{H}} \\
&\qquad + (c_{\mathcal{R}} + \beta_\varphi c_{\mathcal{S}}) \int_0^t \|x_1(s) - x_2(s)\|_{\mathcal{H}}\,\mathrm{d}s\,\|x_1(t) - x_2(t)\|_{\mathcal{H}}, \quad \text{a.e. } t \in I,
\end{aligned}$$

hence,

$$\begin{aligned}
&\frac{1}{2}\|x_1(t) - x_2(t)\|_{\mathcal{H}}^2 - \frac{1}{2}\|x_0^1 - x_0^2\|_{\mathcal{H}}^2 \\
&\quad \leqslant \int_0^t \|x_1(s) - x_2(s)\|_{\mathcal{H}}\|f_1(s) - f_2(s)\|_{\mathcal{H}}\,\mathrm{d}s \\
&\qquad + (c_{\mathcal{R}} + \beta_\varphi c_{\mathcal{S}})\left(\int_0^t \|x_1(s) - x_2(s)\|_{\mathcal{H}}\,\mathrm{d}s\right)^2, \quad t \in I.
\end{aligned}$$

So, we have

$$\begin{aligned}
&\frac{1}{2}\|x_1(t) - x_2(t)\|_{\mathcal{H}}^2 \\
&\quad \leqslant \frac{1}{2}\|x_0^1 - x_0^2\|_{\mathcal{H}}^2 + \frac{1}{2}\int_0^t \|x_1(s) - x_2(s)\|_{\mathcal{H}}^2\,\mathrm{d}s \\
&\qquad + \frac{1}{2}\int_0^t \|f_1(s) - f_2(s)\|_{\mathcal{H}}^2\,\mathrm{d}s + (c_{\mathcal{R}} + \beta_\varphi c_{\mathcal{S}})T\int_0^t \|x_1(s) - x_2(s)\|_{\mathcal{H}}^2\,\mathrm{d}s, \quad t \in I.
\end{aligned}$$

Applying Grönwall's inequality, it finds therefore

$$\left\|x_1(t) - x_2(t)\right\|_{\mathcal{H}}^2 \leqslant M\left(\left\|x_0^1 - x_0^2\right\|_{\mathcal{H}}^2 + \int_0^t \left\|f_1(s) - f_2(s)\right\|_{\mathcal{H}}^2 \,\mathrm{d}s\right), \quad t \in I,$$

where the constant $M > 0$ only depends on $c_{\mathcal{R}}$, $c_{\mathcal{S}}$, β_φ, and T. Consequently, we have

$$\left\|x_1(t) - x_2(t)\right\|_{\mathcal{H}} \leqslant L_M\left(\left\|x_0^1 - x_0^2\right\|_{\mathcal{H}} + \left(\int_0^t \left\|f_1(s) - f_2(s)\right\|_{\mathcal{H}}^2 \,\mathrm{d}s\right)^{1/2}\right), \quad t \in I,$$

which proves the Lipschitzianity of the solution operator. □

4.3 Noninitial boundary value problems

In this subsection, we consider the existence of periodic solutions for the following differential inclusion problem:

$$\begin{aligned} &\dot{x}(t) \in f(t) - \partial J\big(t, x(t)\big) - A\big(t, x(t)\big) - \partial_c\psi\big(t, x(t)\big), \quad \text{a.e. } t \in I,\\ &x(0) = x(T). \end{aligned} \tag{7}$$

Theorem 4. *Assume that* ($\mathcal{H}_A$), ($\mathcal{H}_J$), *and* ($\mathcal{H}_\psi$) *hold. If* $m_A > m_J$, *then problem* (7) *has a unique solution* $x_\pi \in W^{1,2}(I;\mathcal{H})$.

Proof. Let us consider the operator $\mathcal{F} : \mathcal{H} \to \mathcal{H}$ defined by $\mathcal{F}(x_0) = x(T; x_0)$, where $x(\cdot, x_0)$ is the unique solution of (4) with the initial condition $x(0) = x_0$. Our goal is to prove that $\mathcal{F}$ has a unique fixed point in $\mathcal{H}$. Let $x_1 := x(\cdot, x_0^1)$ and $x_2 = x(\cdot, x_0^2)$. Keeping in mind that $m_A > m_J$ and the operator $x \mapsto \partial f(t,x) + A(t,x) + \partial_c\psi(t,x)$ is m strongly monotone with $m := m_A - m_J$, it gives

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|x_1(t) - x_2(t)\right\|_{\mathcal{H}}^2 &= \big\langle x_1(t) - x_2(t), \dot{x}_1(t) - \dot{x}_2(t)\big\rangle \\ &\leqslant -m\left\|x_1(t) - x_2(t)\right\|_{\mathcal{H}}^2, \quad \text{a.e. } t \in I. \end{aligned}$$

Employing Grönwall's inequality yields $\|x_1(t) - x_2(t)\|_{\mathcal{H}} \leqslant \mathrm{e}^{-mt}\|x_0^1 - x_0^2\|_{\mathcal{H}}$ for all $t \in I$. Then we have $\|\mathcal{F}(x_0^1) - \mathcal{F}(x_0^2)\|_{\mathcal{H}} \leqslant \kappa\|x_0^1 - x_0^2\|_{\mathcal{H}}$, where $\kappa := \mathrm{e}^{-mT} < 1$. Therefore, the operator $\mathcal{F}$ has a unique fixed point $x_{0,T} \in \mathcal{H}$ by the contractive fixed point theory. It is clear that $x_\pi := x(\cdot, x_{0,T})$ is the unique solution of (7). □

Likewise, we can consider the existence of antiperiodic solutions to the following differential inclusion problem:

$$\begin{aligned} &\dot{x}(t) \in f(t) - \partial J\big(t, x(t)\big) - A\big(t, x(t)\big) - \partial_c\psi\big(t, x(t)\big), \quad \text{a.e. } t \in I,\\ &x(0) = -x(T). \end{aligned} \tag{8}$$

Using the same argument given in the proof of Theorem 4, it is easy to conclude the following result.

