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Exponential synchronization for second-order switched quaternion-valued neural networks with neutral-type and mixed time-varying delays*

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Abstract. This article focuses on the global exponential synchronization (GES) for second-order state-dependent switched quaternion-valued neural networks (SOSDSQVNNs) with neutral-type and mixed delays. By proposing some new Lyapunov–Krasovskii functionals (LKFs) and adopting some inequalities, several new criteria in the shape of algebraic inequalities are proposed to ensure the GES for the concerned system by using hybrid switched controllers (HSCs). Different from the common reducing order and separation ways, this article presents some new LKFs to straightway discuss the GES of the concerned system based on non-reduction order and non-separation strategies. Ultimately, an example is provided to validate the effectiveness of the theoretical outcomes.

Keywords: quaternion-valued neural network, second-order model, exponential synchronization, mixed delay, hybrid switched control.

1 Introduction

In recent years, the dynamics analysis of neural networks (NNs) has acquired the extensive attention in various fields involving image processing [30], associative memory [29] and deep learning [27]. These actual applications are strongly associated to the dynamics of NNs. It is widely known that the synchronization is not only an important dynamic characteristic of NNs, but also is extensively applied in cognitive processes [20] and secure communication [33]. Meanwhile, because of the limited switching speed of amplifiers in the hardware implementation, NNs inevitably possess time delays. Besides,

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the variations of the present states of NNs rely on not only the present specific states, but also the former states and the variations of the former states at each moment. Due to these reasons, the neutral terms are proposed as broader time delays that consults time delays in the derivative of state functions. Thus, the research on the synchronization of NNs with different types of time delays are of great value.

Different from common NNs with only the first derivative of the states, the second-order NNs (SONNs) with inertial terms not only have obvious engineering and biological contexts [1], but also their dynamics are more complex [3]. So far, the dynamical behaviors of different types of SONNs have been extensively studied in [6, 19]. In most of the previous studies, the reducing order method [6, 19] was usually used to discuss SONNs. But this way causes the increase of system dimension and the difficulty in theory analysis. Nowadays, the non-reduced order strategies [24, 31, 32] are adopted to handle SONNs.

The memristive nervous systems could afford better applications for simulating the human brain [16]. Thus, there has been a growing attention to the study of memristive NNs [4, 8]. On account of the features of contractive hysteresis loop of memristors [5], the memristive NNs should be a sort of state-dependent switched NNs (SDSNNs). Then numerous studies have been done on these SDSNNs. Recently, some scholars have also done many investigations on second-order SDSNNs (SOSDSNNs). For examples, in the real domain, Hua et al. [7] discussed the finite-time synchronization of SOSDSNNs. The authors [17] investigated the stability and the GES of SOSDSNNs. Meanwhile, there are many NNs, which are actually unstable. As a consequence, some controllers need be appended to NNs to assure the related dynamics. Now, various control tactics such as state feedback control [7], pinning control [17] and intermittent control [18] are accepted to achieve the stabilization and synchronization of NNs. As pointed out in [15], the switching control can effectively improve the performance of control strategy. Nowadays, Zhang et al. [31] investigated the GES of SOSDSNNs with distributed delays by using HSC.

It is worth stressing that the real world data tend to be multidimensional. The quaternion systems could deal with high-dimensional information better in practical applications such as 3D wind signal modeling [25] and color face recognition [34]. So, the quaternion systems show remarkable advantage in dealing with multidimensional information. Recently, there has been an increasing attention to the study on quaternion-valued NNs (QVNNs), and some excellent results were obtained in [9–11, 21, 22, 26]. With the extensive research on various second-order QVNNs (SOQVNNs) [12–14, 28], SOSDSQVNNs have gradually been noticed [23]. In [13, 14, 21–23, 26], the decomposition means is used to study the dynamics of QVNNs and SOSDSNNs. But a distinct disadvantage of the decomposition means is that the computational complexity increases. To overcome this shortcoming, the non-separation method is proposed in [10–12, 28]. But, to our knowledge, there is hardly any paper that concerned the synchronization of SOSDSQVNNs via non-separation and non-reduced order methods and applying HSCs. These constitute the incentive for the current research.

Inspired by the discussion above, this paper intends to discuss the GES of SOSDSQVNNs via HSCs. The primary contributions of this article consist of three aspects: (i) Quaternion, state-dependent parameters, neutral and mixed delays are all concerned in the synchronization analysis for the presented SONNs. (ii) Different from the common reduced order means and separation strategies, the GES for SOSDSQVNNs is directly discussed as a whole by presenting some new LKFs and using HSCs, and some sufficient conditions for the GES of the concerned SONNs are acquired in the form of algebraic inequalities. (iii) Compared with the existing studies, the obtained results here include those [31] as special situations and are more universal, which indicate that the results here are applicable to not only SOQVNNs, but also other SONNs.

The rest of the paper is composed as follows: Section 2 provides some preliminaries and model description. In Section 3, the GES is analyzed, and some new conditions are derived. A numerical simulation is given in Section 4 to explain the effectiveness of the outcomes. Ultimately, a brief summary is stated in Section 5.

In this paper, $\mathbb{R}^m$, $\mathbb{Q}^m$ represent m-dimensional real-valued and quaternion-valued spaces, respectively. $\mathbb{Q} \triangleq \{\vartheta = \vartheta^r + \imath\vartheta^\imath + \jmath\vartheta^\jmath + \kappa\vartheta^\kappa | \vartheta^r, \vartheta^\imath, \vartheta^\jmath, \vartheta^\kappa \in \mathbb{R}\}$. $\overline{h} = h^r - \imath h^\imath - \jmath h^\jmath - \kappa h^\kappa$, $|h| = \sqrt{h\overline{h}} = \sqrt{\overline{h}h} = \sqrt{(h^r)^2 + (h^\imath)^2 + (h^\jmath)^2 + (h^\kappa)^2}$ and $\mathrm{Re}(h) = h^r$ are the conjugate, module and real part of quaternion $h = h^r + \imath h^\imath + \jmath h^\jmath + \kappa h^\kappa$, respectively. For $\eta = (\eta_1, \eta_2, \ldots, \eta_m)^{\mathrm{T}} \in \mathbb{Q}^m$, $\|\eta\| = \sqrt{\sum_{q=1}^m |\eta_q|^2}$. $\mathrm{co}\{Q_1, Q_2\}$ is closure of the convex hull of $\mathbb{Q}$ generated by quaternion numbers Q_1, Q_2. $\Upsilon = \{1, 2, \ldots, m\}$, $\Theta = \{r, \imath, \jmath, \kappa\}$.

2 Preliminaries

Consider the following SOSDSQVNNs for $q \in \Upsilon$:

$$\ddot{\vartheta}_q(t) = G_q\bigl(\vartheta(t)\bigr) + \sum_{l=1}^m d_{ql}\bigl(\vartheta_q(t)\bigr)\mathbf{x}_l\bigl(\dot{\vartheta}_l\bigl(t - \varepsilon(t)\bigr)\bigr) + \sum_{l=1}^m z_{ql}\bigl(\vartheta_q(t)\bigr) \int_{t-\sigma(t)}^{t} \mathbf{x}_l\bigl(\vartheta_l(w)\bigr)\,\mathrm{d}w, \tag{1}$$

where

$$G_q\bigl(\vartheta(t)\bigr) = -\delta_q\dot{\vartheta}_q(t) - \xi_q\vartheta_q(t) + \sum_{l=1}^m b_{ql}(\vartheta_q(t))\mathbf{y}_l\bigl(\vartheta_l(t)\bigr) + \sum_{l=1}^m c_{ql}\bigl(\vartheta_q(t)\bigr)\mathbf{y}_l\bigl(\vartheta_l\bigl(t - \rho(t)\bigr)\bigr) + g_q(t),$$

