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Applications of variational methods to some three-point boundary value problems with instantaneous and noninstantaneous impulses*

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Abstract. In this paper, we study the multiple solutions for some second-order p-Laplace differential equations with three-point boundary conditions and instantaneous and noninstantaneous impulses. By applying the variational method and critical point theory the multiple solutions are obtained in a Sobolev space. Compared with other local boundary value problems, the three-point boundary value problem is less studied by variational method due to its variational structure. Finally, two examples are given to illustrate the results of multiplicity.

Keywords: three-point BVPs, variational method, critical point theory, multiple solutions, noninstantaneous impulse.

1 Introduction

In recent years, the research on impulsive differential equations have attracted widespread attention. This is because many phenomena in life are not a continuous process and will change suddenly due to the influence of external factors. Therefore, it is more appropriate to use impulsive differential equations to describe such situations instead of simply using differential equations or difference equations. Impulsive differential equations have been widely used in recent years, especially, in the field of biological mathematics. For example, in the study of pharmacokinetic model, since oral and injected drugs often enter the human body in the form of impulse, it is more reasonable to use impulse differential equations to describe the changes of drug concentration in the human body.

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Moreover, the impulsive differential equations have been fully applied in the fields of pest control and comprehensive environmental management, and many gratifying results have been obtained.

Impulses can be divided into instantaneous and noninstantaneous impulses due to the duration. However, in many applications, instantaneous and noninstantaneous impulses occur at the same time in some dynamic processes such as intravenous injection. Since the drug enters the blood and the subsequent absorption of the body is a sudden and continuous process, this situation can be explained as an impulsive behavior. The impulse suddenly starts to jump at any fixed point in time (drug enters the blood) and continues to occur within a limited time interval (the body absorbs the drug). Due to its wide range of applications, some scholars began to study the existence of solutions of differential equations with instantaneous and noninstantaneous impulses, and through fixed point theorems, upper and lower solution theorems, variational methods, etc., they obtained some excellent results [1, 3, 4, 6–8, 10, 11, 13–19]. Especially, in [4], Bai and Nieto first gave the variational structure of a linear equation with noninstantaneous impulses as the following problem:

$$\begin{aligned}
&-y''(t) = f_j(t), \quad t \in (s_j, t_{j+1}],\ j = 0, 1, \dots, m,\\
&y'(t) = \alpha_j, \quad t \in (t_j, s_j],\ j = 1, 2, \dots, m,\\
&y'\big(s_j^+\big) = y'\big(s_j^-\big), \quad j = 1, 2, \dots, m,\\
&y(0) = y(T) = 0, \qquad y'(0) = \alpha_0.
\end{aligned}$$

It is the first time that the critical point theory has been applied to consider the problems with noninstantaneous impulses. Then Tian and Zhang in [15] studied the existence of classical solutions for differential equations with instantaneous and noninstantaneous impulses, they considered the following problem:

$$\begin{aligned}
&-y''(t) = f_j\big(t, y(t)\big), \quad t \in (s_j, t_{j+1}],\ j = 0, 1, \dots, m,\\
&-\Delta y'(t_j) = I_j\big(y(t_j)\big), \quad j = 1, 2, \dots, m,\\
&y'(t) = y'\big(t_j^+\big), \quad t \in (t_j, s_j],\ j = 1, 2, \dots, m,\\
&y'\big(s_j^+\big) = y'\big(s_j^-\big), \quad j = 1, 2, \dots, m,\\
&y(0) = y(T) = 0.
\end{aligned}$$

In addition, Bai et al. in [9] studied a three-point boundary value problem, they firstly gave the variational structure of the nonlocal boundary value problem, and gave a different idea to deal with the functionals by imposing the boundary value conditions on admissible space rather than the functionals.

Inspired by the above literatures, we try to study the problem, which the impulse suddenly starts to jump at any fixed point t_j and continues to occur within a limited time interval $(t_j, s_j]$. This paper considers the multiple solutions for a class of three-point boundary value problems (BVPs) with instantaneous and noninstantaneous impulses as follows:

$$
\begin{aligned}
& -\left(\mu(t)\Phi_p\left(y'(t)\right)\right)' + \lambda(t)\Phi_p\left(y(t)\right) = f_j\left(t, y(t)\right), \quad t \in (s_j, t_{j+1}], \\
& \quad j = 0, 1, \dots, m, \\
& -\Delta\left(\mu(t_j)\Phi_p\left(y'(t_j)\right)\right) = I_j\left(y(t_j)\right), \quad j = 1, 2, \dots, m, \\
& \mu(t)\Phi_p\left(y'(t)\right) = \mu\left(t_j^+\right)\Phi_p\left(y'\left(t_j^+\right)\right), \quad t \in (t_j, s_j],\ j = 1, 2, \dots, m, \\
& \mu\left(s_j^+\right)\Phi_p\left(y'\left(s_j^+\right)\right) = \mu\left(s_j^-\right)\Phi_p\left(y'\left(s_j^-\right)\right), \quad j = 1, 2, \dots, m, \\
& y(0) = 0, \qquad y(1) = \zeta y(\eta),
\end{aligned}
\tag{1}
$$

