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Robust piecewise adaptive control for an uncertain semilinear parabolic distributed parameter systems*

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Abstract. In this study, we focus on designing a robust piecewise adaptive controller to globally asymptotically stabilize a semilinear parabolic distributed parameter systems (DPSs) with external disturbance, whose nonlinearities are bounded by unknown functions. Firstly, a robust piecewise adaptive control is designed against the unknown nonlinearity and the external disturbance. Then, by constructing an appropriate Lyapunov–Krasovskii functional candidate (LKFC) and using the Wiritinger's inequality and a variant of the Agmon's inequality, it is shown that the proposed robust piecewise adaptive controller not only ensures the globally asymptotic stability of the closed-loop system, but also guarantees a given performance. Finally, two simulation examples are given to verify the validity of the design method.

Keywords: semilinear parabolic distributed parameter systems, robust piecewise adaptive control, globally asymptotic stabilization, spatial L_∞ norm.

1 Introduction

In actual engineering applications, most physical models are widely distributed in space, continuously changing in time, and have the characteristics of spatiotemporal dynamics. Therefore, they cannot be modeled by ordinary differential equations (ODEs), which are precisely determined by partial differential equations (PDEs). The parabolic DPSs has the typical characteristics of a DPSs, so the research of the parabolic DPSs has important theoretical significance and practical value. The parabolic DPSs has been widely used in reaction diffusion problems, such as chemical reaction control, heat conduction control [24] and tube flow control [3].

Over a long period in the past, a large number of literatures about the design of efficient controller for parabolic DPSs have emerged. Generally speaking, according to the control effect, the control models of DPSs mainly includes intradomain control and

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boundary control [22], where Roffel and Betlem discussed the design methods of intradomain control and boundary control for a class of linear/nonlinear parabolic DPSs in [22], and Krstic further introduced the boundary control problem of DPSs in [18]. At present, there are a lot of literatures about the boundary control of PDEs. At first, inspired by the fact that the main dynamics of the parabolic PDEs can be roughly described by a low-dimensional ODE systems, for linear parabolic PDEs, a design method of predictive boundary control and a design method of sampling data boundary control were introduced in [11] and [8], respectively. In [19], a method of boundary control was proposed for semilinear parabolic DPSs with noncollocated observations in which the low-dimensional ordinary differential equation model was obtained by using the Galerkin method, and a finite-dimensional controller was designed based on the model. Similarly, for intradomain control researches, the main idea in literatures [31–33] was that the order of the systems was reduced by using the Galerkin method to obtain the low-dimensional nonlinear ODEs firstly, and then the appropriate controller was designed for the obtained low-dimensional nonlinear ODEs by using the existing fuzzy control technology. This method is generally applicable only to high-dissipation PDE systems with low-dimensional dominant dynamic behavior. The essential defect of the results in references [8, 11, 19, 31–33] is that the method of designing after truncation may cause inherent loss of important information.

In order to achieve a better control effect, it is necessary to directly study the control design method of the original PDEs model. So far, many scholars have made great efforts to study the fuzzy control design method of parabolic DPSs based on the fuzzy PDEs model. In [29, 30], based on the fuzzy PDEs model, the problem of distributed fuzzy control method for a type of nonlinear parabolic DPSs was solved. In [34], a design method of fuzzy boundary control for a type of nonlinear DPSs was introduced, and it could be realized by only a few controllers, but the control design method was conservative and suitable only for weak semilinear parabolic DPSs with small sector bound of nonlinear term. To overcome this defect in [34], a pointwise control design method based on finite dimensional fuzzy observer for a class of parabolic PDEs was proposed in [6]. However, the design method introduced in [6] belongs to the so-called "reduce-then-design" approach of PDEs system control. Then, in [20, 26], the output feedback control design methods of linear and semilinear parabolic PDEs models were studied, respectively. Recently, in [28], for semilinear parabolic PDEs, the author proposed a design method of sampled data output feedback control, which could not only guarantee H_2 performance, but also ensure H_∞ performance index in the sense of L_∞ norm. However, the existing literatures on the control problems of semilinear parabolic DPSs based on fuzzy PDEs models are all local stability results, which is one of the motivations of this study.

It is well known that parabolic PDEs are widely used in many important engineering, medical and other control problems. However, in most situations, some parameters in the equations cannot be obtained accurately, and these parameters are very important for practical engineering applications and biomedical applications. Therefore, the research on adaptive control of parabolic PDEs is of great practical value. Recently, adaptive control method has been widely used in systems modeled by ODEs [7, 35]. In [7, 35], adaptive fuzzy practical fixed-time tracking control and finite-time adaptive fault-tolerant control for nonlinear systems were proposed, respectively. However, in [7,35], fuzzy logic systems and radial basis function neural networks were used to approximate nonlinear terms, respectively, so the semiglobal results were obtained in these cases. Adaptive controller has also been widely used in hyperbolic PDEs [1, 2, 4, 5]. The existing adaptive control design methods for parabolic PDEs mainly include [15]: Lyapunov-based design, design with passive identifiers and design with swapping identifiers (see [16, 17, 23]). Recently, a simple deterministic equivalent adaptive controller was proposed in [14], which was suitable for more general systems. On the basis of [14], a new adaptive scheme for a specific Dirichlet driving benchmark problem was proposed in [15] in which only the response coefficient and high-frequency gain are unknown parameters. Most of the literatures above basically use the adaptive boundary control method for the PDEs. In addition, for the linear parabolic DPSs, the collocated and noncollocated piecewise control were given in [27]. The piecewise control is that the controllers are placed in different spatial segmented areas, and the controllers do not affect each other. Compared with a system controlled by a boundary controller, the piecewise control reduces its conservativeness. As far as the authors know, there are very few literatures about the piecewise adaptive controller designed in the spatial domain that makes the uncertain nonlinear parabolic system globally asymptotically stabilized, which is the other motivation of this study.

In this paper, we mainly focus on studying the globally asymptotic stabilization for an uncertain semilinear parabolic PDEs with external disturbance under the designed robust piecewise adaptive controller. The main contributions of this research are listed as follows:

1. By using the theory of contraction semigroup theory we give a detailed analysis of the well-posedness of the solutions of the open-loop system and the closed-loop system.
2. A robust piecewise adaptive controller designed in this study guarantees the globally asymptotic stability of the uncertain semilinear parabolic DPSs, which overcomes the defects of the existing semiglobal results. Moreover, the designed controller can ensure a given performance in the sense of $|\cdot|_\infty$.
3. A robust piecewise adaptive control is designed against the unknown nonlinearity and the external disturbance in the system. The designed controller meets the requirements of designing different controllers in different spatial segmented areas, and the controllers do not affect each other. Compared with a system controlled by a boundary controller, the controller designed in this study reduces its conservativeness and improves its effectiveness.

The other parts of the study are arranged as follows. Section 2 introduces the system to be studied and some preliminaries. In Section 3, the main results are given. Section 4 gives an actual numerical simulation example to prove the validity of the theoretical design. Finally, a conclusion is given.