Theorem 5. *Assume that* $(\mathcal{H}_A)$, $(\mathcal{H}_J)$, *and* $(\mathcal{H}_\psi)$ *hold. If* $m_A > m_J$, *then problem* (8) *has a unique solution* $x_{-\pi} \in W^{1,2}(I;\mathcal{H})$.

We end this section by showing that the unique solution to (7) can be obtained asymptotically from any solution of (4).

Theorem 6. *Assume, in addition to* $(\mathcal{H}_A)$, $(\mathcal{H}_J)$, $(\mathcal{H}_\psi)$, *that* $m_A > m_J$ *and for all* $(t,x) \in \mathbb{R}_+ \times \mathcal{H}$, $J(t+T, x) = J(t,x)$, $A(t+T, x) = A(t,x)$, $\psi(t+T, x) = \psi(t,x)$, *and* $f(t+T) = f(t)$. *Let* $x(\cdot, x_0)$ *be the unique solution of* (4) *with the initial condition* $x(0) = x_0$ *and define* $x_n(t) := x(t + nT; x_0)$ *for all* $t \in I$ *and* $n \in \mathbb{N}$. *Then, for any* $x_0 \in \mathcal{H}$, $x_n \to x_\pi$ *in* $C(I;\mathcal{H})$, *where* x_π *is the unique periodic solution of* (7).

Proof. Set $m := m_A - m_J > 0$, and let us consider $h(t) := \|x_n(t) - x_\pi(t)\|^2_{\mathcal{H}}/2$ for $t \in I$ and $n \in \mathbb{N}$. Then, for a.e. $t \in I$, it gives $\dot h(t) = \langle x_n(t) - x_\pi(t), \dot x_n(t) - \dot x_\pi(t)\rangle \leqslant -mh(t)$, where we used the monotonicity of the map $x \mapsto \partial J(t,x) + A(t,x) + \partial_c \psi(t,x)$. Therefore, for all $t \in I$, it is true $\|x_n(t) - x_\pi(t)\|_{\mathcal{H}} \leqslant \|x_n(0) - x_\pi(0)\|_{\mathcal{H}} \mathrm{e}^{-mT}$ for all $t \in I$, where we have used the Grönwall inequality. Thus, by using the same inequalities, for all $t \in I$, we have

$$\begin{aligned}
&\|x_n(t) - x_\pi(t)\|_{\mathcal{H}} \\
&\quad \leqslant \|x_n(0) - x_\pi(0)\|_{\mathcal{H}} \mathrm{e}^{-mt} = \|x_{n-1}(T) - x_\pi(T)\|_{\mathcal{H}} \mathrm{e}^{-mt} \\
&\quad \leqslant \|x_{n-1}(0) - x_\pi(0)\|_{\mathcal{H}} \mathrm{e}^{-m(t+T)} \leqslant \cdots \leqslant \|x_0 - x_\pi(0)\|_{\mathcal{H}} \mathrm{e}^{-m(t+nT)},
\end{aligned}$$

which shows that $x_n \to x$ in $C(I;\mathcal{H})$. □

Remark 3. As showed in the previous proof, x_n converges to x_π exponentially.

5 Applications

To illustrate the applicability of the theoretical results established in Section 4, we will present two comprehensive applications. The first one is a fractional evolution inclusion problem involving a multivalued term, which is formulated by the Clarke generalized gradient. The second application is a dynamic semipermeability problem, which is, more precisely, a complicated mixed boundary value problem of parabolic type with history-dependent operators and nonsmooth potential functionals.

5.1 Application to a fractional evolution inclusion problem

In the subsection, we are interested in the study of an evolutionary inclusion problem with a generalized nonlocal space-fractional Laplace operator and a Clarke generalized subgradient operator. Let Ω be a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary, $s \in (0,1)$ be such that $N > 2s$, $\Omega^{\mathrm{C}} := \mathbb{R}^N \setminus \Omega$, $0 < T < \infty$, and $I := [0,T]$. More precisely, the classical form of the evolutionary inclusion problem under consideration is formulated as follows.

Problem 1. Find function $u : \mathbb{R}^N \times [0,T] \to \mathbb{R}$ such that

$$\frac{\partial u}{\partial t}(x,t) + \mathcal{L}_K u(x,t) + \partial j(x,t,u) \ni f(x,t) \quad \text{in } \Omega \times (0,T),$$
$$u(x,t) = 0 \quad \text{in } \Omega^{\mathrm{C}} \times (0,T),$$
$$u(x,0) = u_0(x) \quad \text{in } \Omega,$$

where the operator $\mathcal{L}_K$ stands for the generalized nonlocal space-fractional Laplace operator defined as follows: $\mathcal{L}_K u(x) := -\int_{\mathbb{R}^N} (u(x+y) + u(x-y) - 2u(x))K(y)\,\mathrm{d}y$ for a.e. $x \in \mathbb{R}^N$, for all $u \in X_0$.