$\vartheta_q(t) \in \mathbb{Q}$ is the state of the qth neuron at time t; the item $\ddot{\vartheta}_q(t)$ is referred to as an inertial term; $\mathbf{y}_l(\vartheta(\cdot))$, $\mathbf{x}_l(\vartheta(\cdot)) \in \mathbb{Q}$ express the activation functions; $\varepsilon(t)$ is neutral-type time-varying delay, $\rho(t)$ represents time-varying delays, $\sigma(t)$ is the distributed delay, which meet $\dot{\rho}(t) \leqslant \upsilon_1 < 1$, $\dot{\varepsilon}(t) \leqslant \upsilon_2 < 1$, $0 < \rho(t) \leqslant \varrho$, $0 < \sigma(t) \leqslant \tau$, $0 < \varepsilon(t) \leqslant \epsilon$ and υ_1, υ_2, ϱ, ϵ, τ are constants; $g_q(t) \in \mathbb{Q}$ denotes the external input; $\delta_q > 0, \xi_q > 0$, $\zeta_{ql}(\vartheta_q(t)) \in \mathbb{Q}$ for $\zeta = b, c, d, z$ is the memristive connection weight. The initial values of (1) are $\vartheta_q(s) = \varphi_q(s)$, $\dot{\vartheta}_q(s) = \psi_q(s)$, $-\varsigma \leqslant s \leqslant 0$ with $\varsigma = \max\{\epsilon, \varrho, \tau\}$. State-dependent parameters in (1) satisfy

$$b_{ql}(\vartheta_q(t)) = \begin{cases} \grave{b}_{ql} = \grave{b}^r_{ql} + \imath\grave{b}^\imath_{ql} + \jmath\grave{b}^\jmath_{ql} + \kappa\grave{b}^\kappa_{ql}, & |\vartheta_q(t)| \leqslant \chi_q, \\ \acute{b}_{ql} = \acute{b}^r_{ql} + \imath\acute{b}^\imath_{ql} + \jmath\acute{b}^\jmath_{ql} + \kappa\acute{b}^\kappa_{ql}, & |\vartheta_q(t)| > \chi_q, \end{cases}$$

$$c_{ql}(\vartheta_q(t)) = \begin{cases} \grave{c}_{ql} = \grave{c}^r_{ql} + \imath\grave{c}^\imath_{ql} + \jmath\grave{c}^\jmath_{ql} + \kappa\grave{c}^\kappa_{ql}, & |\vartheta_q(t)| \leqslant \chi_q, \\ \acute{c}_{ql} = \acute{c}^r_{ql} + \imath\acute{c}^\imath_{ql} + \jmath\acute{c}^\jmath_{ql} + \kappa\acute{c}^\kappa_{ql}, & |\vartheta_q(t)| > \chi_q, \end{cases}$$

$$d_{ql}(\vartheta_q(t)) = \begin{cases} \grave{d}_{ql} = \grave{d}^r_{ql} + \imath\grave{d}^\imath_{ql} + \jmath\grave{d}^\jmath_{ql} + \kappa\grave{d}^\kappa_{ql}, & |\vartheta_q(t)| \leqslant \chi_q, \\ \acute{d}_{ql} = \acute{d}^r_{ql} + \imath\acute{d}^\imath_{ql} + \jmath\acute{d}^\jmath_{ql} + \kappa\acute{d}^\kappa_{ql}, & |\vartheta_q(t)| > \chi_q, \end{cases}$$

$$z_{ql}(\vartheta_q(t)) = \begin{cases} \grave{z}_{ql} = \grave{z}^r_{ql} + \imath\grave{z}^\imath_{ql} + \jmath\grave{z}^\jmath_{ql} + \kappa\grave{z}^\kappa_{ql}, & |\vartheta_q(t)| \leqslant \chi_q, \\ \acute{z}_{ql} = \acute{z}^r_{ql} + \imath\acute{z}^\imath_{ql} + \jmath\acute{z}^\jmath_{ql} + \kappa\acute{z}^\kappa_{ql}, & |\vartheta_q(t)| > \chi_q, \end{cases}$$

where $\chi_q > 0$ is the switching jump, $\grave{\zeta}^\theta_{ql}, \acute{\zeta}^\theta_{ql} \in \mathbb{R}$ for $\zeta = b, c, d, z$ are constants, $\theta \in \Theta$, $q, l \in \Upsilon$.

The controlled system of (1) can be described as

$$\begin{aligned} \ddot{\nu}_q(t) &= G_q\bigl(\nu(t)\bigr) + \sum_{l=1}^m d_{ql}\bigl(\nu_q(t)\bigr)\mathbf{x}_l\bigl(\dot{\nu}_l\bigl(t-\varepsilon(t)\bigr)\bigr) \\ &\quad + \sum_{l=1}^m z_{ql}\bigl(\nu_q(t)\bigr) \int\limits_{t-\sigma(t)}^{t} \mathbf{x}_l\bigl(\nu_l(w)\bigr)\,\mathrm{d}w + v_q(t), \end{aligned} \tag{2}$$

where

$$\begin{aligned} G_q\bigl(\nu(t)\bigr) &= -\delta_q\dot{\nu}_q(t) - \xi_q\nu_q(t) + \sum_{l=1}^m b_{ql}\bigl(\nu_q(t)\bigr)\mathbf{y}_l\bigl(\nu_l(t)\bigr) \\ &\quad + \sum_{l=1}^m c_{ql}\bigl(\nu_q(t)\bigr)\mathbf{y}_l\bigl(\nu_l\bigl(t-\rho(t)\bigr)\bigr) + g_q(t), \end{aligned}$$

the parameters $\zeta_{ql}(\nu_q(t))(\zeta = b, c, d, z)$ are the same as (1), the initial values of (2) are $\nu_q(s) = \widetilde{\varphi}_q(s)$, $\dot{\nu}_q(s) = \widetilde{\psi}_q(s)$, $-\varsigma \leqslant s \leqslant 0$. $v_q(t)$ is HSC and shown as

$$\begin{aligned} v_q(t) &= -\alpha_q(t)\mu_q(t) - \beta_q(t)\dot{\mu}_q(t) - \pi_q(t), \\ \dot{\alpha}_q(t) &= \omega_q\mathrm{e}^{2at}\bigl(\mu_q(t)\overline{\mu_q(t)} + \mathrm{Re}\bigl(\overline{\mu_q(t)}\dot{\mu}_q(t)\bigr)\bigr), \\ \dot{\beta}_q(t) &= \varpi_q\mathrm{e}^{2at}\bigl(\dot{\mu}_q(t)\overline{\dot{\mu}_q(t)} + \mathrm{Re}\bigl(\overline{\mu_q(t)}\dot{\mu}_q(t)\bigr)\bigr), \end{aligned} \tag{3}$$

$$\pi_q(t) = \begin{cases} \sum_{l=1}^m \grave{z}_{ql} \int_{t-\sigma(t)}^t (\mathbf{x}_l(\nu_l(w)) - \mathbf{x}_l(\vartheta_l(w)))\,\mathrm{d}w, \\ \quad |\vartheta_q(t)| \leqslant \chi_q,\ |\nu_q(t)| \leqslant \chi_q, \\ \sum_{l=1}^m \acute{z}_{ql} \int_{t-\sigma(t)}^t (\mathbf{x}_l(\nu_l(w)) - \mathbf{x}_l(\vartheta_l(w)))\,\mathrm{d}w, \\ \quad |\vartheta_q(t)| > \chi_q,\ |\nu_q(t)| > \chi_q, \end{cases} \tag{4_1}$$

and

$$\pi_q(t) = \begin{cases} \sum_{l=1}^{m} (A_{ql}(t) + \int_{t-\sigma(t)}^{t} \grave{Z}_{ql}(w)\,\mathrm{d}w), \\ \quad |\vartheta_q(t)| > \chi_q,\ |\nu_q(t)| \leqslant \chi_q, \\ \sum_{l=1}^{m} (-A_{ql}(t) + \int_{t-\sigma(t)}^{t} \acute{Z}_{ql}(w)\,\mathrm{d}w), \\ \quad |\vartheta_q(t)| \leqslant \chi_q,\ |\nu_q(t)| > \chi_q, \end{cases} \tag{4_2}$$

where ω_q, ϖ_q are some positive constants and $\alpha_q(t)$, $\beta_q(t)$ are the control gains, $\mu_q(t) = \nu_q(t) - \vartheta_q(t)$ is the synchronization error,

$$\begin{aligned} A_{ql}(t) &= B_{ql}(t) + (\grave{d}_{ql} - \acute{d}_{ql})\mathbf{x}_l\big(\dot{\vartheta}_l\big(t - \varepsilon(t)\big)\big), \\ B_{ql}(t) &= (\grave{b}_{ql} - \acute{b}_{ql})\mathbf{y}_l\big(\vartheta_l(t)\big) + (\grave{c}_{ql} - \acute{c}_{ql})\mathbf{y}_l\big(\vartheta_l\big(t - \rho(t)\big)\big), \\ \grave{Z}_{ql}(w) &= \grave{z}_{ql}\mathbf{x}_l(\nu_l(w)) - \acute{z}_{ql}\mathbf{x}_l\big(\vartheta_l(w)\big), \\ \acute{Z}_{ql}(w) &= \acute{z}_{ql}\mathbf{x}_l(\nu_l(w)) - \grave{z}_{ql}\mathbf{x}_l\big(\vartheta_l(w)\big). \end{aligned}$$