Notations. $|\cdot|$ is the absolute value sign of scalars. $\mathbb{R}$ is the set of real numbers, $\mathbb{R}^m$ represents m-dimensional Euclidean space, $\mathcal{H} \triangleq (\mathcal{L}_2[0, L]; \mathbb{R})$ is a Hilbert space of square integrable vector functions $\varrho(y) : [0, L] \to \mathbb{R}$ with $\langle \varrho_1(\cdot), \varrho_2(\cdot) \rangle = \int_0^L \varrho_1(y)\varrho_2(y)\,\mathrm{d}y$

and $|\varrho_1|_2 = \sqrt{\langle \varrho_1(\cdot), \varrho_1(\cdot) \rangle}$. For any $\varrho(\cdot) \in \mathcal{H}$, its L_∞ norm is defined as $|\varrho(\cdot)|_\infty \triangleq \max_{y \in [0,L]} |\varrho(y)|$, and its $\mathcal{H}^m$ norm is defined as $|\varrho|_{\mathcal{H}^m} = (\sum_{0 \leqslant |\alpha| \leqslant m} |D^\alpha \varrho|_2^2)^{1/2}$. The superscript T is used for the transpose of a vector or a matrix, and $\eta_t(y,t) = \partial \eta(y,t)/\partial t$, $\eta_y(y,t) = \partial \eta(y,t)/\partial y$, $\eta_{yy}(y,t) = \partial^2 \eta(y,t)/\partial y^2$.

2 System description and some preliminaries

In this study, we consider a second-order semilinear parabolic DPSs described as

$$\begin{aligned} &\eta_t(y,t) = \big(\alpha(y)\eta_y(y,t)\big)_y + h\big(\eta(y,t)\big) + S^{\mathrm{T}}(y)u(t) + c(y,t), \\ &\eta(y,0) = \eta_0(y), \qquad \eta_y(y,t)|_{y=0} = \eta(y,t)|_{y=L} = 0, \end{aligned} \tag{1}$$

where $\eta(\cdot,t)$ is the state variable, which belongs to $\mathcal{H}$. $t \geqslant 0$ represents time, $y \in [0,L] \subset \mathbb{R}$ stands for the spatial position. $\alpha(y)$ belongs to $\mathcal{H}^1$, and suppose that it satisfies $\alpha^* \geqslant \alpha(y) \geqslant \alpha_* > 0$ and $\alpha(0) < 2\alpha(L)$, where α_* and α^* are known constants. $h(\eta(y,t))$ is the unknown nonlinear function, which belongs to the class C^1 and satisfies $h(0) = 0$. $c(y,t)$ indicates an external disturbance and satisfies $\int_0^\infty |c(\cdot,t)|_\infty^2 \,\mathrm{d}t < \infty$. $u(t) \triangleq [u_1(t), u_2(t), \dots, u_m(t)]^{\mathrm{T}} \in \mathbb{R}^m$ is the control input provided by m controllers, where m is a finite positive integer. $S(y) \triangleq [s_1(y), s_2(y), \dots, s_m(y)]^{\mathrm{T}} \in \mathbb{R}^m$ is a known integrable function of y, and $s_i(y)$ represents the ith controller's distribution over the domain $(0,L)$. In this study, we choose

$$s_i(y) = \begin{cases} 1, & y \in [y_i, y_{i+1}], \\ 0, & \text{others}, \end{cases} \tag{2}$$

where $0 = y_1 < y_2 < \cdots < y_m < y_{m+1} = L$, $i \in M \triangleq \{1, 2, \dots, m\}$.

Remark 1. Note that the system described in (1) has a wide range of applications in practical engineering, such as heat and mass transfer, combustion theory and chemistry [10]. Firstly, it can be used to describe the spatiotemporal evolution dynamics of neutron concentration in nuclear reactor when $(\alpha(y)\eta_y(y,t))_y = \nu\eta_{yy}(y,t)$ and $h(\eta(y,t)) = \zeta_1(\zeta_2\eta(y,t) - \eta^2(y,t))$, where $\eta(y,t)$ represents the neutron concentration profile of a nuclear reactor. Secondly, it can describe the mass transfer in a two-component medium at rest with a volume chemical reaction of the quasi-first order when $h(\eta(y,t)) = \zeta_3\eta(y,t) \times (1 - \eta(y,t))$. In addition, the kinetic function $\zeta_3\eta(y,t)(1 - \eta(y,t))$ can also model an autocatalytic chain in combustion theory.

Assumption 1. The nonlinearity $h(\eta(y,t))$ is a C^1 function, and it is Lipschitz continuous, i.e.,

$$\big|h\big(\eta(y,t)\big) - h\big(\eta^*(y,t)\big)\big| \leqslant \beta\big|\eta(y,t) - \eta^*(y,t)\big| \quad \forall \eta, \eta^* \in \mathcal{H},$$

for $\beta > 0$ and $\beta \in \mathbb{R}$.

Lemma 1 [Wirtinger's inequality]. (*See* [28].) *Let $\eta(\cdot,t) \in \mathcal{H}(0,L)$, and it satisfies $\eta(0)=0$ or $\eta(L)=0$. Then we get*

$$\int_0^L \eta^2(y,t)\,\mathrm{d}y \leqslant 4L^2\pi^{-2}\int_0^L \eta_y^2(y,t)\,\mathrm{d}y, \quad t \geqslant 0. \tag{3}$$

Similarly, with the property $\eta_y(0)=0$ or $\eta_y(L)=0$, we can get

$$\int_0^L \eta_y^2(y,t)\,\mathrm{d}y \leqslant 4L^2\pi^{-2}\int_0^L \eta_{yy}^2(y,t)\,\mathrm{d}y, \quad t \geqslant 0, \tag{4}$$

for $\eta(\cdot,t) \in \mathcal{H}^2(0,L)$.

Lemma 2 [Lumer–Phillips theorem]. (*See* [13, 25].) *For any operator Γ, the following statements are equivalent*:

(i) *Operator Γ is the generator of a contraction semigroup on space $\mathcal{H}$*;
(ii) *Operator Γ is the maximum-dissipative.*

Lemma 3 [A variant of Agmon's inequality]. (*See* [28].) *Suppose $\eta(\cdot,t) \in \mathcal{H}(0,L)$ is a scalar function and satisfies $\eta(0,t)=0$ or $\eta(L,t)=0$, $t \geqslant 0$. Then one has*

$$\left|\eta(\cdot,t)\right|_\infty^2 \leqslant 2\left|\eta(\cdot,t)\right|_2\left|\eta_y(\cdot,t)\right|_2 \leqslant \left|\eta(\cdot,t)\right|_{\mathcal{H}^1}^2, \quad t \geqslant 0. \tag{5}$$

Based on (5), *as $|\eta(\cdot,t)|_2^2 \leqslant |\eta(\cdot,t)|_\infty^2$, for all $y \in [0,L]$ and $t \geqslant 0$, we further obtain*

$$\left|\eta(\cdot,t)\right|_2^2 \leqslant 2\left|\eta(\cdot,t)\right|_2\left|\eta_y(\cdot,t)\right|_2 \leqslant \left|\eta(\cdot,t)\right|_{\mathcal{H}^1}^2, \quad t \geqslant 0. \tag{6}$$

From (3)–(6) we derive

$$\begin{aligned} &\left|\eta(\cdot,t)\right|_2^2 \leqslant L\left|\eta(\cdot,t)\right|_\infty^2, \quad \left|\eta_y(\cdot,t)\right|_2^2 \leqslant L\left|\eta_y(\cdot,t)\right|_\infty^2, \\ &\left|\eta(\cdot,t)\right|_\infty^2 \leqslant 2\left|\eta(\cdot,t)\right|_2\left|\eta_y(\cdot,t)\right|_2 \leqslant 4L\pi^{-1}\left|\eta_y(\cdot,t)\right|_2^2 \\ &\qquad \leqslant 16L^3\pi^{-3}\left|\eta_{yy}(\cdot,t)\right|_2^2 \quad \forall y \in [0,L],\ \forall t \geqslant 0. \end{aligned} \tag{7}$$

Lemma 4 [Young's inequality]. (*See* [12].) *Suppose $a, b \geqslant 0 \in \mathbb{R}$, $\epsilon_1 > 1$, $1/\epsilon_1 + 1/\epsilon_2 = 1$, then*

$$ab \leqslant \frac{a^{\epsilon_1}}{\epsilon_1} + \frac{b^{\epsilon_2}}{\epsilon_2}.$$

If and only if $a^{\epsilon_1} = b^{\epsilon_2}$, the equality $ab = a^{\epsilon_1}/\epsilon_1 + b^{\epsilon_2}/\epsilon_2$ holds.

