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Global dynamics of solutions for a sixth-order parabolic equation describing continuum evolution of film-free surface*

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Abstract. This paper is concerned with a sixth-order diffusion equation, which describes continuum evolution of film-free surface. By using the regularity estimates for the semigroups, iteration technique and the classical existence theorem of global attractors we verified the existence of global attractor for this surface diffusion equation in the spaces $H^3(\Omega)$ and fractional-order spaces $H^k(\Omega)$, where $0 \leqslant k < \infty$.

Keywords: global dynamics, sixth-order parabolic equation, absorbing set.

1 Introduction

In order to describe the continuum evolution of the film-free surface, the authors in [5] proposed the following classical surface diffusion equation:

$$\upsilon_n = \mathcal{D}\Delta_S \mu = \mathcal{D}\Delta_S(\mu_\gamma + \mu_\omega) = \mathcal{D}\Delta_S\big(\widetilde{\gamma}_{\alpha\beta} C_{\alpha\beta} + \nu\Delta^2 u + \mu_\omega\big),$$

where υ_n, D_s, S_0, Ω_0, V_0, R and T are the normal surface velocity, the surface diffusivity, the number of atoms per unit area on the surface, the atomic volume, the molar volume of lattice cites in the film, the universal gas constant and the absolute temperature, respectively. $\mathcal{D} = D_S S_0 \Omega_0 V_0/(RT)^{23}$. Moreover, Δ_S is the surface Laplace operator, ν represents the regularization coefficient that measures the energy of edges and corners, $C_{\alpha\beta}$ means the surface curvature tensor, and μ_w being an exponentially decaying function of u that has a singularity at $u \to 0$ (see [5]).

Particularly, in the small-slop approximation, the cases of a crystal, which has cubic symmetry and high-symmetry orientations, arise an evolution equation in the following

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form for the film thickness:

$$\frac{\partial u}{\partial t} = D\{D^5u + D^3u - D[|Du|^2D^2u] + D[w_0(u) + w_2(u)|Du|^2 + w_3(u)D^2u]\}, \tag{1}$$

where $(x,t) \in Q_T \equiv \Omega \times (0,T)$, $\Omega = (0,1)$, $D = \partial/\partial x$, and $w_{0,2,3}(h)$ are three smooth functions, respectively $[w_3(h_0) = 0$, $2w_2 = \mathrm{d}w_3/\mathrm{d}h]$. From the physical consideration, Eq. (1) is supplemented with the following boundary value conditions:

$$Du(x,t) = D^3u(x,t) = D^5u(x,t) = 0, \quad x = 0,1, \tag{2}$$

and initial condition

$$u(x,0) = u_0(x), \quad x \in [0,1]. \tag{3}$$

We remark that Eq. (1) is related to the formation of quantum dots in epitaxially grown thin solid films. This formation has been attracting attention as a very promising area of nanotechnology that can lead to a new generation of electronic devices. According to the mechanism for the formation, the substrate induces the film growth in a certain crystallographic orientation. In the absence of wetting interactions with the substrate, due to a large surface-energy anisotropy, this orientation would be thermodynamically forbidden, and the surface would undergo a long-wave faceting (spinodal decomposition) instability. In [5], the authors show that wetting interactions between the film and the substrate can suppress this instability and qualitatively change its spectrum, leading to the damping of long-wave perturbations and the selection of the preferred wavelength at the instability threshold. This creates a possibility for the formation of stable regular arrays of quantum dots even in the absence of epitaxial stresses.

In [15], on the basis of the Schauder-type estimates and the techniques in Campanato spaces, the author assumed that the smooth functions $w_{0,2,3}(u)$ satisfy

$$\begin{aligned} &w_3(u_0) = 0, \qquad 2w_2(u) = w_3'(u), \\ &W_0(u) = \int_0^s w_0(u)\,\mathrm{d}s \geqslant \frac{3}{4}[w_3(u)]^2, \end{aligned} \tag{4}$$

and studied the existence of classical global solutions.

The dynamic properties of diffusion systems such as the global asymptotical behaviors of solutions and global attractors are important for the study of diffusion models, which ensure the stability of diffusion phenomena and provide the mathematical foundation for the study of diffusion dynamics. For the higher-order diffusion equation, there are many classical results related to its global attractors, see, for example, Dlotko [3], Li and Zhong [7], Schimperna [10], Alouini [1], Cheskidov and Dai [2], Huang, Yang, Lu and Wang [6], Duan and Xu [4] and so on. To the best of our knowledge, the existence of the global attractor for problem (1)–(3) has not been addressed yet, which is the main goal of this article.

In this paper, by using the regularity estimates for the semigroups, iteration technique and the classical existence theorem of global attractors we consider the global attractor of solutions for the initial boundary value problem for Eq. (1). The main results are Theorem 1 in the next section and Theorem 2 in Section 3.

This article is organized as follows. In Section 2, we give some preparations for our consideration and prove the existence of global attractors for problem (1)–(3) in the Sobolev space $H^3(\Omega)$. In Section 3, we prove the existence of global attractors for problem (1)–(3) in the Sobolev space $H^k(\Omega)$ with any $k \geqslant 0$.

2 Global attractor in H^3

In order to consider the global attractors of problem (1)–(3), we select a closed metric space $\mathcal{U}_0$ and prove the existence of global attractors of problem (1)–(3) in this space, where $\mathcal{U}_0 = \{u\colon\ u \in H^3(\Omega),\ Du|_{x=0,1} = 0,\ \int_0^1 u\,\mathrm{d}x = 0\}$.

For convenience, we give the following lemma on global existence and uniqueness of solution to problem (1)–(3).

Lemma 1. *Assume that $u_0 \in \mathcal{U}_0$ and the functions $w_{0,2,3}(h)$ satisfy* (4)*. Then* (1)–(3) *admits a unique global solution $u(x,t)$, which satisfies*

$$u(x,t) \in L^\infty\big(0,T;H^3(\Omega)\big) \cap L^2\big(0,T;H^6(\Omega)\big).$$

The proof of existence and regularity of solutions is based on the Galerkin method and a priori estimates in the following. Thanks to the above existence lemma, we know that there exists a continuous operator semigroup $\{S(t)\}_{t\geqslant 0}$ in $H^3(\Omega)$ satisfying $S(t)u_0 = u(t,t_0)$, $t \geqslant 0$.

Then, by the classical existence theorem of global attractors (see [14]), we give the following theorem on the existence of the global attractor of problem (1)–(3) in $H^3(\Omega)$.

Theorem 1. *Assume $u_0 \in H^3(\Omega)$ and the three smooth functions $w_{0,2,3}(u)$ satisfy* (4)*. Then the solution u of problem* (1)–(3) *possesses a global attractor $\mathscr{A}$ in the space $H^3(\Omega)$, which attracts all bounded set in the space $H^3(\Omega)$.*

Similar to [14], we assume that the semigroup $S(t)_{t\geqslant 0}$ is generated by the solutions of Eq. (1) with initial conditions $u_0 \in H^3(\Omega)$. Then we give two lemmas to prove Theorem 1.