We first impose the following assumptions for the data of Problem 1.

(H_j) $j : \Omega \times I \times \mathbb{R} \to \mathbb{R}$ is such that $j(\cdot,\cdot,0) \in L^1(\Omega \times I)$ and

- (i) for each $r \in \mathbb{R}$, the function $(x,t) \mapsto j(x,t,r)$ is measurable on $\Omega \times I$;
- (ii) for a.e. $(x,t) \in \Omega \times I$, the functional $r \mapsto j(x,t,r)$ is locally Lipschitz continuous;
- (iii) there exist $\alpha_j, \beta_j \in L^2_+(I)$ satisfying $|\xi| \leqslant \alpha_j(t) + \beta_j(t)|r|$ for all $\xi \in \partial j(x,t,r)$ and all $(x,t,r) \in \Omega \times I \times \mathbb{R}$;
- (iv) there exists $m_j \geqslant 0$ such that $(\xi - \eta)(r_1 - r_2) \geqslant -m_j |r_1 - r_2|^2$ for all $\xi \in \partial j(x,t,r_1)$, $\eta \in \partial j(x,t,r_2)$, $(x,t) \in \Omega \times I$, and $r_1, r_2 \in \mathbb{R}$.

(H_0) $f \in L^2(I; X_0)$ and $u_0 \in X_0$.

The weak solutions to Problem 1 are understood as follows.

Definition 3. We say that $u : I \to X_0$ is a weak solution to Problem 1 if $u(x,0) = u_0(x)$ in Ω and the following inequality holds for all $v \in X_0$:

$$\int_{\mathbb{R}^N} \frac{\partial u(x,t)}{\partial t} v(x)\,\mathrm{d}x + \int_{\Omega} j^0\big(x,t,u(x,t); v(x)\big)\,\mathrm{d}x + \int_{\mathbb{R}^N} v(x)\mathcal{L}_K u(x,t)\,\mathrm{d}x$$
$$\geqslant \int_{\mathbb{R}^N} f(x,t)v(x)\,\mathrm{d}x, \quad \text{a.e. } t \in I.$$

Let us define function $J : I \times X_0 \to \mathbb{R}$ by $J(t,u) := \int_\Omega j(x,t,u(x))\,\mathrm{d}x$ for all $(t,u) \in I \times X_0$. For function J, we have the following lemma.

Lemma 4. *Assume that* (H_j) *is fulfilled. Then the following statements hold*:

- (i) $t \mapsto J(t,u)$ *is measurable on* I *for all* $u \in X_0$.
- (ii) *For a.e.* $t \in I$, $X_0 \ni u \mapsto J(t,u) \in \mathbb{R}$ *is locally Lipschitz.*
- (iii) *For all* $(t,u) \in I \times X_0$, *we have* $J^0(t,u) \leqslant \int_\Omega j^0(x,t,u(x))\,\mathrm{d}x$ *and* $\partial J(t,u) \subset \int_\Omega \partial j(x,t,u(x))\,\mathrm{d}x$.
- (iv) *There exists a constant* $c_j > 0$ *such that* $\|\partial J(t,u)\|_{X_0} \leqslant c_j(\alpha_j(t) + \beta_j(t)\|u\|_{X_0})$ *for all* $(t,u) \in I \times X_0$.

(v) *For any* $u, v \in X_0$, $t \in I$, $\xi \in \partial J(t, u)$ *and* $\eta \in \partial J(t, v)$, *the inequality is satisfied* $\langle \xi - \eta, u - v\rangle \geqslant -m_j c(2)^2 \|u - v\|^2_{X_0}$.

Proof. Statements (i)–(iv) are the direct consequences of [13, Thm. 3.47]. It remains us to prove conclusion (v). Let $u, v \in X_0$, $t \in I$, $\xi \in \partial J(t, u)$, and $\eta \in \partial J(t, v)$ be arbitrary. Statement (iii) indicates

$$\begin{aligned}\langle \xi - \eta,\, v - u\rangle &= \langle \xi,\, v - u\rangle + \langle \eta,\, u - v\rangle \\ &\leqslant J^0(t, u; v - u) + J^0(t, v; u - v) \\ &\leqslant \int_\Omega m_j \big|u(x) - v(x)\big|^2 \,\mathrm{d}x = m_j \|u - v\|^2_{L^2(\Omega)} \\ &\leqslant m_j c(2)^2 \|u - v\|^2_{X_0},\end{aligned}$$

where the last inequality is obtained by using Lemma 2. Therefore, the desired inequality is valid. □

The existence and uniqueness of weak solutions to Problem 1 is provided by the following result.

Theorem 7. *Assume that* $H(K)$, $H(j)$, $H(0)$, *and* $1 \geqslant m_j c(2)^2$ *hold, then Problem* 1 *has a unique weak solution* $u(\cdot, f, x_0) \in W^{1,2}(I; X_0)$, *and the solution operator* $\mathcal{T}$: $(f, x_0) \mapsto u(\cdot, f, x_0)$ *is Lipschitz continuous from* $L^2(I; X_0) \times X_0$ *into* $C(I; X_0)$.