Based on the above preliminaries, our goal is to design a robust piecewise adaptive controller to make the system globally asymptotically stable and guarantee a given performance.

3 Main results

In this part, we first give the well-posedness analysis of the system solution, and then give the stability analysis of the system under the designed robust piecewise adaptive controller.

Firstly, we introduce the robust piecewise adaptive controller for the semilinear parabolic DPSs as follows:

$$u_i(t) = -\frac{\hat{\beta}(t)\int_0^L (\eta^2(y,t) + \alpha^*\eta_y^2(y,t))\,\mathrm{d}y + \int_0^L \eta^2(y,t)\,\mathrm{d}y}{\int_{y_i}^{y_{i+1}} (\eta(y,t) - (\alpha_*\eta_y(y,t))_y)\,\mathrm{d}y}, \quad i \in M, \tag{8}$$

where $\hat{\beta}(t)$ is defined as the estimated value of the unknown constant β at time t.

Remark 2. In (8), we can guarantee that $\eta(y,t) - (\alpha_*\eta_y(y,t))_y \neq 0$. When $\eta(y,t) - (\alpha_*\eta_y(y,t))_y = 0$, the obtained $\eta(y,t)$ is no longer the solution of the second-order semilinear parabolic DPSs (1). Moreover, if $\eta(y,t) - (\alpha_*\eta_y(y,t))_y = 0$, the form of the parabolic DPSs becomes

$$\eta_t(y,t) = \eta(y,t) + h\bigl(\eta(y,t)\bigr) + S^{\mathrm{T}}(y)u(t) + c(y,t), \quad t > 0,\ y \in [0,L]. \tag{9}$$

When $\alpha_* = \alpha(y)$, from (9) it is easy to see that its form does not satisfy the form of the second-order parabolic DPSs. However, we focus on studying the second-order parabolic DPSs in this study. According to the above analysis, we can obtain the result $\eta(y,t) - (\alpha_*\eta_y(y,t))_y \neq 0$.

Then, substituting (8) into (1), we can obtain

$$\begin{aligned}
\eta_t(y,t) &= \bigl(\alpha(y)\eta_y(y,t)\bigr)_y + h\bigl(\eta(y,t)\bigr) + c(y,t) \\
&\quad - \sum_{i=1}^{m} s_i(y)\frac{\hat{\beta}(t)\int_0^L (\eta^2(y,t) + \alpha^*\eta_y^2(y,t))\,\mathrm{d}y + \int_0^L \eta^2(y,t)\,\mathrm{d}y}{\int_{y_i}^{y_{i+1}} (\eta(y,t) - (\alpha_*\eta_y(y,t))_y)\,\mathrm{d}y}, \\
\eta(y,0) &= \eta_0(y), \quad y \in [0,L], \qquad \eta_y(y,t)|_{y=0} = \eta(y,t)|_{y=L} = 0, \quad t > 0.
\end{aligned} \tag{10}$$

The chosen adaptive law is

$$\dot{\hat{\beta}}(t) = \int_0^L \eta^2(y,t)\,\mathrm{d}y + \int_0^L \alpha^*\eta_y^2(y,t)\,\mathrm{d}y. \tag{11}$$

3.1 Well-posed analysis of system solution

In this part, we focus on the well-posedness analyses of open-loop system and closed-loop system, respectively. The open-loop system (1) can be expressed as the abstract differential equation

$$\eta_t(t) = \Gamma\eta(t) + h\bigl(\eta(t)\bigr) + \mathfrak{s}^{\mathrm{T}}u(t) + c(t), \qquad \eta(0) = \eta_0(\cdot), \tag{12}$$

where the state variable $\eta(t) \triangleq \eta(y,t)$, $y \in [0,L]$, and Γ is defined as follows:

$$\Gamma\bar{\eta}(y) \triangleq \frac{\mathrm{d}}{\mathrm{d}y}\left(\alpha(y)\frac{\mathrm{d}\bar{\eta}(y)}{\mathrm{d}y}\right).$$

The domain of the differential operator Γ is

$$D(\Gamma) \triangleq \left\{\bar{\eta} \in \mathcal{H}^2(0,L)\colon \frac{\mathrm{d}\bar{\eta}(y)}{\mathrm{d}y}|_{y=0} = \bar{\eta}(y)|_{y=L} = 0\right\},$$

$h(\eta(t)) \triangleq h(\eta(y,t))$, $(\mathfrak{s}^{\mathrm{T}}u(t))(y) \triangleq S^{\mathrm{T}}(y)u(t)$, $\mathfrak{s} \in \mathcal{L}(\mathbb{R}^m; \mathcal{L}_2([0,L]))$, $u(t) \in \mathcal{L}_2([0,\infty);\mathbb{R}^m)$, and $c(t) \triangleq c(y,t)$, $y \in [0,L]$. The following theorem gives the well-posed analysis of system solutions.

Theorem 1. *The open-loop system* (1) *has well-posed solution. What is more, the closed-loop system* (10) *with* (11) *also has well-posed solution.*

Proof. Firstly, we discuss the well-posed solution of system (1). From the definition of inner product $\langle\cdot,\cdot\rangle$ and the integration by parts technology we can get

$$\langle\tilde{\eta}, \Gamma\tilde{\eta}\rangle = -\int_0^L \alpha(y)\left(\frac{\mathrm{d}\tilde{\eta}(y)}{\mathrm{d}y}\right)^2 \mathrm{d}y \quad \forall\tilde{\eta}(y) \in D(\Gamma). \tag{13}$$

From $\alpha(y) \geqslant \alpha_* > 0$ and (13) we obtain

$$\langle\tilde{\eta}, \Gamma\tilde{\eta}\rangle \leqslant -\alpha_* \int_0^L \tilde{\eta}_y^2(y)\,\mathrm{d}y \leqslant 0.$$

In addition, according to the definition of Γ, we can easily know that it is a self-adjoint operator. Though the above analysis, we can know that operator Γ is a dissipative operator.