where $p > 1$, $\Phi_p(y) := |y|^{p-2}y$, $\mu(t), \lambda(t) \in L^p[0,1]$, $0 = s_0 < t_1 < s_1 < \dots < s_{m_1} = \eta < t_{m_1+1} < \dots < s_m < t_{m+1} = 1$, $\zeta > 0$, $0 < \eta < 1$, and

$$\Delta\left(\mu(t_j)\Phi_p\left(y'(t_j)\right)\right) = \mu\left(t_j^+\right)\Phi_p\left(y'\left(t_j^+\right)\right) - \mu\left(t_j^-\right)\Phi_p\left(y'\left(t_j^-\right)\right)$$

for $y'(t_j^{\pm}) = \lim_{t \to t_j^{\pm}} y'(t)$, $j = 1, 2, \dots, m$, and $f_j \in C((s_j, t_{j+1}] \times \mathbb{R}, \mathbb{R})$, $I_j \in C(\mathbb{R}, \mathbb{R})$.

The remainder of the paper is organized as follows. Section 2 will prove that the critical point of functional J is the classical solution of BVPs (1). Section 3 will present the main results with the specific proof. Section 4 will give two examples to verify the results.

2 Preliminaries

In this paper, we assume the following condition:

(A1) $1 \leqslant \lambda(t) \leqslant c$ and $\mu(t) \geqslant 1$ for $t \in (s_j, t_{j+1}]$, $\mu(t), \lambda(t) \in L^p[0,1]$, $p > 1$, $j = 0, 1, \dots, m$, and c is a positive constant.

Let $Z = \{y \in W^{1,p}([0,1], \mathbb{R})\colon\ y(0) = 0,\ y(1) = \zeta y(\eta)\}$ with the norm

$$\|y\|_Z = \left[\int_0^1 \left(\mu(t)\left|y'(t)\right|^p + \lambda(t)\left|y(t)\right|^p\right) \mathrm{d}t\right]^{1/p},$$

then the following norm

$$\|y\| = \left[\int_0^1 \mu(t)\left|y'(t)\right|^p \mathrm{d}t\right]^{1/p} \quad \forall y \in Z$$

is equivalent to the norm $\|y\|_Z$. In fact, for all $y \in Z$, there is $y(t) = \int_0^t y'(s)\,\mathrm{d}s$, then by Hölder inequality

$$\int_0^1 \left|y'(t)\right| \mathrm{d}t \leqslant \left[\int_0^1 \left|y'(t)\right|^p \mathrm{d}t\right]^{1/p},$$

let $(c/\mu(t)+1)^{1/p} \leqslant c_0$, we can obtain

$$\|y\| \leqslant \|y\|_Z \leqslant c_0\|y\|.$$

Define the following functional on Z:

$$J(y) = \frac{1}{p}\|y\|^p + \sum_{j=0}^{m} \int_{s_j}^{t_{j+1}} \lambda(t)\Phi_p\big(y(t)\big)y(t)\,\mathrm{d}t - \sum_{j=0}^{m} \int_{s_j}^{t_{j+1}} F_j\big(t, y(t)\big)\,\mathrm{d}t - \sum_{j=1}^{m} \int_{0}^{y(t_j)} I_j(s)\,\mathrm{d}s, \tag{2}$$

then for all $\omega \in Z$, there is the derivative

$$\begin{aligned}\langle J'(y), \omega\rangle &= \int_0^1 \mu(t)\Phi_p\big(y'(t)\big)\omega'(t)\,\mathrm{d}t - \sum_{j=0}^{m} \int_{s_j}^{t_{j+1}} \big[f_j\big(t, y(t)\big) - \lambda(t)\Phi_p\big(y(t)\big)\big]\omega(t)\,\mathrm{d}t \\ &\quad - \sum_{j=1}^{m} I_j\big(y(t_j)\big)\omega(t_j).\end{aligned} \tag{3}$$

Lemma 1. (*See* [9].) $Z \hookrightarrow C([0,1], \mathbb{R})$.

Lemma 2. *For each $y \in Z$, there is $\|y\|_\infty \leqslant \|y\|$.*

Proof. For each $y(t) \in Z$, there holds $y(0) = 0$ and

$$\big|y(t)\big| \leqslant \int_0^1 \big|y'(t)\big|\,\mathrm{d}t \leqslant \left[\int_0^1 \mu(t)\big|y'(t)\big|^p\,\mathrm{d}t\right]^{1/p},$$

thus $\|y\|_\infty \leqslant \|y\|$. The proof is complete. □

Definition 1. Let Z be a real Banach space, $J \in C^1(Z, \mathbb{R})$. If any sequence $\{y_j\} \subset Z$ with

$$J(y_j) \text{ being bounded and } \lim_{j\to\infty} J'(y_j) \to \theta$$

contains a convergent subsequence, then the functional J is called satisfying the Palais–Smale $(\mathrm{PS})_c$ condition.