Proof. Let $\mathcal{H} := X_0$. Consider the operator $A : \mathcal{H} \to \mathcal{H}$ defined for all $u, v \in \mathcal{H}$ by $\langle Au, v\rangle := \int_{\mathbb{R}^N} v(x)\mathcal{L}_K(u(x))\,\mathrm{d}x$. We now claim that A is a continuous linear operator. For any $u, v \in \mathcal{H}$, it has

$$\begin{aligned}\langle Au, v\rangle &= \int_{\mathbb{R}^N} v(x)\mathcal{L}_K\big(u(x)\big)\,\mathrm{d}x = \int_{\mathbb{R}^{2N}} \big[v(x) - v(y)\big]\big[u(x) - u(y)\big]K(x - y)\,\mathrm{d}y\,\mathrm{d}x \\ &= \langle u, v\rangle_{X_0}.\end{aligned}$$

It obvious to see that A is a linear continuous operator and $\|Au\|_{\mathcal{H}} = \|u\|_{\mathcal{H}}$. In addition, the fact $\langle Au - Av,\, u - v\rangle = \|u - v\|^2_{\mathcal{H}}$ for all $u, v \in \mathcal{H}$ implies that A is strongly monotone with constant $m_A = 1$.

Let us consider the following intermediate problem: find $u : I \to \mathcal{H}$ such that for all $v \in \mathcal{H}$,

$$\begin{aligned}&\big\langle u'(t) + Au(t),\, v\big\rangle + J^0\big(t, u(t); v\big) \geqslant \big\langle f(t), v\big\rangle, \quad \text{a.e. } t \in I, \\ &u(0) = u_0.\end{aligned} \tag{9}$$

Employing Lemma 4(iii), we can see that a solution to problem (9) is also a weak solution to Problem 1. On the other hand, it is not difficult to verify that problem (9) is equivalent to the following evolutionary inclusion problem:

$$\begin{aligned}&u'(t) + Au(t) + \partial J\big(t, u(t)\big) \ni f(t), \quad \text{a.e. } t \in I, \\ &u(0) = u_0.\end{aligned} \tag{10}$$

Observe that $A : \mathcal{H} \to \mathcal{H}$ reads hypothesis ($\mathcal{H}_A$), hence, we are now in a position to invoke Lemma 4 and Theorem 2 that problem (10) has a unique solution $u(\cdot, f, u_0) \in W^{1,2}(I; \mathcal{H})$. So, u is also a weak solution to Problem 1. For the uniqueness part of Problem 1, it can be obtained directly by using the standard procedure (see the proof of Theorem 2). Finally, the Lipschitz continuity of the solution operator $\mathcal{T} : (f, x_0) \mapsto u(\cdot, f, x_0)$ could be verified by employing the same argument with the proof of Theorem 2. □

Furthermore, we are going to apply the results established in Section 4.3 to investigate the fractional evolution inclusion problem, Problem 1, with periodic and antiperiodic boundary value conditions, respectively.

Problem 2. Find function $u : \mathbb{R}^N \times I \to \mathbb{R}$ such that

$$\begin{aligned}
&\frac{\partial u}{\partial t}(x,t) + \mathcal{L}_K u(x,t) + \partial j(x,t,u) \ni f(x,t) \quad \text{in } \Omega \times (0,T),\\
&u(x,t) = 0 \quad \text{in } \Omega^{\mathrm{C}} \times (0,T),\\
&u(x,0) = u(x,T) \quad \text{in } \Omega.
\end{aligned}$$

Likewise, the weak solutions to Problem 2 are given as follows.

Definition 4. We say that $u : I \to X_0$ is a weak solution to Problem 2 if $u(x,0) = u(x,T)$ in Ω and the following inequality holds for all $v \in X_0$:

$$\begin{aligned}
&\int_{\mathbb{R}^N} \frac{\partial u(x,t)}{\partial t} v(x)\,\mathrm{d}x + \int_{\Omega} j^0\bigl(x,t,u(x,t); v(x)\bigr)\,\mathrm{d}x + \int_{\mathbb{R}^N} v(x)\mathcal{L}_K u(x,t)\,\mathrm{d}x\\
&\quad \geqslant \int_{\mathbb{R}^N} f(x,t)v(x)\,\mathrm{d}x, \quad \text{a.e. } t \in I.
\end{aligned}$$

Invoking Theorem 4 and the proof of Theorem 7, we have the following existence and uniqueness result for Problem 2.

Theorem 8. *Assume that* (H_K)*,* (H_j)*,* (H_0)*, and* $1 > m_j c(2)^2$ *are satisfied, then Problem* 2 *has a unique solution* $u(\cdot, f, x_0) \in W^{1,2}(I; X_0)$.

We end the subsection by considering the antiperiodic boundary value problem.