Next, we verify that the other necessary condition $\mathrm{Ran}(\sigma I - \Gamma) = \mathcal{H}$ holds, where $\sigma > 0$ is a constant. To prove the condition $\mathrm{Ran}(\sigma I - \Gamma) = \mathcal{H}$, we firstly introduce its equivalent condition that the following equation

$$(\sigma I - \Gamma)\tilde{\eta} = \hat{\eta}, \quad \hat{\eta} \in \mathcal{H}, \tag{14}$$

has a unique solution. Then, based on the definition of operator Γ, we rewrite (14) as follows:

$$\sigma\tilde{\eta}(y) - \frac{\mathrm{d}}{\mathrm{d}y}\left(\alpha(y)\frac{\mathrm{d}\tilde{\eta}(y)}{\mathrm{d}y}\right) = \hat{\eta}(y), \quad y \in [0,L], \tag{15}$$

with the following boundary condition:

$$\left.\frac{\mathrm{d}\tilde{\eta}(y)}{\mathrm{d}y}\right|_{y=0} = \tilde{\eta}(y)|_{y=L} = 0. \tag{16}$$

By simply deforming equation (15) we can get

$$\frac{\mathrm{d}^2\tilde{\eta}(y)}{\mathrm{d}^2 y} + \frac{\alpha'(y)}{\alpha(y)}\frac{\mathrm{d}\tilde{\eta}(y)}{\mathrm{d}y} - \frac{\sigma}{\alpha(y)}\tilde{\eta}(y) = -\frac{1}{\alpha(y)}\hat{\eta}(y). \tag{17}$$

Then we can obtain the following general solution to (17):

$$\begin{aligned}\tilde{\eta}(y) = \exp(ry)\Bigg[&\int_0^y \exp(-2r\gamma)\exp\left(-\int_0^y \frac{\alpha'(\gamma)}{\alpha(\gamma)}\,\mathrm{d}\gamma\right)\mathrm{d}\gamma\\ &\times\left(\int_0^y -\frac{1}{\alpha(\gamma)}\hat{\eta}(\gamma)\exp(r\gamma)\exp\left(\int_0^y \frac{\alpha'(\gamma)}{\alpha(\gamma)}\,\mathrm{d}\gamma\right)\mathrm{d}\gamma\right)\Bigg]\\ &+ c_1\exp(ry)\int_0^y \exp(-2r\gamma)\exp\left(-\int_0^y \frac{\alpha'(\gamma)}{\alpha(\gamma)}\,\mathrm{d}\gamma\right)\mathrm{d}\gamma\\ &+ c_2\exp(ry),\end{aligned} \tag{18}$$

where c_1, c_2 are arbitrary constants, and $r \in \mathbb{R}$ satisfies the following equality:

$$r^2 + \frac{\alpha'(y)}{\alpha(y)}r - \frac{\sigma}{\alpha(y)} \equiv 0.$$

By using the boundary condition (16) we deduce

$$\begin{aligned}&\frac{1}{2r}\frac{\alpha(0)}{\alpha(L)}\big(\exp(rL) - c_1\exp(-rL)\big) + c_2\exp(rL) = \tilde{c},\\ &c_1 + rc_2 = 0,\end{aligned} \tag{19}$$

where

$$\tilde{c} = \exp(rL)\frac{\alpha^2(0)}{\alpha^2(L)}\int_0^L \exp(-2r\gamma)\,\mathrm{d}\gamma\int_0^L -\frac{1}{\alpha(\gamma)}\hat{\eta}(\gamma)\exp(r\gamma)\,\mathrm{d}\gamma.$$

Then the coefficient matrix of c_1 and c_2 in (19) is expressed as follows:

$$\bar{c} = \begin{pmatrix}\frac{1}{2r}\frac{\alpha(0)}{\alpha(L)}(\exp(rL) - \exp(-rL)) & \exp(rL)\\ 1 & r\end{pmatrix}, \quad i \in M.$$

The determinant of matrix $\bar{c}$ is

$$\begin{aligned}|\bar{c}| &= \begin{vmatrix}\frac{1}{2r}\frac{\alpha(0)}{\alpha(L)}(\exp(rL) - \exp(-rL)) & \exp(rL)\\ 1 & r\end{vmatrix}\\ &= \frac{\alpha(0)}{2\alpha(L)}(\exp(rL) - \exp(-rL)) - \exp(rL).\end{aligned}$$

In order to proof $|\bar{c}| \neq 0$, we define $\exp(rL) = x$. Because of $r, L \in \mathbb{R}$, it can be easily get $x \in \mathbb{R}$. Firstly, we assume

$$|\bar{c}| = \frac{\alpha(0)}{2\alpha(L)}\big(\exp(rL) - \exp(-rL)\big) - \exp(rL) = 0,$$

namely,

$$\frac{\alpha(0)}{2\alpha(L)}\big(x - x^{-1}\big) - x = 0. \tag{20}$$

Because of $0 < \alpha(0) < 2\alpha(L)$, equation (20) obviously has no real number solution, i.e., $|\bar{c}| \neq 0$. Thus, the matrix $\bar{c}$ is invertible. Then we obtain that the solution of (19) is unique. Furthermore, we obtain that the general solution form (18) is unique under the boundary condition (16). Therefore, (15) with (16) has unique solution for any given $\sigma > 0$ and $\tilde{\eta} \in \mathcal{H}$. Though the above analysis, we obtain the result that $\mathrm{Ran}(\sigma I - \Gamma) = \mathcal{H}$ for all $\sigma > 0$.

From the results that operator Γ is a dissipative operator and $\mathrm{Ran}(\sigma I - \Gamma) = \mathcal{H}$ for all $\sigma > 0$ we obtain that Γ is the maximum-dissipative operator. According to the Lumer–Philips theorem, it is proved that operator Γ is the generator of the contractive semigroup $\exp(\Gamma t)$ on $\mathcal{H}$. Because of $h(\eta(t))$ belongs to the class of C^1 and $\int_0^\infty |c(\cdot,t)|_\infty^2\,\mathrm{d}t < \infty$ and by using Theorem 3.13 in [9] and Theorem 1.5 in [21, Chap. 6] we can obtain that system (1) has a unique solution for $t \in [0, t_f]$, $t_f \geqslant 0$ and $\eta(0) \in D(\Gamma)$.

Then we further consider the well-posedness of the closed-loop system (10), which can be expressed as (12). Firstly, from the analysis of the well-posedness on open-loop system we get that the operator Γ can generate a contraction semigroup $\exp(\Gamma t)$ on $\mathcal{H}$. Then from the definition of $s_i(y)$, $i \in M$, in (2) and the definition of operator $\mathfrak{s}$, for $t \geqslant 0$, $u(t) \in \mathcal{L}_2([0,\infty); \mathbb{R}^m)$, it follows that

$$\big|\big(\mathfrak{s}^{\mathrm{T}} u(t)\big)(y)\big|_2 = \big|S^{\mathrm{T}}(y)u(t)\big|_2 \leqslant \sqrt{mL}\big|u(t)\big|_2.$$

It is clearly that operator $\mathfrak{s}$ is bounded and $\mathfrak{s} \in \mathcal{L}(\mathbb{R}^m, \mathcal{L}^2([0,L]))$. In addition, because of $h(\eta(t))$ belongs to the class of C^1 and $\int_0^\infty |c(\cdot,t)|_\infty^2\,\mathrm{d}t < \infty$ and by using Theorem 3.13 in [9] and Theorem 1.5 in [21, Chap. 6] we can get that system (10) has a unique solution for $t \in [0, t_f]$, $t_f \geqslant 0$ and $\eta(0) \in D(\Gamma)$. □

Remark 3. It can be further shown that system (10) has a unique solution for all $\eta(0) \in \mathcal{H}$, $t \in [0, t_f]$, $t_f \geqslant 0$. To get the result, it is only to show that system (10) has a unique solution for all $\eta(0) \in D(\Gamma)$, where $D(\Gamma)$ is a dense subspace of $\mathcal{H}$. Therefore, we will first show that $D(\Gamma)$ is a dense subspace of $\mathcal{H}$. In $D(\Gamma)$, we define a special inner product $(y_1, y_2)_{D(\Gamma)} \triangleq (y_1, y_2) + (\Gamma y_1, \Gamma y_2)$ for all $y_1, y_2 \in D(\Gamma)$ and with the norm $|\eta(\cdot)|_{D(\Gamma)} \triangleq |\eta(\cdot)|_2 + \Gamma|\eta(\cdot)|_2$. In addition, because of Γ is a closed operator, it can be obtained that $D(\Gamma)$ is a complete inner product space. Therefore, $D(\Gamma)$ is another inner product space, which can be continuously embedded in $\mathcal{H}$. Thus, we have that $D(\Gamma)$ is a dense subspace of $\mathcal{H}$ and further obtain that system (10) has a unique solution $\eta(\cdot,t)$ for any $t \in [0, t_f]$, $t_f \geqslant 0$ and $\eta(0) \in \mathcal{H}$.