Lemma 3. (*See* [5].) *Let Z be a real Banach space, and let $J \in C^1(Z, \mathbb{R})$ be a lower bounded functional, which satisfies the* $(\mathrm{PS})_c$ *condition. Then J have the minimum value in Z, that is, there exists $y_0 \in Z$ such that $J(y_0) = \inf_{y\in Z} J(y)$, then y_0 is a critical point of J.*

Lemma 4. (*See* [2].) *Let Z be a real Banach space, and let $J \in C^1(Z,\mathbb{R})$ satisfy the* $(\mathrm{PS})_c$ *condition. Assume $J(\theta) = 0$ and*

(P1) *there are constants $\rho, \alpha > 0$ such that $J|_{\partial B_\rho} \geqslant \alpha$, and*

(P2) *there is an $e \in Z \setminus B_\rho$ such that $J(e) \leqslant 0$.*

Then there exists a critical point y^* of J such that

$$c = J(y^*) = \inf_{h\in\Gamma} \max_{s\in[0,1]} J\big(\sigma(s)\big) \geqslant \delta,$$

where $\Gamma = \{\sigma \in C([0,1], Z)\colon\ \sigma(0) = y_0,\ \sigma(1) = y_1\}$.

Lemma 5. (*See* [11].) *Let E be a real Banach space, $J \in C^1(E,\mathbb{R})$ be even, and let the functional J satisfy the* $(\mathrm{PS})_c$ *condition. Assume that J satisfies the following*:

(i) $J(\theta) = 0$;

(ii) *There exist $\alpha, \tau > 0$ such that $J|_{\partial B_\tau \cap Z} \geqslant \alpha$;*

(iii) *If $E_1 \subset E$, where E_1 is a finite dimensional subspace, there exists $r = r(E_1)$ such that $J(y) \leqslant 0$ for all $y \in E_1$ with $\|y\| > r$.*

Then J possesses an unbounded sequence of critical values.

Lemma 6. *The weak solution $y \in Z$ is the classical solution of problem* (1).

Proof. Let y is a weak solution of (1), then $\langle J'(y), \omega\rangle = 0$, that is, for all $\omega \in Z$,

$$\int_0^1 \mu(t)\Phi_p\big(y'(t)\big)\omega'(t)\,\mathrm{d}t - \sum_{j=0}^{m} \int_{s_j}^{t_{j+1}} \big[f_j\big(t, y(t)\big) - \lambda(t)\Phi_p\big(y(t)\big)\big]\omega(t)\,\mathrm{d}t - \sum_{j=1}^{m} I_j\big(y(t_j)\big)\omega(t_j) = 0. \tag{4}$$

Without loss of generality, let $\omega \in C_0^\infty(s_j, t_{j+1}]$, and let $\omega(t) \equiv 0$ as $t \in [0, s_j] \cup (t_{j+1}, 1]$, $j = 0, 1, \dots, m$. Then substitute $\omega(t)$ into equation (4). For $j = 0, 1, \dots, m$, there is

$$\int_{s_j}^{t_{j+1}} \mu(t)\Phi_p\big(y'(t)\big)\omega'(t)\,\mathrm{d}t = \int_{s_j}^{t_{j+1}} \big[f_j\big(t, y(t)\big) - \lambda(t)\Phi_p\big(y(t)\big)\big]\omega(t)\,\mathrm{d}t,$$

thus

$$-\big(\mu(t)\Phi\big(y'(t)\big)\big)' = f_j\big(t, y(t)\big) - \lambda(t)\Phi_p\big(y(t)\big), \quad t \in (s_j, t_{j+1}]. \tag{5}$$

Then substitute (5) into (4), there is

$$\sum_{j=0}^{m} \mu\big(t_{j+1}^-\big)\Phi_p\big(y'(t_{j+1}^-)\big)\omega\big(t_{j+1}^-\big) - \sum_{j=0}^{m} \mu\big(s_j^+\big)\Phi_p\big(y'(s_j^+)\big)\omega(s_j^+) + \sum_{j=1}^{m} \int_{t_j}^{s_j} \mu(t)\Phi_p\big(y'(t)\big)\omega'(t)\,\mathrm{d}t - \sum_{j=1}^{m} I_j\big(y(t_j)\big)\omega(t_j) = 0 \quad \forall \omega(t) \in Z. \tag{6}$$