Problem 3. Find function $u : \mathbb{R}^N \times I \to \mathbb{R}$ such that

$$\begin{aligned}
&\frac{\partial u}{\partial t}(x,t) + \mathcal{L}_K u(x,t) + \partial j(x,t,u) \ni f(x,t) \quad \text{in } \Omega \times (0,T),\\
&u(x,t) = 0 \quad \text{in } \Omega^{\mathrm{C}} \times (0,T),\\
&u(x,0) = -u(x,T) \quad \text{in } \Omega.
\end{aligned}$$

Definition 5. We say that $u : I \to X_0$ is a weak solution to Problem 3 if $u(x,0) = -u(x,T)$ in Ω and the following inequality holds for all $v \in X_0$:

$$\int_{\mathbb{R}^N} \frac{\partial u(x,t)}{\partial t} v(x)\,\mathrm{d}x + \int_{\Omega} j^0(x,t,u(x,t);v(x))\,\mathrm{d}x + \int_{\mathbb{R}^N} v(x)\mathcal{L}_K u(x,t)\,\mathrm{d}x \geqslant \int_{\mathbb{R}^N} f(x,t)v(x)\,\mathrm{d}x \quad \text{a.e. } t \in I.$$

Analogously, from Theorem 5 and the proof of Theorem 7, we have the following theorem.

Theorem 9. *Assume that* (H_K), (H_j), (H_0), *and* $1 > m_j c(2)^2$, *then Problem* 3 *has a unique solution* $u(\cdot, f, x_0) \in W^{1,2}(I; X_0)$.

5.2 Application to a dynamic semipermeability problem

The semipermeability boundary conditions can describe exactly behavior of various types of membranes, natural and artificial ones, and arise in models of heat conduction, electrostatics, hydraulics and in the description of the flow of a Bingham fluid in which the solution represents temperature, electric potential, pressure, and so forth. The current subsection is devoted to exploring a comprehensive semipermeability problem of parabolic type involving Volterra-type integral terms and nonsmooth potential functions.

Let $0 < T < +\infty$ and Ω be a bounded domain in $\mathbb{R}^d$ with Lipschitz continuous boundary $\Gamma := \partial\Omega$. Denote by ν the unit outward normal on the boundary Γ. The boundary Γ is decomposed into three mutually disjoint and relatively open subsets Γ_1, Γ_2 and Γ_3 such that $\Gamma = \bar{\Gamma}_1 \cup \bar{\Gamma}_2 \cup \bar{\Gamma}_3$ and $\mathrm{meas}(\Gamma_1) > 0$. In the sequel, we denote by $\mathcal{Q} = \Omega \times (0,T)$, $\Sigma_1 = \Gamma_1 \times (0,T)$, $\Sigma_2 = \Gamma_2 \times (0,T)$, and $\Sigma_3 = \Gamma_3 \times (0,T)$. The classical formulation of semipermeability problem is described as follows.

Problem 4. Find $u : \mathcal{Q} \to \mathbb{R}$ such that

$$\frac{\partial u(x,t)}{\partial t} + Lu(x,t) + \partial j\big(x,t,u(x,t)\big) + \int_0^t E(t-s)u(x,s)\,\mathrm{d}s \ni f_0(x,t) \quad \text{in } \mathcal{Q},$$

$$u(x,t) = 0 \quad \text{on } \Sigma_1,$$

$$-\frac{\partial u(x,t)}{\partial \nu_a} = f_2(x,t) \quad \text{on } \Sigma_2, \qquad -\frac{\partial u}{\partial \nu_a} = F\left(\int_0^t \big|u(x,s)\big|\,\mathrm{d}s\right)\mathrm{sgn}(u(x,t)) \quad \text{on } \Sigma_3,$$

$$u(x,0) = u_0(x) \quad \text{in } \Omega,$$

where $\partial u/\partial \nu_a$ denotes the conormal derivative with respect to the second-order differential operator $L := -\sum_{i,j=1}^d (\partial/\partial x_i)(a_{ij}(x)\partial/\partial x_j))$, and sgn stands for the sign function.

In order to deliver the weak formulation of Problem 4, we are now in a position to introduce the following function spaces $\mathcal{H} = \{v \in H^1(\Omega)\colon v = 0 \text{ on } \Gamma_1\}$ and $V = L^2(\Omega)$. However, from Korn's inequality and the condition $\operatorname{meas}(\Gamma_1) > 0$, it finds that $\mathcal{H}$ is a Hilbert space endowed with the inner product $\langle u, v\rangle = \int_\Omega (\nabla u(x), \nabla v(x))_{\mathbb{R}^d}\,\mathrm{d}x$ for all $u, v \in \mathcal{H}$.

Also, we impose the following assumptions.

(H_a) $a : \Omega \to \mathbb{R}$ is such that $a_{ij} \in L^\infty(\Omega)$ for $i, j = 1, \dots, d$, and there exists a constant $m_a > 0$ such that $\sum_{i,j=1}^d a_{ij}(x)\xi_i\xi_j \geqslant m_a\|\xi\|^2$ for all $\xi \in \mathbb{R}^d$ and a.e. $x \in \Omega$.

