Nonlinear Analysis: Modelling and Control, Vol. 27, No. 4, 740–765
https://doi.org/10.15388/namc.2022.27.27446

Vilnius University Press

Modeling and analysis of SIR epidemic dynamics in immunization and cross-infection environments: Insights from a stochastic model*

Zhengbo Chang[a,b], **Xinzhu Meng**[b,c,1], **Tasawar Hayat**[d], **Aatef Hobiny**[c]

[a]College of Electrical Engineering and Automation,
Shandong University of Science and Technology,
Qingdao 266590, China
zbchang98@126.com

[b]College of Mathematics and Systems Science,
Shandong University of Science and Technology,
Qingdao 266590, China
mxz721106@sdust.edu.cn

[c]Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group,
Department of Mathematics, King Abdulaziz University,
Jeddah 21589, Saudi Arabia

[d]Department of Mathematics,
Quaid-i-Azam University 45320,
Isamabad 44000, Pakistan

Received: July 27, 2021 / **Revised:** January 27, 2022 / **Published online:** May 2, 2022

Abstract. We propose a stochastic SIR model with two different diseases cross-infection and immunization. The model incorporates the effects of stochasticity, cross-infection rate and immunization. By using stochastic analysis and Khasminski ergodicity theory, the existence and boundedness of the global positive solution about the epidemic model are firstly proved. Subsequently, we theoretically carry out the sufficient conditions of stochastic extinction and persistence of the diseases. Thirdly, the existence of ergodic stationary distribution is proved. The results reveal that white noise can affect the dynamics of the system significantly. Finally, the numerical simulation is made and consistent with the theoretical results.

Keywords: stochastic SIR model, nonlinear incidence rates, disease cross-infection, persistence in mean, ergodic stationary distribution.

*This work was supported by the SDUST Research Fund (2014TDJH102), the Research Fund for the Taishan Scholar Project of Shandong Province of China, Shandong Provincial Natural Science Foundation of China (ZR2019MA003).

[1]Corresponding author.

1 Introduction

Mathematical modeling is an important tool that can help us understand the transmission of an infectious disease. Many scholars [1–3, 6–11, 13, 15, 16, 18, 21, 24, 30] have put forward mathematical models and have made contributions to disease control.

As far as we know, a classic and important SIR model was early investigated by Kermazk and McKendrick [11] in 1927. In classical SIR models, the infected patients can recover health with treatment. Many scholars have also investigated the SIR model in different situations. Capasso et al. [3] summarized Kermazk–McKendrick model and took into account nonlinear incidence phenomena for large numbers of infectives. Meanwhile, Capasso et al. expanded the threshold theory and laid a foundation for solving the stochastic nonlinear threshold. Hethcote [10] gave a qualitative analysis of nonlinear incidence SIR model, which is appropriate for viral agent diseases and considered social impact. Liu [18] investigated a deterministic and modified nonlinear SIR model with periodic solutions. In [18], the author also explored the corresponding stochastic epidemic models and the asymptotic behavior of the solution. Ghosh et al. [7] gave an SIR model with nonmonotonic incidence and logistic growth. In [7], authors studied the condition for backward bifurcation and Hopf bifurcation and solved the optimal control problem. Dieu et al. [6] classified a stochastic SIR model and developed ergodicity of the underlying system.

In classical epidemiological models, bilinear and standard incidence rates [28] are suitable for a small number of people in a short time, so many scholars use nonlinear incidence [2, 6, 7, 10, 16, 18, 21]. Logistic model [7] is more in line with the law of social population growth. In references [16, 21], the authors introduced the stochastic epidemic model with cross-infection of diseases. It has become a common phenomenon that people are infected with different diseases at the same time. On the basis of reference [2, 6, 7, 10, 16, 18, 21], a deterministic SIR model with cross-infection and permanent immunization is proposed in which the nonlinear incidence is used. The corresponding model is as follows:

$$\begin{aligned}
\mathrm{d}S(t) &= \left[rS\left(1-\frac{S}{K}\right)-\frac{p\beta_1 SI_1}{\alpha_1+S}-\frac{(1-p)\beta_2 SI_2}{\alpha_2+I_2}-\mu_1 S-\delta S\right]\mathrm{d}t,\\
\mathrm{d}I_1(t) &= \left[\frac{p\beta_1 SI_1}{\alpha_1+S}-(\mu_2+\gamma_1)I_1\right]\mathrm{d}t,\\
\mathrm{d}I_2(t) &= \left[\frac{(1-p)\beta_2 SI_2}{\alpha_2+I_2}-(\mu_3+\gamma_2)I_2\right]\mathrm{d}t,\\
\mathrm{d}R(t) &= \left[\gamma_1 I_1+\gamma_2 I_2-\mu_4 R+\delta S\right]\mathrm{d}t,
\end{aligned}\tag{1}$$

where $S(t)$ and $R(t)$ with the natural mortality rate μ_1 and μ_4, respectively, are the susceptible class and the removed class, respectively. $I_1(t)$ and $I_2(t)$ are the individuals with cross-infection at time t. β_1 and β_2 are the contact rates. p is the proportion of patients infected by two diseases. μ_2 and μ_3 are the mortality of cross-infection diseases, which include natural mortality and mortality due to diseases. γ_1 and γ_2 are recovery rates of cross-infection diseases, respectively. δ is constant vaccination rate of susceptible class.

The susceptible class will have permanent immunity after vaccination. All parameters are positive. Functions $\beta_1 S I_1/(\alpha_1 + S)$ and $\beta_2 S I_2/(\alpha_2 + I_2)$ represent nonlinear incidence rates for cross-infection diseases.

However, the spread of diseases is often affected by environmental noise [1,4,8,9,13,15,16,21,24,28,30], a lot of literatures add randomness to reflect real life more accurately. The properties for stationary distribution of random variables were proved in [8]. [15] gave an SIR epidemic model that a susceptible person is infected with a disease and temporarily immunized. All references [1,4,9,13,24,28,30] are stochastic epidemic models with nonlinear incidence that has been used in chemostat model [25]. We assume that the mortality rates of $S(t)$, $I_1(t)$, $I_2(t)$, $R(t)$ are disturbed by white noise in system (1), then we have

$$\begin{aligned}
\mathrm{d}S(t) &= \left[rS\left(1 - \frac{S}{K}\right) - \frac{p\beta_1 S I_1}{\alpha_1 + S} - \frac{(1-p)\beta_2 S I_2}{\alpha_2 + I_2} - \mu_1 S - \delta S\right]\mathrm{d}t \\
&\quad + \sigma_1 S\,\mathrm{d}B_1(t), \\
\mathrm{d}I_1(t) &= \left[\frac{p\beta_1 S I_1}{\alpha_1 + S} - (\mu_2 + \gamma_1)I_1\right]\mathrm{d}t + \sigma_2 I_1\,\mathrm{d}B_2(t), \\
\mathrm{d}I_2(t) &= \left[\frac{(1-p)\beta_2 S I_2}{\alpha_2 + I_2} - (\mu_3 + \gamma_2)I_2\right]\mathrm{d}t + \sigma_3 I_2\,\mathrm{d}B_3(t), \\
\mathrm{d}R(t) &= \left[\gamma_1 I_1 + \gamma_2 I_2 - \mu_4 R + \delta S\right]\mathrm{d}t + \sigma_4 R\,\mathrm{d}B_4(t).
\end{aligned}$$

Thus, we can consider the following system:

$$\begin{aligned}
\mathrm{d}S &= \left[rS\left(1 - \frac{S}{K}\right) - \frac{p\beta_1 S I_1}{\alpha_1 + S} - \frac{(1-p)\beta_2 S I_2}{\alpha_2 + I_2} - \mu_1 S - \delta S\right]\mathrm{d}t \\
&\quad + \sigma_1 S\,\mathrm{d}B_1(t), \\
\mathrm{d}I_1 &= \left[\frac{p\beta_1 S I_1}{\alpha_1 + S} - (\mu_2 + \gamma_1)I_1\right]\mathrm{d}t + \sigma_2 I_1\,\mathrm{d}B_2(t), \\
\mathrm{d}I_2 &= \left[\frac{(1-p)\beta_2 S I_2}{\alpha_2 + I_2} - (\mu_3 + \gamma_2)I_2\right]\mathrm{d}t + \sigma_3 I_2\,\mathrm{d}B_3(t).
\end{aligned} \tag{2}$$

In the next, we only consider the dynamic properties of system (2) through differential equation theories.