Based on (1) and (2), the error system is obtained as follows:

$$\begin{aligned} \ddot{\mu}_q(t) = {}& -\delta_q\dot{\mu}_q(t) - \xi_q\mu_q(t) \\ & + \sum_{l=1}^{m} \big[b_{ql}\big(\nu_q(t)\big)\mathbf{y}_l\big(\nu_l(t)\big) - b_{ql}\big(\vartheta_q(t)\big)\mathbf{y}_l\big(\vartheta_l(t)\big)\big] \\ & + \sum_{l=1}^{m} \big[c_{ql}\big(\nu_q(t)\big)\mathbf{y}_l\big(\nu_l\big(t - \rho(t)\big)\big) - c_{ql}\big(\vartheta_q(t)\big)\mathbf{y}_l\big(\vartheta_l\big(t - \rho(t)\big)\big)\big] \\ & + \sum_{l=1}^{m} \left[z_{ql}\big(\nu_q(t)\big) \int_{t-\sigma(t)}^{t} \mathbf{x}_l\big(\nu_l(w)\big)\,\mathrm{d}w - z_{ql}\big(\vartheta_q(t)\big) \int_{t-\sigma(t)}^{t} \mathbf{x}_l\big(\vartheta_l(w)\big)\,\mathrm{d}w\right] \\ & + \sum_{l=1}^{m} \big[d_{ql}\big(\nu_q(t)\big)\mathbf{x}_l\big(\dot{\nu}_l\big(t - \varepsilon(t)\big)\big) - d_{ql}\big(\vartheta_q(t)\big)\mathbf{x}_l\big(\dot{\vartheta}_l\big(t - \varepsilon(t)\big)\big)\big] + \upsilon_q(t). \end{aligned} \tag{5}$$

Since the connective weights $\zeta_{ql}(\vartheta_q(t))$, $\zeta_{ql}(\nu_q(t))$ for $\zeta = b, c, d, z$ can be discontinuous, the solutions of (5) have the meaning of Filippovs. With the help of the differential inclusion theory [2] and combining (3) with (4), one has

$$\begin{aligned} \ddot{\mu}_q(t) \in {}& -\big(\delta_q + \beta_q(t)\big)\dot{\mu}_q(t) - \big(\xi_q + \alpha_q(t)\big)\mu_q(t) \\ & + \sum_{l=1}^{m} \mathrm{co}\big\{\grave{b}_{ql}, \acute{b}_{ql}\big\}\mathbf{y}_l\big(\mu_l(t)\big) + \sum_{l=1}^{m} \mathrm{co}\big\{\grave{c}_{ql}, \acute{c}_{ql}\big\}\mathbf{y}_l\big(\mu_l\big(t - \rho(t)\big)\big) \\ & + \sum_{l=1}^{m} \mathrm{co}\big\{\grave{d}_{ql}, \acute{d}_{ql}\big\}\mathbf{x}_l\big(\dot{\mu}_l\big(t - \varepsilon(t)\big)\big), \end{aligned}$$

where

$$\mathbf{y}_l\big(\mu_l(\cdot)\big) = \mathbf{y}_l\big(\nu_l(\cdot)\big) - \mathbf{y}_l\big(\vartheta_l(\cdot)\big), \qquad \mathbf{x}_l\big(\dot{\mu}_l(\cdot)\big) = \mathbf{x}_l\big(\dot{\nu}_l(\cdot)\big) - \mathbf{x}_l\big(\dot{\vartheta}_l(\cdot)\big).$$

So, there exist $b_{ql} \in co\{\grave{b}_{ql}, \acute{b}_{ql}\}, c_{ql} \in \mathrm{co}\{\grave{c}_{ql}, \acute{c}_{ql}\}, d_{ql} \in \mathrm{co}\{\grave{d}_{ql}, \acute{d}_{ql}\}$ such that

$$\ddot{\mu}_q(t) = -\big(\delta_q + \beta_q(t)\big)\dot{\mu}_q(t) - \big(\xi_q + \alpha_q(t)\big)\mu_q(t) + \sum_{l=1}^{m} b_{ql}\mathbf{y}_l\big(\mu_l(t)\big) + \sum_{l=1}^{m} c_{ql}\mathbf{y}_l\big(\mu_l\big(t-\rho(t)\big)\big) + \sum_{l=1}^{m} d_{ql}\mathbf{x}_l\big(\dot{\mu}_l\big(t-\varepsilon(t)\big)\big). \tag{6}$$

Assumption A. There are constants $Y_l > 0$ and $X_l > 0$ such that

$$\big|\mathbf{y}_l(\vartheta_l) - \mathbf{y}_l(\nu_l)\big| \leqslant Y_l|\vartheta_l - \nu_l|, \qquad \big|\mathbf{x}_l(\vartheta_l) - \mathbf{x}_l(\nu_l)\big| \leqslant X_l|\vartheta_l - \nu_l|$$

for all $\vartheta_l, \nu_l \in \mathbb{Q}$ and $\mathbf{y}_l(0) = \mathbf{x}_l(0) = 0$, $l \in \Upsilon$.

Definition 1. Let $\vartheta(t) = (\vartheta_1, \vartheta_2, \dots, \vartheta_m)^{\mathrm{T}}$, $\nu(t) = (\nu_1, \nu_2, \dots, \nu_m)^{\mathrm{T}}$ be arbitrary solutions of systems (1) and (2) that satisfy the corresponding initial values, respectively. If there are constants $\eta > 0$, $a > 0$ such that

$$\big\|\mu(t)\big\| \leqslant \eta \sup_{-\varsigma\leqslant s\leqslant 0} \big\|\mu(s), \dot{\mu}(s)\big\|\mathrm{e}^{-at}, \quad t \geqslant 0,$$

then systems (1) and (2) are said to reach the GES, where $\mu(t) = \nu(t) - \vartheta(t)$.

For simplicity, some denotations are given as follows for $\theta \in \Theta$, $q, l \in \Upsilon$:

$$|b_{ql}^{\theta}| = \max\big\{\big|\grave{b}_{ql}^{\theta}\big|, \big|\acute{b}_{ql}^{\theta}\big|\big\}, \qquad |c_{ql}^{\theta}| = \max\big\{\big|\grave{c}_{ql}^{\theta}\big|, \big|\acute{c}_{ql}^{\theta}\big|\big\}, \qquad |d_{ql}^{\theta}| = \max\big\{\big|\grave{d}_{ql}^{\theta}\big|, \big|\acute{d}_{ql}^{\theta}\big|\big\},$$

$$b_{ql} = \big|b_{ql}^{r}\big| + \imath\big|b_{ql}^{\imath}\big| + \jmath\big|b_{ql}^{\jmath}\big| + \kappa\big|b_{ql}^{\kappa}\big|, \qquad c_{ql} = \big|c_{ql}^{r}\big| + \imath\big|c_{ql}^{\imath}\big| + \jmath\big|c_{ql}^{\jmath}\big| + \kappa\big|c_{ql}^{\kappa}\big|,$$

$$d_{ql} = \big|d_{ql}^{r}\big| + \imath\big|d_{ql}^{\imath}\big| + \jmath\big|d_{ql}^{\jmath}\big| + \kappa\big|d_{ql}^{\kappa}\big|, \qquad \Pi_q = \sum_{l=1}^{m}\left(|b_{lq}| + \frac{\mathrm{e}^{2a\varrho}}{1-\upsilon_1}|c_{lq}|\right)Y_q,$$

$$\Omega_q = \frac{1}{2}\sum_{l=1}^{m}\big(|b_{ql}|Y_l + |c_{ql}|Y_l + |d_{ql}|X_l\big), \qquad \widetilde{\Omega}_q = \frac{1}{2}\sum_{l=1}^{m}\big(|b_{ql}| + |c_{ql}|\big)Y_l,$$

$$\Xi_q = a + 1 + \Omega_q + \sum_{l=1}^{m}\frac{\mathrm{e}^{2a\epsilon}}{1-\upsilon_2}|d_{lq}|X_q, \qquad \widetilde{\Xi}_q = a + 1 + \widetilde{\Omega}_q,$$

$$\Gamma_q = a(\eta_q + 1) + \Omega_q + \Pi_q, \qquad \widetilde{\Gamma}_q = a(\eta_q + 1) + \widetilde{\Omega}_q + \Pi_q,$$

$$\eta_q = \xi_q + \delta_q + \alpha_q + \beta_q - 2a - 1 > 0,$$

where a, α_q, β_q are some positive numbers to need be determined.