Lemma 2. *There exists a bounded set $\mathscr{B}$ whose size depends only on Ω such that for all $u_0 \in B \subset U$, there exists $t > t_0 = t_0(B) \geqslant 0$ satisfying $S(t)u_0 \in \mathscr{B}$.*

Proof. It suffices to prove that there is a positive constant C such that for large t, there holds $\|u(t)\|_{H^3} \leqslant C$. Now we begin to prove the lemma.

Let

$$F(t) = \int_0^1 \left(\frac{1}{2}\big|D^2u\big|^2 - \frac{1}{2}|Du|^2 + \frac{1}{12}|Du|^4 + W_0(u) - \frac{1}{2}w_3(u)|Du|^2\right)\mathrm{d}x.$$

Integrating by parts and by (2) we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}F(t) &= \int_0^1 \biggl(D^2uD^2u_t - DuDu_t \\ &\qquad + \frac{1}{3}|Du|^3Du_t + W_0(u)u_t - \frac{1}{2}w_3'(u)u_t|Du|^2 - w_3(u)DuDu_t\biggr)\,\mathrm{d}x \\ &= \int_0^1 \bigl(D^4u + D^2u - |Du|^2D^2u + W_0(u) + w_2(u)|Du|^2 + w_3(u)D^2u\bigr)u_t\,\mathrm{d}x \\ &= -\int_0^1 \bigl|D^5u + D^3u - D\bigl[|Du|^2D^2u\bigr] \\ &\qquad + D\bigl[w_0(u) + w_2(u)|Du|^2 + w_3(u)D^2u\bigr]\bigr|^2\,\mathrm{d}x \\ &\leqslant 0.\end{aligned}$$

Hence, $F(t) \leqslant F(0)$, then

$$\begin{aligned}&\int_0^1 \bigl|D^2u\bigr|^2\,\mathrm{d}x + \frac{1}{12}\int_0^1 |Du|^4\,\mathrm{d}x + \int_0^1 W_0(u)\,\mathrm{d}x \\ &\qquad \leqslant F(0) + \frac{1}{2}\int_0^1 |Du|^2\,\mathrm{d}x + \frac{1}{2}\int_0^1 w_3(u)|Du|^2\,\mathrm{d}x.\end{aligned} \tag{5}$$

Then by Poincaré's inequality and (2) we derive that

$$\int_0^1 |Du|^2\,\mathrm{d}x \leqslant \frac{1}{\pi^2}\int_0^1 \bigl|D^2u\bigr|^2\,\mathrm{d}x. \tag{6}$$

Combining (5) and (6) together gives

$$\begin{aligned}&\frac{1}{2}\int_0^1 \bigl|D^2u\bigr|^2\,\mathrm{d}x + \frac{1}{12}\int_0^1 |Du|^4\,\mathrm{d}x + \int_0^1 W_0(u)\,\mathrm{d}x \\ &\qquad \leqslant F(0) + \frac{1}{2\pi^2}\int_0^1 \bigl|D^2u\bigr|^2\,\mathrm{d}x + \frac{1}{12}\int_0^1 |Du|^4\,\mathrm{d}x + \frac{3}{4}\int_0^1 \bigl[w_3(u)\bigr]^2\,\mathrm{d}x.\end{aligned}$$

Notice that $W_0(u) \geqslant (3/4)[w_3(u)]^2$, then there exists $C = 2\pi^2F(0)/(\pi^2 - 1)$ such that

$$\int_0^1 \bigl|D^2u\bigr|^2\,\mathrm{d}x \leqslant C. \tag{7}$$

It then follows from (6) and (7) that

$$\int_0^1 |Du|^2 \,\mathrm{d}x \leqslant C. \tag{8}$$

Note that $u_0 \in \mathcal{U}_0$. By Poincaré's inequality and (8) we deduce that for $t \geqslant t_1(B) \geqslant 0$,

$$\int_0^1 |u|^2 \,\mathrm{d}x \leqslant C' \int_0^1 |Du|^2 \,\mathrm{d}x \leqslant C. \tag{9}$$

Hence, by (7), (8) and (9) we get $\sup_{x\in\Omega} |u| \leqslant C$ and $\sup_{x\in\Omega} |Du| \leqslant C$. Multiplying (1) with D^6u and integrating it over Ω, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \left|D^3u\right|^2 \mathrm{d}x + \int_0^1 \left|D^6u\right|^2 \mathrm{d}x \\
&\quad= \int_0^1 \left|D^5u\right|^2 \mathrm{d}x + \int_0^1 D^2\left[|Du|^2 D^2u\right] D^6u \,\mathrm{d}x \\
&\qquad - \int_0^1 D^2\left[w_0(u) + w_2(u)|Du|^2 + w_3(u)D^2u\right] D^6u \,\mathrm{d}x \\
&\quad\leqslant \int_0^1 \left|D^5u\right|^2 \mathrm{d}x + 2\int_0^1 \left|D^2u\right|^3 D^6u \,\mathrm{d}x + 6\int_0^1 DuD^2uD^3uD^6u \,\mathrm{d}x \\
&\qquad + \int_0^1 \left[|Du|^2 + w_3(u)\right] D^4uD^6u \,\mathrm{d}x \\
&\qquad + \int_0^1 \left[w_0''(u)|Du|^2 + w_2''(u)|Du|^4\right] D^6u \,\mathrm{d}x \\
&\qquad + \int_0^1 \left[w_0'(u) + \left(3w_2'(u) + 2w_3''(u)\right)|Du|^2\right] D^2uD^6u \,\mathrm{d}x \\
&\qquad + 2\int_0^1 w_3'(u)\left|D^2u\right|^2 D^6u \,\mathrm{d}x + 3\int_0^1 w_3'(u)DuD^3uD^6u \,\mathrm{d}x \\
&\quad= \int_0^1 \left|D^5u\right|^2 \mathrm{d}x + I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7.
\end{aligned}$$