Some notations of stochastic differential equations can be seen in [19]. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbf{P})$ be a complete probability space, which has a filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ and satisfies the usual conditions. The functions $B_1(t)$, $B_2(t)$, $B_3(t)$ are Brownian motion defined on this complete probability space. We define $\mathbb{R}^m_+ = \{y \in \mathbb{R}^m\colon\ y_i > 0,\ 1 \leqslant i \leqslant m\}$. Let $f(t)$ be an integrable function on $[0, +\infty)$ and define $\langle f(t)\rangle = (1/t)\int_0^t f(\theta)\,\mathrm{d}\theta$. This paper mainly studies mathematical modeling and theoretical proof as well as the influence of parameter changes on the model.

We arrange the article as follows: Section 2 proves that system (2) has a global positive solution, which is unique. In Section 3, stochastic boundedness of the solution is explored. Section 4 investigates the extinction and persistence conditions of the stochastic system (2). We show the maximum value at point S^0 and the existence of a unique ergodic stationary distribution in Section 5. At last, we give the numerical simulations and a brief conclusion.

2 Global positive solution

Theorem 1. *For any given initial value $(S(0), I_1(0), I_2(0)) \in \mathbb{R}^3_+$ and $t \geqslant 0$, there is a unique positive solution $(S(t), I_1(t), I_2(t))$ of system (2), which belongs to $\mathbb{R}^3_+$ with probability one.*

Proof. By standard arguments, there is a unique positive local solution $(S(t), I_1(t), I_2(t))$ on $t \in [0, \tau_e)$, provided that $(S(0), I_1(0), I_2(0)) \in \mathbb{R}^3_+$. Here τ_e denotes the explosion time. Next, we need to prove the global property of the solution.

Define a C^2 function as

$$\begin{aligned} V(S, I_1, I_2) = S &- \frac{\alpha_1(\mu_2+\gamma_1)}{p\beta_1} - \frac{\alpha_1(\mu_2+\gamma_1)}{p\beta_1} \ln \frac{p\beta_1 S}{\alpha_1(\mu_2+\gamma_1)} \\ &+ I_1 - 1 - \ln I_1 + I_2 - 1 - \ln I_2. \end{aligned}$$

Since for any $u > 0$, we have $u - 1 - \ln u \geqslant 0$, it follows that V is a positive definite function. Through Itô's formula, we have

$$\begin{aligned} \mathrm{d}V(S, I_1, I_2) = \mathcal{L}V(S, I_1, I_2)\,\mathrm{d}t &+ \sigma_1\left(S - \frac{\alpha_1(\mu_2+\gamma_1)}{p\beta_1}\right)\mathrm{d}B_1(t) \\ &+ \sigma_2(I_1 - 1)\,\mathrm{d}B_2(t) + \sigma_3(I_2 - 1)\,\mathrm{d}B_3(t), \end{aligned}$$

where

$$\begin{aligned} &\mathcal{L}V(S, I_1, I_2) \\ &\quad = \left(1 - \frac{\alpha_1(\mu_2+\gamma_1)}{p\beta_1 S}\right)\left[rS\left(1 - \frac{S}{K}\right) - \frac{p\beta_1 S I_1}{\alpha_1 + S} - \frac{(1-p)\beta_2 S I_2}{\alpha_2 + I_2} - \mu_1 S - \delta S\right] \\ &\qquad + \frac{\alpha_1(\mu_2+\gamma_1)\sigma_1^2}{2p\beta_1} + \left(1 - \frac{1}{I_1}\right)\left[\frac{p\beta_1 S I_1}{\alpha_1 + S} - (\mu_2+\gamma_1)I_1\right] + \frac{\sigma_2^2}{2} \\ &\qquad + \left(1 - \frac{1}{I_2}\right)\left[\frac{(1-p)\beta_2 S I_2}{\alpha_2 + I_2} - (\mu_3+\gamma_2)I_2\right] + \frac{\sigma_3^2}{2} \\ &\quad \leqslant -\frac{r}{K}S^2 + \left(r + \frac{r\alpha_1(\mu_2+\gamma_1)}{p\beta_1 K}\right)S + \frac{(1-p)\alpha_1\beta_2(\mu_2+\gamma_1)}{p\beta_1} \\ &\qquad + \frac{\alpha_1(\mu_2+\gamma_1)(\mu_1+\delta)}{p\beta_1} + \mu_2 + \mu_3 + \gamma_1 + \gamma_2 + \frac{\alpha_1(\mu_2+\gamma_1)\sigma_1^2}{2p\beta_1} + \frac{\sigma_2^2}{2} + \frac{\sigma_3^2}{2} \\ &\quad \leqslant \sup_{S\in\mathbb{R}_+}\left\{-\frac{r}{K}S^2 + \left(r + \frac{r\alpha_1(\mu_2+\gamma_1)}{p\beta_1 K}\right)S\right\} + \frac{(1-p)\alpha_1\beta_2(\mu_2+\gamma_1)}{p\beta_1} \\ &\qquad + \frac{\alpha_1(\mu_2+\gamma_1)(\mu_1+\delta)}{p\beta_1} + \mu_2 + \mu_3 + \gamma_1 + \gamma_2 + \frac{\alpha_1(\mu_2+\gamma_1)\sigma_1^2}{2p\beta_1} + \frac{\sigma_2^2}{2} + \frac{\sigma_3^2}{2} \\ &\quad := \lambda. \end{aligned}$$

Here λ is a positive constant. Through [20], we can get the global property of the unique positive solution. $\square$

3 Stochastically ultimate boundedness

Theorem 2. *The solution* $(S(t), I_1(t), I_2(t))$ *of system* (2) *with any initial value* $(S(0), I_1(0), I_2(0)) \in \mathbb{R}^3_+$ *is stochastically ultimate bounded.*

