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New discussion concerning to optimal control for semilinear population dynamics system in Hilbert spaces

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Abstract. The objective of our paper is to investigate the optimal control of semilinear population dynamics system with diffusion using semigroup theory. The semilinear population dynamical model with the nonlocal birth process is transformed into a standard abstract semilinear control system by identifying the state, control, and the corresponding function spaces. The state and control spaces are assumed to be Hilbert spaces. The semigroup theory is developed from the properties of the population operators and Laplacian operators. Then the optimal control results of the system are obtained using the C_0-semigroup approach, fixed point theorem, and some other simple conditions on the nonlinear term as well as on operators involved in the model.

Keywords: population dynamics, diffusion, optimal control, Gronwall's inequality.

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1 Introduction

Let us consider ζ, a bounded domain in $\mathbb{R}^n$ ($n \in \{1,2,3\}$) along the smooth boundary region $\partial\zeta$. We considered that a biological population is independent to move in the ζ environment. Let $p(t,g,r)$ be the population dissemination of human beings of age $g \geqslant 0$ at location $r \in \zeta$ for time $t \geqslant 0$. We considered that the population flow is $k\nabla p(t,g,r)$, where ∇ is the gradient vector with respect to the spatial variable r, and $k > 0$ is a constant. The life expectancy of individual is denoted by g_+, $\sigma(t,g,r)$ is known as the natural mortality rate, and $\alpha(t,g,r)$ is known as the natural fertility rate corresponding to individual of age g at location $r \in \zeta$ for the time $t \geqslant 0$. In this paper, we have considered that the population distribution $p(t,g,r)$ is differentiable in variables t and g. Moreover, the natural fertility and mortality rate depends only on age.

Let $X = L_2((0,g_+) \times \zeta)$ be a Hilbert space, and let $Y = L_2([0,b];X)$ be a function space. The symbol $\|\cdot\|$ is norm in X. We have considered the following semilinear population dynamics model with nonlocal birth process:

$$
\begin{aligned}
&\frac{\partial p}{\partial t} + \frac{\partial p}{\partial g} + \sigma(g)p - k\Delta p = B\chi_w(r)u(t) + F\bigl(t,p(t)\bigr), \\
&\qquad (t,g,r) \in (0,b) \times (0,g_+) \times \zeta, \\
&p(t,g,r) = 0, \quad (t,g,r) \in (0,b) \times (0,g_+) \times \partial\zeta, \\
&p(t,0,r) = \int_0^{g_+} \alpha(g)p(t,g,r)\,\mathrm{d}g, \quad (t,r) \in (0,b) \times \zeta, \\
&p(0) = p_0, \quad p_0 \in X,
\end{aligned}
\tag{1}
$$

where Δ is Laplacian operator, u is the control function in Y, which is a supply or removal of population, and χ_w is the characteristic function of w, $B : Y \to Z$ is a bounded linear operator, the nonlinear function $F : (0,b) \times X \to X$ represents an infusion of population due to some natural or unnatural reasons, and p_0 is the initial population distribution. In some models, $\partial/\partial Fp = 0$ is assumed instead of zero population at the boundary region, where $\partial/\partial F$ denotes the exterior normal derivatives on $\partial\zeta$, and $\partial/\partial Fp = 0$ means that the population inflow or outflow of the region ζ is zero.

In recent years, the existence and uniqueness of mild solution, optimal control, time-optimal control approximate control, and exact control for fractional-order, integer-order, integro –differential system, neutral system, etc. have been studied by many researcher's articles [1–3, 7–9, 11–24, 26–30, 32–35]. In [6], the authors obtained the existence and optimal control results using Krasnoselskii's fixed point theorem and minimizing sequence concept for the second-order stochastic differential equations having mixed fractional Brownian motion. In [22], the authors discussed the existing result for the mild solution and optimal control for fractional-order $\alpha \in (1,2]$ semilinear control system by using α-order sine and cosine family theory, Banach fixed point theorem, and certain assumptions on nonlinearity.

Motivations and contributions

In [1, 11, 19], the authors discussed the optimal control for the population of dynamics by different techniques. Taking the consideration of ideas from the above literature review and motivated by the fact, the optimal control of semilinear population dynamics system (1) is studied in the semigroup framework approach.

- We have converted the population dynamic system into a abstract control problem.
- The mild solution is defined in terms of integral equation.
- We have discussed the existence and uniqueness of the assumed system by employing the contraction principle.
- Optimal control results are obtained using Cauchy–Schwarz's inequality and Gronwall's lemma.

The structure of this article is as follows. Section 1 discusses some literature related to population dynamics and optimal control theory. Some new notations, important facts, lemmas, vital definitions, and theoretical results are recalled and problem formulation is done in Section 2. In Section 3, we assumed some conditions and proved the existence and uniqueness of mild solution for system (1). Section 4 deals with the optimal control for the population dynamics system with diffusion. Finally, in the last Section 5, we discussed the time optimal control.

2 Preliminaries

Definition 1. (See [31].) A one-parameter family $\{C(\tau),\ \tau \in \mathbb{R}\}$ of bounded linear operators mapping the Hilbert space V into itself is said to be a strongly continuous cosine family if and only if

(i) $C(0) = I$;
(ii) $C(s+\tau) + C(s-\tau) = 2C(s)C(\tau)$;
(iii) $C(\tau)x$ is continuous in τ on $\mathbb{R}$ for each fixed $x \in V$;
(iv) $\|C(\tau)\| \leqslant M$ for all $\tau \in [0, T]$.

If $\{C(\tau),\ \tau \in \mathbb{R}\}$ is a strongly continuous cosine family in V, then $S(\tau)$, $\tau \in \mathbb{R}$, is the one-parameter family of operators in V defined by

$$S(\tau)w = \int_0^\tau C(s)w\,\mathrm{d}s, \quad w \in V,\ \tau \in \mathbb{R}.$$

The infinitesimal generator of a strongly continuous cosine family $\{C(\tau),\ \tau \in \mathbb{R}\}$ is the operator $A : D(A) \subseteq V \to V$ defined by

$$Aw = \frac{\mathrm{d}^2}{\mathrm{d}t^2} C(0)w,$$

where $D(A)$ is defined as $D(A) = \{w \in V\colon\ C(\tau)w$ is a twice continuously differentiable function of $\tau\}$.