3 Main results

Theorem 1. *Under Assumption* A, *if there are constants* $a \in (0,1)$, $\alpha_q > 0$, $\beta_q > 0$ *such that*

$$\Gamma_q - (\xi_q + \alpha_q) \leqslant 0, \qquad \Xi_q - (\delta_q + \beta_q) \leqslant 0,$$

then systems (2) *and* (1) *achieve the GES via HSCs* (3) *and* (4).

Proof. See Appendix A.

Corollary 1. *Under the conditions of Theorem* 1*, let* $\alpha_q = \Gamma_q$*,* $\beta_q = \Xi_q$*, then systems* (1) *and* (2) *reach the GES via HSCs* (3) *and* (4)*.*

Remark 1. If $d_{ql}(\vartheta_l(t)) = 0$, then system (1) can be transformed into

$$\ddot{\vartheta}_q(t) = G_q(\vartheta(t)) + \sum_{l=1}^{m} z_{ql}\big(\vartheta_q(t)\big) \int\limits_{t-\sigma(t)}^{t} \mathbf{x}_l\big(\nu_l(w)\big)\,\mathrm{d}w. \tag{7}$$

If $z_{ql}(\vartheta_l(t)) = 0$, then system (1) is reduced into

$$\ddot{\vartheta}_q(t) = G_q\big(\vartheta(t)\big) + \sum_{l=1}^{m} d_{ql}\big(\vartheta_q(t)\big)\mathbf{x}_l\big(\dot{\vartheta}_l\big(t-\varepsilon(t)\big)\big). \tag{8}$$

If $d_{ql}(\vartheta_l(t)) = 0$ and $z_{ql}(\vartheta_l(t)) = 0$, then system (1) can be simplified into

$$\ddot{\vartheta}_q(t) = G_q\big(\vartheta(t)\big). \tag{9}$$

Remark 2. It is obvious that system (9) is just system (1) in [23] and includes systems considered in [13, 28] with $b_{ql}(\vartheta_q(t)) = b_{ql}(t)$, $c_{ql}(\vartheta_q(t)) = c_{ql}(t)$ as special cases. Thus, system (1) here is more general.

The response system of model (7) is

$$\ddot{\nu}_q(t) = G_q\big(\nu(t)\big) + \sum_{l=1}^{m} z_{ql}\big(\nu_q(t)\big) \int\limits_{t-\sigma(t)}^{t} \mathbf{x}_l\big(\nu_l(w)\big)\,\mathrm{d}w + v_q(t), \tag{10}$$

where $v_q(t)$ is given in (3), and $\pi_q(t)$ is designed as

$$\pi_q(t) = \begin{cases} \sum_{l=1}^{m} \grave{z}_{ql} \int_{t-\sigma(t)}^{t} (\mathbf{x}_l(\nu_l(w)) - \mathbf{x}_l(\vartheta_l(w)))\,\mathrm{d}w, \\ \quad |\vartheta_q(t)| \leqslant \chi_q,\ |\nu_q(t)| \leqslant \chi_q, \\ \sum_{l=1}^{m} \acute{z}_{ql} \int_{t-\sigma(t)}^{t} (\mathbf{x}_l(\nu_l(w)) - \mathbf{x}_l(\vartheta_l(w)))\,\mathrm{d}w, \\ \quad |\vartheta_q(t)| > \chi_q,\ |\nu_q(t)| > \chi_q, \\ \sum_{l=1}^{m} (B_{ql}(t) + \int_{t-\sigma(t)}^{t} \grave{Z}_{ql}(w)\,\mathrm{d}w), \\ \quad |\vartheta_q(t)| > \chi_q,\ |\nu_q(t)| \leqslant \chi_q, \\ \sum_{l=1}^{m} (-B_{ql}(t) + \int_{t-\sigma(t)}^{t} \acute{Z}_{ql}(w)\,\mathrm{d}w), \\ \quad |\vartheta_q(t)| \leqslant \chi_q,\ |\nu_q(t)| > \chi_q, \end{cases} \tag{11}$$

where $B_{ql}(t)$, $\grave{Z}_{ql}(w)$, $\acute{Z}_{ql}(w)$ are defined in (4).

The response system of model (8) is

$$\ddot{\nu}_q(t) = G_q\big(\nu(t)\big) + \sum_{l=1}^{m} d_{ql}\big(\nu_q(t)\big)\mathbf{x}_l\big(\dot{\nu}_l\big(t-\varepsilon(t)\big)\big) + v_q(t), \tag{12}$$

where $v_q(t)$ is given in (3), and $\pi_q(t)$ is of the form

$$\pi_q(t) = \begin{cases} 0, & |\vartheta_q(t)| \leqslant \chi_q,\ |\nu_q(t)| \leqslant \chi_q \\ & \text{or } |\vartheta_q(t)| > \chi_q,\ |\nu_q(t)| > \chi_q, \\ \sum_{l=1}^m A_{ql}(t), & |\vartheta_q(t)| > \chi_q,\ |\nu_q(t)| \leqslant \chi_q, \\ -\sum_{l=1}^m A_{ql}(t), & |\vartheta_q(t)| \leqslant \chi_q,\ |\nu_q(t)| > \chi_q, \end{cases} \tag{13}$$

where $A_{ql}(t)$ is defined in (4).

The response system of model (9) is

$$\ddot{\nu}_q(t) = G_q\bigl(\nu(t)\bigr) + v_q(t), \tag{14}$$

where $v_q(t)$ is given in (3), and $\pi_q(t)$ is of the form

$$\pi_q(t) = \begin{cases} 0, & |\vartheta_q(t)| \leqslant \chi_q,\ |\nu_q(t)| \leqslant \chi_q \\ & \text{or } |\vartheta_q(t)| > \chi_q,\ |\nu_q(t)| > \chi_q, \\ \sum_{l=1}^m B_{ql}(t), & |\vartheta_q(t)| > \chi_q,\ |\nu_q(t)| \leqslant \chi_q, \\ -\sum_{l=1}^m B_{ql}(t), & |\vartheta_q(t)| \leqslant \chi_q,\ |\nu_q(t)| > \chi_q, \end{cases} \tag{15}$$

where $B_{ql}(t)$ is defined in (4).

Theorem 2. *Under Assumption* A, *if there are constants* $a \in (0,1)$, $\alpha_q > 0$, $\beta_q > 0$ *such that*

$$\widetilde{\varGamma}_q - (\xi_q + \alpha_q) \leqslant 0, \qquad \widetilde{\varXi}_q - (\delta_q + \beta_q) \leqslant 0,$$

then systems (7) *and* (10) *achieve the GES under HSCs* (3) *and* (11).

Proof. See Appendix B. □

Corollary 2. *Under the conditions of Theorem* 2, *let* $\alpha_q = \widetilde{\varGamma}_q$, $\beta_q = \widetilde{\varXi}_q$, *then systems* (7) *and* (10) *achieve the GES under HSCs* (3) *and* (11).

Remark 3. If $\mathbf{x}_l(\vartheta(t)) = \mathbf{y}_l(\vartheta(t))$ and quaternion-valued system (7) is reduced to real-valued one, and hybrid controller (3) is degenerated to the following form:

$$\begin{aligned} v_q(t) &= -\alpha_q(t)\mu_q(t) - \beta_q(t)\dot{\mu}_q(t) - \pi_q(t), \\ \dot{\alpha}_q(t) &= \omega_q \mathrm{e}^{2at}\bigl(\mu_q(t)\dot{\mu}_q(t) + \mu_q^2(t)\bigr), \\ \dot{\beta}_q(t) &= \varpi_q \mathrm{e}^{2at}\bigl(\mu_q(t)\dot{\mu}_q(t) + \dot{\mu}_q^2(t)\bigr), \end{aligned} \tag{16}$$

where $\pi_q(t)$ has the similar form as (4) with $\grave{d}_{ql} = \acute{d}_{ql} = 0$, then Theorem 2 here degrades into Theorem 3.1 in [31]. If $\rho(t), \sigma(t)$ are constants, then Corollary 2 becomes Corollary 3.1 in [31].