By Nirenberg's inequality we have

$$\begin{aligned}
\int_0^1 \left|D^5u\right|^2 \mathrm{d}x &\leqslant \left[C_1\left(\int_0^1 \left|D^6u\right|^2 \mathrm{d}x\right)^{3/8}\left(\int_0^1 \left|D^2u\right|^2 \mathrm{d}x\right)^{1/8} + C_2\left(\int_0^1 \left|D^2u\right|^2 \mathrm{d}x\right)^{1/2}\right]^2 \\
&\leqslant \varepsilon' \int_0^1 \left|D^6u\right|^2 \mathrm{d}x + C'_\varepsilon, \\
\int_0^1 \left|D^2u\right|^6 \mathrm{d}x &\leqslant \left[C_1\left(\int_0^1 \left|D^6u\right|^2 \mathrm{d}x\right)^{1/24}\left(\int_0^1 \left|D^2u\right|^2 \mathrm{d}x\right)^{11/24} + C_2\left(\int_0^1 \left|D^2u\right|^2 \mathrm{d}x\right)^{1/2}\right]^6 \\
&\leqslant \varepsilon \int_0^1 \left|D^6u\right|^2 \mathrm{d}x + C_\varepsilon, \\
\int_0^1 \left|D^2u\right|^4 \mathrm{d}x &\leqslant \left[C_1\left(\int_0^1 \left|D^6u\right|^2 \mathrm{d}x\right)^{1/32}\left(\int_0^1 \left|D^2u\right|^2 \mathrm{d}x\right)^{15/32} + C_2\left(\int_0^1 \left|D^2u\right|^2 \mathrm{d}x\right)^{1/2}\right]^4 \\
&\leqslant \varepsilon \int_0^1 \left|D^6u\right|^2 \mathrm{d}x + C_\varepsilon, \\
\int_0^1 \left|D^3u\right|^4 \mathrm{d}x &\leqslant \left[C_1\left(\int_0^1 \left|D^6u\right|^2 \mathrm{d}x\right)^{5/32}\left(\int_0^1 \left|D^2u\right|^2 \mathrm{d}x\right)^{11/32} + C_2\left(\int_0^1 \left|D^2u\right|^2 \mathrm{d}x\right)^{1/2}\right]^4 \\
&\leqslant \varepsilon \int_0^1 \left|D^6u\right|^2 \mathrm{d}x + C_\varepsilon, \\
\int_0^1 \left|D^3u\right|^2 \mathrm{d}x &\leqslant \left[C_1\left(\int_0^1 \left|D^6u\right|^2 \mathrm{d}x\right)^{1/8}\left(\int_0^1 \left|D^2u\right|^2 \mathrm{d}x\right)^{3/8} + C_2\left(\int_0^1 \left|D^2u\right|^2 \mathrm{d}x\right)^{1/2}\right]^2 \\
&\leqslant \varepsilon \int_0^1 \left|D^6u\right|^2 \mathrm{d}x + C_\varepsilon, \\
\int_0^1 \left|D^4u\right|^2 \mathrm{d}x &\leqslant \left[C_1\left(\int_0^1 \left|D^6u\right|^2 \mathrm{d}x\right)^{1/4}\left(\int_0^1 \left|D^2u\right|^2 \mathrm{d}x\right)^{1/4} + C_2\left(\int_0^1 \left|D^2u\right|^2 \mathrm{d}x\right)^{1/2}\right]^2 \\
&\leqslant \varepsilon \int_0^1 \left|D^6u\right|^2 \mathrm{d}x + C_\varepsilon.
\end{aligned}$$

Then

$$I_1 \leqslant C \int_0^1 \left|D^2u\right|^6 \mathrm{d}x + C \int_0^1 \left|D^6u\right|^2 \mathrm{d}x \leqslant \varepsilon' \int_0^1 \left|D^6u\right|^2 \mathrm{d}x + C'_\varepsilon,$$

$$I_2 \leqslant 6\sup_{x\in\Omega}|Du|\int_0^1 D^2uD^3uD^6u\,\mathrm{d}x$$
$$\leqslant C\int_0^1 |D^2u|^4\,\mathrm{d}x + C\int_0^1 |D^3u|^4\,\mathrm{d}x + C\int_0^1 |D^6u|^2\,\mathrm{d}x \leqslant \varepsilon'\int_0^1 |D^6u|^2\,\mathrm{d}x + C'_\varepsilon,$$

$$I_3 \leqslant \sup_{x\in\Omega}\big|(Du)^2 + w_3(u)\big|\int_0^1 D^4uD^6u\,\mathrm{d}x$$
$$\leqslant C\int_0^1 |D^4u|^2\,\mathrm{d}x + C\int_0^1 |D^6u|^2\,\mathrm{d}x \leqslant \varepsilon'\int_0^1 |D^6u|^2\,\mathrm{d}x + C'_\varepsilon,$$

$$I_4 \leqslant \sup_{x\in\Omega}\big|w_0''(u)(Du)^2 + w_2''(u)(Du)^4\big|\int_0^1 D^6u\,\mathrm{d}x \leqslant \varepsilon'\int_0^1 |D^6u|^2\,\mathrm{d}x + C'_\varepsilon,$$

$$I_5 \leqslant \sup_{x\in\Omega}\big|w_0'(u) + \big(3w_2'(u) + 2w_3''(u)\big)(Du)^2\big|\int_0^1 D^2uD^6u\,\mathrm{d}x$$
$$\leqslant C\int_0^1 |D^2u|^2\,\mathrm{d}x + C\int_0^1 |D^6u|^2\,\mathrm{d}x \leqslant \varepsilon'\int_0^1 |D^6u|^2\,\mathrm{d}x + C'_\varepsilon,$$

$$I_6 \leqslant 2\sup_{x\in\Omega}\big|w_3'(u)\big|\int_0^1 |D^2u|^2D^6u\,\mathrm{d}x \leqslant C\int_0^1 |D^2u|^4\,\mathrm{d}x + C\int_0^1 |D^6u|^2\,\mathrm{d}x$$
$$\leqslant \varepsilon'\int_0^1 |D^6u|^2\,\mathrm{d}x + C'_\varepsilon,$$

$$I_7 \leqslant 3\sup_{x\in\Omega}\big|w_3'(u)Du\big|\int_0^1 D^3uD^6u\,\mathrm{d}x \leqslant C\int_0^1 |D^3u|^2\,\mathrm{d}x + C\int_0^1 |D^6u|^2\,\mathrm{d}x$$
$$\leqslant \varepsilon'\int_0^1 |D^6u|^2\,\mathrm{d}x + C'_\varepsilon.$$

Summing up, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 |D^3u|^2\,\mathrm{d}x + (1-8\varepsilon')\int_0^1 |D^6u|^2\,\mathrm{d}x \leqslant 8C'_\varepsilon. \tag{10}$$

Assuming that ε' is small enough and it satisfies $1 - 8\varepsilon' > 0$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \left|D^3u\right|^2 \mathrm{d}x + C'\int_0^1 \left|D^3u\right|^2 \mathrm{d}x \leqslant C''.$$

Applying Gronwall's inequality, we have

$$\int_0^1 \left|D^3u\right|^2 \mathrm{d}x \leqslant \mathrm{e}^{-C't}\int_0^1 \left|D^3u_0\right|^2 \mathrm{d}x + \frac{C''}{C'}.$$

Therefore, for initial data in any bounded set $B \subset H^3(\Omega)$, there is a uniform time $t_2(B)$ depending on B such that for $t \geqslant t_2(B) \geqslant 0$,

$$\int_0^1 \left|D^3u\right|^2 \mathrm{d}x \leqslant \frac{2C''}{C'}. \tag{11}$$

Sobolev embedding theorem gives $\sup_{x\in\Omega} |D^2u| \leqslant C$. Adding (7), (8), (9) and (10), we have $\|u(x,t)\|_{H^3} \leqslant C$. Let $t_0(B) = \max\{t_1(B), t_2(B)\}$, then the lemma is proved. □

From the above lemma we know that $\{S(t)\}_{t\geqslant 0}$ has a bounded absorbing set in $H^3(\Omega)$. In what follows, we prove the precompactness of the orbit in $H^3(\Omega)$.