Definition 2. The characteristic function of a subset ω of a set Ω is a function $\chi_\omega : \Omega \to \{0,1\}$ defined as

$$\chi_\omega(x) := \begin{cases} 1, & x \in \omega, \\ 0, & x \notin \omega. \end{cases}$$

We are focusing here on the subsequent two linear population dynamics models with diffusion from the literature.

In [5], the behavior of the semigroup of the following linear population dynamics model with Dirichlet boundary condition type was studied:

$$\begin{aligned}
&\frac{\partial p}{\partial t} + \frac{\partial p}{\partial g} + \sigma(g)p - k\Delta p = 0, \quad (t,g,r) \in (0,b) \times (0,g_+) \times \zeta, \\
&p(t,g,r) = 0, \quad (t,g,r) \in (0,b) \times (0,g_+) \times \partial\zeta, \\
&p(t,0,r) = \int_0^{g_+} \alpha(g)p(t,g,r)\,\mathrm{d}g, \quad (t,r) \in (0,b) \times \zeta, \\
&p(0,g,r) = p_0(g,r), \quad (g,r) \in (0,g_+) \times \zeta.
\end{aligned}$$

Suppose that $A : X \to X$ is defined as

$$Ay = \frac{\partial}{\partial g} + \sigma(g)p - k\Delta p$$

with domain of A denoted by

$$D(A) = \left\{ \psi \in X\colon\ A\psi \in X,\ \psi|_{\partial\zeta} = 0,\ \psi(0,r) = \int_0^{g_+} \alpha(g)\psi(t,g,r)\,\mathrm{d}g \right\},$$

and k is the diffusion constant.

In this paper, it has been proven that A is the infinitesimal generator of a strongly continuous semigroup (C_0-semigroup) $\mathcal{S}(t)$, $t \geqslant 0$. This enables us to reformulate system (2) in the following abstract form:

$$\begin{aligned}
\frac{\partial p(t,g,r)}{\partial t} &= Ap(t,g,r), \quad (t,g,r) \in (0,b) \times (0,g_+) \times \zeta, \\
p(0,g,r) &= p_0(g,r), \quad (g,r) \in (0,g_+) \times \zeta.
\end{aligned}$$

Note that by Theorem 2 of [5] the C_0-semigroup $\mathcal{S}(t)$, $t \geqslant 0$, generated by A is compact when $t \geqslant g_+$, otherwise it is not compact because the C_0-semigroup generated by the population operator

$$A_1 = -\frac{\partial}{\partial g}p - \sigma(g)p$$

is not compact when $t < g_+$.

Similarly, in [10] the formulation of C_0-semigroup for the following linear population dynamics system with diffusion in Neuman boundary condition type is studied:

$$\begin{aligned}
&\frac{\partial p}{\partial t} + \frac{\partial p}{\partial g} + \sigma(g)p - k(g)\Delta_r p = 0, \quad t > 0,\ g > 0,\ r \in \zeta, \\
&\frac{\partial}{\partial v} p(t,g,r) = 0, \quad t > 0,\ g > 0,\ r \in \partial\zeta \\
&p(t,0,r) = \int_0^{g_+} \alpha(g)p(t,g,r)\,\mathrm{d}g, \quad (t,r) \in (0,b) \times \zeta, \\
&p(0,g,r) = p_0(g,r), \quad (g,r) \in (0,g_+) \times \zeta,\ p_0 \in X.
\end{aligned} \tag{2}$$

We now determine $A : X \to X$ as

$$Ap = -\frac{\partial}{\partial g}p - \sigma(g)p + k(g)\Delta_x p.$$

In that paper, it is shown that the operator A is the infinitesimal generator of a C_0-semigroup $\mathcal{S}(t)$, $t \geqslant 0$, using m-dissipativity of the diffusion operator Δ and the population operator without diffusion. That means the operator A is characterized in the following way.

Let the operator $A_1 : D(A_1) \subseteq X \to X$ be defined by

$$A_1\psi(g,r) = -\frac{\partial\psi(g,r)}{\partial g} - \sigma(g)\psi(g,r), \quad \psi \in D(A_1),$$

where

$$\begin{aligned}
D(A_1) = \biggl\{ &\psi \in X\colon\ \psi(\cdot,r) \text{ follows the condition of local continuity on } (0,\infty), \\
&\psi(0,r) = \int_0^{g_+} \alpha(g)\psi(g,r)\,\mathrm{d}g,\ \text{a.e. } r \in \zeta,\ \psi_g + \sigma\psi \in X \biggr\}.
\end{aligned}$$

Then it is shown that A_1 is linear and m-dissipative.

Similarly, let the operator $A_2 : D(A_2) \subseteq X \to X$ be defined by

$$A_2\psi = \Delta\psi(g,r), \quad \psi \in D(A_2),$$

where

$$D(A_2) = \biggl\{ \psi \in X\colon\ \psi(g,\cdot) \in H^2(\zeta,\mathfrak{R}),\ \frac{\partial}{\partial v}\psi\bigg|_{\partial\zeta} = 0,\ \text{a.e. } \Delta\psi \in X \biggr\}.$$

Then it is shown that A_2 is linear and m-dissipative operator. From the above two operators $A : D(A) \subseteq X \to X$ is presented as

$$A\psi = A_1\psi + A_2\psi, \quad \psi \in D(A),$$

where $D(A) = D(A_1) \cap D(A_2)$.

Therefore, A is linear and m-dissipative since a sum of two m-dissipative operators is m-dissipative. Hence, A is an infinitesimal generator of the C_0-semigroup $\mathcal{S}(t)$, $t \geqslant 0$ [25].