Proof. Let $u = S+I_1+I_2$, $\mu = \min\{\mu_1+\delta,\ \mu_2+\gamma_1,\ \mu_3+\gamma_2\}$. Define $W(u) = (1+u)^\nu$, where constant $\nu > 0$ will be given later. Then

$$\mathrm{d}W(u) = \mathcal{L}W(u)\,\mathrm{d}t + \nu(1+u)^{\nu-1}\bigl[\sigma_1 S\,\mathrm{d}B_1(t) + \sigma_2 I_1\,\mathrm{d}B_1(t) + \sigma_1 I_2\,\mathrm{d}B_1(t)\bigr],$$

where

$$\begin{aligned}
\mathcal{L}W(u) &= \nu(1+u)^{\nu-1}\biggl[rS\biggl(1-\frac{S}{K}\biggr) - (\mu_1+\delta)S - (\mu_2+\gamma_1)I_1 - (\mu_3+\gamma_2)I_2\biggr] \\
&\quad + \frac{\nu(\nu-1)}{2}(1+u)^{\nu-2}\bigl(\sigma_1^2 S^2 + \sigma_2^2 I_1^2 + \sigma_3^2 I_2^2\bigr) \\
&\leqslant \nu(1+u)^{\nu-1}\biggl[\frac{Kr}{4} - \mu S - \mu I_1 - \mu I_2\biggr] \\
&\quad + \frac{\nu(\nu-1)}{2}(1+u)^{\nu-2}\bigl(\sigma_1^2 S^2 + \sigma_2^2 I_1^2 + \sigma_3^2 I_2^2\bigr) \\
&\leqslant \nu(1+u)^{\nu-1}\biggl[\frac{Kr}{4} - \mu u\biggr] + \frac{\nu(\nu-1)}{2}(1+u)^{\nu-2}\bigl(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2\bigr)u^2 \\
&= \nu(1+u)^{\nu-2}\biggl((1+u)\biggl(\frac{Kr}{4} - \mu u\biggr) + \biggl(\frac{\nu-1}{2} \vee 0\biggr)\bigl(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2\bigr)u^2\biggr) \\
&= \nu(1+u)^{\nu-2}\biggl[-\biggl(\mu - \biggl(\frac{\nu-1}{2} \vee 0\biggr)\bigl(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2\bigr)u^2\biggr) \\
&\quad + \biggl(\frac{Kr}{4} - \mu\biggr)u + \biggl(\frac{Kr}{4}\biggr)\biggr].
\end{aligned}$$

Choose $\nu > 0$ such that

$$\mu - \biggl(\frac{\nu-1}{2} \vee 0\biggr)\bigl(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2\bigr) := \varphi > 0,$$

then we obtain

$$\mathcal{L}W(u) \leqslant \nu(1+u)^{\nu-2}\biggl[-\varphi u^2 + \biggl(\frac{Kr}{4} - \mu\biggr)u + \frac{Kr}{4}\biggr]$$

and

$$\begin{aligned}
\mathrm{d}W(u) &\leqslant \nu(1+u)^{\nu-2}\biggl[-\varphi u^2 + \biggl(\frac{Kr}{4} - \mu\biggr)u + \frac{Kr}{4}\biggr]\mathrm{d}t \\
&\quad + \nu(1+u)^{\nu-1}\bigl[\sigma_1 S\,\mathrm{d}B_1(t) + \sigma_2 I_1\,\mathrm{d}B_1(t) + \sigma_1 I_2\,\mathrm{d}B_1(t)\bigr].
\end{aligned}$$

Then we get

$$\begin{aligned}\mathrm{d}\bigl[\mathrm{e}^{kt}W\bigl(u(t)\bigr)\bigr] &= \mathcal{L}\bigl[\mathrm{e}^{kt}W\bigl(u(t)\bigr)\bigr]\,\mathrm{d}t\\ &\quad + \mathrm{e}^{kt}\nu(1+u)^{\nu-1}\bigl[\sigma_1 S\,\mathrm{d}B_1(t) + \sigma_2 I_1\,\mathrm{d}B_2(t) + \sigma_1 I_2\,\mathrm{d}B_3(t)\bigr],\end{aligned}$$

where $0 < k < \nu\varphi$.

Thus, one has

$$\mathbf{E}\bigl[\mathrm{e}^{kt}W\bigl(u(t)\bigr)\bigr] = W\bigl(u(0)\bigr) + \mathbf{E}\int_0^t \mathcal{L}\bigl(\mathrm{e}^{ks}W\bigl(u(s)\bigr)\,\mathrm{d}s, \tag{3}$$

where

$$\begin{aligned}&L\bigl(\mathrm{e}^{kt}W\bigl(u(t)\bigr)\bigr)\\ &\quad = k\mathrm{e}^{kt}W\bigl(u(t)\bigr) + \mathrm{e}^{kt}LW(u)\\ &\quad \leqslant \nu\mathrm{e}^{kt}(1+u)^{\nu-2}\left[\frac{k}{\nu}(1+u)^2 - \varphi u^2 + \left(\frac{Kr}{4} - \mu\right)v + \frac{Kr}{4}\right]\\ &\quad = \nu\mathrm{e}^{kt}(1+u)^{\nu-2}\left[-\left(\varphi - \frac{k}{\nu}\right)u^2 + \left(\frac{Kr}{4} - \mu + \frac{2k}{\nu}\right)u + \frac{Kr}{4} + \frac{k}{\nu}\right]\\ &\quad < \nu\mathrm{e}^{kt}Q.\end{aligned}$$

Here

$$Q := \sup_{u\in\mathbb{R}_+}(1+u)^{\nu-2}\left[-\left(\varphi - \frac{k}{\nu}\right)u^2 + \left(\frac{Kr}{4} - \mu + \frac{2k}{\nu}\right)u + \frac{Kr}{4} + \frac{k}{\nu}\right] + 1.$$

Following (3), we obtain

$$\mathbf{E}\bigl[\mathrm{e}^{kt}W\bigl(u(t)\bigr)\bigr] \leqslant W\bigl(u(0)\bigr) + \frac{\nu Q}{k}\mathrm{e}^{kt}.$$

Consequently, we have

$$\lim_{t\to\infty}\sup\mathbf{E}\bigl[\bigl(1+u(t)\bigr)^{\nu}\bigr] \leqslant \frac{\nu Q}{k} := Q_0 \quad \text{a.s.}$$

For any small constant $\varepsilon > 0$ and letting $H = [\nu Q/(k\varepsilon)]^{1/\nu}$, the Chebyshev's inequality [28] implies that

$$\mathbf{P}\bigl\{(1+u) > H\bigr\} \leqslant \frac{\mathbf{E}(1+u)^{\nu}}{H^{\nu}}.$$

Then

$$\mathbf{P}\bigl\{(1+u) > H\bigr\} \leqslant \frac{\mathbf{E}(1+u)^{\nu}}{H^{\nu}} \leqslant \frac{\frac{\nu Q}{k}}{\frac{\nu Q}{k\varepsilon}} = \varepsilon.$$

Consequently,

$$\mathbf{P}\bigl\{(1+u) \leqslant H\bigr\} \geqslant 1 - \varepsilon,$$

so $u(t)$ is ultimately bounded. Therefore, $S(t)$, $I_1(t)$, $I_2(t)$ are ultimately bounded. $\square$

4 Extinction and persistence in mean

From system (2) we get

$$\begin{aligned}\mathrm{d}S &= \left[rS\left(1-\frac{S}{K}\right)-\frac{p\beta_1 SI_1}{\alpha_1+S}-\frac{(1-p)\beta_2 SI_2}{\alpha_2+I_2}-\mu_1 S-\delta S\right]\mathrm{d}t\\ &\quad+\sigma_1 S\,\mathrm{d}B_1(t)\\ &\leqslant \left[rS\left(1-\frac{S}{K}\right)-\mu_1 S-\delta S\right]\mathrm{d}t+\sigma_1 S\,\mathrm{d}B_1(t)\\ &= \left[(r-\mu_1-\delta)S(t)-\frac{r}{K}S^2(t)\right]\mathrm{d}t+\sigma_1 S(t)\,\mathrm{d}B_1(t).\end{aligned}$$

We consider the stochastic equation

$$\mathrm{d}X(t)=\left[(r-\mu_1-\delta)X(t)-\frac{r}{K}X^2(t)\right]\mathrm{d}t+\sigma_1 X(t)\,\mathrm{d}B_1(t),\qquad X(0)=X_0. \quad (4)$$

From stochastic comparison theory we know that $S(t)\leqslant X(t)$.