Lemma 3. *For any initial data u_0 in any bounded set $B \subset H^3(\Omega)$, there is a $T(B) > 0$ such that*

$$\|u(t)\|_{H^4} \leqslant C \quad \forall t \geqslant T > 0.$$

Proof. The uniform boundedness of $H^3(\Omega)$ norm of $u(t)$ has been achieved in Lemma 2. In what follows, we give the estimate on H^4-norm.

Differentiating (1) with respect to x, multiplying the resultant by D^7u and integrating on Ω, using the boundary conditions, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \left|D^4u\right|^2 \mathrm{d}x + \int_0^1 \left|D^7u\right|^2 \mathrm{d}x \\
&\leqslant \int_0^1 \left|D^6u\right|^2 \mathrm{d}x \\
&\quad + \int_0^1 \left[12\left|D^2u\right|^2 + w_0'(u) + \left(3w_2'(u) + 5w_3''(u)\right)|Du|^2 \right. \\
&\qquad \left. + 7w_3'(u)D^2u\right] D^3uD^7u \,\mathrm{d}x \\
&\quad + 6\int_0^1 Du\left|D^3u\right|^2 D^7u \,\mathrm{d}x + \int_0^1 \left[8DuD^2u + 4w_3'(u)Du\right] D^4uD^7u \,\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^1 \bigl[|Du|^2 + w_3(u)\bigr] D^5u D^7u \,\mathrm{d}x \\
&+\int_0^1 \bigl[w_0'''(u)|Du|^3 + 3w_0''(u)DuD^2u + w_2'''(u)|Du|^5 \\
&\quad + \bigl(7w_2''(u) + 2w_3'''(u)\bigr)|Du|^3 D^2u \\
&\quad + 6\bigl(w_2'(u) + w_3''(u)\bigr)Du\bigl|D^2u\bigr|^2\bigr] D^7u \,\mathrm{d}x \\
= &\int_0^1 \bigl|D^6u\bigr|^2 \,\mathrm{d}x + J_1 + J_2 + J_3 + J_4 + J_5. \qquad (12)
\end{aligned}
$$

By Nirenberg's inequality we have

$$
\begin{aligned}
\int_0^1 \bigl|D^6u\bigr|^2 \,\mathrm{d}x &\leqslant \biggl[C_3\biggl(\int_0^1 \bigl|D^7u\bigr|^2 \,\mathrm{d}x\biggr)^{3/8}\biggl(\int_0^1 \bigl|D^3u\bigr|^2 \,\mathrm{d}x\biggr)^{1/8} + C_4\biggl(\int_0^1 \bigl|D^3u\bigr|^2 \,\mathrm{d}x\biggr)^{1/2}\biggr]^2 \\
&\leqslant \varepsilon' \int_0^1 \bigl|D^7u\bigr|^2 \,\mathrm{d}x + C_\varepsilon', \\
\int_0^1 \bigl|D^3u\bigr|^4 \,\mathrm{d}x &\leqslant \biggl[C_3\biggl(\int_0^1 \bigl|D^7u\bigr|^2 \,\mathrm{d}x\biggr)^{1/32}\biggl(\int_0^1 \bigl|D^3u\bigr|^2 \,\mathrm{d}x\biggr)^{15/32} + C_4\biggl(\int_0^1 \bigl|D^3u\bigr|^2 \,\mathrm{d}x\biggr)^{1/2}\biggr]^4 \\
&\leqslant \varepsilon \int_0^1 \bigl|D^7u\bigr|^2 \,\mathrm{d}x + C_\varepsilon, \\
\int_0^1 \bigl|D^4u\bigr|^2 \,\mathrm{d}x &\leqslant \biggl[C_3\biggl(\int_0^1 \bigl|D^7u\bigr|^2 \,\mathrm{d}x\biggr)^{1/8}\biggl(\int_0^1 \bigl|D^3u\bigr|^2 \,\mathrm{d}x\biggr)^{3/8} + C_4\biggl(\int_0^1 \bigl|D^3u\bigr|^2 \,\mathrm{d}x\biggr)^{1/2}\biggr]^2 \\
&\leqslant \varepsilon \int_0^1 \bigl|D^7u\bigr|^2 \,\mathrm{d}x + C_\varepsilon, \\
\int_0^1 \bigl|D^5u\bigr|^2 \,\mathrm{d}x &\leqslant \biggl[C_3\biggl(\int_0^1 \bigl|D^7u\bigr|^2 \,\mathrm{d}x\biggr)^{1/4}\biggl(\int_0^1 \bigl|D^3u\bigr|^2 \,\mathrm{d}x\biggr)^{1/4} + C_4\biggl(\int_0^1 \bigl|D^3u\bigr|^2 \,\mathrm{d}x\biggr)^{1/2}\biggr]^2 \\
&\leqslant \varepsilon \int_0^1 \bigl|D^7u\bigr|^2 \,\mathrm{d}x + C_\varepsilon.
\end{aligned}
$$

Hence, we have

$$
J_1 \leqslant C\int_0^1 \bigl|D^3u\bigr|^2 \,\mathrm{d}x + C\int_0^1 \bigl|D^7u\bigr|^2 \,\mathrm{d}x \leqslant \varepsilon' \int_0^1 \bigl|D^7u\bigr|^2 \,\mathrm{d}x + C_\varepsilon',
$$