Now, we reformulate the abstract form of system (2) as

$$\begin{aligned} \frac{\partial p(t,g,r)}{\partial t} &= Ap(t,g,r), \quad (t,g,r) \in (0,b) \times (0,g_+) \times \zeta, \\ p(0,g,r) &= p_0(g,r), \quad (g,r) \in (0,g_+) \times \zeta. \end{aligned}$$

This article deals mainly the optimal control of (1) using C_0-semigroup theory.

Now, we consider system (1) as an extension of model (2) to semilinear population dynamics control system with finite time interval and fixed life expectancy $g_+ > 0$. Therefore, the abstract form of the population dynamics system (1) can be rewritten as follows:

$$\begin{aligned} \frac{\partial p(t)}{\partial t} &= Ap(t) + B\chi_w(r)u(t) + F\bigl(t, p(t)\bigr), \quad t \in (0,b], \\ p(0) &= p_0, \quad p_0 \in X,\ p_0 \geqslant 0, \end{aligned} \tag{3}$$

where

$$\begin{aligned} D(A) = \biggl\{ &\psi \in X\colon \psi(\cdot,r) \text{ follows the condition of local continuity on } [0,a_+), \\ &\psi(g,\cdot) \in H^2(\zeta,\mathfrak{R}),\ \psi(0)|_{\partial\zeta} = 0,\ \text{a.e. } a > 0, \\ &A\psi = \Delta\psi + \psi_a + \sigma(a) \in X,\ \psi(0,r) = \int_0^{a_+} \alpha(a)\psi(a,r)\,\mathrm{d}g,\ \text{a.e. } r \in \zeta \biggr\}. \end{aligned}$$

Assume that the integral cost function is presented as

$$J(p,u) := \int_0^b \mathcal{T}\bigl(t, p(t), u(t)\bigr)\,\mathrm{d}t$$

subject to

$$\begin{aligned} \frac{\partial p(t)}{\partial t} &= Ap(t) + B\chi_w(r)u(t) + F\bigl(t, p(t)\bigr), \quad t \in (0,b], \\ p(0) &= p_0, \quad p_0 \in X,\ p_0 \geqslant 0. \end{aligned}$$

$U_{\mathrm{ad}} := \{\upsilon(\cdot)\colon [0,b] \to X \text{ is measurable, } u(t) \in Y \text{ a.e.}\}$ is defined as admissible set.

Also, it is clear that U_{ad} is nonvoid. $U_{\mathrm{ad}} \subset L_2([0,b];X)$ is closed, convex, and bounded. Also, for every $u \in U_{\mathrm{ad}}$, $Bu \in L_2([0,b];Y)$.

Let p represent the mild solution, and let $u \in U_{\mathrm{ad}}$ represent the control, then $\mathcal{A}_{\mathrm{ad}}$ is a set of state-control pairs (p,u), which are admissible. Hence, the optimal control problem is given by

Determine a pair $(p^0, u^0) \in \mathcal{A}_{\mathrm{ad}}$ such that

$$J\bigl(p^0, u^0\bigr) := \inf\bigl\{J(p,u)\colon (p,u) \in \mathcal{A}_{\mathrm{ad}} = \delta\bigr\}.$$

The linear system corresponding to (3) is given by

$$\begin{aligned}\frac{\partial p(t)}{\partial t} &= Ap(t) + B\chi_w(r)u(t), \quad t \in (0, b],\\ p(0) &= p_0, \quad p_0 \in X,\ p_0 \geqslant 0.\end{aligned}$$

Then we will make use of the C_0-semigroup $\mathcal{S}(t)$, $t \geqslant 0$, operator for the mild solution of (3) defined below:

$$p(t) = \mathcal{S}(t)p_0 + \int_0^t \mathcal{S}(t-s)\bigl[B\chi_w u(s) + F\bigl(s, p(s)\bigr)\bigr]\,\mathrm{d}s, \quad 0 \leqslant t \leqslant b. \tag{4}$$

Suppose $p(t) \in X$ is mild solution of (3), then we define the population distribution $p(t, g, r)$ in the following manner:

$$\begin{aligned}p(t, g, r) &= \mathcal{S}(t)p_0(g, r)\\ &\quad + \int_0^t \mathcal{S}(t-s)\bigl[B\chi_w u(s, g, r) + F\bigl(s, p(s, g, r)\bigr)\bigr]\,\mathrm{d}s, \quad 0 \leqslant t \leqslant b.\end{aligned}$$

Here we give the verification of the natural fertility rate of the population in relation to the mild solution of (1):

$$\begin{aligned}p(t, 0, r) &= \int_0^{g_+} \alpha(g)p(t, g, r)\,\mathrm{d}g\\ &= \int_0^{g_+} \alpha(g)\left\{\mathcal{S}(t)p_0 + \int_0^t \mathcal{S}(t-s)\bigl[B\chi_w u(s) + F\bigl(s, p(s)\bigr)\bigr]\,\mathrm{d}s\right\}\mathrm{d}g\\ &= \int_0^{g_+} \alpha(g)\mathcal{S}(t)p_0\,\mathrm{d}g + \int_0^{g_+} \alpha(g)\int_0^t \mathcal{S}(t-s)B\chi_w u(s)\,\mathrm{d}s\,\mathrm{d}g\\ &\quad + \int_0^{g_+} \alpha(g)\int_0^t \mathcal{S}(t-s)F\bigl(s, p(s)\bigr)\,\mathrm{d}s\,\mathrm{d}g.\end{aligned} \tag{5}$$

On the other hand, when we put $g = 0$ in the integral equation (5), we have

$$\begin{aligned}p(t, 0, r) &= \mathcal{S}(t)p_0(0, r)\\ &\quad + \int_0^t \mathcal{S}(t-s)\bigl[B\chi_w u(s, 0, r) + F\bigl(s, p(s, 0, r)\bigr)\bigr]\,\mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
&= \mathcal{S}(t)\int_0^{g_+}\alpha(g)p_0(g,r)\,\mathrm{d}g + \int_0^t \mathcal{S}(t-s)B\chi_w u(s,0,r)\,\mathrm{d}s \\
&\quad + \int_0^t \mathcal{S}(t-s)F\left(s, \int_0^{g_+}\alpha(g)p(s,g,r)\,\mathrm{d}g\right)\mathrm{d}s. \qquad (6)
\end{aligned}
$$