From Pasquali [22] we get Lemma 1.

Lemma 1. *Define* $R_1=r/(\mu_1+\delta+\sigma_1^2/2)$. *If* $R_1<1$, *then we have* $\lim_{t\to+\infty}X(t)=0$ *a.s., while if* $R_1>1$, *system* (4) *has a unique ergodic stationary distribution* $s(\cdot)$ *with probability density*

$$\pi(X)=\frac{\kappa_2^{\kappa_1+1}}{\Gamma(\kappa_1)}X^{\kappa_1}\mathrm{e}^{-\kappa_2 X},$$

where $\kappa_1=2(r-\mu_1-\delta)/\sigma_1^2$, $\kappa_2=2r/(K\sigma_1^2)$, $\Gamma(s)=\int_0^\infty t^{s-1}\mathrm{e}^{-t}\,\mathrm{d}t$, *and*

$$\mathbf{P}\left\{\lim_{t\to\infty}\frac{1}{t}\int_0^t\omega\bigl(X(s)\bigr)\,\mathrm{d}s=\int_{R_+}\omega(X)\pi(X)\,\mathrm{d}X\right\}=1,$$

where ω *is an integrable function with measure* ρ.

Proof. From system (4) we get the stationary Fokker–Plank equation

$$\frac{\mathrm{d}}{\mathrm{d}X}\left((r-\mu_1-\delta)X(t)-\frac{r}{K}X^2(t)\right)\pi(X)-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}X^2}\bigl(\sigma_1^2X^2\pi(X)\bigr)=0$$

with probability density $\pi(X)$. Let $h(X)=\sigma_1^2X^2\pi(X)$, then we can simplify the equation in the following form:

$$\frac{\mathrm{d}h(X)}{\mathrm{d}X}-g(X)h(X)=-d,$$

where d is a constant, $g(X)=(\kappa_1X-\kappa_2X^2)/X^2$.

Then we can get

$$h(X) = H(X)\left[J - d\int_1^X \frac{1}{H(\tau)}\,\mathrm{d}\tau\right],$$

where $H(X) = \mathrm{e}^{\int_1^X g(\tau)d\tau}$, J is a constant. We can calculate

$$H(X) = \mathrm{e}^{\int_1^X g(\tau)\,\mathrm{d}\tau} = \mathrm{e}^{\int_1^X (\kappa_1\tau - \kappa_2\tau^2)\tau^2\,\mathrm{d}\tau} = \mathrm{e}^{\kappa_1 lnX - \kappa_2 X + c_1} = c_2 X^{\kappa_1}\mathrm{e}^{-\kappa_2 X},$$

where c_1 and c_2 are constants. We have

$$\pi(X) = \frac{H(X)}{\sigma_1^2 X^2}\left[J - d\int_1^X \frac{1}{H(\tau)}\,\mathrm{d}\tau\right].$$

From the conditions $\pi(X) \geqslant 0$ and $\int_0^\infty \pi(X)\,\mathrm{d}X = 1$ we integrate the above formula, and let $d = 0$, $J = (\int_0^\infty (c_2/\sigma_1^2)X^{\kappa_1-2}\mathrm{e}^{-\kappa_2 X}\,\mathrm{d}X)^{-1}$.

We can get

$$\pi(X) = \frac{\kappa_2^{\kappa_1-1}}{\Gamma(\kappa_1 - 1)}X^{\kappa_1-2}\mathrm{e}^{-\kappa_2 X}. \qquad \square$$

For the following proof, we define

$$\mathfrak{R}_1 = \frac{p\beta_1}{\mu_2 + \gamma_1 + \frac{\sigma_2^2}{2}}\int_0^{+\infty} \frac{x}{\alpha_1 + x}\pi(x)\,\mathrm{d}x, \qquad \mathfrak{R}_2 = \frac{(1-p)\beta_2}{\alpha_2(\mu_3 + \gamma_2 + \frac{\sigma_3^2}{2})}\int_0^{+\infty} x\pi(x)\,\mathrm{d}x.$$

Lemma 2. (*See* [29].) *If* $(S(t), I_1(t), I_2(t))$ *is the solution of system* (2)*, then we have*

$$\lim_{t\to\infty} \frac{S(t) + I_1(t) + I_2(t)}{t} = 0 \quad \textit{a.s.}$$

and

$$\lim_{t\to\infty} \frac{S(t)}{t} = 0, \quad \lim_{t\to\infty} \frac{I_1(t)}{t} = 0, \ \lim_{t\to\infty} \frac{I_2(t)}{t} = 0 \quad \textit{a.s.}$$

Lemma 3. (*See* [14, 29].) *Assume* $\mu > (\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)/2$. *If* $(S(t), I_1(t), I_2(t))$ *is the solution of* (2)*, then*

$$\lim_{t\to\infty} \frac{\int_0^t S(s)\,\mathrm{d}B_1(s)}{t}, \lim_{t\to\infty} \frac{\int_0^t I_1(s)\,\mathrm{d}B_2(s)}{t}, \lim_{t\to\infty} \frac{\int_0^t I_2(s)\,\mathrm{d}B_3(s)}{t} = 0 \quad \textit{a.s.}$$

Theorem 3. *The diseases* $I_1(t)$ *and* $I_2(t)$ *are said to be extinctive if* $\mathfrak{R}_1 < 1$ *and* $\mathfrak{R}_2 < 1$*, respectively.*

Proof. We first prove the extinction of disease $I_1(t)$. Applying Itô's formula to system (2), one has

$$\begin{aligned} \mathrm{d}\ln I_1 &= \left[\frac{p\beta_1 S}{\alpha_1+S} - \left(\mu_2+\gamma_1+\frac{\sigma_2^2}{2}\right)\right]\mathrm{d}t + \sigma_2\,\mathrm{d}B_2(t) \\ &\leqslant \left[\frac{p\beta_1 X}{\alpha_1+X} - \left(\mu_2+\gamma_1+\frac{\sigma_2^2}{2}\right)\right]\mathrm{d}t + \sigma_2\,\mathrm{d}B_2(t). \end{aligned}$$

From system (4) we learn $S \leqslant X$. Then we have

$$\frac{\ln I_1(t)}{t} - \frac{\ln I_1(0)}{t} \leqslant p\beta_1 \frac{1}{t}\int_0^t \frac{X}{\alpha_1+X}\,\mathrm{d}X - \left(\mu_2+\gamma_1+\frac{\sigma_2^2}{2}\right) + \frac{1}{t}\int_0^t \sigma_2\,\mathrm{d}B_2(t).$$

Then we get the following inequality by Lemmas 1 and 3

$$\begin{aligned} \limsup_{t\to+\infty}\frac{\ln I_1(t)}{t} &\leqslant p\beta_1 \lim_{t\to+\infty}\frac{1}{t}\int_0^t \frac{X}{\alpha_1+X}\,\mathrm{d}X - \left(\mu_2+\gamma_1+\frac{\sigma_2^2}{2}\right) \\ &= p\beta_1 \int_0^{+\infty} \frac{x}{\alpha_1+x}\pi(x)\,\mathrm{d}x - \left(\mu_2+\gamma_1+\frac{\sigma_2^2}{2}\right) \\ &= \left(\mu_2+\gamma_1+\frac{\sigma_2^2}{2}\right)(\mathfrak{R}_1-1) < 0, \end{aligned}$$

which implies $\lim_{t\to+\infty} I_1(t) = 0$ a.s. if $\mathfrak{R}_1 < 1$.