Theorem 3. *Under Assumption* A*, if there exist constants* $a \in (0,1)$*,* $\alpha_q > 0, \beta_q > 0$ *such that*

$$\varGamma_q - (\xi_q + \alpha_q) \leqslant 0, \qquad \varXi_q - (\delta_q + \beta_q) \leqslant 0,$$

then systems (8) *and* (12) *can achieve the GES via HSCs* (3) *and* (13)*.*

Corollary 3. *Under the conditions of Theorem* 3*, let* $\alpha_q = \varGamma_q$*,* $\beta_q = \varXi_q$*, then systems* (8) *and* (12) *achieve the GES via HSCs* (3) *and* (13)*.*

Theorem 4. *Under Assumption* A*, if there are constants* $a \in (0,1)$*,* $\alpha_q > 0$*,* $\beta_q > 0$ *such that*

$$\widetilde{\varGamma}_q - (\xi_q + \alpha_q) \leqslant 0, \qquad \widetilde{\varXi}_q - (\delta_q + \beta_q) \leqslant 0,$$

then systems (9) *and* (14) *achieve the GES under HSCs* (3) *and* (15)*.*

Corollary 4. *Under the conditions of Theorem* 4*, let* $\alpha_q = \widetilde{\varGamma}_q$*,* $\beta_q = \widetilde{\varXi}_q$*, then systems* (9) *and* (14) *achieve the GES under HSCs* (3) *and* (15)*.*

Remark 4. If system (9) is reduced to real-valued one, and HSC (3) is degenerated to the form of (16), then Theorem 4 here turns into Theorem 3.2 in [31]. If $\rho(t)$ is constant, then Corollary 4 here becomes Corollary 3.2 in [31]. From Remarks 2, 3 the HSC scheme (3) here is more general and flexible.

Remark 5. If system (1) is reduced to complex-valued one, one can also get corresponding results. It is worth noting that there is no outstanding criteria that can handle the synchronization of SOSDSQVNNs with neutral-type and mixed delays in the existing studies. The results here as new synchronization conditions can fill in the gaps.

Remark 6. Different from the reduced order method in [7, 12, 13, 17, 18, 23, 28] and the decomposition method in [9, 13, 14, 22, 23, 26], the paper directly study the GES for the considered systems as a whole by adopting some new LKFs. Evidently, the approach here is more practical and simpler.

Remark 7. The various dynamical properties for SOQVNNs [12–14, 28] and SOSDSQVNNs [23] without neutral-type and distributed delays were presented. However, SOSDSQVNNs (1) and (2) here are all with neutral-type and distributed time delays. Thus, the results in [12–14, 23, 28] cannot be applied here. As a result, the results here replenish and enrich some earlier works, which makes clear that the results here are new.

4 Numerical simulation

Consider SOSDSQVNN (1) with two neurons and the following parameters: $\delta_1 = \delta_2 = 0.1$, $\xi_1 = \xi_2 = 5$, $\chi_1 = \chi_q = 2 = 0.5$,

$$\rho(t) = 0.3\sin^2(t) + 0.7, \qquad \varepsilon(t) = 1.2 - 0.8\cos(t), \qquad \sigma(t) = 0.1\sin^2(t),$$

Table 1. State-dependent parameter $b^{\theta}_{qh}(\vartheta^{\theta}_{q}(t))$ with $\theta \in \Theta$.

Subscript	Parameter							
	$\grave{b}^{r}_{qh}$	$\acute{b}^{r}_{qh}$	$\grave{b}^{\imath}_{qh}$	$\acute{b}^{\imath}_{qh}$	$\grave{b}^{\jmath}_{qh}$	$\acute{b}^{\jmath}_{qh}$	$\grave{b}^{\kappa}_{qh}$	$\acute{b}^{\kappa}_{qh}$
11	1.0	1.3	0.6	0.4	0.3	0.6	−1.2	−1.4
12	−0.4	−0.3	1.4	1.2	1.1	0.9	−0.6	−0.3
21	1.2	1.0	0.4	0.3	−0.8	−0.6	0.7	0.5
22	−0.8	−0.3	1.2	0.9	−1.0	−1.3	1.2	1.1

Table 2. State-dependent parameter $c^{\theta}_{qh}(\vartheta^{\theta}_{q}(t))$ with $\theta \in \Theta$.

Subscript	Parameter							
	$\grave{c}^{r}_{qh}$	$\acute{c}^{r}_{qh}$	$\grave{c}^{\imath}_{qh}$	$\acute{c}^{\imath}_{qh}$	$\grave{c}^{\jmath}_{qh}$	$\acute{c}^{\jmath}_{qh}$	$\grave{c}^{\kappa}_{qh}$	$\acute{c}^{\kappa}_{qh}$
11	0.6	0.3	−1.4	−1.2	1.0	1.1	−0.5	−0.3
12	1.3	1.4	0.5	0.7	−1.4	−1.1	0.6	0.4
21	1.2	1.3	−0.8	−1.0	−0.3	−0.5	1.2	1.0
22	0.7	0.6	0.5	0.4	1.2	1.3	−1.3	−1.1

Table 3. State-dependent parameter $d^{\theta}_{qh}(\vartheta^{\theta}_{q}(t))$ with $\theta \in \Theta$.

Subscript	Parameter							
	$\grave{d}^{r}_{qh}$	$\acute{d}^{r}_{qh}$	$\grave{d}^{\imath}_{qh}$	$\acute{d}^{\imath}_{qh}$	$\grave{d}^{\jmath}_{qh}$	$\acute{d}^{\jmath}_{qh}$	$\grave{d}^{\kappa}_{qh}$	$\acute{d}^{\kappa}_{qh}$
11	0.6	0.3	−1.5	−1.2	1.1	1.0	0.5	0.3
12	1.4	1.3	0.6	0.8	1.4	1.3	0.6	0.3
21	1.3	1.2	−0.7	−0.4	0.6	0.8	1.5	1.4
22	0.6	0.4	0.5	0.3	−1.3	−1.5	1.1	1.3

Table 4. State-dependent parameter $z^{\theta}_{qh}(\vartheta^{\theta}_{q}(t))$ with $\theta \in \Theta$.

Subscript	Parameter							
	$\grave{z}^{r}_{qh}$	$\acute{z}^{r}_{qh}$	$\grave{z}^{\imath}_{qh}$	$\acute{z}^{\imath}_{qh}$	$\grave{z}^{\jmath}_{qh}$	$\acute{z}^{\jmath}_{qh}$	$\grave{z}^{\kappa}_{qh}$	$\acute{z}^{\kappa}_{qh}$
11	1.4	1.5	1.1	1.0	−0.3	−0.5	0.8	0.6
12	0.9	0.7	1.2	1.3	−1.4	−1.3	0.7	0.8
21	0.5	0.6	−0.8	−0.6	1.1	1.4	−0.7	−0.5
22	1.3	1.2	0.9	0.8	−1.1	−1.0	0.7	0.8

$$
\begin{aligned}
\mathbf{y}_l\big(\vartheta(t)\big) &= \mathbf{x}_l\big(\vartheta(t)\big) \\
&= \tanh\big(\vartheta^{r}(t)\big) + \imath\tanh\big(\vartheta^{\imath}(t)\big) + \jmath\tanh\big(\vartheta^{\jmath}(t)\big) + \kappa\tanh\big(\vartheta^{\kappa}(t)\big), \\
g_1(t) &= 3(1+\imath+\jmath+\kappa)\sin(t), \qquad g_2(t) = 5(1+\imath+\jmath+\kappa)\cos(t),
\end{aligned}
$$

other parameters are shown in Tables 1–4, respectively.

By simple calculation, one can obtain $\varrho = 1$, $\epsilon = 2$, $\tau = 0.1$, $\upsilon_1 = 0.3$, $\upsilon_2 = 0.8$, $Y_1 = Y_2 = X_1 = X_2 = 1$. Figures 1–2 manifest time responses of states of systems (1) and (2) without controller and show that systems (1) and (2) have not reached the synchronization.

Take $a = 0.005$, $\omega_1 = \varpi_1 = \omega_2 = \varpi_2 = 0.5$, $\alpha_1(0) = \alpha_2(0) = \beta_1(0) = \beta_2(0) = 0$, $\alpha_1 = 17$, $\alpha_2 = 18$, $\beta_1 = 25$, $\beta_2 = 26$, one can obtain $\Gamma_1 = 15.9909$, $\Gamma_2 = 16.7501$, $\Xi_1 = 30.3644$, $\Xi_2 = 29.4216$,

$$\Gamma_1 - (\alpha_1 + \xi_1) = -6.0091 < 0, \qquad \Gamma_2 - (\alpha_2 + \xi_2) = -6.2499 < 0,$$
$$\Xi_1 - (\delta_1 + \beta_1) = -5.2644 < 0, \qquad \Xi_2 - (\delta_2 + \beta_2) = -3.3216 < 0.$$

Consequently, all the conditions of Theorem 1 hold. The synchronization time curves between systems (1) and (2) with HSCs (3) and (4) are presented in Figs. 3–4.