(H_j) $j : \mathcal{Q} \times \mathbb{R} \to \mathbb{R}$ is such that:

- (i) $j(\cdot,\cdot,r)$ is measurable on $\mathcal{Q}$ for all $r \in \mathbb{R}$ and there exists $e \in L^2(\Omega)$ such that $j(\cdot,\cdot,e(\cdot)) \in L^1(\mathcal{Q})$;
- (ii) $j(x,t,\cdot)$ is locally Lipschitz for a.e. $(x,t) \in \mathcal{Q}$;
- (iii) $|\partial j(x,t,r)| \leqslant c_{0j}(t) + c_{1j}|r|$ for all $r \in \mathbb{R}$, a.e. $(x,t) \in \mathcal{Q}$ with $c_{0j} \in L^2_+(I)$ and $c_{1j} \geqslant 0$;
- (iv) there exists $m_j \geqslant 0$ satisfying $(\xi_1 - \xi_2)(r_1 - r_2) \geqslant -m_j|r_1 - r_2|^2$ for all $\xi_1 \in \partial j(x,t,r_1)$, $\xi_2 \in \partial j(x,t,r_2)$, $r_1, r_2 \in \mathbb{R}$ and for a.e. $(x,t) \in \mathcal{Q}$.

(H_F) $F : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$ is such that:

- (i) $F(\cdot, r)$ is measurable on Γ_3 for all $r \in \mathbb{R}$.
- (ii) there is $L_F > 0$ such that $|F(x,r_1) - F(x,r_2)| \leqslant L_F|r_1 - r_2|$ for all r_1, $r_2 \in \mathbb{R}$, a.e. $x \in \Gamma_3$.
- (iii) $F(\cdot, 0) \in L^2(\Gamma_3)$.

(H_E) $E \in C(I)$.

(H_0) $f_0 \in L^2(\mathcal{Q})$, $f_2 \in L^2(\Sigma_2)$, $u_0 \in \mathcal{H}$.

It follows from Riesz's representation theorem that there is a function $f : I \to H^{-1}(\Omega)$ such that $\langle f(t), v\rangle = \int_\Omega f_0(t)v\,\mathrm{d}x + \int_{\Gamma_1} f_2(t)v\,d\Gamma$ for all $v \in \mathcal{H}$ and all $t \in I$.

Using a standard procedure, it is not difficult to get the weak formulation of Problem 4 as follows.

Problem 5. Find a function $u : I \to \mathcal{H}$ such that $u(0) = u_0$ and for all $v \in \mathcal{H}$,

$$\left\langle u'(t) + Au(t) + \int_0^t E(t-s)u(s)\,\mathrm{d}s - f(t),\ v - u(t)\right\rangle + \int_\Omega j^0\big(t, u(t); v - u(t)\big)\,\mathrm{d}x$$
$$+ \int_{\Gamma_3} F\left(\int_0^t \big|u(s)\big|\,\mathrm{d}s\right)\big(|v| - \big|u(t)\big|\big)\,\mathrm{d}\Gamma \geqslant 0, \quad \text{a.e. } t \in I,$$

where the operator $A : \mathcal{H} \to \mathcal{H}$ is defined by

$$\langle Au, v\rangle = \int_\Omega \sum_{i,j=1}^d a_{ij}(x)\frac{\partial u(x)}{\partial x_i}\frac{\partial v(x)}{\partial x_j}\,\mathrm{d}x, \quad u, v \in \mathcal{H}.$$

We also consider a function $J : I \times \mathcal{H} \to \mathbb{R}$ defined by $J(t,u) := \int_\Omega j(x,t,u(x))\,\mathrm{d}x$ for all $(t,u) \in I \times \mathcal{H}$. It is obvious that under hypothesis (H_j) the following lemma is available.

Lemma 5. *Assume that* (H_j) *is fulfilled. Then the following statements hold*:

(i) $t \mapsto J(t,u)$ *is measurable on I for all* $u \in \mathcal{H}$.
(ii) *For a.e.* $t \in I$, $\mathcal{H} \ni u \mapsto J(t,u) \in \mathbb{R}$ *is locally Lipschitz.*
(iii) *For all* $(t,u) \in I \times \mathcal{H}$, *we have* $J^0(t,u) \leqslant \int_\Omega j^0(x,t,u(x))\,\mathrm{d}x$ *and* $\partial J(t,u) \subset \int_\Omega \partial j(x,t,u(x))\,\mathrm{d}x$.
(iv) *There exists a constant* $c_j > 0$ *such that* $\|\partial J(t,u)\|_{\mathcal{H}} \leqslant c_j(c_{0j}(t) + c_{1j}\|u\|_{\mathcal{H}})$ *for all* $(t,u) \in I \times \mathcal{H}$.
(v) *We have* $\langle \xi - \eta, u - v\rangle \geqslant -m_j c_{\mathcal{H}}^2 \|u - v\|_{\mathcal{H}}^2$ *for any* $u, v \in \mathcal{H}$, $t \in I$, $\xi \in \partial J(t,u)$, *and* $\eta \in \partial J(t,v)$. *Here* $c_{\mathcal{H}} > 0$ *is such that* $\|u\|_{L^2(\Omega)} \leqslant c_{\mathcal{H}}\|u\|_{\mathcal{H}}$ *for all* $u \in \mathcal{H}$.

Let us define the operators $\mathcal{R} : L^2(I;\mathcal{H}) \to L^2(I;\mathcal{H})$ and $\mathcal{S} : L^2(I;\mathcal{H}) \to L^2(I;Y)$ by $\mathcal{R}u(t) := \int_0^t E(t-s)u(s)\,\mathrm{d}s$ and $\mathcal{S}u(t) := \int_0^t |u(s)|\,\mathrm{d}s$ for all $u \in L^2(I;\mathcal{H})$, where $Y = L^2(\Gamma_3)$.