From (5) and (6) we establish the following conditions:

$$
\int_0^t \mathcal{S}(t-s)B\chi_w u(s,0,r)\,\mathrm{d}s = \int_0^{g_+}\alpha(g)\int_0^t \mathcal{S}(t-s)B\chi_w u(s)\,\mathrm{d}s\,\mathrm{d}g,
$$

$$
\int_0^t \mathcal{S}(t-s)F\left(s, \int_0^{g_+}\alpha(g)p(s,g,r)\,\mathrm{d}g\right)\mathrm{d}s = \int_0^{g_+}\alpha(g)\int_0^t \mathcal{S}(t-s)F\bigl(s,p(s)\bigr)\,\mathrm{d}s\,\mathrm{d}g,
$$

which can be simplified as

$$
B\chi_w u(s,0,r) = \int_0^{g_+}\alpha(g)B\chi_w u(s)\,\mathrm{d}g, \quad 0 \leqslant t \leqslant b,
$$

$$
F\left(s, \int_0^{g_+}\alpha(g)p(s,g,r)\,\mathrm{d}g\right) = \int_0^{g_+}\alpha(g)F\bigl(s,p(s)\bigr)\,\mathrm{d}g.
$$

3 Basic assumptions, existence and uniqueness results

In this article the following assumptions are important to obtain existence and uniqueness results.

- (A1) (i) $\alpha \in L^\infty(0,g_+)$, $\alpha(g) \geqslant 0$, a.e. $g \in (0,g_+)$. There exists $g_0, g_1 \in (0,g_+)$, $g_0 \leqslant g_1$, such that $\alpha(g) = 0$, a.e. $g \in (0,g_+) \cup (g_1,g_+)$, and $\alpha(g) > 0$ a.e. in (g_0,g_1).
  - (ii) $\sigma \in C([0,g_+])$, $\sigma(g) \geqslant 0$, a.e. $\int_0^{g_+}\sigma(g)\,\mathrm{d}g = \infty$ and $\int_0^{g}\sigma(g)\,\mathrm{d}g < \infty$.
  - (iii) $p_0(g,r) \geqslant 0$ a.e. in $(0,g_+) \times \zeta$.
  - (iv) $F(t,p) : X \to [0,\infty)$.
- (A2) The C_0-semigroup $\mathcal{S}(t)$ is uniformly bounded, i.e., there exists $M > 0$ such that $\|\mathcal{S}(t)\| \leqslant M$ for all $t \geqslant 0$.
- (A3) For each $t \geqslant 0$, F follows the Lipschitz continuity with respect to the population distribution p, i.e., $\|F(t,p_1) - F(t,p_2)\| \leqslant l\|p_1 - p_2\|$ for some constant number l.
- (A4) $t \to F(t,p)$ for all t is measurable.
- (A5) There exist $k > 0$ such that $\|F(t,p)\| \leqslant k\{1 + \|p\|\}$.

For the biological significance of assumption (A1), one can refer to [1, 11, 19].

The interpretation of the integral condition in (A1)(ii) is that each individual dies before age g_+. Hence, the approximate controllability could not hold at age g_+.

Lemma 1. (*See* [28].) *According to the control* u*, the mild solution of* (3) *is assumed to be* (4) *on* $[0, b]$*. Then there exists* $M_{ph} = M_{ph}(u) > 0$ *such that*

$$\|p(t)\| \leqslant M_{ph}, \quad t \in [0, b].$$

Proof. Let $p(t)$ described by (4) be the mild solution (3). Provided that $t \in [0, b]$, by applying assumption (A5) and Cauchy–Schwartz inequality we have

$$p(t) = \mathcal{S}(t)p_0 + \int_0^t \mathcal{S}(t-s)\big[B\chi_w u(s) + F\big(s, p(s)\big)\big]\,\mathrm{d}s.$$

Then

$$\begin{aligned}
\|p(t)\| &\leqslant \|\mathcal{S}(t)\|\|p_0\| + \int_0^t \|\mathcal{S}(t-s)\|\big\|\big[B\chi_w u(s) + F\big(s, p(s)\big)\big]\big\|\,\mathrm{d}s \\
&\leqslant M\|p_0\| + M\|B\|\|u\| + Mk\int_0^t \big[1 + \|p(s)\|\big]\,\mathrm{d}s \\
&\leqslant M\|p_0\| + M\cdot M_B\|u\| + Mkb + Mk\int_0^t \|p(s)\|\,\mathrm{d}s \\
&\leqslant M' + Mk\int_0^t \|p(s)\|\,\mathrm{d}s.
\end{aligned}$$

By referring Gronwall's inequality we have $\|p(t)\| \leqslant M_{ph}$ for all $t \in [0, b]$. □

Theorem 1. *Assume that* (A1)–(A5) *are fulfilled, then corresponding to each control function* $u \in U_{\mathrm{ad}}$*, system* (3) *has a unique mild solution in* $L_2([0, b]; X)$.

Proof. Let $p_0 \in X$ and consider $W = C\{0, b; X\}$, the Banach space of all continuous functions from $[0, b]$ to X along with supremum norm.

We now define $\Phi : C[0, b; X] \to C[0, b; X]$ such that

$$(\Phi p)(t) = \mathcal{S}(t)p_0 + \int_0^t \mathcal{S}(t-s)\big[B\chi_w u(s) + F\big(s, p(s)\big)\big]\,\mathrm{d}s.$$

Clearly, Φ is well defined. We need to verify that (4) represents the mild solution on $[0, b_1]$. It is enough to verify that Φ has a unique fixed point in $C([0, b]; X)$. By applying the fixed point technique we are able to verify this discussion.