$$J_2 \leqslant 6 \sup_{x\in\Omega} |Du| \int_0^1 |D^3u|^2 D^7u \, \mathrm{d}x \leqslant C \int_0^1 |D^3u|^4 \, \mathrm{d}x + C \int_0^1 |D^7u|^2 \, \mathrm{d}x$$
$$\leqslant \varepsilon' \int_0^1 |D^7u|^2 \, \mathrm{d}x + C'_\varepsilon,$$
$$J_3 \leqslant \sup_{x\in\Omega} |8DuD^2u + 4w_3'(u)Du| \int_0^1 D^4uD^7u \, \mathrm{d}x$$
$$\leqslant C \int_0^1 |D^4u|^2 \, \mathrm{d}x + C \int_0^1 |D^7u|^2 \, \mathrm{d}x \leqslant \varepsilon' \int_0^1 |D^7u|^2 \, \mathrm{d}x + C'_\varepsilon,$$
$$J_4 \leqslant \sup_{x\in\Omega} |(Du)^2 + w_3(u)| \int_0^1 D^5uD^7u \, \mathrm{d}x \leqslant C \int_0^1 |D^5u|^2 \, \mathrm{d}x + C \int_0^1 |D^7u|^2 \, \mathrm{d}x$$
$$\leqslant \varepsilon' \int_0^1 |D^7u|^2 \, \mathrm{d}x + C'_\varepsilon,$$
$$J_5 \leqslant \sup_{x\in\Omega} \big| w_0'''(u)(Du)^3 + 3w_0''(u)DuD^2u + w_2'''(u)(Du)^5$$
$$+ \big(7w_2''(u) + 2w_3'''(u)\big)(Du)^3D^2u + 6\big(w_2'(u) + w_3''(u)\big)Du(D^2u)^2\big| \int_0^1 D^7u \, \mathrm{d}x$$
$$\leqslant \varepsilon' \int_0^1 |D^7u|^2 \, \mathrm{d}x + C'_\varepsilon.$$

Summing up, we derive that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_0^1 |D^4u|^2 \, \mathrm{d}x + (1 - 6\varepsilon') \int_0^1 |D^7u|^2 \, \mathrm{d}x \leqslant 6C'_\varepsilon.$$

Assuming that ε' is small enough and it satisfies $1 - 6\varepsilon' > 0$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_0^1 |D^4u|^2 \, \mathrm{d}x \leqslant -C \int_0^1 |D^7u|^2 \, \mathrm{d}x + C' \leqslant C \int_0^1 |D^4u|^2 \, \mathrm{d}x + C'. \tag{13}$$

By (10) we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \|D^3u\|^2 \, \mathrm{d}x + C\|D^6u\|^2 \leqslant C'. \tag{14}$$

Integrating (14) between t and $t+1$ and by (11), we have

$$\int_t^{t+1} \left\|D^6 u\right\|^2 \mathrm{d}\tau \leqslant \left\|D^3 u(t)\right\|^2 + C \leqslant C'.$$

By Poincaré's inequality we have

$$\int_t^{t+1} \left\|D^4 u\right\|^2 \mathrm{d}\tau \leqslant \int_t^{t+1} \left\|D^5 u\right\|^2 \mathrm{d}\tau \leqslant \int_t^{t+1} \left\|D^6 u\right\|^2 \mathrm{d}\tau \leqslant C. \tag{15}$$

Therefore, by (13), (15) and the uniform Gronwall inequality we have

$$\int_0^1 \left|D^4 u\right|^2 \mathrm{d}x \leqslant C, \quad t \geqslant 1.$$

The lemma is proved. □

Now we give the proof of Theorem 1.

Proof of Theorem 1. Combining Lemma 2 with Lemma 3, by [14] we have completed the proof of Theorem 1. □

3 Global attractor in H^k

In order to consider the global attractors for Eq. (1) in the H^k space, we introduce the definition as follows:

$$H = \left\{ u \in L^2(\Omega)\colon \int_0^1 u \,\mathrm{d}x = 0 \right\}, \tag{16}$$

$$H_{1/2} = \left\{ u \in H^3(\Omega) \cap H\colon Du|_{\partial\Omega} = 0, \int_0^1 u \,\mathrm{d}x = 0 \right\}, \tag{17}$$

$$H_1 = \left\{ u \in H^6(\Omega) \cap H\colon Du|_{\partial\Omega} = D^3 u|_{\partial\Omega} = D^5 u|_{\partial\Omega} = 0, \int_0^1 u \,\mathrm{d}x = 0 \right\}. \tag{18}$$

In this article, we let

$$G(u) = D^4 - D^2\left[|Du|^2 D^2 u\right] + D^2\left[w_0(u) + w_2(u)|Du|^2 + w_3(u) D^2 u\right]$$

be a nonlinear function, and we assume that the linear operator $L = D^6 : H_1 \to H$ in (16)–(18) is a sectorial operator, which generates an analytic semigroup e^{tL}, and L induces the fractional-power operators and fractional-order spaces as follows:

$$\mathcal{L}^\alpha = (-L)^\alpha : H_\alpha \to H, \quad \alpha \in \mathbb{R}, \tag{19}$$

where $H_\alpha = D(\mathcal{L}^\alpha)$ is the domain of $\mathcal{L}^\alpha$. By the semigroup theory of linear operators, $H_\beta \subset H_\alpha$ is a compact inclusion for any $\beta > \alpha$.

Now we give the main theorem of this article, which provides the existence of global attractors of Eq. (1) in any kth space H^k.

Theorem 2. *Assume $u_0 \in H^3(\Omega)$ and the smooth functions $w_{0,2,3}(h)$ satisfy* (4)*, then the solution u of problem* (1)–(3) *possesses a global attractor $\mathscr{A}$ in the space H^k, which attracts all the bounded set of H^k in the H^k-norm, where $k \in [0, \infty)$.*

On the basis of Ma and Wang [9], it is well known that the solution $u(t, u_0)$ of problem (1)–(3) can be expressed as

$$u(t, u_0) = \mathrm{e}^{tL} u_0 + \int_0^t \mathrm{e}^{(t-\tau)L} G(u)\, \mathrm{d}\tau, \tag{20}$$

where $L = D^6$ and

$$\begin{aligned} G(u) &= D^2 g(u) \\ &= D^4 - D^2\bigl[|Du|^2 D^2 u\bigr] + D^2\bigl[w_0(u) + w_2(u)|Du|^2 + w_3(u) D^2 u\bigr]. \end{aligned}$$

Then (20) means

$$u(t, u_0) = \mathrm{e}^{tL} u_0 + \int_0^t \mathrm{e}^{(t-\tau)L} D^2 g(u)\, \mathrm{d}\tau = \mathrm{e}^{tL} u_0 + \int_0^t (-L)^{1/3} \mathrm{e}^{(t-\tau)L} g(u)\, \mathrm{d}\tau.$$

In order to prove Theorem 2, we first prove the following lemma.

Lemma 4. *For any bounded set $U \in H_\alpha$, there exists a constant $C > 0$ such that*

$$\bigl\|u(t, u_0)\bigr\|_{H_\alpha} \leqslant C \quad \forall t \geqslant 0,\ u_0 \in U \subset H_\alpha,\ \alpha > 0. \tag{21}$$

Proof. For $\alpha = 1/2$, this follows from Lemma 2, i.e., for any bounded set $U \subset H_{1/2}$, there exists a constant C, $C > 0$, such that

$$\bigl\|u(t, u_0)\bigr\|_{H_{1/2}} \leqslant C \quad \forall t \geqslant 0,\ u_0 \in U \subset H_{1/2}.$$

Then we only need to prove (21) for any $\alpha \geqslant 1/2$.