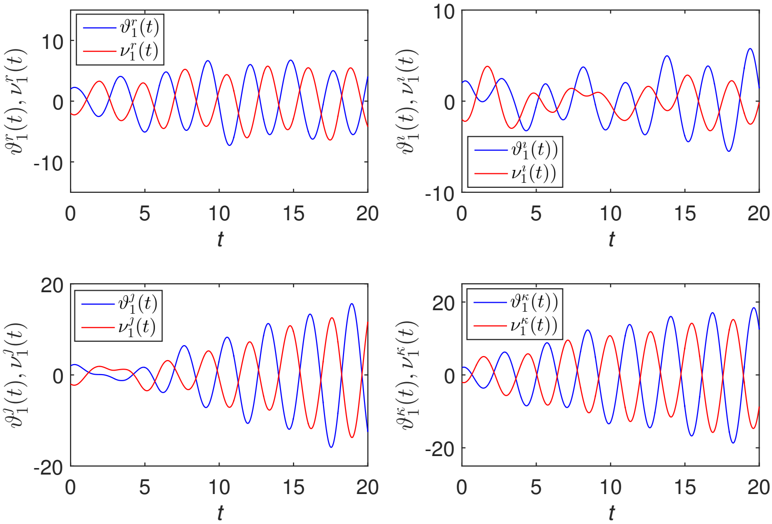

Figure 1. Evaluations of states $\vartheta_1^\theta(t), \nu_1^\theta(t)$ with $v_1(t) = v_2(t) = 0$.

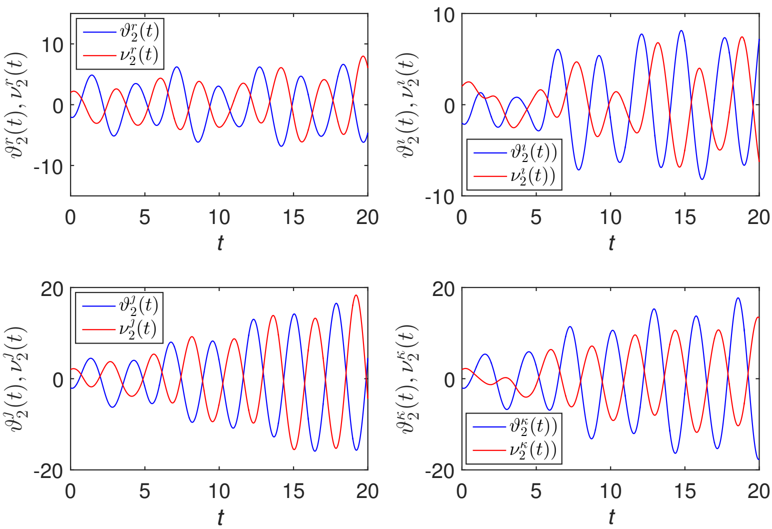

Figure 2. Evaluations of states $\vartheta_2^\theta(t), \nu_2^\theta(t)$ with $v_1(t) = v_2(t) = 0$.

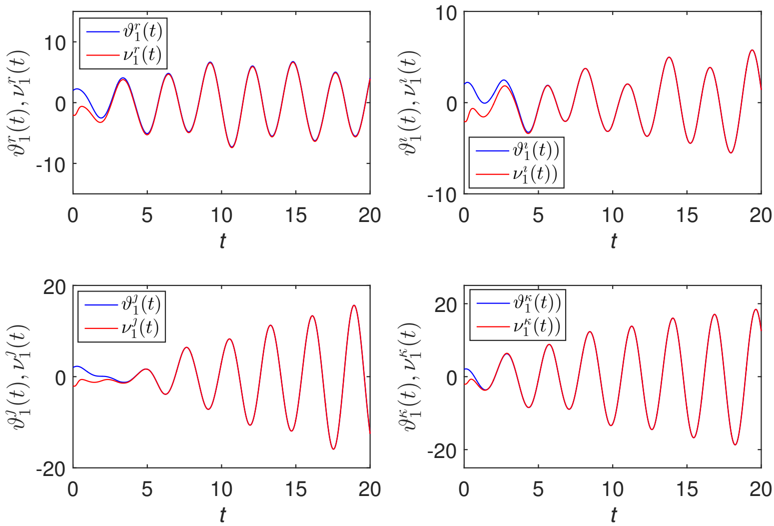

Figure 3. Synchronization curves of states $\vartheta_1^{\theta}(t), \nu_1^{\theta}(t)$ with HSCs (3) and (4).

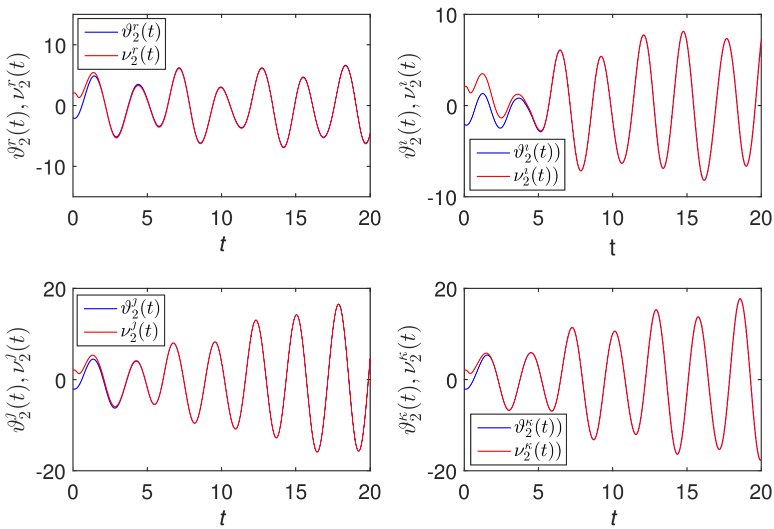

Figure 4. Synchronization curves of states $\vartheta_2^{\theta}(t), \nu_2^{\theta}(t)$ with HSCs (3) and (4).

Figure 6 reveals the variation for the control gain parameters $\alpha_1(t)$, $\alpha_2(t)$, $\beta_1(t)$, $\beta_2(t)$ for HSC (3). Thus, Figs. 1–5 confirm the effectiveness of Theorem 1. Choose synchronization errors $\mu_1^{\theta}(t)$, $\mu_2^{\theta}(t)$ with randomly 20 initial conditions in $[-4, 4]$, Fig. 5 manifests the trajectories of synchronization errors $\mu_1^{\theta}(t)$, $\mu_2^{\theta}(t)$ between models (1) and (2).

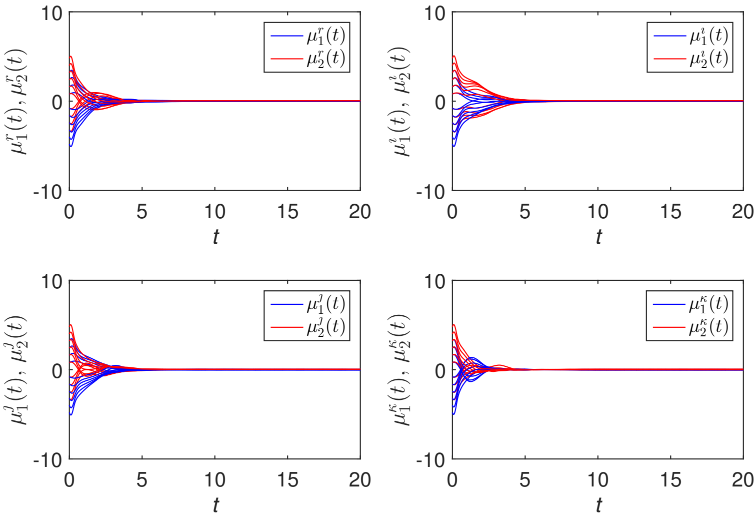

Figure 5. Evaluations of errors $\mu_1^\theta(t), \mu_2^\theta(t)$ with HSCs (3) and (4).

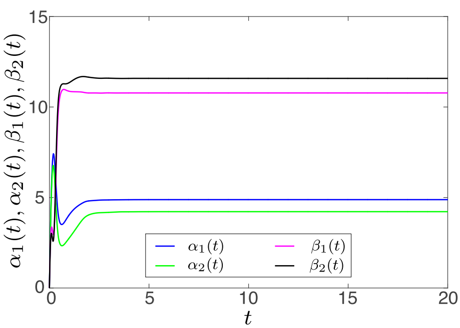

Figure 6. Time curves of the control gains $\alpha_1(t), \alpha_2(t), \beta_1(t), \beta_2(t)$ for HSC (3).