Remark 4. It follows from [21, Exs. 4 and 6] that $\mathcal{R}$ and $\mathcal{S}$ are two history-dependent operators.

Moreover, let us consider the function $\varphi : Y \times \mathcal{H} \to \mathbb{R}$ defined by $\varphi(y,u) = \int_{\Gamma_3} F(y)|v|\,d\Gamma$ for all $y \in Y$ and $u \in \mathcal{H}$. The following result establishes the well-posedness for Problem 5.

Theorem 10. *Assume that* (H_a), (H_j), (H_E), (H_F), (H_0), *and* $m_a > m_j c_{\mathcal{H}}^2$ *are fulfilled, then Problem* 5 *has a unique solution* $u(\cdot, f_0, f_2, u_0) \in W^{1,2}(I;\mathcal{H})$. *Moreover, the solution operator* $(f_0, f_2, u_0) \mapsto u(\cdot, f_0, f_2, u_0)$ *is Lipschitz continuous from* $L^2(I;V) \times L^2(I;L^2(\Gamma_2)) \times \mathcal{H}$ *into* $C(I;\mathcal{H})$.

Proof. We first study the intermediate problem: find $u : I \to \mathcal{H}$ such that $u(0) = u_0$ and for all $v \in \mathcal{H}$,

$$\begin{aligned}&\big\langle u'(t) + Au(t) + \mathcal{R}u(t) - f(t),\, v - u(t)\big\rangle + J^0\big(t, u(t); v - u(t)\big)\\ &\quad \geqslant \varphi\big(\mathcal{S}u(t), v\big) - \varphi\big(\mathcal{S}u(t), u(t)\big), \quad \text{a.e. } t \in I.\end{aligned}$$

In fact, the inequality above could be rewritten to the following inclusion problem: find $u : I \to \mathcal{H}$ such that $u(0) = u_0$ and

$$u'(t) + Au(t) + \mathcal{R}u(t) + \partial J\big(t, u(t)\big) + \partial_c\varphi\big(\mathcal{S}u(t), u(t)\big) \ni f(t), \quad \text{a.e. } t \in I. \quad (11)$$

However, Lemma 5 reveals the fact that a solution of problem (11) is also a solution of Problem 5. Based on this fact, we are going to utilize Theorem 3 for concluding the existence of solutions of Problem 5.

From hypothesis (H_a) it is not difficult to prove that A is a continuous and strongly monotone operator with constant $m_A := m_a$. Notice that $a_{ij} \in L^\infty(\Omega)$ for $i, j =$

$1, \ldots, d$, we can obtain the inequality $\|Au\|_{\mathcal{H}} \leqslant c_A \|u\|_{\mathcal{H}}$ for all $u \in \mathcal{H}$ with some $c_A > 0$. Besides, by virtue of condition (H_F) and the definition of φ, it gives that φ satisfies condition ($\mathcal{H}_\varphi$) (see [21, p. 251, Thm. 113,]).

Therefore, all conditions in Theorem 3 are verified. This theorem implies that problem (11) has a unique solution $u(\cdot, f_0, f_2, u_0) \in W^{1,2}(I;\mathcal{H})$, which is a solution to Problem 5 as well. On the other hand, from the smallness condition $m_a > c_{\mathcal{H}}^2 m_j$ it follows a standard procedure to obtain that $u(\cdot, f_0, f_2, u_0) \in W^{1,2}(I;\mathcal{H})$ is also the unique solution to Problem 5. Moreover, the Lipschitz continuity of the solution operator $\mathcal{T} : (f_0, f_2, x_0) \mapsto u(\cdot, f_0, f_2, x_0)$ could be verified by employing the same argument with the proof of Theorem 2. □

6 Conclusions

In this paper, a class of nonlinear evolutionary variational-hemivariational inequalities involving history-dependent operators is introduced and studied. We propose a new methodology, which is based on a hidden maximal monotonicity, to deliver the well-posedness results of the inequality problems under the periodic and antiperiodic boundary conditions, respectively. These theoretical results extend the recent one obtained by Han, Migórski, and Sofonea [7]. Moreover, to illustrate the applicability of the abstract results established in the paper, a fractional evolution inclusion and a dynamic semipermeability problem are investigated, respectively.

Acknowledgment. The authors wish to thank the referees for providing several helpful suggestions which help to improve the paper.

References

1. C. Baiocchi, A. Capelo, *Variational and Quasivariational Inequalities*, John Wiley & Sons, New York, 1984. Translated from the Italian by L. Jayakar.
2. H.H. Bauschke, P.L. Combettes, *Convex analysis and monotone operator theory in Hilbert spaces,* 2nd ed., CMS Books Math./Ouvrages Math. SMC, Springer, 2017, https://doi.org/10.1007/978-3-319-48311-5.
3. F.H. Clarke, *Optimization and Nonsmooth Analysis,* 2nd ed., Classics Appl. Math., Vol. 5, SIAM, Philadelphia, PA, 1990, https://doi.org/10.1137/1.9781611971309.
4. G. Duvaut, J.-L. Lions, *Inequalities in Mechanics and Physics*, Grundlehren Math. Wiss., Vol. 219, Springer, Berlin, New York, 1976, https://doi.org/10.1007/978-3-642-66165-5. Translated from the French by C.W. John.
5. W. Han, Singular perturbations of variational-hemivariational inequalities, *SIAM J. Math. Anal.*, **52**(2):1549–1566, 2020, https://doi.org/10.1137/19M1282490.
6. W. Han, S. Migórski, M. Sofonea (Eds.), *Advances in Variational and Hemivariational Inequalities*, Adv. Mech. Math., Vol. 33, Springer, Cham, 2015, https://doi.org/10.1007/978-3-319-14490-0.