3.2 Globally asymptotic stabilization

In this section, we mainly focus on the design of a robust piecewise adaptive controller so that the semilinear parabolic DPSs with external disturbance achieve globally asymptotic stability under the action of the controller. Firstly, the parameter estimated error is defined as $\tilde{\beta}(t) = \beta - \hat{\beta}(t)$. Then we choose a LKFC for system (10)

$$\mathbb{V}(t) = \mathbb{V}_1(t) + \mathbb{V}_2(t) + \mathbb{V}_3(t), \tag{21}$$

where $\mathbb{V}_1(t) = \int_0^L \eta^2(y,t)\,\mathrm{d}y/2$, $\mathbb{V}_2(t) = \int_0^L \alpha(y)\eta_y^2(y,t)\,\mathrm{d}y/2$, $\mathbb{V}_3(t) = \tilde{\beta}^2(t)/2$. From (21) it is obvious that $\mathbb{V}(t)$ is continuous in time.

From (21) and $\alpha^* \geqslant \alpha(y) \geqslant \alpha_* > 0$ it is easy to get

$$\mathbb{V}(t) \leqslant \frac{1}{2}\int_0^L \eta^2(y,t)\,\mathrm{d}y + \frac{1}{2}\alpha^*\int_0^L \eta_y^2(y,t)\,\mathrm{d}y + \frac{1}{2}\tilde{\beta}^2(t). \tag{22}$$

Theorem 2. *Consider the semilinear parabolic DPSs described by* (1) *with* $c(y,t) = 0$, $\alpha^* \geqslant \alpha(y) \geqslant \alpha_* > 0$ *and* $\alpha(0) < 2\alpha(L)$. *Suppose Assumption* 1 *holds, then there exists a piecewise adaptive controller* (8)*, and a adaptive law* (11) *ensuing the solution of system described by* (10) *is globally asymptotically stable, namely,* $\lim_{t\to\infty} |\eta(y,t)|_2 = 0$.

Proof. Along the solution of the closed-loop system, the derivative of $\mathbb{V}_1(t)$ with respect to time t is expressed as

$$\begin{aligned}\dot{\mathbb{V}}_1(t) &= \int_0^L \eta(y,t)\eta_t(y,t)\,\mathrm{d}y \\ &= \int_0^L \eta(y,t)\big(\alpha(y)\eta_y(y,t)\big)_y\,\mathrm{d}y + \int_0^L \eta(y,t)h\big(\eta(y,t)\big)\,\mathrm{d}y \\ &\quad + \int_0^L \eta(y,t)S^{\mathrm{T}}(y)u(t)\,\mathrm{d}y.\end{aligned} \tag{23}$$

For the first item of (23), from the boundary condition and integration by parts we derive

$$\int_0^L \eta(y,t)\big(\alpha(y)\eta_y(y,t)\big)_y\,\mathrm{d}y = -\int_0^L \alpha(y)\eta_y^2(y,t)\,\mathrm{d}y. \tag{24}$$

For the second item of (23), from Assumption 1 we obtain

$$\begin{aligned}\int_0^L \eta(y,t)h\big(\eta(y,t)\big)\,\mathrm{d}y &\leqslant \int_0^L \big|\eta(y,t)\big|\big|h\big(\eta(y,t)\big) - h(0)\big|\,\mathrm{d}y \\ &\leqslant \beta\int_0^L \eta^2(y,t)\,\mathrm{d}y.\end{aligned} \tag{25}$$

For the third item of (23), by the definition of function $S(y)$ and (2) we have

$$\int_0^L \eta(y,t)S^{\mathrm{T}}(y)u(t)\,\mathrm{d}y = \int_{y_i}^{y_{i+1}} \eta(y,t)\,\mathrm{d}y\,u_i(t), \quad i \in M. \tag{26}$$

Substituting (24)–(26) into (23) yields

$$\dot{\mathbb{V}}_1(t) \leqslant -\alpha_* \int_0^L \eta_y^2(y,t)\,\mathrm{d}y + \beta \int_0^L \eta^2(y,t)\,\mathrm{d}y + \int_{y_i}^{y_{i+1}} \eta(y,t)\,\mathrm{d}y\,u_i(t). \tag{27}$$

Then, along the solution to the closed-loop system, the derivative of $\mathbb{V}_2(t)$ and using the integration by parts, one gets

$$\begin{aligned}\dot{\mathbb{V}}_2(t) &= \int_0^L \alpha(y)\eta_y(y,t)\eta_{yt}(y,t)\,\mathrm{d}y \\ &\leqslant -\int_0^L \big(\alpha_*\eta_y(y,t)\big)_y^2\,\mathrm{d}y + \beta\int_0^L \big(\alpha^*\eta_y^2(y,t)\big)\,\mathrm{d}y \\ &\quad - \int_{y_i}^{y_{i+1}} \big(\alpha_*\eta_y(y,t)\big)_y\,\mathrm{d}y\,u_i(t), \quad i \in M. \end{aligned} \tag{28}$$

Finally, the derivative of $\mathbb{V}_3(t)$ with respect to time t is given as follows:

$$\dot{\mathbb{V}}_3(t) = \tilde{\beta}(t)\dot{\tilde{\beta}}(t) = -\tilde{\beta}(t)\dot{\hat{\beta}}(t). \tag{29}$$

From (27), (28), (29) and (21) we obtain

$$\begin{aligned}\dot{\mathbb{V}}(t) \leqslant{}& -\alpha_* \int_0^L \eta_y^2(y,t)\,\mathrm{d}y + \beta \int_0^L \eta^2(y,t)\,\mathrm{d}y + \int_{y_i}^{y_{i+1}} \eta(y,t)\,\mathrm{d}y\,u_i(t) \\ & - \int_0^L \big(\alpha_*\eta_y(y,t)\big)_y^2\,\mathrm{d}y + \beta\int_0^L \big(\alpha^*\eta_y^2(y,t)\big)\,\mathrm{d}y \\ & - \int_{y_i}^{y_{i+1}} \big(\alpha_*\eta_y(y,t)\big)_y\,\mathrm{d}y\,u_i(t) - \tilde{\beta}(t)\dot{\hat{\beta}}(t), \quad i \in M. \end{aligned} \tag{30}$$

Substituting (8) and (11) into (30) and using $\tilde{\beta} = \beta - \hat{\beta}$, we derive

$$\dot{\mathbb{V}}(t) \leqslant -\int_0^L \eta^2(y,t)\,\mathrm{d}y - \alpha_* \int_0^L \eta_y^2(y,t)\,\mathrm{d}y - \int_0^L \big(\alpha_*\eta_y(y,t)\big)_y^2\,\mathrm{d}y.$$

By LaSalle–Yoshizawa theorem we obtain controller (8), and the adaptive law (11) ensuing the solution of system (10) is globally asymptotically stable, that is, $\lim_{t\to\infty} |\eta(y,t)|_2 = 0$. □

Remark 4. $s_i(y)$ represents the spatial distribution of the ith controller, and $u_i(t)$ stands for the control input of the ith segment domain. Different segmented areas need different controllers to perform work, and different controllers do not interact with each other. Therefore, compared with the system controlled by a boundary controller, the controller designed in this study reduces its conservativeness and improves its effectiveness.