In the same way as in the proof of $I_1(t)$, we can get the following inequality from system (2):

$$\begin{aligned} \mathrm{d}\ln I_2 &= \left[\frac{(1-p)\beta_2 S}{\alpha_2+I_2} - \left(\mu_3+\gamma_2+\frac{\sigma_3}{2}\right)\right]\mathrm{d}t + \sigma_3\,\mathrm{d}B_3(t) \\ &\leqslant \left[\frac{(1-p)\beta_2 X}{\alpha_2} - \left(\mu_3+\gamma_2+\frac{\sigma_3}{2}\right)\right]\mathrm{d}t + \sigma_3\,\mathrm{d}B_3(t), \end{aligned}$$

where $S \leqslant X$. Then we get

$$\frac{\ln I_2(t)}{t} - \frac{\ln I_2(0)}{t} \leqslant \frac{(1-p)\beta_2}{\alpha_2}\frac{1}{t}\int_0^t X\,\mathrm{d}X - \left(\mu_3+\gamma_2+\frac{\sigma_3}{2}\right) + \frac{1}{t}\int_0^t \sigma_3\,\mathrm{d}B_2(t).$$

So we have

$$\begin{aligned} \limsup_{t\to+\infty}\frac{\ln I_2(t)}{t} &\leqslant \frac{(1-p)\beta_2}{\alpha_2}\int_0^{\infty} x\pi(x)\,\mathrm{d}x - \left(\mu_3+\gamma_2+\frac{\sigma_3^2}{2}\right) \\ &= \left(\mu_3+\gamma_2+\frac{\sigma_3^2}{2}\right)(\mathfrak{R}_2-1) < 0, \end{aligned}$$

thus, $\lim_{t\to+\infty} I_2(t) = 0$ a.s. if $\mathfrak{R}_2 < 1$.

Theorem 4. *For any given initial value* $(S(0), I_1(0), I_2(0)) \in \mathbb{R}^3_+$, *we have the following results of system* (2):

(i) *If* $\Re_1 > 1$ *and* $\Re_2 < 1$, *then* $I_1(t)$ *will be persistent in mean, and* $I_2(t)$ *will be extinct. Besides, we have*

$$\liminf_{t\to+\infty}\langle I_1(t)\rangle \geqslant \frac{r\alpha_1^2}{p^2\beta_1^2 K}\left(\mu_2+\gamma_1+\frac{\sigma_2^2}{2}\right)(\Re_1-1) > 0 \quad \text{a.s.}$$

(ii) *If* $\Re_1 < 1$ *and* $\Re_2 > 1$, *then* $I_1(t)$ *will be extinct, and* $I_2(t)$ *will be persistent in mean. Besides, we have*

$$\liminf_{t\to+\infty}\bigl\langle I_2(t)\bigr\rangle \geqslant \frac{r\alpha_2^2(\mu_3+\gamma_2+\frac{\sigma_3}{2})}{r\alpha_2(\mu_3+\gamma_2)+(1-p)^2\beta_2^2 K}(\Re_2-1) > 0 \quad \text{a.s.}$$

(iii) *If* $\Re_1 > 1$ *and* $\Re_2 > 1$, *then* $I_1(t)$ *and* $I_2(t)$ *will be persistent in mean and satisfy*

$$\begin{aligned}&\liminf_{t\to+\infty}\bigl\langle I_1(t)+I_2(t)\bigr\rangle\\&\quad\geqslant \frac{1}{\Theta}\left[\left(\mu_2+\gamma_1+\frac{\sigma_2^2}{2}\right)(\Re_1-1)+\alpha_2\left(\mu_3+\gamma_2+\frac{\sigma_3^2}{2}\right)(\Re_2-1)\right] > 0 \quad \text{a.s.},\end{aligned}$$

where

$$\Theta = \max\left\{\frac{Kp\beta_1(p\beta_1+\alpha_1\beta_2-p\alpha_1\beta_2)}{r\alpha_1^2}, \frac{K(1-p)\beta_2+r\alpha_2(\mu_3+\gamma_2)}{r\alpha_2}\right\}.$$

Proof. From the first equations of system (2) and system (4) one gets

$$\begin{aligned}\frac{\ln S(t)-\ln S(0)}{t} &= r-\mu_1-\delta-\frac{\sigma_1^2}{2}-\frac{r}{K}\frac{1}{t}\int_0^t S(s)\,\mathrm{d}s-\frac{1}{t}\int_0^t\frac{p\beta_1 I_1(s)}{\alpha_1+S(s)}\,\mathrm{d}s\\&\quad-\frac{1}{t}\int_0^t\frac{(1-p)\beta_2 I_2(s)}{\alpha_2+I_2(s)}\,\mathrm{d}s+\frac{\sigma_1 B_1(t)}{t}\\&\geqslant r-\mu_1-\delta-\frac{\sigma_1^2}{2}-\frac{r}{K}\frac{1}{t}\int_0^t S(s)\,\mathrm{d}s-\frac{1}{t}\int_0^t\frac{p\beta_1}{\alpha_1}I_1(s)\,\mathrm{d}s\\&\quad-\frac{1}{t}\int_0^t\frac{(1-p)\beta_2}{\alpha_2}I_2(s)\,\mathrm{d}s+\frac{\sigma_1 B_1(t)}{t}\end{aligned}\tag{5}$$

and

$$\frac{\ln X(t)-\ln X(0)}{t} = r-\mu_1-\delta-\frac{\sigma_1^2}{2}-\frac{r}{K}\frac{1}{t}\int_0^t X(s)\,\mathrm{d}s+\frac{\sigma_1 B_1(t)}{t}.\tag{6}$$

From (5) and (6) one has

$$\begin{aligned} 0 &\geqslant \frac{\ln S(t) - \ln X(t)}{t} \\ &\geqslant -\frac{r}{K}\frac{1}{t}\int_0^t \big(S(s) - X(s)\big)\,\mathrm{d}s - \frac{1}{t}\int_0^t \frac{p\beta_1}{\alpha_1} I_1(s)\,\mathrm{d}s - \frac{1}{t}\int_0^t \frac{(1-p)\beta_2}{\alpha_2} I_2(s)\,\mathrm{d}s, \end{aligned}$$

that is,

$$\frac{1}{t}\int_0^t (S(s) - X(s))\,\mathrm{d}s \geqslant -\frac{p\beta_1 K}{r\alpha_1}\frac{1}{t}\int_0^t I_1(s)\,\mathrm{d}s - \frac{(1-p)\beta_2 K}{r\alpha_2}\frac{1}{t}\int_0^t I_2(s)\,\mathrm{d}s.$$