Step 1. We prove that for any bounded set $U \subset H_\alpha$ ($1/2 \leqslant \alpha < 2/3$), there exists a constant $C > 0$ such that

$$\bigl\|u(t, u_0)\bigr\|_{H^\alpha} \leqslant C \quad \forall t \geqslant 0,\ u_0 \in U,\ \frac{1}{2} \leqslant \alpha < \frac{2}{3}. \tag{22}$$

We claim that $g : H_{1/2} \to H$ is bounded, by Sobolev embedding theorem we have

$$\begin{aligned} &H_{1/2} \hookrightarrow H^2(\Omega), \qquad &&H_{1/2} \hookrightarrow W^{1,8}(\Omega), \\ &H_{1/2} \hookrightarrow W^{2,4}(\Omega), \qquad &&H_{1/2} \hookrightarrow L^\infty(\Omega). \end{aligned}$$

Then we obtain

$$\begin{aligned}
\|g(u)\|_{H^2} &\leqslant C\int_0^1 \bigl(|D^2u|^2+|Du|^8+|D^2u|^4 \\
&\qquad + |w_0(u)|^2+|w_2(u)|^4+|Du|^8+|w_3(u)|^4+|D^2u|^4\bigr)\,\mathrm{d}x \\
&\leqslant C\bigl(\|u\|_{H^2}^2+\|u\|_{W^{1,8}}^8+\|w_0(u)\|_{L^\infty}^2+\|w_2(u)\|_{L^\infty}^4 \\
&\quad + \|u\|_{W^{1,8}}^8+\|w_3(u)\|_{L^\infty}^4+\|u\|_{W^{2,4}}^4\bigr) \\
&\leqslant C\bigl(\|u\|_{H^2}^2+\|u\|_{W^{1,8}}^8+\|u\|_{W^{2,4}}^4+C\bigr) \\
&\leqslant C\bigl(\|u\|_{H_{1/2}}^2+\|u\|_{H_{1/2}}^8+\|u\|_{H_{1/2}}^4+1\bigr),
\end{aligned} \tag{23}$$

which means that $g: H_{1/2}\to H$ is bounded. By (19), (20) and (23) we find that

$$\begin{aligned}
\|u(t,u_0)\|_{H^\alpha} &= \biggl\|\mathrm{e}^{tL}u_0+\int_0^t (-L)^{1/3}\mathrm{e}^{(t-\tau)L}g(u)\,\mathrm{d}\tau\biggr\|_{H_\alpha} \\
&\leqslant C\|u_0\|_{H^\alpha}+\int_0^t \bigl\|(-L)^{1/3+\alpha}\mathrm{e}^{(t-\tau)L}g(u)\bigr\|_H\,\mathrm{d}\tau \\
&\leqslant C\|u_0\|_{H^\alpha}+\int_0^t \bigl\|(-L)^{1/3+\alpha}\mathrm{e}^{(t-\tau)L}\bigr\|\cdot\bigl\|g(u)\bigr\|_H\,\mathrm{d}\tau \\
&\leqslant C\|u_0\|_{H^\alpha}+C\int_0^t (t-\tau)^{-\beta}\mathrm{e}^{-\delta(t-\tau)}\,\mathrm{d}\tau \\
&\leqslant C\|u_0\|_{H^\alpha}+C\int_0^t \tau^{-\beta}\mathrm{e}^{-\delta\tau}\,\mathrm{d}\tau \leqslant C \quad \forall t\geqslant 0,\ u_0\in U\subset H_\alpha,
\end{aligned}$$

where $\beta=1/3+\alpha$ $(0<\beta<1)$. Hence, (22) is valid.

Step 2. We prove that for any bounded set $U\subset H_\alpha$ $(2/3\leqslant\alpha<5/6)$, there exists a constant $C>0$ such that

$$\|u(t,u_0)\|_{H_\alpha}\leqslant C \quad \forall t\geqslant 0,\ u_0\in U,\ \frac{2}{3}\leqslant\alpha<\frac{5}{6}. \tag{24}$$

We claim that $g:\ H_\alpha\to\ H_{1/6}$ is bounded, by Sobolev embedding theorem we have

$$\begin{aligned}
&H_\alpha\hookrightarrow H^3(\Omega), && H_\alpha\hookrightarrow W^{1,4}(\Omega), && H_\alpha\hookrightarrow W^{2,8}(\Omega), && H_\alpha\hookrightarrow W^{1,8}(\Omega),\\
&H_\alpha\hookrightarrow W^{3,4}(\Omega), && H_\alpha\hookrightarrow W^{1,6}(\Omega), && H_\alpha\hookrightarrow W^{2,4}(\Omega), && H_\alpha\hookrightarrow L^\infty(\Omega),
\end{aligned}$$

where $1/2\leqslant\alpha<2/3$.

Then we obtain

$$
\begin{aligned}
\|g(u)\|_{H_{1/6}}^2 \leqslant C \int_0^1 \bigl(&|D^3u|^2 + |Du|^2|D^2u|^4 + |Du|^4|D^3u|^2 \\
&+ |w_0'(u)|^2|Du|^2 + |w_2'(u)|^2|Du|^6 + |w_2(u)|^2|Du|^2|D^2u|^2 \\
&+ |w_3'(u)|^2|Du|^2|D^2u|^2 + |w_3(u)|^2|D^3u|^2\bigr)\,\mathrm{d}x \\
\leqslant C\bigl(&\|u\|_{H^3}^2 + \|u\|_{W^{1,4}}^4 + \|u\|_{W^{2,8}}^8 + \|u\|_{W^{1,8}}^8 + \|u\|_{W^{3,4}}^4 \\
&+ \|w_0'(u)\|_{L^\infty}^2\|u\|_H^2 + \|w_2'(u)\|_{L^\infty}^2\|u\|_{W^{1,6}}^6 + \|w_2(u)\|_{L^\infty}^4\|u\|_{W^{1,4}}^4 \\
&+ \|u\|_{W^{2,4}}^4 + \|w_3'(u)\|_{L^\infty}^4\|u\|_{W^{1,4}}^4 + \|u\|_{W^{2,4}}^4 + \|w_3(u)\|_{L^\infty}^2\|u\|_{H^3}^2\bigr) \\
\leqslant C\bigl(&\|u\|_{H_\alpha}^2 + \|u\|_{H_\alpha}^4 + \|u\|_{H_\alpha}^8 + \|u\|_{H_\alpha}^6\bigr),
\end{aligned}
\tag{25}
$$