5 Conclusions

This article investigated the GES of SOSDSQVNNs with neutral-type and mixed time-varying delays by HSCs. Different from the usual reduced-order method of SONNs [7, 12, 13, 17, 18, 23, 28] and separation method of QVNNs [9, 13, 14, 23], some new LKFs are introduced to straight analyze the GES for the drive and response systems in quaternion field, and some sufficient conditions are established in the form of algebraic inequalities. The obtained results here perfect the existing studies on the GES as extra cases. Finally, the numerical simulation results check on the validity of the theoretical analysis. In the coming work, some other dynamical behaviors, such as convergence and passivity, can be taken into account for model (1) by non-reduced order and non-separation strategies.

Appendix A: Proof of Theorem 1

Consider a LKF as follows:

$$V(t) = \frac{1}{2}\mathrm{e}^{2at}\sum_{q=1}^{m}\eta_q\mu_q(t)\overline{\mu_q(t)} + V_1(t) + V_2(t) + V_3(t), \tag{A.1}$$

where

$$\begin{aligned}
V_1(t) &= \frac{1}{2}\mathrm{e}^{2at}\sum_{q=1}^{m}(\mu_q(t)+\dot{\mu}_q(t))\overline{(\mu_q(t)+\dot{\mu}_q(t))},\\
V_2(t) &= \frac{1}{1-\upsilon_1}\sum_{q=1}^{m}\sum_{l=1}^{m}|c_{ql}|Y_l\int\limits_{t-\rho(t)}^{t}\mathrm{e}^{2a(w+\varrho)}\mu_l(w)\overline{\mu_l(w)}\,\mathrm{d}w\\
&\quad + \frac{1}{1-\upsilon_2}\sum_{q=1}^{m}\sum_{l=1}^{m}|d_{ql}|X_l\int\limits_{t-\varepsilon(t)}^{t}\mathrm{e}^{2a(w+\epsilon)}\dot{\mu}_l(w)\overline{\dot{\mu}_l(w)}\,\mathrm{d}w,\\
V_3(t) &= \frac{1}{2}\sum_{q=1}^{m}\frac{1}{\omega_q}(\alpha_q-\alpha_q(t))^2 + \frac{1}{2}\sum_{q=1}^{m}\frac{1}{\varpi_q}(\beta_q-\beta_q(t))^2.
\end{aligned}$$

Computing the derivatives of $V(t)$, $V_1(t)$, $V_2(t)$, $V_3(t)$ along system (6), one can get

$$\begin{aligned}
\dot{V}(t)|_{(6)} &= a\mathrm{e}^{2at}\sum_{q=1}^{m}\eta_q\mu_q(t)\overline{\mu_q(t)} + \frac{1}{2}\mathrm{e}^{2at}\sum_{q=1}^{m}\eta_q\big[\mu_q(t)\overline{\dot{\mu}_q(t)} + \dot{\mu}_q(t)\overline{\mu_q(t)}\big]\\
&\quad + \dot{V}_1(t) + \dot{V}_2(t) + \dot{V}_3(t)\\
&= \mathrm{e}^{2at}\sum_{q=1}^{m}\big[a\eta_q\mu_q(t)\overline{\mu_q(t)} + \eta_q\operatorname{Re}\big(\overline{\mu_q(t)}\dot{\mu}_q(t)\big)\big]\\
&\quad + \dot{V}_1(t) + \dot{V}_2(t) + \dot{V}_3(t),
\end{aligned} \tag{A.2}$$

$$\begin{aligned}
\dot{V}_1(t)|_{(6)} &= a\mathrm{e}^{2at}\sum_{q=1}^{m}\big(\mu_q(t)+\dot{\mu}_q(t)\big)\overline{\big(\mu_q(t)+\dot{\mu}_q(t)\big)}\\
&\quad + \frac{1}{2}\mathrm{e}^{2at}\sum_{q=1}^{m}\big(\dot{\mu}_q(t)+\ddot{\mu}_q(t)\big)\overline{\big(\dot{\mu}_q(t)+\mu_q(t)\big)}\\
&\quad + \frac{1}{2}\mathrm{e}^{2at}\sum_{q=1}^{m}\big(\mu_q(t)+\dot{\mu}_q(t)\big)\overline{\big(\dot{\mu}_q(t)+\ddot{\mu}_q(t)\big)}\\
&= \mathrm{e}^{2at}\sum_{q=1}^{m}\big[a\mu_q(t)\overline{\mu_q(t)} + (a+1)\dot{\mu}_q(t)\overline{\dot{\mu}_q(t)} + (2a+1)\operatorname{Re}\big(\overline{\mu_q(t)}\dot{\mu}_q(t)\big)\\
&\quad + \frac{1}{2}\big(\ddot{\mu}_q(t)\overline{\dot{\mu}_q(t)} + \ddot{\mu}_q(t)\overline{\mu_q(t)} + \dot{\mu}_q(t)\overline{\ddot{\mu}_q(t)} + \mu_q(t)\overline{\ddot{\mu}_q(t)}\big)\big]
\end{aligned}$$

$$
\begin{aligned}
&= \mathrm{e}^{2at} \sum_{q=1}^{m} \Bigg\{ \big[a - \big(\xi_q + \alpha_r(t)\big)\big] \mu_q(t)\overline{\mu_q(t)} \\
&\quad + \big[a + 1 - \big(\delta_q + \beta_q(t)\big)\big] \dot{\mu}_q(t)\overline{\dot{\mu}_q(t)} \\
&\quad + \big[2a + 1 - \big(\xi_q + \delta_q + \alpha_q(t) + \beta_q(t)\big)\big] \operatorname{Re}\big(\overline{\mu_q(t)}\dot{\mu}_q(t)\big) \\
&\quad + \sum_{l=1}^{m} \big[\operatorname{Re}\big(b_{ql}\overline{\dot{\mu}_q(t)}\mathbf{y}_l\big(\mu_l(t)\big)\big) + \operatorname{Re}\big(c_{ql}\overline{\dot{\mu}_q(t)}\mathbf{y}_l\big(\mu_l\big(t - \rho(t)\big)\big)\big) \\
&\quad + \operatorname{Re}\big(d_{ql}\overline{\dot{\mu}_q(t)}\mathbf{x}_l\big(\dot{\mu}_l\big(t-\varepsilon(t)\big)\big)\big) + \operatorname{Re}\big(b_{ql}\overline{\mu_q(t)}\mathbf{y}_l\big(\mu_l(t)\big)\big) \\
&\quad + \operatorname{Re}\big(c_{ql}\overline{\mu_q(t)}\mathbf{y}_l\big(\mu_l\big(t-\rho(t)\big)\big)\big) + \operatorname{Re}\big(d_{ql}\overline{\mu_q(t)}\mathbf{x}_l\big(\dot{\mu}_l\big(t-\varepsilon(t)\big)\big)\big)\big] \Bigg\}. \qquad \text{(A.3)}
\end{aligned}
$$

From Assumption A, using the properties of norm of quaternion and mean-value inequality, one obtains

$$
\begin{aligned}
&\sum_{q=1}^{m}\sum_{l=1}^{m} \operatorname{Re}\big(b_{ql}\overline{\dot{\mu}_q(t)}\mathbf{y}_l\big(\mu_l(t)\big)\big) \\
&\quad \leqslant \sum_{q=1}^{m}\sum_{l=1}^{m} |b_{ql}|Y_l\big|\overline{\dot{\mu}_q(t)}\big|\big|\mu_l(t)\big| \\
&\quad \leqslant \sum_{q=1}^{m}\sum_{l=1}^{m} \frac{1}{2}\big(|b_{lq}|Y_q\mu_q(t)\overline{\mu_q(t)} + |b_{ql}|Y_l\dot{\mu}_q(t)\overline{\dot{\mu}_q(t)}\big). \qquad \text{(A.4)}
\end{aligned}
$$