(i) By Theorem 3, since $\mathfrak{R}_2 < 1$, then $\lim_{t\to+\infty} I_2(t) = 0$ a.s. Since $\mathfrak{R}_1 > 1$, then $0 < I_2(t) < \epsilon$ and $\epsilon > 0$ small enough such that

$$p\beta_1 \frac{1}{t}\int_0^t \frac{X(s)}{\alpha_1 + X(s)}\,\mathrm{d}s - \left(\mu_2 + \gamma_1 + \frac{\sigma_2}{2}\right) - \frac{p(1-p)\beta_1\beta_2 K\epsilon}{r\alpha_1\alpha_2} > 1.$$

Applying Itô's formula to the the Lyapunov function $\ln I_1$, it follows that

$$\begin{aligned} \mathrm{d}\ln I_1 &= \left[\frac{p\beta_1 S}{\alpha_1 + S} - \left(\mu_2 + \gamma_1 + \frac{\sigma_2^2}{2}\right)\right]\mathrm{d}t + \sigma_2\,\mathrm{d}B_2(t) \\ &= \left[\frac{p\beta_1 X}{\alpha_1 + X} - \left(\mu_2 + \gamma_1 + \frac{\sigma_2^2}{2}\right) + \frac{p\alpha_1\beta_1(S - X)}{(\alpha_1 + S)(\alpha_1 + X)}\right]\mathrm{d}t + \sigma_2\,\mathrm{d}B_2(t) \\ &\geqslant \left[\frac{p\beta_1 X}{\alpha_1 + X} - \left(\mu_2 + \gamma_1 + \frac{\sigma_2^2}{2}\right) + \frac{p\beta_1(S - X)}{\alpha_1}\right]\mathrm{d}t + \sigma_2\,\mathrm{d}B_2(t). \end{aligned} \tag{7}$$

Calculating (7) directly, we can obtain

$$\begin{aligned} \frac{\ln I_1(t)}{t} \geqslant{}& \frac{1}{t}\int_0^t \frac{p\beta_1 X(s)}{\alpha_1 + X(s)}\,\mathrm{d}s - \left(\mu_2 + \gamma_1 + \frac{\sigma_2^2}{2}\right) - \frac{p^2\beta_1^2 K}{r\alpha_1^2}\frac{1}{t}\int_0^t I_1(s)\,\mathrm{d}s \\ & - \frac{p(1-p)\beta_1\beta_2 K\epsilon}{r\alpha_1\alpha_2} + \frac{\sigma_2 B_2(t)}{t} + \frac{\ln I_1(0)}{t}. \end{aligned}$$

Simplifying the inequality above, we can obtain

$$\begin{aligned} \langle I_1(t)\rangle \geqslant{}& \frac{r\alpha_1^2}{p^2\beta_1^2 K}\left[\frac{1}{t}\int_0^t \frac{p\beta_1 X(s)}{\alpha_1 + X(s)}\,\mathrm{d}s - \left(\mu_2 + \gamma_1 + \frac{\sigma_2^2}{2}\right) - \frac{p(1-p)\beta_1\beta_2 K\epsilon}{r\alpha_1\alpha_2}\right. \\ & \left. + \frac{\sigma_2 B_2(t)}{t} - \frac{\ln I_1(t)}{t} + \frac{\ln I_1(0)}{t}\right] \end{aligned}$$

$$\geqslant \begin{cases} \frac{r\alpha_1^2}{p^2\beta_1^2 K}\big[\frac{1}{t}\int_0^t \frac{p\beta_1 X(s)}{\alpha_1+X(s)}\,\mathrm{d}s - (\mu_2+\gamma_1+\frac{\sigma_2^2}{2}) - \frac{p(1-p)\beta_1\beta_2 K\epsilon}{r\alpha_1\alpha_2} \\ \quad + \frac{\sigma_2 B_2(t)}{t} + \frac{\ln I_1(0)}{t}\big], \quad 0 < I_1(t) < 1; \\ \frac{r\alpha_1^2}{p^2\beta_1^2 K}\big[\frac{1}{t}\int_0^t \frac{p\beta_1 X(s)}{\alpha_1+X(s)}\,\mathrm{d}s - (\mu_2+\gamma_1+\frac{\sigma_2^2}{2}) - \frac{p(1-p)\beta_1\beta_2 K\epsilon}{r\alpha_1\alpha_2} \\ \quad + \frac{\sigma_2 B_2(t)}{t} - \frac{\ln I_1(t)}{t}\big], \quad 1 \leqslant I_1(t), \end{cases}$$

where $\varepsilon \to 0$ as $t \to +\infty$. According to Lemma 4, one sees that $\lim_{t\to+\infty} I_1(t)/t = 0$ and $\lim_{t\to+\infty} \ln I_1(t)/t = 0$ as $I_1(t) \geqslant 1$.

Then one can calculate that

$$\liminf_{t\to+\infty}\langle I_1(t)\rangle \geqslant \frac{r\alpha_1^2}{p^2\beta_1^2 K}\left(\mu_2+\gamma_1+\frac{\sigma_2^2}{2}\right)(\mathfrak{R}_1 - 1) > 0.$$

(ii) By Theorem 3, since $\mathfrak{R}_1 < 1$, then $\lim_{t\to+\infty} I_1(t) = 0$ a.s. Since $\mathfrak{R}_2 > 1$, then $0 < I_1(t) < \epsilon$ and $\epsilon > 0$ small enough such that

$$(1-p)\beta_2 \int_0^{+\infty} x\pi(x)\,\mathrm{d}x - \alpha_2\left(\mu_3+\gamma_2+\frac{\sigma_3^2}{2}\right) - \frac{p(1-p)\beta_1\beta_2 K\epsilon}{r\alpha_1} > 1.$$

Using Itô's formula to the Lyapunov function $\alpha_2 \ln I_2(t) + I_2(t)$ gets

$$\begin{aligned} &\mathrm{d}\big(\alpha_2 \ln I_2(t) + I_2(t)\big) \\ &\quad = \left[(1-p)\beta_2 S - \alpha_2\left(\mu_3+\gamma_2+\frac{\sigma_3^2}{2}\right) - \left(\mu_3+\gamma_2\right)I_2\right]\mathrm{d}t + \sigma_3\,\mathrm{d}B_3(t) \\ &\quad = \left[(1-p)\beta_2 X - \alpha_2\left(\mu_3+\gamma_2+\frac{\sigma_3^2}{2}\right) - (\mu_3+\gamma_2)I_2 + (1-p)\beta_2(S-X)\right]\mathrm{d}t \\ &\qquad + \sigma_3\,\mathrm{d}B_3(t). \end{aligned}$$

Then we get

$$\begin{aligned} &\frac{\alpha_2 \ln I_2(t) + I_2(t)}{t} \\ &\quad \geqslant \frac{1}{t}\int_0^t (1-p)\beta_2 X(s)\,\mathrm{d}s - \alpha_2\left(\mu_3+\gamma_2+\frac{\sigma_3^2}{2}\right) - \frac{p(1-p)\beta_1\beta_2 K\epsilon}{r\alpha_1} \\ &\qquad - \left(\mu_3+\gamma_2+\frac{(1-p)^2\beta_2^2 K}{r\alpha_2}\right)\frac{1}{t}\int_0^t I_2(s)\,\mathrm{d}s + \frac{\sigma_3 B_3(t)}{t} + \frac{\alpha_2 \ln I_2(0) + I_2(0)}{t}. \end{aligned}$$