Assume $\omega \in C_0^\infty(t_j, s_j]$, and $\omega(t) \equiv 0$ as $t \in [0, t_j] \cup (s_j, 1]$, $j = 1, 2, \dots, m$. Then substituting $\omega(t)$ into equation (6), we get that $\mu(t)\Phi_p(y'(t))$ is a constant, that is, for $j = 1, 2, \dots, m$,

$$\mu(t)\Phi_p\big(y'(t)\big) = \mu(t_j^+)\Phi_p\big(y'\big(t_j^+\big)\big) = \mu(s_j^-)\Phi_p\big(y'\big(s_j^-\big)\big), \quad t \in (t_j, s_j]. \tag{7}$$

Substitute (7) into (6), there is

$$\begin{aligned}&\sum_{j=0}^{m} \mu\big(t_{j+1}^-\big)\Phi_p\big(y'\big(t_{j+1}^-\big)\big)\omega\big(t_{j+1}^-\big) - \sum_{j=0}^{m} \mu\big(s_j^+\big)\Phi_p\big(y'\big(s_j^+\big)\big)\omega\big(s_j^+\big)\\&\quad + \sum_{j=1}^{m} \mu\big(t_j^+\big)\Phi_p\big(y'\big(t_j^+\big)\big)\omega(s_j) \sum_{j=1}^{m} \mu\big(t_j^+\big)\Phi_p\big(y'\big(t_j^+\big)\big)\omega(t_j)\\&\quad - \sum_{j=1}^{m} I_j\big(y(t_j)\big)\omega(t_j) = 0 \quad \forall \omega(t) \in Z,\end{aligned}$$

that is,

$$\begin{aligned}&\sum_{j=1}^{m} \big[\mu\big(t_j^-\big)\Phi_p\big(y'\big(t_j^-\big)\big) - \mu\big(t_j^+\big)\Phi_p\big(y'\big(t_j^+\big)\big) - I_j\big(y(t_j)\big)\big]\omega(t_j)\\&\quad + \sum_{j=1}^{m} \big[\mu\big(t_j^+\big)\Phi_p\big(y'\big(t_j^+\big)\big) - \mu\big(s_j^+\big)\Phi_p\big(y'\big(s_j^+\big)\big)\big]\omega(s_j) = 0.\end{aligned}$$

Then for $j = 1, 2, \dots, m$,

$$\begin{gathered}\mu\big(t_j^-\big)\Phi_p\big(y'\big(t_j^-\big)\big) - \mu\big(t_j^+\big)\Phi_p\big(y'\big(t_j^+\big)\big) - I_j\big(y(t_j)\big) = 0,\\ \mu\big(t_j^+\big)\Phi_p\big(y'\big(t_j^+\big)\big) - \mu\big(s_j^+\big)\Phi_p\big(y'\big(s_j^+\big)\big) = 0,\end{gathered}$$

so there is

$$\mu\big(s_j^+\big)\Phi_p\big(y'\big(s_j^+\big)\big) = \mu\big(s_j^-\big)\Phi_p\big(y'\big(s_j^-\big)\big),$$

and for $j = 1, 2, \dots, m$, there is

$$I_j\big(y(t_j)\big) = \mu\big(t_j^-\big)\Phi_p\big(y'\big(t_j^-\big)\big) - \mu\big(t_j^+\big)\Phi_p\big(y'\big(t_j^+\big)\big) = -\Delta\big(\mu(t_j)\Phi_p\big(y'(t_j)\big)\big).$$

Then we can get y is a classical solution of problem (1), the proof is completed. □

3 Main results

Theorem 1. *Assume that $N_1 = \{0, 1, \dots, m\}$, $N_2 = \{1, 2, \dots, m\}$ and the following conditions hold*:

(A2) *There are constants $\alpha_j, \beta_j \in \mathbb{R}$ and $M \geqslant 0$ such that for $|y| \geqslant M$,*

(i) $0 < \alpha_j F_j(t, y) \leqslant y f_j(t, y)$, $t \in (s_j, t_{j+1}]$, $j \in N_1$;

(ii) $0 < \beta_j \int_0^{y(t)} I_j(s)\,\mathrm{d}s \leqslant y I_j(y)$, $(t, y) \in [0, 1] \times \mathbb{R}$, $j \in N_2$, *where* $F_j(t, y) = \int_0^y f_j(t, s)\,\mathrm{d}s$ *for* $(t, y) \in (s_j, t_{j+1}] \times \mathbb{R}$, $j \in N_1$, *and* $1 < p < \beta = \min\{\inf_{j \in N_1} \alpha_j, \inf_{j \in N_2} \beta_j\}$ *as* $\beta \in \mathbb{R}$.