Assume that $r \in \mathbb{R}$ and $r > 0$. Assume that B_r denotes the closed ball in $C[0, b_1; X]$ with radius r:

$$B_r := \big\{p(\cdot) \in C[0, b_1; X]\colon\ \|p\|_{C([0,b_1];X)} \leqslant r\big\}.$$

Clearly, B_r is bounded and closed subset of $C[0, b_1; X]$. For any $p(\cdot) \in B_r$, we have

$$\|\Phi p\|_X \leqslant M\|p_0\| + M\|B\|\|u\| + Mkb_1 + Mk \int_0^{b_1} \|p(s)\| \,\mathrm{d}s$$
$$\leqslant M\|p_0\| + M\|B\|\|u\| + Mkb_1 + Mkb_1 r.$$

Now, let $M[\|p_0\| + \|B\|\|u\| + kb_1 + kb_1 r] < r$. Then

$$M\big[\|p_0\| + \|B\|\|u\| + kb_1\big] < r(1 - Mkb_1).$$

For positive right-hand side,

$$Mkb_1 < 1. \tag{7}$$

Therefore, if (7) holds, the Φ maps B_r into itself.

Now, we will prove that Φ^n is a contraction on B_r. Let $p_1, p_2 \in B_r$. With the help of definition of B_r, there exists $r > 0$ such that $\|p_1\|, \|p_2\| < r$. Hence, condition (A3) yields

$$\big\|(\Phi p_1)(t) - (\Phi p_2)(t)\big\| \leqslant MLb\big\{\|p_1 - p_2\|\big\}.$$

Following the same method n times, we have

$$\big\|\Phi^n p_1 - \Phi^n p_2\big\| \leqslant \frac{(MLb)^n}{n!}\big\{\|p_1 - p_2\|\big\}. \tag{8}$$

Conclusion (8) is achieved easily by continuing n times the above procedure. There exist n such that $(MLb)^n/n! < 1$. Hence, Φ^n is contraction for sufficiently large n. Using contraction principle in $C[0, b_1; X]$, Φ^n has a unique fixed point p, which represents mild solution of (3). In the same manner, we will show that (4) is the mild solution on $[b_1, b_2]$, $b_1 < b_2$. Continuing this process, we will achieve that (4) represents mild solution on the maximal existence interval $[0, b)$, $b < \infty$. □

Our aim is here to verify the uniqueness. Let us consider any two solutions p_1 and p_2, we have

$$\big\|p_1(t) - p_2(t)\big\| \leqslant M \int_0^t \big\|F(s, p_1) - F(s, p_2)\big\| \,\mathrm{d}s$$
$$\leqslant Ml \int_0^t \big\|p_1(s) - p_2(s)\big\| \,\mathrm{d}s.$$

By applying Gronwall's inequality we conclude that $p_1(t) = p_2(t)$ for all $t \in [0, b]$.

4 Optimal control

The existence of optimal control for the considered system with diffusion is discussed in this section.

Define

$$L : [0, b] \times X \times Y \to \mathbb{R} \cup \infty$$

with the following conditions:

(C1) L is Borel measurable.
(C2) For all $t \in [0, b]$, the function $\mathcal{T}(t, p, u)$ is sequentially lower semicontinuous on $X \times Y$.
(C3) For $p \in X$, convexity is fulfilled by $\mathcal{T}(t, p, u)$ on Y.
(C4) There exists $l \in [0, \infty)$, $j \in (0, \infty)$, $\alpha \in [0, \infty)$, and $\alpha \in L^1([0, T]; \mathbb{R})$ with the condition $\mathcal{T}(t, p, u) \geqslant l\|p\| + j\|u\|_Y + \alpha(t)$.

Theorem 2. *If* (C1)–(C4) *and* (A1)–(A5) *hold, then for system* (3)*, there exists at least an optimal pair* $(p^0, u^0) \in \mathcal{A}_{\mathrm{ad}}$*, and*

$$J(p^0, u^0) := \int_0^t \mathcal{T}\big(t, p^0(t), u^0(t)\big)\, \mathrm{d}t \leqslant J(p, u), \quad (p, u) \in \mathcal{A}_{\mathrm{ad}}.$$

Proof. If greatest lower bound of $\{J(p, u) | (p, u) \in \mathcal{A}_{\mathrm{ad}}\}$ is $+\infty$, then, obviously, we obtain the result. So, we will assume that greatest lower bound of $\{J(p, u) | (p, u) \in \mathcal{A}_{\mathrm{ad}}\}$ as $\delta < +\infty$. From above conditions (C1)–(C4) it is clear that $\delta > -\infty$. With the help of greatest lower bound, there exist a sequence of state-control pair $(p^m, u^m) \in \mathcal{A}_{\mathrm{ad}}\}$ such that $J(p^m, u^m) \to \delta$ as $m \to +\infty$ assuming $(p^m, u^m) \in \mathcal{A}_{\mathrm{ad}}\}$ as minimizing sequence. We know that $L_p([0, b]; U)$ is a reflexive separable Banach space, $\{u^m\}$ is a bounded subset of $L_2([0, b]; X)$, and also $\{u^m\} \subseteq U_{\mathrm{ad}}$, $m \in \mathbb{N}$, so there is relabeled sequence $\{u^m\}$ and $u^0 \in L_p([0, b]; X)$ such that $u^m \to u^0$ (weakly converges as $m \to +\infty$) in $L_p([0, b]; X)$. As we know that the admissible set $U_{\mathrm{ad}} \subset L_p([0, b]; X)$ is bounded, closed, and convex, so Mazur's lemma forces us to conclude that $u^0 \in U_{\mathrm{ad}}$.