According to the above main results, we will give another main result, that is, the controller (8) and the adaptive law (11) can ensure the globally asymptotic stability of the solution of the system with external disturbance and also guarantee a given performance in $|\cdot|_\infty$.

Theorem 3. *Consider the semilinear parabolic DPSs* (1) *with Assumption* 1, $\alpha^* \geqslant \alpha(y) \geqslant \alpha_* > 0$ *and* $\alpha(0) < 2\alpha(L)$. *Given constants* $\mu > 0$, $p = q + L/2 > 0$ *such that* $(8L^4 + 16qL^3)/\pi^3 - \alpha_*^2/2 < 0$ *and* $1 - \mu^2/L < 0$ *hold. Then there exists a robust piecewise adaptive controller* (8)*, and a adaptive law* (11) *can not only ensure that system* (10) *is globally asymptotically stable, but also guarantee the following given performance*:

$$q\int_0^\infty \left|\eta(y,t)\right|_\infty^2 \mathrm{d}t \leqslant \frac{L}{2}\left|\eta_0(y)\right|_\infty^2 + \frac{L\alpha^*}{2}\left|\eta_{0,y}(y)\right|_\infty^2 + \frac{1}{2}\tilde{\beta}^2(0) + \mu^2\int_0^\infty \left|c(\cdot,t)\right|_\infty^2 \mathrm{d}t. \tag{31}$$

Proof. Based on the process of Theorem 2, we get the time derivative of $\mathbb{V}(t)$

$$\begin{aligned}\dot{\mathbb{V}}(t) &= \int_0^L \eta(y,t)\eta_t(y,t)\,\mathrm{d}y + \int_0^L \alpha(y)\eta_y(y,t)\eta_{yt}(y,t)\,\mathrm{d}y - \tilde{\beta}(t)\dot{\hat{\beta}}(t)\\ &\leqslant -\int_0^L \eta^2(y,t)\,\mathrm{d}y - \alpha_*\int_0^L \eta_y^2(y,t)\,\mathrm{d}y - \alpha_*^2\int_0^L \eta_{yy}(y,t)^2\,\mathrm{d}y\\ &\quad + \int_0^L \eta(y,t)c(y,t)\,\mathrm{d}y - \alpha_*\int_0^L \eta_{yy}(y,t)c(y,t)\,\mathrm{d}y.\end{aligned} \tag{32}$$

By using Lemmas 3 and 4 the forth and fifth item of (32) become

$$\begin{aligned}\int_0^L \eta(y,t)c(y,t)\,\mathrm{d}y &\leqslant \frac{1}{2}\left|\eta(y,t)\right|_2^2 + \frac{1}{2}\left|c(y,t)\right|_2^2\\ &\leqslant \frac{L}{2}\left|\eta(y,t)\right|_\infty^2 + \frac{1}{2}\left|c(y,t)\right|_2^2,\end{aligned} \tag{33}$$

$$-\alpha_* \int_0^L \eta_{yy}(y,t)c(y,t)\,\mathrm{d}y \leqslant \frac{\alpha_*^2}{2}\left|\eta_{yy}(y,t)\right|_2^2 + \frac{1}{2}\left|c(y,t)\right|_2^2. \tag{34}$$

Substituting (33) and (34) into (32), we derive

$$\begin{aligned}\dot{\mathbb{V}}(t) &\leqslant -\left|\eta(y,t)\right|_2^2 - \alpha_*\left|\eta_y(y,t)\right|_2^2 - \alpha_*^2\left|\eta_{yy}(y,t)\right|_2^2 \\ &\quad + \frac{L}{2}\left|\eta(y,t)\right|_\infty^2 + \frac{\alpha_*^2}{2}\left|\eta_{yy}(y,t)\right|_2^2 + \left|c(y,t)\right|_2^2.\end{aligned} \tag{35}$$

Based on (35) and (7), one can have the inequality as follows:

$$\begin{aligned}&\dot{\mathbb{V}}(t) + p\left|\eta(y,t)\right|_\infty^2 - \mu^2\left|c(y,t)\right|_\infty^2 \\ &\quad \leqslant -\left|\eta(y,t)\right|_2^2 - \alpha_*\left|\eta_y(y,t)\right|_2^2 - \alpha_*^2\left|\eta_{yy}(y,t)\right|_2^2 + \frac{L}{2}\left|\eta(y,t)\right|_\infty^2 \\ &\qquad + \frac{\alpha_*^2}{2}\left|\eta_{yy}(y,t)\right|_2^2 + \left|c(y,t)\right|_2^2 + \frac{16pL^3}{\pi^3}\left|\eta_{yy}(y,t)\right|_2^2 - \frac{\mu^2}{L}\left|c(y,t)\right|_2^2.\end{aligned} \tag{36}$$

From the relations $p = q + L/2 > 0$, $(8L^4 + 16qL^3)/\pi^3 - \alpha_*^2/2 < 0$ and $1 - \mu^2/L < 0$ (36) becomes

$$\dot{\mathbb{V}}(t) + q\left|\eta(y,t)\right|_\infty^2 - \mu^2\left|c(y,t)\right|_\infty^2 \leqslant -\left|\eta(y,t)\right|_2^2 - \alpha_*\left|\eta_y(y,t)\right|_2^2 \leqslant 0. \tag{37}$$

For any given $N \gg 1$, by integrating (37) from $t = 0$ to $t = t_N$, using Lemma 3 and (22), one obtains

$$q\int_0^{t_N} \left|\eta(y,t)\right|_\infty^2 \mathrm{d}t \leqslant \mathbb{V}(0) - \mathbb{V}(t_N) + \mu^2 \int_0^{t_N} \left|c(y,t)\right|_\infty^2 \mathrm{d}t.$$

Due to $\mathbb{V}(t_N) \geqslant 0$, when $N \to \infty$, we further obtain the following result:

$$\begin{aligned}q\int_0^\infty \left|\eta(y,t)\right|_\infty^2 \mathrm{d}t &\leqslant \frac{L}{2}\left|\eta_0(y)\right|_\infty^2 + \frac{L\alpha^*}{2}\left|\eta_{0,y}(y)\right|_\infty^2 + \frac{1}{2}\tilde{\beta}^2(0) \\ &\quad + \mu^2\int_0^\infty \left|c(\cdot,t)\right|_\infty^2 \mathrm{d}t.\end{aligned}$$

Thus, we obtain that there exists controller (8), and the adaptive law (11) ensuing the solution of system (10) is globally asymptotically stable with the given performance. □

Remark 5. To improve the given performance in (31), we can construct the LKFC for system (10) as follows:

$$\mathbb{V}(t) = \frac{1}{2}\theta_1\int_0^L \eta^2(y,t)\,\mathrm{d}y + \frac{1}{2}\theta_2\int_0^L \alpha(y)\eta_y^2(y,t)\,\mathrm{d}y + \frac{1}{2}\theta_3\tilde{\beta}^2(t), \quad \theta_j > 0,$$

$j = 1, 2, 3$, and the given performance described by (31) should be changed as

$$q \int_0^\infty \big|\eta(y,t)\big|_\infty^2 \,\mathrm{d}t \leqslant \frac{\theta_1 L}{2}\big|\eta_0(y)\big|_\infty^2 + \frac{\theta_2 L \alpha^*}{2}\big|\eta_{0,y}(y)\big|_\infty^2 + \frac{\theta_3}{2}\tilde{\beta}^2(0) + \mu^2 \int_0^\infty \big|c(\cdot,t)\big|_\infty^2 \,\mathrm{d}t.$$

Remark 6. The design method proposed in this study is developed for homogeneous Dirichlet–Neumann mixed boundary condition, but it is also suitable for homogeneous Dirichlet and Neumann boundary conditions, respectively.