(A3) *For* $p > 1$, *there is*

$$\lim_{y\to 0}\frac{F_j(t,y)}{|y|^p} = 0, \quad (t,y)\in(s_j,t_{j+1}]\times\mathbb{R},\ j\in N_1,$$

$$\lim_{y\to 0}\frac{\int_0^{y(t)} I_j(s)\,\mathrm{d}s}{|y|^p} = 0, \quad (t,y)\in[0,1]\times\mathbb{R},\ j\in N_2.$$

Then BVPs (1) *has at least two classical solutions.*

Proof. Firstly, let $\{y_k\}$ be a sequence in Z such that $\{J(y_k)\}$ is bounded and $\lim_{k\to\infty} J'(y_k) = 0$. Then there exists nonnegative constants D such that $|J(y_k)| \leqslant D$. By (3) there is

$$\begin{aligned}
&\sum_{j=0}^{m}\int_{s_j}^{t_{j+1}} f_j\big(t,y_k(t)\big)y_k(t)\,\mathrm{d}t + \sum_{j=1}^{m} I_j\big(y_k(t_j)\big)y_k(t_j)\\
&\quad = \|y_k\|^p + \sum_{j=0}^{m}\int_{s_j}^{t_{j+1}} \lambda(t)\Phi_p\big(y_k(t)\big)y_k(t)\,\mathrm{d}t - \big\langle J'(y_k), y_k\big\rangle,
\end{aligned}$$

then

$$\begin{aligned}
D &\geqslant J(y_k)\\
&= \frac{1}{p}\|y_k\|^p + \sum_{j=0}^{m}\int_{s_j}^{t_{j+1}} \lambda(t)\Phi_p\big(y_k(t)\big)y_k(t)\,\mathrm{d}t - \sum_{j=0}^{m}\int_{s_j}^{t_{j+1}} F_j\big(t,y_k(t)\big)\,\mathrm{d}t\\
&\quad - \sum_{j=1}^{m}\int_{0}^{y_k(t_j)} I_j(s)\,\mathrm{d}s\\
&\geqslant \frac{\|y_k\|^p}{p} + \sum_{j=0}^{m}\int_{s_j}^{t_{j+1}} \lambda(t)\Phi_p\big(y_k(t)\big)y_k(t)\,\mathrm{d}t - \sum_{j=0}^{m}\frac{1}{\alpha_j}\int_{s_j}^{t_{j+1}} f_j\big(t,y_k(t)\big)y_k(t)\,\mathrm{d}t\\
&\quad - \sum_{j=1}^{m}\frac{1}{\beta_j} I_j\big(y_k(t_j)\big)y_k(t_j)\\
&\geqslant \frac{\|y_k\|^p}{p} + \sum_{j=0}^{m}\int_{s_j}^{t_{j+1}} \lambda(t)\Phi_p\big(y_k(t)\big)y_k(t)\,\mathrm{d}t - \frac{1}{\beta}\Bigg[\sum_{j=0}^{m}\int_{s_j}^{t_{j+1}} f_j\big(t,y_k(t)\big)y_k(t)\,\mathrm{d}t\Bigg]\\
&\quad - \frac{1}{\beta}\Bigg[\sum_{j=1}^{m} I_j\big(y_k(t_j)\big)y_k(t_j)\Bigg]
\end{aligned}$$

$$\geqslant \left(\frac{1}{p}-\frac{1}{\beta}\right)\|y_k\|^p + \left(1-\frac{1}{\beta}\right)\sum_{j=0}^{m}\int\limits_{s_j}^{t_{j+1}} \lambda(t)\Phi_p\big(y_k(t)\big)y_k(t)\,\mathrm{d}t + \frac{1}{\beta}\big\langle J'(y_k), y_k\big\rangle$$
$$\geqslant \left(\frac{1}{p}-\frac{1}{\beta}\right)\|y_k\|^p + \frac{1}{\beta}\big\langle J'(y_k), y_k\big\rangle,$$

by (A2), $\beta > p > 1$, and $\langle J'(y_k), y_k\rangle \to 0$ as $k \to +\infty$, thus $\{y_k\}$ is bounded in Z. Then there exist a subsequence $\{y_i\}$ of sequence $\{y_k\}$ such that $y_i \rightharpoonup y$ in Z, so

$$\begin{aligned}
&\big\langle J'(y_i) - J'(y),\, y_i - y\big\rangle \\
&\quad = \int\limits_0^1 \mu(t)\big(\Phi_p\big(y_i'(t)\big) - \Phi_p\big(y'(t)\big)\big)\big(y_i'(t) - y'(t)\big)\,\mathrm{d}t \\
&\qquad + \sum_{j=0}^{m}\int\limits_{s_j}^{t_{j+1}} \lambda(t)\big(\Phi_p\big(y_i(t)\big) - \Phi_p\big(y(t)\big)\big)\big(y_i(t) - y(t)\big)\,\mathrm{d}t \\
&\qquad + \sum_{j=0}^{m}\int\limits_{s_j}^{t_{j+1}} \big[f_j\big(t, y_i(t)\big) - f_j\big(t, y(t)\big)\big]\big(y_i(t) - y(t)\big)\,\mathrm{d}t \\
&\qquad + \sum_{j=1}^{m}\big[I_j\big(y_i(t_j)\big) - I_j\big(y(t_j)\big)\big]\big(y_i(t_j) - y(t_j)\big),
\end{aligned}$$