which means that $g:\ H_\alpha \to H_{1/6}$ is bounded. On the basis of Step 1 and (25), we deduce that

$$
\begin{aligned}
\|u(t,u_0)\|_{H^\alpha} &= \left\| \mathrm{e}^{tL}u_0 + \int_0^t (-L)^{1/3}\mathrm{e}^{(t-\tau)L}g(u)\,\mathrm{d}\tau \right\|_{H}^{\alpha} \\
&\leqslant C\|u_0\|_{H_\alpha} + \int_0^t \|(-L)^{1/6+\alpha}\mathrm{e}^{(t-\tau)L}g(u)\|_{H_{1/6}}\,\mathrm{d}\tau \\
&\leqslant C\|u_0\|_{H^\alpha} + \int_0^t \|(-L)^{1/6+\alpha}\mathrm{e}^{(t-\tau)L}\|\cdot\|g(u)\|_{H_{1/6}}\,\mathrm{d}\tau \\
&\leqslant C\|u_0\|_{H^\alpha} + C\int_0^t (t-\tau)^{-\beta}\mathrm{e}^{-\delta(t-\tau)}\,\mathrm{d}\tau \\
&\leqslant C\|u_0\|_{H^\alpha} + C\int_0^t \tau^{-\beta}\mathrm{e}^{-\delta\tau}\,\mathrm{d}\tau \leqslant C \quad \forall t \geqslant 0,\ u_0 \in U \subset H_\alpha,
\end{aligned}
$$

where $\beta = 1/6 + \alpha$ ($0 < \beta < 1$). Hence, (24) is valid.

Step 3. We prove that for any bounded set $U \subset H_\alpha$ ($5/6 \leqslant \alpha < 1$), there exists a constant $C > 0$ such that

$$
\|u(t,u_0)\|_{H_\alpha} \leqslant C \quad \forall t \geqslant 0,\ u_0 \in U,\ \frac{5}{6} \leqslant \alpha < 1. \tag{26}
$$

We claim that $g : H_\alpha \to H_{1/3}$ is bounded, by Sobolev embedding theorem we have

$$
\begin{aligned}
&H_\alpha \hookrightarrow H^4(\Omega), && H_\alpha \hookrightarrow W^{2,6}(\Omega), && H_\alpha \hookrightarrow W^{2,8}(\Omega), && H_\alpha \hookrightarrow W^{3,4}(\Omega), \\
&H_\alpha \hookrightarrow W^{4,4}(\Omega), && H_\alpha \hookrightarrow W^{1,4}(\Omega), && H_\alpha \hookrightarrow W^{2,4}(\Omega), && H_\alpha \hookrightarrow L^\infty(\Omega)
\end{aligned}
$$

where $2/3 \leqslant \alpha < 5/6$.

Then we obtain

$$
\begin{aligned}
\|g(u)\|_{H_{1/3}}^2 \leqslant C \int_0^1 \big(&|D^4u|^2 + |D^2u|^6 + |DuD^2uD^3u|^2 + |Du|^4|D^4u|^2 \\
&+ |w_0''(u)|^2|Du|^4 + |w_0'(u)|^2|D^2u|^2 \\
&+ |w_2'(u)|^2|Du|^8 + |w_2'(u)|^2|Du|^4|D^2u|^2 \\
&+ |w_2(u)|^2|D^2u|^4 + |w_2(u)|^2|Du|^2|D^3u|^2 \\
&+ |w_3''(u)|^2|Du|^4|D^2u|^2 + |w_3'(u)|^2|D^2u|^4 \\
&+ |w_3'(u)|^2|Du|^2|D^3u|^2 + |w_3(u)|^2|D^4u|^2\big)\,\mathrm{d}x \\
\leqslant C\big(&\|u\|_{H^4}^2 + \|u\|_{W^{2,6}}^6 + \|u\|_{W^{1,8}}^8 + \|u\|_{W^{2,8}}^8 + \|u\|_{W^{3,4}}^4 + \|u\|_{W^{4,4}}^4 \\
&+ \|w_0''(u)\|_{L^\infty}^2\|u\|_{W^{1,4}}^4 + \|w_0'(u)\|_{L^\infty}^2\|u\|_{H^2}^2 + \|w_2'(u)\|_{L^\infty}^2\|u\|_{W^{1,8}}^8 \\
&+ \|u\|_{W^{2,4}}^4 + \|w_2(u)\|_{L^\infty}^2\|u\|_{W^{2,4}}^4 + \|w_2(u)\|_{L^\infty}^4\|u\|_{W^{1,4}}^4 \\
&+ \|w_3''(u)\|_{L^\infty}^4\|u\|_{W^{1,8}}^8 + \|w_3'(u)\|_{L^\infty}^2\|u\|_{W^{2,4}}^4 + \|w_3'(u)\|_{L^\infty}^4\|u\|_{W^{1,4}}^4\big) \\
&+ \|u\|_{W^{3,4}}^4 + \|w_3(u)\|_{L^\infty}^2\|u\|_{H^4}^2\big) \\
\leqslant C\big(&\|u\|_{H_\alpha}^2 + \|u\|_{H_\alpha}^6 + \|u\|_{H_\alpha}^8 + \|u\|_{H_\alpha}^4\big), \qquad (27)
\end{aligned}
$$

which means that $g: H_\alpha \to H_{1/3}$ is bounded. On the basis of Step 2 and (27), we deduce that

$$
\begin{aligned}
\|u(t,u_0)\|_{H^\alpha} &= \left\| \mathrm{e}^{tL}u_0 + \int_0^t (-L)^{1/3}\mathrm{e}^{(t-\tau)L}g(u)\,\mathrm{d}\tau \right\|_{H_\alpha} \\
&\leqslant C\|u_0\|_{H_\alpha} + \int_0^t \big\|(-L)^\alpha \mathrm{e}^{(t-\tau)L}g(u)\big\|_{H_{1/3}}\,\mathrm{d}\tau \\
&\leqslant C\|u_0\|_{H_\alpha} + \int_0^t \big\|(-L)^\alpha \mathrm{e}^{(t-\tau)L}\big\| \cdot \big\|g(u)\big\|_{H_1/3}\,\mathrm{d}\tau \\
&\leqslant C\|u_0\|_{H_\alpha} + C\int_0^t \tau^{-\alpha}\mathrm{e}^{-\delta\tau}\,\mathrm{d}\tau \leqslant C \quad \forall t \geqslant 0,\ u_0 \in U \subset H_\alpha.
\end{aligned}
$$

Hence, (26) is valid.