4 Numerical simulation

In this section, a numerical simulation example and a practical example will be given to illustrate the rationality and effectiveness of the algorithm.

Example 1. Consider the robust piecewise adaptive control problem of the following system:

$$\begin{aligned} &\eta_t(y,t) = \eta_{yy}(y,t) + S^{\mathrm{T}}(y)u(t) + 3\sin\big(\eta(y,t)\big) + c(y,t), \\ &\eta(y,0) = \eta_0(y), \qquad \eta_y(y,t)|_{y=0} = \eta(y,t)|_{y=L} = 0, \end{aligned} \tag{38}$$

where $\eta(y,t) \in \mathcal{H}$ is the state variable, $u(t) \triangleq [u_1, u_2, \ldots, u_m]^{\mathrm{T}} \in \mathbb{R}^m$ is the control input, and $S(y) \triangleq [s_1(y), s_2(y), \ldots, s_m(y)]^{\mathrm{T}} \in \mathbb{R}^m$, where $s_i(y)$ denotes the distribution of the ith controller in the spatial domain and satisfies (2). Set $L = 1$, $\eta_0(y) = 5\sin(\pi y)$, $y \in [0,1]$.

Firstly, considering system (38) without control input and external disturbance, the evolution trend of $\eta(y,t)$ can be obtained as shown in Fig. 1. It can be seen from Fig. 1 that the equilibrium profile $\eta(\cdot,t) = 0$ of system (38) without control input and external disturbance is unstable.

Next, we will consider the control performance of the adaptive controller in Theorems 2 and 3. In this example, we divide the domain $(0,L)$ into two local spatial regions $(0,0.5]$ and $(0.5,1)$. For the semilinear parabolic PDEs model (38), the adaptive controller replaced in each segment is expressed as

$$u_i(t) = -\frac{\hat{\beta}(t)\int_0^1 (\eta^2(y,t) + \eta_y^2(y,t))\,\mathrm{d}y + \int_0^1 \eta^2(y,t)\,\mathrm{d}y}{\int_{y_i}^{y_{i+1}} (\eta(y,t) - (\eta_y(y,t))_y)\,\mathrm{d}y} \tag{39}$$

with the following adaptive law:

$$\dot{\hat{\beta}}(t) = \int_0^1 \eta^2(y,t)\,\mathrm{d}y + \int_0^1 \eta_y^2(y,t)\,\mathrm{d}y. \tag{40}$$

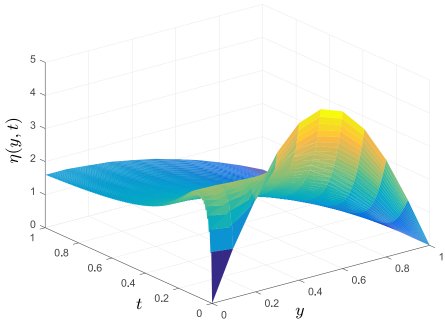

Figure 1. The evolution trend of the solution $\eta(y,t)$ of the open-loop system.

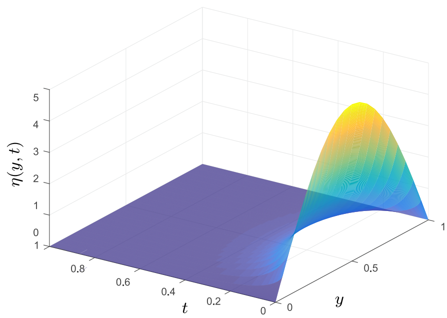

Figure 2. The evolution trend of the solution $\eta(y,t)$ of the closed-loop system with $c(y,t)=0$.

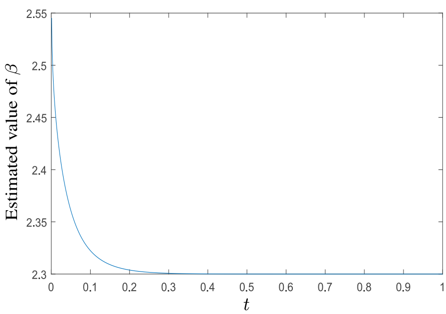

Figure 3. The evolution of the estimated value of β with $c(y,t)=0$.

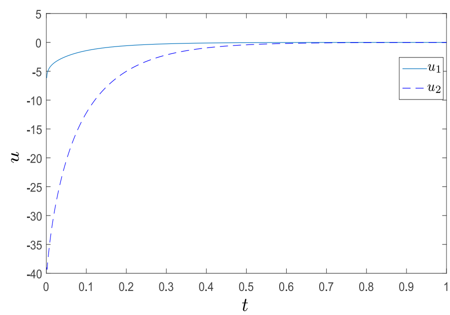

Figure 4. The evolution of $u(t)$ with $c(y,t)=0$.

Using controller (39) with the adaptive law (40) to the system described by (38) with $c(y,t)=0$, we obtain the evolution trend of the solution $\eta(y,t)$ of the closed-loop system as shown in Fig. 2. Combined with Figs. 1 and 2, it can be seen that controller (39) with the adaptive law (40) can stabilize the PDEs system (38) with $c(y,t)=0$. In addition, the result shown in Fig. 3 illustrates that the parameter estimate $\hat{\beta}$ is bounded, and the result shown in Fig. 4 indicates that $u(t)$ is also bounded.

Secondly, consider system (38) with $c(y,t)=\cos(\pi y)\exp(-t)$ and $u(t)=0$, the evolution trend of $\eta(y,t)$ can be obtained as shown in Fig. 5. Similarly, from Fig. 5 we can see that the equilibrium profile $\eta(\cdot,t)=0$ of the open-loop system described by (38) with $c(y,t)=\cos(\pi y)\exp(-t)$ is still unstable when $t\in[0,1]$.

Next, we further consider the stabilization effect of the robust adaptive controller (39) with the adaptive law (40) on system (38) when the disturbance $c(y,t)=\cos(\pi y)\exp(-t)$. The trajectories of $\eta(y,t)$, parameter estimation $\hat{\beta}$ and $u(t)$ are shown in Figs. 6, 7 and 8, respectively.

By comparing the results shown in Figs. 5 and 6, we get that under the action of the piecewise adaptive robust controller (39) with the adaptive law (40), the equilibrium profile $\eta(\cdot,t)=0$ of (38) with $c(y,t)=\cos(\pi y)\exp(-t)$ is stable before $t=1$. Thus,

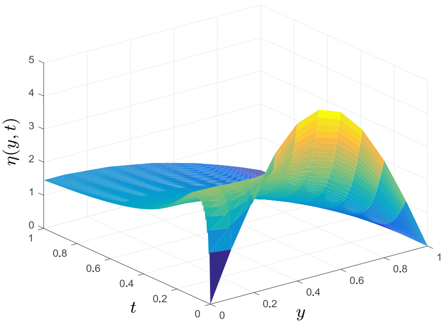

Figure 5. The evolution trend of the solution $\eta(y,t)$ of the open-loop system with $c(y,t) = \cos(\pi y)\exp(-t)$.

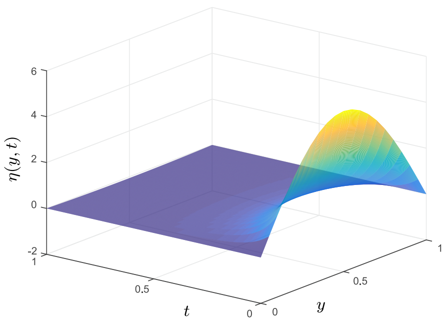

Figure 6. The evolution trend of the solution $\eta(y,t)$ of the closed-loop system with $c(y,t) = \cos(\pi y)\exp(-t)$.