by Lemma 1 there is $y_i \to y$ in $C[0,1]$, then

$$\sum_{j=0}^{m}\int\limits_{s_j}^{t_{j+1}} \big[f_j\big(t, y_i(t)\big) - f_j\big(t, y(t)\big)\big]\big(y_i(t) - y(t)\big)\,\mathrm{d}t \to 0,$$
$$\sum_{j=1}^{m}\big[I_j\big(y_i(t_j)\big) - J_j\big(y(t_j)\big)\big]\big(y_i(t_j) - y(t_j)\big) \to 0,$$

and by $J'(y_i) \to 0$, $y_i \rightharpoonup y$ in Z there is $\langle J'(y_i) - J'(y),\, y_i - y\rangle \to 0$ as $i \to +\infty$. By [12, Eq. (2.2)] there exist $c_p, d_p > 0$ and $v_1(t), v_2(t) \in Z$ such that

$$\begin{aligned}
&\int\limits_0^1 \mu(t)\big(\Phi_p\big(v_1'(t)\big) - \Phi_p\big(v_2'(t)\big)\big)\big(v_1'(t) - v_2'(t)\big)\,\mathrm{d}t \\
&\quad + \sum_{j=0}^{m}\int\limits_{s_j}^{t_{j+1}} \lambda(t)\big(\Phi_p\big(v_1(t)\big) - \Phi_p\big(v_2(t)\big)\big)\big(v_1(t) - v_2(t)\big)\,\mathrm{d}t \\
&\geqslant \begin{cases} c_p\int_0^1 \mu(t)|v_1'(t) - v_2'(t)|^p\,\mathrm{d}t + \sum_{j=0}^m \int_{s_j}^{t_{j+1}} \lambda(t)|v_1(t) - v_2(t)|^p\,\mathrm{d}t, & p \geqslant 2 \\ d_p\int_0^1 \frac{\mu(t)|v_1'(t)-v_2'(t)|^2}{(|v_1'(t)|+|v_2'(t)|)^{2-p}}\,\mathrm{d}t + \sum_{j=0}^m \int_{s_j}^{t_{j+1}} \frac{\lambda(t)|v_1(t)-v_2(t)|^2}{(|v_1(t)|+|v_2(t)|)^{2-p}}\,\mathrm{d}t, & 1 < p < 2, \end{cases}
\end{aligned}$$

and by [14, Lemma 2.7], for $p > 1$, there is $\|y_i - y\| \to 0$ as $i \to +\infty$. Therefore, $y_i \to y$ in Z, that is, the sequence $\{y_k\} \subset Z$ has a convergent subsequence, then J satisfies the $(\mathrm{PS})_c$ condition.

Secondly, by (2) and assumption (A2) we get $J(\theta) = 0$. For any $\tau > 0$, let $B_\tau = \{y \in Z\colon \|y\| < \tau\}$. By assumption (A3), given $\sum_{j=0}^{m} \varepsilon_{1j} + \sum_{j=1}^{m} \varepsilon_{2j} = 1/(2p) > 0$, where $\varepsilon_{1j} > 0$ ($j = 0, 1, \dots, m$), $\varepsilon_{2j} > 0$ ($j = 1, 2, \dots, m$), there exists $\delta > 0$ such that $F_j(t, y) \leqslant \varepsilon_{1j}|y|^p$, $j \in N_1$, and $\int_0^y I_j(s)\,\mathrm{d}s \leqslant \varepsilon_{2j}|y|^p$, $j \in N_2$, for $\|y\| \leqslant \delta$. Then for all $y \in \partial B_\delta$, $\|y\| = \delta$, there is

$$
\begin{aligned}
J(y) &= \frac{1}{p}\|y\|^p + \sum_{j=0}^{m} \int_{s_j}^{t_{j+1}} \lambda(t)\Phi_p\big(y(t)\big)y(t)\,\mathrm{d}t - \sum_{j=0}^{m} \int_{s_j}^{t_{j+1}} F_j\big(t, y(t)\big)\,\mathrm{d}t - \sum_{j=1}^{m} \int_0^{y(t_j)} I_j(s)\,\mathrm{d}s \\
&\geqslant \frac{1}{p}\|y\|^p - \sum_{j=0}^{m} \int_{s_j}^{t_{j+1}} \varepsilon_{1j}|y|^p\,\mathrm{d}t - \sum_{j=1}^{m} \varepsilon_{2j}|y|^p\,\mathrm{d}t \\
&\geqslant \frac{1}{p}\|y\|^p - \left(\sum_{j=0}^{m} \varepsilon_{1j} + \sum_{j=1}^{m} \varepsilon_{2j}\right)\|y\|_\infty^p \geqslant \frac{\|y\|^p}{p} - \left(\sum_{j=0}^{m} \varepsilon_{1j} + \sum_{j=1}^{m} \varepsilon_{2j}\right)\|y\|^p \\
&= \left[\frac{1}{p} - \left(\sum_{j=0}^{m} \varepsilon_{1j} + \sum_{j=1}^{m} \varepsilon_{2j}\right)\right]\|y\|^p = \frac{1}{2p}\|y\|^p.
\end{aligned}
$$