Then one can calculate that

$$\liminf_{t\to+\infty}\big\langle I_2(t)\big\rangle \geqslant \frac{r\alpha_2^2(\mu_3+\gamma_2+\frac{\sigma_3}{2})}{r\alpha_2(\mu_3+\gamma_2)+(1-p)^2\beta_2^2 K}(\mathfrak{R}_2 - 1) > 0.$$

(iii) Define

$$V = \ln I_1 + \ln I_2^{\alpha_2} + I_2,$$

then we have

$$\begin{aligned}
D^+V &= \left[\frac{p\beta_1 S}{\alpha_1 + S} - \left(\mu_2 + \gamma_1 + \frac{\sigma_2^2}{2}\right)\right] \mathrm{d}t + \sigma_2 \,\mathrm{d}B_2(t) \\
&\quad + \left[(1-p)\beta_2 S - \alpha_2\left(\mu_3 + \gamma_2 + \frac{\sigma_3^2}{2}\right) - \left(\mu_3 + \gamma_2\right) I_2\right] \mathrm{d}t + \sigma_3 \,\mathrm{d}B_3(t) \\
&\geqslant \left[\frac{p\beta_1 X}{\alpha_1 + X} - \left(\mu_2 + \gamma_1 + \frac{\sigma_2^2}{2}\right) + \frac{p\beta_1(S-X)}{\alpha_1}\right] \mathrm{d}t + \sigma_2 \,\mathrm{d}B_2(t) + \sigma_3 \,\mathrm{d}B_3(t) \\
&\quad + \left[(1-p)\beta_2 X - \alpha_2\left(\mu_3 + \gamma_2 + \frac{\sigma_3^2}{2}\right)\right. \\
&\quad \left. - (\mu_3 + \gamma_2) I_2 + (1-p)\beta_2 (S - X)\right] \mathrm{d}t.
\end{aligned} \tag{8}$$

From (8) one gets

$$\begin{aligned}
\frac{V(t) - V(0)}{t} &\geqslant \frac{1}{t}\int_0^t \frac{p\beta_1 X(s)}{\alpha_1 + X(s)} \,\mathrm{d}s - \left(\mu_2 + \gamma_1 + \frac{\sigma_2}{2}\right) + \frac{1}{t}\int_0^t (1-p)\beta_2 X(s) \,\mathrm{d}s \\
&\quad - \alpha_2\left(\mu_3 + \gamma_2 + \frac{\sigma_3}{2}\right) - \frac{Kp\beta_1(p\beta_1 + \alpha_1\beta_2 - p\alpha_1\beta_2)}{r\alpha_1^2}\frac{1}{t}\int_0^t I_1(s) \,\mathrm{d}s \\
&\quad - \frac{K(1-p)\beta_2 + r\alpha_2(\mu_3 + \gamma_2)}{r\alpha_2}\frac{1}{t}\int_0^t I_2(s) \,\mathrm{d}s + \frac{\sigma_2 B_2(t)}{t} + \frac{\sigma_3 B_3(t)}{t}.
\end{aligned}$$

Sorting out the above inequalities results in

$$\begin{aligned}
&\liminf_{t\to+\infty}\langle I_1(t) + I_2(t)\rangle \\
&\quad \geqslant \frac{1}{\Theta}\left[\left(\mu_2 + \gamma_1 + \frac{\sigma_2^2}{2}\right)(\mathfrak{R}_1 - 1) + \alpha_2\left(\mu_3 + \gamma_2 + \frac{\sigma_3}{2}\right)(\mathfrak{R}_2 - 1)\right] > 0, \\
&\Theta = \max\left\{\frac{Kp\beta_1(p\beta_1 + \alpha_1\beta_2 - p\alpha_1\beta_2)}{r\alpha_1^2}, \frac{K(1-p)\beta_2 + r\alpha_2(\mu_3 + \gamma_2)}{r\alpha_2}\right\}. \qquad \square
\end{aligned}$$

5 Stationary distribution

In this section, we use the Has'minskii theory [12] to prove the stationary distribution of system (2).

Theorem 5. *System* (2) *has a unique ergodic stationary distribution* $\varrho(\cdot)$, *if*

$$\mathcal{R} = \frac{(1-p)\beta_2 S^0}{\alpha_2} + \frac{p\beta_1 S^0}{\alpha_1 + S^0} - \left(\mu_2 + \mu_3 + \gamma_1 + \gamma_2 + \frac{S^0\sigma_1^2}{2} + \frac{\sigma_2^2}{2} + \frac{\sigma_3^2}{2}\right) > 0,$$

$$b_1 > \frac{K}{rS^0}\left(\frac{p\alpha_1\beta_1}{(\alpha_1+S^0)^2} + \frac{(1-p)\beta_2}{\alpha_2}\right) + \frac{p\alpha_1\beta_1 K}{r(\alpha_1+S^0)^3},$$

and

$$\mu_2+\mu_3+\gamma_1+\gamma_2-2(\vartheta+1)\big(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\big)>0.$$

Proof. System (2) has a diffusion matrix

$$A=\begin{pmatrix}\sigma_1^2S^2 & 0 & 0\\ 0 & \sigma_2^2I_1^2 & 0\\ 0 & 0 & \sigma_3^2I_2^2\end{pmatrix}.$$

Let $H=\min_{(S,I_1,I_2)\in\overline{U}}\{\sigma_1^2S^2,\sigma_2^2I_1^2,\sigma_3^2I_2^2\}$ such that

$$\sum_{i,j=1}^{2} a_{ij}\xi_i\xi_j=\sigma_1^2S^2\xi_1^2+\sigma_2^2I_1^2\xi_2^2+\sigma_3^2I_2^2\xi_3^2\geqslant H|\xi|^2,$$

$(S,I_1,I_2)\in\overline{U}$, $\xi=(\xi_1,\xi_2,\xi_3)\in\mathbb{R}^3$, then we get that all the eigenvalues of diffusion matrix are greater than zero.

Define $\overline{V}\colon\mathbb{R}_+^3\to\mathbb{R}$,

$$\overline{V}(S,I_1,I_2)=M\left[b_1\left(S-S^0-S^0\ln\frac{S}{S_0}\right)-b_2S-\ln I_1-\ln I_2+b_3I_1+b_4I_2\right] + \frac{1}{\vartheta+2}(S+I_1+I_2g)^{\vartheta+2}.$$

Here $S^0=(K(r-\mu_1-\delta))/r$, $b_2=K/(rS^0)(p\alpha_1\beta_1/(\alpha_1+S^0)^2+(1-p)\beta_2/\alpha_2)$, $b_3=b_1p\beta_1S^0/(\alpha_1(\mu_2+\gamma_1))$, $b_4=b_2(1-p)\beta_2S^0/(\alpha_2(\mu_3+\gamma_2))$, $0<\vartheta<1$. Defining a sufficiently large $M>0$ such that