Now, let us assume that corresponding to sequence of controls $\{u^m\}$, the sequence of solutions of system (3) be given by $\{p^m\}$, that is,

$$p^m(t) = \mathcal{S}(t)p_0 + \int_0^t \mathcal{S}(t - s)\big[B\chi_w u^m(s) + F\big(s, p^m(s)\big)\big]\, \mathrm{d}s.$$

From definition of B_r we can easily prove that there exists $r > 0$ for which

$$\|p^m\| \leqslant r, \quad m = 0, 1, \dots .$$

Let corresponding to control $u^0 \in U_{\mathrm{ad}}$, p^0 denotes the mild solution, which is given by

$$p^0(t) = \mathcal{S}(t)p_0 + \int_0^t \mathcal{S}(t - s)\big[B\chi_w u^0(s) + F\big(s, p^0(s)\big)\big]\, \mathrm{d}s.$$

Using (A3) and Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}
&\left\|p^m(t)-p^0(t)\right\| \\
&\quad\leqslant \int_0^t \left\|\mathcal{S}(t-s)\left[B\chi_w u^m(s)-B\chi_w u^0(s)\right]\right\| \mathrm{d}s \\
&\qquad+\int_0^t \left\|\mathcal{S}(t-s)\left[F\left(s,p^m(s)\right)-F\left(s,p^0(s)\right)\right]\right\| \mathrm{d}s \\
&\quad\leqslant M\int_0^t \left\|B(s)u^m(s)-B(s)u^0(s)\right\| \mathrm{d}s + Ml\int_0^t \left\|p^m(s)-p^0(s)\right\| \mathrm{d}s.
\end{aligned}$$

By referring Gronwall's lemma

$$\begin{aligned}
\left\|p^m(t)-p^0(t)\right\| &\leqslant M^*\left(\int_0^t \left\|B(s)u^m(s)-B(s)u^0(s)\right\| \mathrm{d}s\right) \\
&\leqslant M^*\left\|Bu^m-Bu^0\right\|_Y,
\end{aligned}\tag{9}$$

where $M^*>0$ is a constant.

Because B is strongly continuous,

$$\left\|Bu^m-Bu^0\right\|_Y \to 0 \quad \text{as } m\to\infty. \tag{10}$$

From (9) and (10) we conclude that

$$\left\|p^m(t)-p^0(t)\right\|_X \to 0 \quad \text{as } m\to\infty.$$

This implies that $\|p^m-p^0\|_X \to 0$ as $m\to\infty$, i.e., $p^m\to p^0$ when $m\to\infty$ (all converge strongly).

By referring [4, Thm. 2.1], under (C1)–(C4), assumptions of Balder are fulfilled. So, with the help of theorem of Balder's, we get

$$(p,u)\to\int_0^t \mathcal{T}\left(t,p(t),u(t)\right) \mathrm{d}t, \tag{11}$$

$L_2([0,b];X)\subset L_1([0,b];X)$ in the weak topology and strong topology of $L_1([0,b];Y)$, (11) is sequentially lower semicontinuous. Hence, on $L_2([0,b];X)$, with condition (C4) and weakly lower semicontinuity of J, $J>-\infty$. Greatest lower bound of J is achieved at $u^0\in U_{\mathrm{ad}}$, i.e.,

$$\begin{aligned}
\delta &:= \lim_{m\to\infty} \mathcal{T}\left(t,p^m(t),u^m(t)\right) \mathrm{d}t \geqslant \int_0^t \mathcal{T}\left(t,p^0(t),u^0(t)\right) \mathrm{d}t \\
&= J\left(p^0,u^0\right) \geqslant \delta.
\end{aligned}$$

□

5 Time optimal control

Let us assume two distinct members $p_0, p_1 \in X$ with the condition $p(t;F;u) = p_1$, $p(0) = p_0$ such that $u \in U_{\mathrm{ad}}$. Define the transition time t_u satisfying $p(t_u;F;u) = p_1$.

The time value t_0 of t_u for which there exists an admissible control satisfying $p(t_u;F;u) = p_1$ is said to be the optimal time, and a control $u_0 \in U_{\mathrm{ad}}$ such that $p(t_0;F;u^0) = p_1$ is said to be the time optimal control. Now, we need to verify that there exist admissible control such that $p(t_0;F;u^0) = p_1$ with respect to $\{p_0, p_1\}$.

Theorem 3. *Assume that* (A1)–(A5) *are fulfilled, then corresponding to* $\{p_0, p_1\}$*, there exist a time optimal control.*

Proof. By referring the methodologies presented in [14, 21, 35], with certain modifications, we present the existence of time optimal control.

Consider
$$t_0 := \inf\bigl\{t\colon\ p(t;F;u) = p_1,\ u \in U_{\mathrm{ad}}\bigr\}.$$
Then there is a nonincreasing sequence $\{t_m\}$, which converges to $\{t_0\}$, and we consider $\{u^m\} \subseteq U_{\mathrm{ad}}$, a sequence of controls having state trajectories as
$$\begin{aligned} &p^m(t_m;F;u) \\ &\quad := \mathcal{S}(t)p_0 + \int_0^t \mathcal{S}(t-s)\bigl[B\chi_w u^m(s) + F\bigl(s, p^m(s)\bigr)\bigr]\,\mathrm{d}s \quad t \in [0,T], \end{aligned}$$
satisfying $p^m(t_m;F;u^m) = p_1$ for all $m = 1, 2, \dots$.

Note that $p^m(t_m;F;u^m) \in L_2([0,T];X)$ since $\{u^m\}$ is bounded, weakly sequentially compact. Therefore, we fix $u^m \to u^0$ in $L_2([0,T];Y)$.