In the same fashion as in the proof of (26), by iteration we can prove that for any bounded set $U \subset H_\alpha$ ($\alpha > 0$), there exists a constant $C > 0$ such that

$$\|u(t,u_0)\|_{H_\alpha} \leqslant C \quad \forall t \geqslant 0,\ u_0 \in U \subset H_\alpha,\ \alpha \geqslant 0.$$

That is, for all $\alpha \geqslant 0$, the semigroup $S(t)$ generated by problem (1)–(3) is uniformly compact in H_α. Lemma 4 is proved. □

Lemma 5. *For any $\alpha \geqslant 0$, problem* (1)–(3) *has a bounded absorbing set in H_α. That is, for any bounded set $U \in H_\alpha$, there exists $T > 0$ and a constant $C > 0$ independent of u_0 such that*

$$\|u(t,u_0)\|_{H_\alpha} \leqslant C \quad \forall t \geqslant T,\ u_0 \in U \subset H_\alpha. \tag{28}$$

Proof. For $\alpha = 1/2$, this follows from Lemma 3. Then we prove (28) for any $\alpha > 1/2$. We proceed in the following steps.

Step 1. We prove that for any $1/2 \leqslant \alpha < 2/3$, (1)–(3) has a bounded absorbing set in H_α.

By (19) we have

$$u(t,u_0) = \mathrm{e}^{tL}u_0 + \int\limits_T^t (-L)^{1/3}\mathrm{e}^{(t-T)L}g(u)\,\mathrm{d}\tau. \tag{29}$$

Assume that B is the bounded absorbing set of problem (1)–(3) and B satisfies $B \subset H_{1/2}$. In addition, we also assume the time $t_0 > 0$ such that

$$u(t,u_0) \in B \quad \forall t \geqslant t_0,\ u_0 \in U \subset H_\alpha, \alpha \geqslant \frac{1}{2}.$$

Note that $\|\mathrm{e}^{tL}\| \leqslant C\mathrm{e}^{-\lambda_1^3 t}$, where $\lambda_1 > 0$ is the first eigenvalue of the equation

$$-\Delta u = \lambda u, \qquad \frac{\partial u}{\partial n} = 0.$$

Then for any given $T > 0$ and $u_0 \in U \subset H_\alpha$ ($\alpha \geqslant 1/2$), we can obtain

$$\lim_{t\to\infty} \|\mathrm{e}^{(t-T)L}u(T,u_0)\|_{H_\alpha} = 0. \tag{30}$$

Adding (23) and (29) together, we have

$$\begin{aligned}
&\|u(t,u_0)\|_{H^\alpha} \\
&\quad\leqslant \|\mathrm{e}^{(t-t_0)L}u(t_0,u_0)\|_{H_\alpha} + \int\limits_{t_0}^t \|(-L)^{1/3+\alpha}\mathrm{e}^{(t-T)L}\| \cdot \|g(u)\|_H\,\mathrm{d}\tau \\
&\quad\leqslant \|\mathrm{e}^{(t-t_0)L}u(t_0,u_0)\|_{H_\alpha} + C\int\limits_{t_0}^t \|(-L)^{1/3+\alpha}\mathrm{e}^{(t-T)L}\|\,\mathrm{d}\tau \\
&\quad\leqslant \|\mathrm{e}^{(t-t_0)L}u(t_0,u_0)\|_{H_\alpha} + C\int\limits_0^{T-t_0} \tau^{-1/3-\alpha}\mathrm{e}^{-\delta\tau}\,\mathrm{d}\tau \\
&\quad\leqslant \|\mathrm{e}^{(t-t_0)L}u(t_0,u_0)\|_{H_\alpha} + C,
\end{aligned} \tag{31}$$

where $C > 0$ is a constant independent of u_0. Then by (30) and (31) we have that (28) holds for all $1/2 \leqslant \alpha < 2/3$.

Step 2. We prove that for any $2/3 \leqslant \alpha < 5/6$, problem (1)–(3) has a bounded absorbing set in H_α.

Adding (25) and (29) together, we have

$$\begin{aligned}
&\|u(t,u_0)\|_{H^\alpha} \\
&\quad\leqslant \|e^{(t-t_0)L}u(t_0,u_0)\|_{H_\alpha} + \int_{t_0}^{t} \|(-L)^{1/6+\alpha}e^{(t-T)L}\| \cdot \|g(u)\|_{H_{1/6}}\,d\tau \\
&\quad\leqslant \|e^{(t-t_0)L}u(t_0,u_0)\|_{H_\alpha} + C\int_{t_0}^{t} \|(-L)^{1/6+\alpha}e^{(t-T)L}\|\,d\tau \\
&\quad\leqslant \|e^{(t-t_0)L}u(t_0,u_0)\|_{H_\alpha} + C\int_{0}^{T-t_0} \tau^{-1/6-\alpha}e^{-\delta\tau}\,d\tau \\
&\quad\leqslant \|e^{(t-t_0)L}u(t_0,u_0)\|_{H_\alpha} + C, \qquad (32)
\end{aligned}$$

where $C > 0$ is a constant independent of u_0. Then by (30) and (32) we have that (28) holds for all $2/3 \leqslant \alpha < 5/6$.

Step 3. We can use the same method as the above step to prove that for any $5/6 \leqslant \alpha < 1$, problem (1)–(3) has a bounded absorbing set in H_α. By the iteration method we can obtain that (28) holds for all $\alpha \geqslant 1/2$. □

Now we give the proof of Theorem 2.

Proof. Combining Lemmas 4 and 5, we have completed the proof of Theorem 2. □

4 Conclusion

In the study of a mechanism for the formation of quantum dots on the surface of thin solid films, there arise a sixth-order parabolic equation (1) (see [5]). Mathematically, Eq. (1) is a nonlinear evolution equation. Studying the properties of solutions for Eq. (1) is so interesting and maybe useful for the study of the surface of thin solid films. Recently, Zhao [15] studied the existence of classical solutions of the initial boundary value problem of such equation. In order to study the long-time behavior of solutions, we prove the existence of global attractor of Eq. (1) with boundary and initial value conditions. The main idea comes from Temam [14] and Ma and Wang [9]. By using the properties of sectorial operator we define a semigroup related to the fractional-order Sobolev space $H^k(0,1)$ ($k \in [0,\infty)$). Then applying Sobolev embedding theorem and iteration technique, we obtain some useful a priori estimates and prove the existence of absorbing sets and asymptotic compactness of semigroup, obtain our main result on the existence of global attractor.

It is worth pointing out that there are also some other papers that studied the existence of global attractor for dissipative equations in fractional-order Sobolev spaces (see, e.g., semilinear parabolic equation [11], fourth-row Cahn–Hilliard equations [12,16], modified Swift–Hohenberg equations [13], sixth-row Cahn–Hilliard equations [8] and the reference cited therein). We also remark that although both [8] and this paper are focus on the sixth-order diffusion equations, the equation considered in [8] is a sixth-order convective Cahn–Hilliard equation, which is different from equation (1). Due to the different terms of both papers, the calculations are different from the paper of Liu and Liu [8].

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