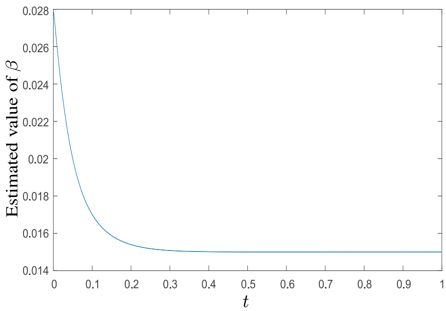

Figure 7. The evolution of the estimated value of β with $c(y,t) = \cos(\pi y)\exp(-t)$.

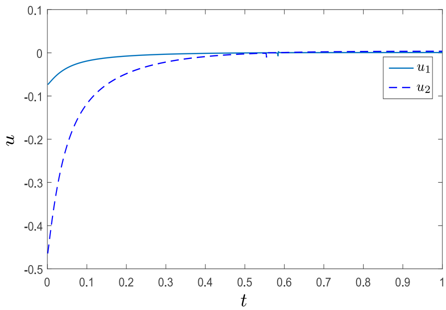

Figure 8. The evolution of $u(t)$ with $c(y,t) = \cos(\pi y)\exp(-t)$.

we obtain the result that the piecewise adaptive robust controller can still stabilize the PDEs system (38) with the disturbance $c(y,t) = \cos(\pi y)\exp(-t)$. Moreover, we obtain that the result shown in Fig. 7 illustrates that the parameter estimate $\hat{\beta}$ is bounded and the result shown in Fig. 8 indicates that the input information $u(t)$ is also bounded.

Finally, through the following actual calculation, we verify that the controller designed in this study can guarantee the given performance

$$\begin{aligned}0.45\int_0^\infty \left|\eta(\cdot,t)\right|_\infty^2 \mathrm{d}t &\leqslant \frac{1}{2}\left|\eta_0(y)\right|_\infty^2 + \frac{1}{2}\left|\eta_{0,y}(y)\right|_\infty^2 + \frac{1}{2}\tilde{\beta}^2(0) \\ &\quad + (2)^2\int_0^\infty \left|c(\cdot,t)\right|_\infty^2 \mathrm{d}t \\ &\approx 68.1341,\end{aligned}$$

which indicates that the robust piecewise adaptive controller (8) with the adaptive law (11) proposed in this study can guarantee (31) with $q = 0.45$ and $\mu = 2$.

Example 2. Consider the spatiotemporal evolution dynamics of neutron concentration in nuclear reactor [10], which can be described by the following parabolic nonlinear PDEs under the piecewise adaptive control architecture:

$$\begin{aligned} &\eta_t(y,t) = \nu\eta_{yy}(y,t) + \beta_1\bigl(\beta_2\eta(y,t) - \eta^2(y,t)\bigr) + S^{\mathrm{T}}(y)u(t), \\ &\eta(y,0) = \eta_0(y), \qquad \eta_y(y,t)|_{y=0} = \eta(y,t)|_{y=L} = 0, \end{aligned} \tag{41}$$

where $\eta(y,t)$ represents the neutron concentration distribution of a nuclear reactor. The initial value and process parameters are selected as $L = 1$, $\nu = 1$, $\beta_1 = 1$, $\beta_2 = 5$, $\eta_0(y) = \sin(\pi y)$, $y \in [0,1]$. Therefore, the evolution trend of the neutron concentration distribution of the open-loop system is shown in Fig. 9, which can be clearly seen that system (41) is unstable at equilibrium point $\eta(y,t) = 0$.

Next, we will consider the control performance of the adaptive controller in Theorem 2. Similar to the previous example, the domain $(0,L)$ is also divided into two local spatial regions $(0,0.5]$ and $(0.5,1)$. Then the evolution trend of closed-loop system is shown in Fig. 10. From Figs. 9 and 10 it can be seen that the controller with the adaptive law can stabilize the PDEs system (41). In addition, the result shown in Fig. 11 illustrates that the parameter estimate $\hat{\beta}$ is bounded, and the result shown in Fig. 12 indicates that $u(t)$ is also bounded.

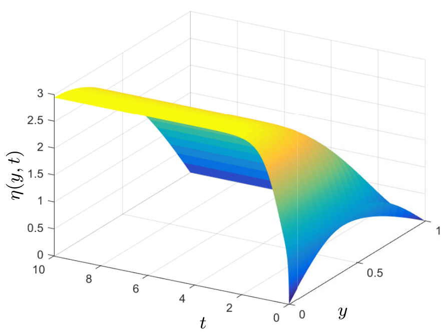

Figure 9. The evolution trend of the solution $\eta(y,t)$ of the open-loop system.

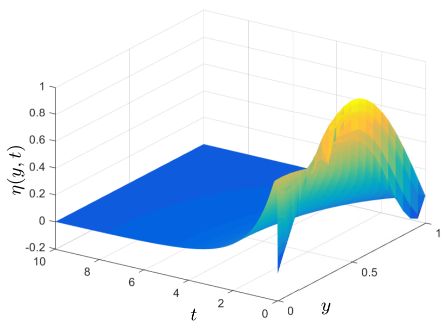

Figure 10. The evolution trend of the solution $\eta(y,t)$ of the closed-loop system.

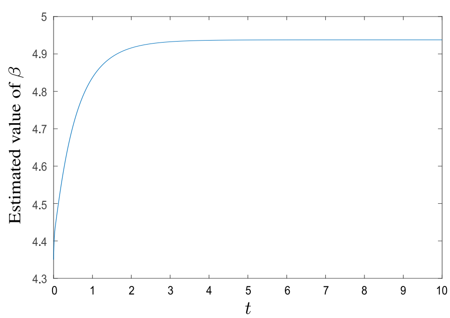

Figure 11. The evolution of the estimated value of β.

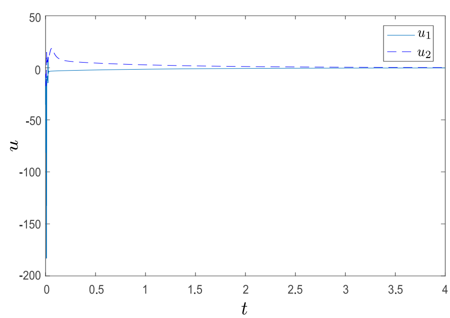

Figure 12. The evolution of $u(t)$.

5 Conclusion

In this study, we have researched the globally asymptotic stabilization for the semilinear parabolic DPSs. Firstly, detailed analyses are given for the well-posedness of the solutions to the open-loop system and the resulting closed-loop system. Then, by constructing an appropriate LKCF, using integration by parts and the Wirtinger's inequality, it is proved that the proposed robust piecewise adaptive controller with the selected adaptive law cannot only make the semilinear parabolic DPSs globally asymptotic stability, but also ensure the given performance defined in the sense of $|\cdot|_\infty$. Finally, the simulation result has shown the rationality and effectiveness of the designed controller.

In the future research work, authors may do further research on the following two aspects. On the one hand, authors will be devoted to the study of more general systems with complex practical applications, such as systems with time delays or systems with uncertain disturbances. On the other hand, in this work, an adaptive control method based on constant parameter estimation is used to design a controller to stabilize the system and make the adaptive parameter estimation bounded. However, this does not lead to convergence of the parameter estimate to its real value. Therefore, the convergence of adaptive parameters is also a research direction of the future work.

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