Let X has the dual space as X^* and $p^* \in X^*$. So, the dual pair (p^m, p^*) is given by
$$\begin{aligned} &\bigl(p^m\bigl(t_m;g;u^m\bigr), p^*\bigr) \\ &\quad = \left(\mathcal{S}(t)p_0 + \int_0^{t_0} \mathcal{S}(t_m - s)\bigl[B\chi_w u^m(s) + F\bigl(s, p^m(s)\bigr)\bigr]\,\mathrm{d}s, p^*\right) \\ &\qquad + \int_{t_0}^{t_m} \bigl(\mathcal{S}(t_m - s)\bigl[B\chi_w u^m(s) + F\bigl(s, p^m(s)\bigr)\bigr]\,\mathrm{d}s, p^*\bigr). \end{aligned} \tag{12}$$
In view of (A5) and Lemma 1, we have
$$\begin{aligned} &\left\|\left(\int_{t_0}^{t_m} \mathcal{S}(t-s)\bigl[B\chi_w u^m(s) + F\bigl(s, p^m(s)\bigr)\bigr]\,\mathrm{d}s, p^*\right)\right\| \\ &\quad \leqslant M\left[\|B\| \int_{t_0}^{t_m} \bigl\|u^m(s)\bigr\|\,\mathrm{d}s + \int_{t_0}^{t_m} k\bigl(\bigl\|p^m(s)\bigr\| + 1\bigr)\,\mathrm{d}s\right]\|p^*\| \\ &\quad \leqslant \bigl[M\|B\|\|u\|_{L_2([0,T];Y)} + Mk(M_{ph} + 1)\bigr]\|p^*\|. \end{aligned}$$

In the right-hand side of equation (12), the first and third term converge strongly to $\mathcal{S}(t)p_0$ and 0, respectively. The convergence of second term is given with the help of weak convergence of u^m to u^0 and assumption (A4), we achieve

$$\int_0^{t_0} \mathcal{S}(t_m - s)\big[B\chi_w u^m(s) + F\big(s, p^m(s)\big)\big]\,\mathrm{d}s \to \int_0^{t_0} \mathcal{S}(t_0 - s)\big[B\chi_w u^0(s) + F\big(s, p^0(s)\big)\big]\,\mathrm{d}s$$

strongly in $L_2([0,T];X)$. As $m \to \infty$, it becomes

$$(p_1, p^*) = \left(\mathcal{S}(t)p_0 + \int_0^{t_0} \mathcal{S}(t-s)\big[B\chi_w u^m(s) + F\big(s, p^m(s)\big)\big]\,\mathrm{d}s,\ p^*\right).$$

Since p^* was arbitrary, we have

$$p_1 = \mathcal{S}(t)p_0 + \int_0^{t_0} \mathcal{S}(t-s)\big[B\chi_w u^m(s) + F\big(s, p^m(s)\big)\big]\,\mathrm{d}s = p\big(t_0; F; u^0\big).$$

Thus, the time optimal control is u^0, and the corresponding trajectory is $p(t_0; F; u^0)$. □

6 Conclusion

The primary focus of our article is to prove the optimal control of the semilinear population dynamics system using C_0-semigroup theory. The main outcomes are proved by applying Lipschitz continuity, Banach contraction principle, and well-known Gronwall's inequality. One may extend the optimal control outcomes of the assumed system, nonlocal and impulsive with suitable modifications. We can also study the optimal control of assumed population dynamics systems in stochastic and fractional-order systems by utilizing stochastic and fractional calculus.

References

1. S. Anita, Optimal control of a nonlinear population dynamics with diffusion, *J. Math. Anal. Appl.*, **152**(1):176–208, 1990, `https://doi.org/10.1006/jmaa.2000.7239`.
2. U. Arora, V. Vijayakumar, A. Shukla, K.S. Nisar, S. Rezapour, W. Jamshed, Results on exact controllability of second-order semilinear control system in Hilbert spaces, *Adv. Difference Equ.*, **2021**:455, 2012, `https://doi.org/10.1186/s13662-021-03620-5`.
3. A.V. Balakrishnan, Optimal control problems in Banach spaces, *J. Soc. Ind. Appl. Math., Ser. A, Control*, **3**:15–180, 1965, `https://doi.org/10.1137/0303014`.

4. E.J. Balder, Necessary and sufficient conditions for L_1-strong-weak lower semicontinuity of integral functionals, *Nonlinear Anal., Theory Methods Appl.*, **11**(12):1399–1404, 1987, https://doi.org/10.1016/0362-546x(87)90092-7.
5. W.L. Chan, B.Z. Guo, On the semigroups of age-size dependent population dynamics with spatial diffusion, *Manuscr. Math.*, **66**(2):161–181, 1989, https://doi.org/10.1007/BF02568489.
6. R. Dayal, M. Malik, S. Abbas, A. Debbouche, Optimal controls for second-order stochastic differential equations driven by mixed-fractional Brownian motion with impulses, *Math. Methods Appl. Sci.*, **43**(7):4107–4124, 2020, https://doi.org/10.1002/mma.6177.
7. C. Dineshkumar, R. Udhayakumar, V. Vijayakumar, K.S. Nisar, A. Shukla, A note on approximate controllability for nonlocal fractional evolution stochastic integrodifferential inclusions of order $r \in (1, 2)$ with delay, *Chaos Solitons Fractals*, **153**(1):111565, 2021, https://doi.org/10.1016/j.chaos.2021.111565.
8. C. Dineshkumar, R. Udhayakumar, V. Vijayakumar, K.S. Nisar, A. Shukla, A note on the approximate controllability of Sobolev type fractional stochastic integro-differential delay inclusions with order $1 < r < 2$, *Math. Comput. Simul.*, **190**:1003–1026, 2021, https://doi.org/10.1016/j.matcom.2021.06.026.
9. H. Frankowska, H. Zhang, X. Zhang, Necessary optimality conditions for local minimizers of stochastic optimal control problems with state constraints, *Trans. Am. Math. Soc.*, **372**(2):1289–1331, 2019, https://doi.org/10.1090/tran/7669.
10. W. Huyer, Semigroup formulation and approximation of a linear age-dependent population problem with spatial diffusion, *Semigroup Forum*, **49**(1):99–114, 1994, https://doi.org/10.1007/BF02573475.
11. B. Jacob, A. Omrane, Optimal control for age-structured population dynamics of incomplete data, *J. Math. Anal. Appl.*, **370**(1):42–48, 2010, https://doi.org/10.1016/j.jmaa.2010.04.042.
12. J.M. Jeong, H.J. Hwang, Optimal control problems for semilinear retarded functional differential equations,, *J. Optim. Theory Appl.*, **167**(1):49–67, 2015, https://doi.org/10.1007/s10957-015-0726-8.
13. J.M. Jeong, J.R. Kim, H.H. Roh, Optimal control problems for semilinear evolution equations, *J. Korean Math. Soc.*, **45**(23):757–769, 2008, https://doi.org/10.4134/JKMS.2008.45.3.757.
14. J.M. Jeong, S.J. Son, Time optimal control of semilinear control systems involving time delays, *J. Optim. Theory Appl.*, **165**(3):793–811, 2015, https://doi.org/10.1007/s10957-014-0639-y.
15. K. Kavitha, V. Vijayakumar, A. Shukla, K.S. Nisar, R. Udhayakumar, Results on approximate controllability of Sobolev-type fractional neutral differential inclusions of Clarke subdifferential type, *Chaos Solitons Fractals*, **151**:111264, 2021, https://doi.org/10.1016/j.chaos.2021.111264.
16. K. Kavitha, V. Vijayakumar, R. Udhayakumar, N. Sakthivel, K.S. Nisar, A note on approximate controllability of the Hilfer fractional neutral differential inclusions with infinite delay, *Math. Methods Appl. Sci.*, **44**(6):4428–4447, 2020, https://doi.org/10.1002/mma.7040.