Then taking into Lemma 5, setting $\tau = \delta$, $\alpha = \delta^p/(2p) > 0$, one has that (P1) holds. Moreover, J is a lower bounded functional, which satisfies the $(\mathrm{PS})_c$ condition in $\bar{B}_\delta$, and $\bar{B}_\delta$ is a real Banach space by $\bar{B}_\delta \subset Z$. So by Lemma 3 there exists $\hat{y} \in \bar{B}_\delta$ such that $J(\hat{y}) = \inf_{y \in \bar{B}_\delta} J(y)$, then $\hat{y}$ is a critical point of J. Moreover, $J(\hat{y}) \leqslant J(\theta) = 0 < \alpha$ as $\theta \in \bar{B}_\delta$.

Thirdly, by assumption (A2), for all $y \in Z \setminus B_\delta$, that is, $\|y\| \geqslant \delta$, one has

$$
\begin{aligned}
&F_j(t, y) \geqslant a_j|y|^{\alpha_j} - d_j, \quad y \in (s_j, t_{j+1}],\ j \in N_1; \\
&\int_0^y I_j(s)\,\mathrm{d}s \geqslant b_j|y|^{\beta_j} - c_j, \quad t \in [0, 1],\ j \in N_2,
\end{aligned}
$$

where a_j, b_j, d_j, c_j are positive constants. Then for $r \in \mathbb{R} \setminus \{0\}$, one has

$$
\begin{aligned}
J(ry) = {} & \frac{|r|^p}{p}\|y\|^p + \sum_{j=0}^{m} \int_{s_j}^{t_{j+1}} \lambda(t)\Phi_p\big(ry(t)\big)ry(t)\,\mathrm{d}t - \sum_{j=0}^{m} \int_{s_j}^{t_{j+1}} F_j\big(t, ry(t)\big)\,\mathrm{d}t \\
& - \sum_{j=1}^{m} \int_0^{ry(t_j)} I_j(s)\,\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant \frac{|r|^p}{p}\|y\|^p + c\sum_{j=0}^{m}\int_{s_j}^{t_{j+1}} |r|^p\|y\|^p \,\mathrm{d}t - \sum_{j=0}^{m}\int_{s_j}^{t_{j+1}} \bigl(a_j\bigl|ry(t)\bigr|^{\alpha_j} - d_j\bigr)\,\mathrm{d}t \\
&\quad - \sum_{j=1}^{m}\bigl(b_j\bigl|ry(t_j)\bigr|^{\beta_j} - c_j\bigr) \\
&\leqslant \left(\frac{1}{p}+c\right)|r|^p\|y\|^p - |r|^\beta\left[\sum_{j=0}^{m} a_j\int_{s_j}^{t_{j+1}}\bigl|y(t)\bigr|^{\alpha_j}\,\mathrm{d}t + \sum_{j=1}^{m} b_j\bigl|ry(t_j)\bigr|^{\beta_j}\right] \\
&\quad + \sum_{j=0}^{m} d_j + \sum_{j=1}^{m} c_j,
\end{aligned}
$$

where $\beta > p$, then $J(ry) \to -\infty$ as $r \to \infty$, that is, when y_1 is sufficiently large (that is, away from the origin), one has $J(y_1) < 0$, thus (P2) holds. Then by Lemma 4 there is a critical point y^* such that $J(y^*) \geqslant \delta > 0 \geqslant J(\hat{y})$. Therefore, $\hat{y}$ and y^* are two different critical points of J, that is, BVPs (1) has at least two classical solutions. □

Proposition 1. *Under the assumptions of Theorem* 1*, if* $F_j(t, y)$ $(j = 0, 1, \dots, m)$ *and* $I_j(t)$ $(j = 1, 2, \dots, m)$ *are odd functions with respect to* y *and* p *is an odd number, then BVPs* (1) *has infinitely many classical solutions.*

Proof. J is a even functional by assumptions, and let $Z_1 \subset Z$ be a finite dimensional subspace. By Theorem 1 it is easy to prove (i) and (ii) in Lemma 5 and choose an $r_0 = r_0(Z_1)$ such that $J(y) < 0$ for all $y \in Z_1 \setminus B_{r_0}$ with $\|y\| > r_0$. By Lemma 5 BVPs (1) possesses infinitely many classical solutions. □