Analogously, one can get

$$
\begin{aligned}
&\operatorname{Re}\big(c_{ql}\overline{\dot{\mu}_q(t)}\mathbf{y}_l\big(\mu_l\big(t-\rho(t)\big)\big)\big) \\
&\quad \leqslant \frac{1}{2}|c_{ql}|Y_l\big(\dot{\mu}_q(t)\overline{\dot{\mu}_q(t)} + \mu_l\big(t-\rho(t)\big)\overline{\mu_l\big(t-\rho(t)\big)}\big), \qquad \text{(A.5)}
\end{aligned}
$$

$$
\begin{aligned}
&\operatorname{Re}\big(d_{ql}\overline{\dot{\mu}_q(t)}\mathbf{x}_l\big(\dot{\mu}_l\big(t-\varepsilon(t)\big)\big)\big) \\
&\quad \leqslant \frac{1}{2}|d_{ql}|X_l\big(\dot{\mu}_q(t)\overline{\dot{\mu}_q(t)} + \dot{\mu}_l\big(t-\varepsilon(t)\big)\overline{\dot{\mu}_l\big(t-\varepsilon(t)\big)}\big), \qquad \text{(A.6)}
\end{aligned}
$$

$$
\sum_{q=1}^{m}\sum_{l=1}^{m} \operatorname{Re}\big(b_{ql}\overline{\mu_q(t)}\mathbf{y}_l\big(\mu_l(t)\big)\big) \leqslant \sum_{q=1}^{m}\sum_{l=1}^{m} \frac{1}{2}\big(|b_{ql}|Y_l + |b_{lq}|Y_q\big)\mu_q(t)\overline{\mu_q(t)}, \qquad \text{(A.7)}
$$

$$
\begin{aligned}
&\operatorname{Re}\big(c_{ql}\overline{\mu_q(t)}\mathbf{y}_l\big(\mu_l\big(t-\rho(t)\big)\big)\big) \\
&\quad \leqslant \frac{1}{2}|c_{ql}|Y_l\big(\mu_q(t)\overline{\mu_q(t)} + \mu_l\big(t-\rho(t)\big)\overline{\mu_l\big(t-\rho(t)\big)}\big), \qquad \text{(A.8)}
\end{aligned}
$$

$$
\begin{aligned}
&\operatorname{Re}\big(d_{ql}\overline{\mu_q(t)}\mathbf{x}_l\big(\dot{\mu}_l\big(t-\varepsilon(t)\big)\big)\big) \\
&\quad \leqslant \frac{1}{2}|d_{ql}|X_l\big(\mu_q(t)\overline{\mu_q(t)} + \dot{\mu}_l\big(t-\varepsilon(t)\big)\overline{\dot{\mu}_l\big(t-\varepsilon(t)\big)}\big). \qquad \text{(A.9)}
\end{aligned}
$$

Substituting (A.4)–(A.9) into (A.3), one has

$$
\begin{aligned}
\dot{V}_1(t)|_{(6)} \leqslant \mathrm{e}^{2at} \sum_{q=1}^{m} \Biggl\{ & \bigl[a - \bigl(\xi_q + \alpha_q(t)\bigr) \\
& + \frac{1}{2} \sum_{l=1}^{m} \bigl(|b_{ql}|Y_l + |c_{ql}|Y_l + |d_{ql}|X_l\bigr) + \sum_{l=1}^{m} |b_{lq}|Y_q\bigr] \mu_q(t)\overline{\mu_q(t)} \\
& + \Biggl[a + 1 - \bigl(\delta_q + \beta_q(t)\bigr) + \frac{1}{2} \sum_{l=1}^{m} \bigl(|b_{ql}|Y_l + |c_{ql}|Y_l + |d_{ql}|X_l\bigr)\Biggr] \dot{\mu}_q(t)\overline{\dot{\mu}_q(t)} \\
& + \bigl[2a + 1 - \bigl(\xi_q + \delta_q + \alpha_q(t) + \beta_q(t)\bigr)\bigr] \operatorname{Re}\bigl(\overline{\mu_q(t)}\dot{\mu}_q(t)\bigr) \\
& + \sum_{l=1}^{m} \bigl[|c_{ql}|Y_l \mu_l\bigl(t - \rho(t)\bigr)\overline{\mu_l(t - \rho(t))} \\
& + |d_{ql}|X_l \dot{\mu}_l\bigl(t - \varepsilon(t)\bigr)\overline{\dot{\mu}_l\bigl(t - \varepsilon(t)\bigr)}\bigr] \Biggr\}.
\end{aligned} \tag{A.10}
$$

In addition, one can get

$$
\begin{aligned}
\dot{V}_2(t)|_{(6)} \leqslant \mathrm{e}^{2at} \Biggl\{ & \sum_{q=1}^{m} \sum_{l=1}^{m} \Biggl[\frac{|c_{lq}|Y_q \mathrm{e}^{2a\varrho}}{1 - \upsilon_1} \mu_q(t)\overline{\mu_q(t)} - |c_{ql}|Y_l \mu_l\bigl(t - \rho(t)\bigr)\overline{\mu_l\bigl(t - \rho(t)\bigr)} \\
& + \frac{|d_{lq}|X_q \mathrm{e}^{2a\epsilon}}{1 - \upsilon_2} \dot{\mu}_q(t)\overline{\dot{\mu}_q(t)} - |d_{ql}|X_l \dot{\mu}_l\bigl(t - \varepsilon(t)\bigr)\overline{\dot{\mu}_l\bigl(t - \varepsilon(t)\bigr)}\Biggr] \Biggr\},
\end{aligned} \tag{A.11}
$$

$$
\begin{aligned}
\dot{V}_3(t)|_{(6)} &= -\sum_{q=1}^{m} \frac{1}{\omega_q} \bigl[\alpha_q - \alpha_q(t)\bigr] \dot{\alpha}_q(t) - \sum_{q=1}^{m} \frac{1}{\varpi_q} \bigl[\beta_q - \beta_q(t)\bigr] \dot{\beta}_q(t) \\
&\leqslant \mathrm{e}^{2at} \sum_{q=1}^{m} \bigl\{ \bigl[\alpha_q(t) - \alpha_q\bigr] \mu_q(t)\overline{\mu_q(t)} + \bigl[\beta_q(t) - \beta_q\bigr] \dot{\mu}_q(t)\overline{\dot{\mu}_q(t)} \\
&\quad + \bigl[\alpha_q(t) + \beta_q(t) - (\alpha_q + \beta_q)\bigr] \operatorname{Re}\bigl(\overline{\mu_q(t)}\dot{\mu}_q(t)\bigr) \bigr\}.
\end{aligned} \tag{A.12}
$$

Put (A.10)–(A.12) into (A.2), one can obtain

$$
\begin{aligned}
\dot{V}(t)|_{(6)} \leqslant \mathrm{e}^{2at} \sum_{q=1}^{m} \Biggl\{ & \Biggl[a(\eta_q + 1) - (\xi_q + \alpha_q) + \frac{1}{2} \sum_{l=1}^{m} \bigl(|b_{ql}|Y_l + |c_{ql}|Y_l + |d_{ql}|X_l\bigr) \\
& + \sum_{l=1}^{m} \biggl(|b_{lq}| + \frac{\mathrm{e}^{2a\varrho}}{1 - \upsilon_1}|c_{lq}|\biggr) Y_q \Biggr] \mu_q(t)\overline{\mu_q(t)} + \Biggl[a + 1 - (\delta_q + \beta_q) \\
& + \frac{1}{2} \sum_{l=1}^{m} \bigl(|b_{ql}|Y_l + |c_{ql}|Y_l + |d_{ql}|X_l\bigr) + \sum_{l=1}^{m} \frac{\mathrm{e}^{2a\epsilon}}{1 - \upsilon_2} |d_{lq}|X_q \Biggr] \dot{\mu}_q(t)\overline{\dot{\mu}_q(t)}
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \big[\eta_q + 2a + 1 - (\xi_q + \delta_q + \alpha_q + \beta_q)\big]\operatorname{Re}\big(\overline{\mu_q(t)}\dot{\mu}_q(t)\big)\Bigg\} \\
&= \mathrm{e}^{2at}\sum_{q=1}^{m}\big\{\big[\varGamma_q - (\xi_q + \alpha_q)\big]\mu_q(t)\overline{\mu_q(t)} + \big[\varXi_q - (\delta_q + \beta_q)\big]\dot{\mu}_q(t)\overline{\dot{\mu}_q(t)}\big\} \\
&\leqslant 0,
\end{aligned}
$$

which means $V(t) \leqslant V(0)$ and $\|\mu(t)\| \leqslant \sqrt{2V(0)/\underline{\eta}}\mathrm{e}^{-at}$ with $\underline{\eta} = \min_{q\in\varUpsilon}\{\eta_q\}$ for all $t \geqslant 0$. From Definition 1 systems (1) and (2) get the GES by HSCs (3) and (4). The proof is completed.

Appendix B: Proof of Theorem 2

Take a LKF as same to (A.1) except for

$$
V_2(t) = \frac{1}{1-\upsilon_1}\sum_{r=1}^{m}\sum_{l=1}^{m}|c_{rl}|Y_l\int_{t-\rho(t)}^{t}\mathrm{e}^{2a(w+\varrho)}\mu_l(w)\overline{\mu_l(w)}\,\mathrm{d}w,
$$

the rest of the proof is analogous to those of Theorem 1 above.

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