17. S. Kumar, Mild solution and fractional optimal control of semilinear system with fixed delay, *J. Optim. Theory Appl.*, **174**(1):108–121, 2017, https://doi.org/10.1007/s10957-015-0828-3.
18. S. Kumar, The solvability and fractional optimal control for semilinear stochastic systems, *Cubo*, **19**(3), 2017, https://doi.org/10.4067/S0719-06462017000300001.
19. Z. Luo, Optimal control for a population of dynamics with age-dependent and diffusion in a periodic environment, *J. Appl. Math. Comput.*, **27**(1–2):77–84, 2008, https://doi.org/10.1007/s12190-008-0063-2.
20. K.S. Nisar, V. Vijayakumar, Results concerning to approximate controllability of non-densely defined Sobolev-type Hilfer fractional neutral delay differential system, *Math. Methods Appl. Sci.*, **44**(17):13615–13632, 2021, https://doi.org/10.1002/mma.7647.
21. N.S. Papageorgiou, On the optimal control of strongly nonlinear evolution equations, *J. Math. Anal. Appl.*, **164**(1):83–103, 1992, https://doi.org/10.1016/0022-247X(92)90146-5.
22. R. Patel, A. Shukla, S.S. Jadon, Existence and optimal control problem for semilinear fractional order $(1,2]$ control system, *Math. Methods Appl. Sci.*, 2020, https://doi.org/10.1002/mma.6662.
23. R. Patel, A. Shukla, S.S. Jadon, Existence and optimal control problem for semilinear fractional order $(1,2]$ control system, *Math. Methods Appl. Sci.*, 2021, https://doi.org/10.1002/mma.6662.
24. R. Patel, A. Shukla, S.S. Jadon, R. Udhayakumar, A novel increment approach for optimal control problem of fractional-order $(1,2]$ nonlinear systems, *Math. Methods Appl. Sci.*, 2021, https://doi.org/10.1002/mma.7681.
25. A. Pazy, *Semigroup of Linear Operators and Application to Partial Differential Equations*, Spinger, New York, 1983, https://doi.org/10.1007/978-1-4612-5561-1.
26. M. Mohan Raja, V. Vijayakumar, R. Udhayakumar, A new approach on approximate controllability of fractional evolution inclusions of order $1 < r < 2$ with infinite delay, *Chaos Solitons Fractals*, **141**:110343, 2020, https://doi.org/10.1016/j.chaos.2020.110343.
27. M. Mohan Raja, V. Vijayakumar, R. Udhayakumar, Y. Zhou, A new approach on the approximate controllability of fractional differential evolution equations of order $1 < r < 2$ in Hilbert spaces, *Chaos Solitons Fractals*, **141**:110310, 2020, https://doi.org/10.1016/j.chaos.2020.110310.
28. A. Shukla, R. Patel, Existence and optimal control results for second-order semilinear system in Hilbert spaces, *Circuits Syst. Signal Process.*, **40**(9):4246–4258, 2021, https://doi.org/10.1007/s00034-021-01680-2.
29. A. Shukla, N. Sukavanam, D.N. Pandey, Approximate Controllability of semilinear fractional control systems of order $\alpha \in (1,2]$ with infinite delay, *Mediterr. J. Math.*, **13**:2539–2550, 2016, https://doi.org/10.1007/s00009-015-0638-8.
30. A. Shukla, V. Vijayakumar, K.S. Nisar, A new exploration on the existence and approximate controllability for fractional semilinear impulsive control systems of order $r \in (1,2)$, *Chaos Solitons Fractals*, **2021**:111615, 2021, https://doi.org/10.1016/j.chaos.2021.111615.

31. C.C. Travis, G.F. Webb, Cosine families and abstract nonlinear second order differential equations, *Acta Math. Acad. Sci. Hung.*, **32**:76–96, 1978, https://doi.org/10.1007/BF01902205.

32. V. Vijayakumar, Approximate controllability for a class of second-order stochastic evolution inclusions of Clarke's subdifferential type, *Results Math.*, **73**(1):42, 2018, https://doi.org/10.1007/s00025-018-0807-8.

33. V. Vijayakumar, S.K. Panda, K.S. Nisar, H.M. Baskonus, Results on approximate controllability results for second-order Sobolev-type impulsive neutral differential evolution inclusions with infinite delay, *Numer. Methods Partial Differ. Equations*, **37**(2):1200–1221, 2020, https://doi.org/10.1002/num.22573.

34. V. Vijayakumar, A. Shukla, K.S. Nisar, W. Jamshed, S. Rezapour, A note on the approximate controllability of second-order integro-differential evolution control systems via resolvent operators, *Adv. Difference Equ.*, **2021**:484, 2021, https://doi.org/10.1186/s13662-021-03639-8.

35. J. Wang, Y. Zhou, M. Medved, On the solvability and optimal controls of fractional integrodifferential evolution systems with infinite delay, *J. Optim. Theory Appl.*, **152**(1):31–50, 2012, https://doi.org/10.1007/s10957-011-9892-5.