4 Examples

Example 1. Consider the following boundary value problem:

$$
\begin{aligned}
&-y''(t) + (1+t)y(t) = y^3, \quad t \in (s_j, t_{j+1}],\ j = 0, 1, \dots, m, \\
&-\Delta\bigl((1+t_j)y'(t_j)\bigr) = y^5, \quad j = 1, 2, \dots, m, \\
&(1+t)y'(t) = \bigl(1+t_j^+\bigr)y'\bigl(t_j^+\bigr), \quad t \in (t_j, s_j],\ j = 1, 2, \dots, m, \\
&\bigl(1+s_j^+\bigr)y'\bigl(s_j^+\bigr) = \bigl(1+s_j^-\bigr)y'\bigl(s_j^-\bigr), \quad j = 1, 2, \dots, m, \\
&y(0) = 0, \quad y(1) = 2y(0.5).
\end{aligned} \tag{8}
$$

Compare with (1), $p = 2$, $\zeta = 2$, $\eta = 0.5$, $\mu(t) = \lambda(t) = (1+t) \geqslant 1$ for $0 \leqslant t \leqslant 1$, then (A1) holds. When $\alpha_j = 3$ $(j = 0, 1, \dots, m)$, $\beta_j = 4$ $(j = 1, 2, \dots, m)$, we can get

$$
\begin{aligned}
&0 < \frac{3}{4}y^4 = \alpha_j F_j(t, y) \leqslant y f_j(t, y) = y^4, \quad j = 0, 1, \dots, m, \\
&0 < \frac{2}{3}y^6 = \beta_j \int_0^y I_j(s)\,\mathrm{d}s \leqslant y I_j(y) = y^6, \quad j = 1, 2, \dots, m,
\end{aligned}
$$

then (A2) holds, and

$$\lim_{y\to 0}\frac{F_j(t,y)}{|y|^p}=\lim_{y\to 0}\frac{y^4}{4|y|^2}=0,\quad j=0,1,\dots,m,$$
$$\lim_{y\to 0}\frac{\int_0^y I_j(s)\,\mathrm{d}s}{|y|^p}=\frac{y^6}{6|y|^2}=0,\quad j=1,2,\dots,m.$$

Then (A3) holds. Thus, by Theorem 1, problem (8) has at least two classical solutions.

Example 2. Consider the following boundary value problem:

$$\begin{aligned}
&-\big((1+t)\Phi_{1.1}\big(y'(t)\big)\big)'+(1+t)\Phi_p\big(y(t)\big)=y^{0.6},\quad t\in(s_j,t_{j+1}],\\
&\quad j=0,1,\dots,m,\\
&-\Delta\big((1+t_j)\Phi_{1.1}\big(y'(t_j)\big)\big)=y^{1/3},\quad j=1,2,\dots,m,\\
&(1+t)\Phi_{1.1}\big(y'(t)\big)=\big(1+t_j^+\big)\Phi_{1.1}\big(y'\big(t_j^+\big)\big),\quad t\in(t_j,s_j],\ j=1,2,\dots,m,\\
&\big(1+s_j^+\big)\Phi_{1.1}\big(y'\big(s_j^+\big)\big)=\big(1+s_j^-\big)\Phi_{1.1}\big(y'\big(s_j^-\big)\big),\quad j=1,2,\dots,m,\\
&y(0)=0,\qquad y(1)=2y(0.5).
\end{aligned}\tag{9}$$

Compare with (1), $p=1.1$, $\zeta=2$, $\eta=0.5$, $\mu(t)=\lambda(t)=(1+t)\geqslant 1$ for $0\leqslant t\leqslant 1$, then (A1) holds. When $\alpha_j=1.2$ ($j=0,1,\dots,m$), $\beta_j=1.25$ ($j=1,2,\dots,m$), we can get

$$0<\frac{3}{4}y^{1.6}=\alpha_j F_j(t,y)\leqslant yf_j(t,y)=y^{1.6},\quad j=0,1,\dots,m,$$
$$0<\frac{15}{16}y^{4/3}=\beta_j\int_0^y I_j(s)\,\mathrm{d}s\leqslant yI_j(y)=y^{4/3},\quad j=1,2,\dots,m,$$

then (A2) holds, and

$$\lim_{y\to 0}\frac{F_j(t,y)}{|y|^p}=\lim_{y\to 0}\frac{5y^{1.6}}{8|y|^{1.1}}=0,\quad j=0,1,\dots,m;$$
$$\lim_{y\to 0}\frac{\int_0^y I_j(s)\,\mathrm{d}s}{|y|^p}=\frac{3y^{4/3}}{4|y|^{1.1}}=0,\quad j=1,2,\dots,m,$$

then (A3) holds. Thus, by Theorem 1, problem (9) has at least two classical solutions.

5 Conclusion

The interesting points of this paper are the following:

(i) By using the variational method we study a three-point boundary value problem instead of the local boundary value problem such as [15];
(ii) We study the second-order p-Laplace differential equations with instantaneous and noninstantaneous impulses;

(iii) The nonlinear term f_j and the impulsive term I_j in this paper do not have to meet the sublinear growth conditions in [15];
(iv) We choose an appropriate space instead of functional to contain the boundary conditions.

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