7. W. Han, S. Migórski, M. Sofonea, Analysis of a general dynamic history-dependent variational-hemivariational inequality, *Nonlinear Anal. Real World Appl.*, **36**:69–88, 2017, `https://doi.org/10.1016/j.nonrwa.2016.12.007`.
8. W. Han, M. Sofonea, Numerical analysis of hemivariational inequalities in contact mechanics, *Acta Numerica*, **28**:175–286, 2019, `https://doi.org/10.1017/s0962492919000023`.
9. W. Han, M. Sofonea, M. Barboteu, Numerical analysis of elliptic hemivariational inequalities, *SIAM J. Numer. Anal.*, **55**(2):640–663, 2017, `https://doi.org/10.1137/16M1072085`.
10. D. Kinderlehrer, G. Stampacchia, *An introduction to variational inequalities and their applications*, Classics Appl. Math., Vol. 31, SIAM, Philadelphia, PA, 2000, `https://doi.org/10.1137/1.9780898719451`. Reprint of the 1980 original.
11. Z. Liu, D. Motreanu, S. Zeng, Positive solutions for nonlinear singular elliptic equations of p-Laplacian type with dependence on the gradient, *Calc. Var. Partial Differ. Equ.*, **58**:28, 2019, `https://doi.org/10.1007/s00526-018-1472-1`.
12. Z.H. Liu, D. Motreanu, S.D. Zeng, Generalized penalty and regularization method for differential variational-hemivariational inequalities, *SIAM J. Optim.*, **31**(2):1158–1183, 2021, `https://doi.org/10.1137/20M1330221`.
13. S. Migórski, A. Ochal, M. Sofonea, *Nonlinear Inclusions and Hemivariational Inequalities*, Adv. Mech. Math.,Vol. 26, Springer, New York, 2013, `https://doi.org/10.1007/978-1-4614-4232-5`.
14. S. Migórski, M. Sofonea, S. Zeng, Well-posedness of history-dependent sweeping processes, *SIAM J. Math. Anal.*, **51**(2):1082–1107, 2019, `https://doi.org/10.1137/18M1201561`.
15. F. Miranda, J.F. Rodrigues, L. Santos, Evolutionary quasi-variational and variational inequalities with constraints on the derivatives, *Adv. Nonlinear Anal.*, **9**(1):250–277, 2020, `https://doi.org/10.1515/anona-2018-0113`.
16. G. Molica Bisci, D. Repovš, Some hemivariational inequalities in the Euclidean space, *Adv. Nonlinear Anal.*, **9**(1):958–977, 2020, `https://doi.org/10.1515/anona-2020-0035`.
17. Z. Naniewicz, P.D. Panagiotopoulos, *Mathematical Theory of Hemivariational Inequalities and Applications*, Pure Appl. Math., Vol. 188, Marcel Dekker, New York, 1995.
18. P.D. Panagiotopoulos, *Hemivariational Inequalities, Applications in Mechanics and Engineering*, Springer, Berlin, 1993, `https://doi.org/10.1007/978-3-642-51677-1`.
19. N.S. Papageorgiou, V.D. Rădulescu, Periodic solutions for time-dependent subdifferential evolution inclusions, *Evol. Equ. Control Theory*, **6**(2):277–297, 2017, `https://doi.org/10.3934/eect.2017015`.
20. P. Pucci, M. Xiang, B.L. Zhang, Existence and multiplicity of entire solutions for fractional p-Kirchhoff equations, *Adv. Nonlinear Anal.*, **5**(1):27–55, 2016, `https://doi.org/10.1515/anona-2015-0102`.
21. M. Sofonea, S. Migórski, *Variational-Hemivariational Inequalities with Applications*, Monogr. Res. Notes Math., CRC Press, Boca Raton, FL, 2018, `https://doi.org/10.1201/9781315153261`.

22. E. Vilches, B.T. Nguyen, Evolution inclusions governed by time-dependent maximal monotone operators with a full domain, *Set-Valued Var. Anal*, **28**:569–581, 2020, https://doi.org/10.1007/s11228-020-00533-5.
23. S. Zeng, Y. Bai, L. Gasiński, P. Winkert, Existence results for double phase implicit obstacle problems involving multivalued operators, *Calc. Var. Partial Differ. Equ.*, **59**(176), 2020, https://doi.org/10.1007/s00526-020-01841-2.
24. S. Zeng, E. Vilches, Well-posedness of history/state-dependent implicit sweeping processes, *J. Optim. Theory Appl.*, **186**(3):960–984, 2020, https://doi.org/10.1007/s10957-020-01730-w.
25. S.D. Zeng, S. Migórski, A.A. Khan, Nonlinear quasi-hemivariational inequalities: Existence and optimal control, *SIAM J. Control Optim.*, **59**(3):1246–1274, 2021, https://doi.org/10.1137/19M1282210.