$$-M\mathcal{R}+\max\{B,C,D,E,F,J\}\leqslant -2, \tag{9}$$

where

$$\begin{aligned}
B&=\sup_{(S,I_1,I_2)\in\mathbb{R}_+^3}\left\{-\frac{r}{2K}S^{\vartheta+3}-\frac{(\mu_2+\gamma_1)}{2}I_1^{\vartheta+2}+Mp\beta_1(b_2+b_3)I_1\right.\\
&\quad\left.-\left(\frac{\mu_3+\gamma_2}{2}-\frac{M(1-p)\beta_2(1+\alpha_2(b_2+b_4))\epsilon}{\alpha_2^2}\right)I_2^{\vartheta+2}+O\right\},\\
C&=\sup_{(S,I_1,I_2)\in\mathbb{R}_+^3}\left\{-\frac{r}{2K}S^{\vartheta+3}-\frac{\mu_2+\gamma_1}{2}I_1^{\vartheta+2}-\frac{\mu_3+\gamma_2}{2}I_2^{\vartheta+2}\right.\\
&\quad\left.+\frac{M(1-p)\beta_2(1+\alpha_2(b_2+b_4))}{\alpha_2^2}SI_2+O\right\},\\
D&=\sup_{(S,I_1,I_2)\in\mathbb{R}_+^3}\left\{-\left(\frac{r}{2K}-\frac{M(1-p)\beta_2(1+\alpha_2(b_2+b_4))\epsilon}{\alpha_2^2(\vartheta+3)}\right)S^{\vartheta+3}\right.\\
&\quad\left.-\frac{\mu_2+\gamma_1}{2}I_1^{\vartheta+2}+Mp\beta_1(b_2+b_3)I_1-\frac{\mu_3+\gamma_2}{2}I_2^{\vartheta+2}+O\right\},
\end{aligned}$$

$$
\begin{aligned}
E = \sup_{(S,I_1,I_2)\in\mathbb{R}^3_+} \bigg\{ & -\frac{r}{4K}S^{\vartheta+3} - \frac{\mu_2+\gamma_1}{2}I_1^{\vartheta+2} - \frac{\mu_3+\gamma_2}{2}I_2^{\vartheta+2} \\
& + \frac{M(1-p)\beta_2(1+\alpha_2(b_2+b_4))}{\alpha_2^2}SI_2 + Mp\beta_1(b_2+b_3)I_1 + O \bigg\}, \\
F = \sup_{(S,I_1,I_2)\in\mathbb{R}^3_+} \bigg\{ & -\frac{r}{2K}S^{\vartheta+3} - \frac{\mu_2+\gamma_1}{4}I_1^{\vartheta+2} - \frac{\mu_3+\gamma_2}{2}I_2^{\vartheta+2} \\
& + \frac{M(1-p)\beta_2(1+\alpha_2(b_2+b_4))}{\alpha_2^2}SI_2 + Mp\beta_1(b_2+b_3)I_1 + O \bigg\}, \\
J = \sup_{(S,I_1,I_2)\in\mathbb{R}^3_+} \bigg\{ & -\frac{r}{2K}S^{\vartheta+3} - \frac{\mu_2+\gamma_1}{2}I_1^{\vartheta+2} - \frac{\mu_3+\gamma_2}{4}I_2^{\vartheta+2} \\
& + \frac{M(1-p)\beta_2(1+\alpha_2(b_2+b_4))}{\alpha_2^2}SI_2 + Mp\beta_1(b_2+b_3)I_1 + O \bigg\}, \\
O = \sup_{(S,I_1,I_2)\in\mathbb{R}^3_+} \bigg\{ & -\frac{r}{2K}S^{\vartheta+3} - \frac{(\mu_2+\gamma_1)}{2}I_1^{\vartheta+2} - \frac{(\mu_3+\gamma_2)}{2}I_2^{\vartheta+2} \\
& + rS(S+I_1+I_2)^{\vartheta+1} + \frac{\vartheta+1}{2}(S+I_1+I_2)^{\vartheta+2}\big(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\big) \bigg\}.
\end{aligned}
$$

It is clear that for $\overline{V}(S,I_1,I_2)$, there exists a unique minimum point $(\underline{S},\underline{I_1},\underline{I_2})$. Then denote
a positive-definite function $V(S,I_1,I_2):\mathbb{R}^3_+\to\mathbb{R}_+$:

$$
V(S,I_1,I_2) = \overline{V}(S,I_1,I_2) - \overline{V}(\underline{S},\underline{I_1},\underline{I_2}) := MV_1 + V_2,
$$

where

$$
\begin{aligned}
V_1 &= b_1\left(S - S^0 - S^0\ln\frac{S}{S^0}\right) - b_2S - \ln I_1 - \ln I_2 + b_3I_1 + b_4I_2, \\
V_2 &= \frac{1}{\vartheta+2}(S+I_1+I_2)^{\vartheta+2} - \overline{V}(\underline{S},\underline{I_1},\underline{I_2}).
\end{aligned}
$$

Applying the Itô's formula yields

$$
\begin{aligned}
\mathcal{L}V_1 = & -\frac{b_1r}{K}\big(S-S^0\big)^2 - \frac{b_1p\beta_1SI_1}{\alpha_1+S} - \frac{b_1(1-p)\beta_2SI_2}{\alpha_2+I_2} + \frac{b_1p\beta_1S^0I_1}{\alpha_1+S} \\
& + \frac{b_1(1-p)\beta_2S^0I_2}{\alpha_2+I_2} + \frac{S^0\sigma_1^2}{2} + \frac{b_2r}{K}S\big(S-S^0\big) + \frac{b_2p\beta_1SI_1}{\alpha_1+S} + \frac{b_2(1-p)\beta_2SI_2}{\alpha_2+I_2} \\
& - \frac{p\beta_1S}{\alpha_1+S} + \mu_2 + \gamma_1 + \frac{\sigma_2^2}{2} - \frac{(1-p)\beta_2S}{\alpha_2+I_2} + \mu_3 + \gamma_2 + \frac{\sigma_3^2}{2} + \frac{b_3p\beta_1SI_1}{\alpha_1+S} \\
& - b_3(\mu_2+\gamma_1)I_1 + \frac{b_4(1-p)\beta_2SI_2}{\alpha_2+I_2} - b_4(\mu_3+\gamma_2)I_2 \\
\leqslant & -\frac{b_1r}{K}\big(S-S^0\big)^2 + \frac{b_2r}{K}S\big(S-S^0\big) - \frac{p\beta_1S}{\alpha_1+S} + \frac{p\beta_1S^0}{\alpha_1+S^0} - \frac{(1-p)\beta_2S}{\alpha_2}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{(1-p)\beta_2 S^0}{\alpha_2}+\frac{(1-p)\beta_2 S I_2}{(\alpha_2+\zeta)^2}-\frac{(1-p)\beta_2 S^0}{\alpha_2}-\frac{p\beta_1 S^0}{\alpha_1+S^0} \\
&+\left(\frac{b_1 p\beta_1 S^0}{\alpha_1}-b_3(\mu_2+\gamma_1)\right)I_1+\left(\frac{b_2(1-p)\beta_2 S^0}{\alpha_2}-b_4(\mu_3+\gamma_2)\right)I_2 \\
&+\frac{b_3 p\beta_1 S I_1}{\alpha_1+S}+\frac{b_4(1-p)\beta_2 S I_2}{\alpha_2+I_2}+\mu_2+\mu_3+\gamma_1+\gamma_2+\frac{S^0\sigma_1^2}{2}+\frac{\sigma_2^2}{2}+\frac{\sigma_3^2}{2} \\
=\,& G(S)+\frac{(1-p)\beta_2 S I_2}{(\alpha_2+\zeta)^2}+\frac{b_3 p\beta_1 S I_1}{\alpha_1+S}+\frac{b_4(1-p)\beta_2 S I_2}{\alpha_2+I_2}-\frac{(1-p)\beta_2 S^0}{\alpha_2} \\
&-\frac{p\beta_1 S^0}{\alpha_1+S^0}+\mu_2+\mu_3+\gamma_1+\gamma_2+\frac{S^0\sigma_1^2}{2}+\frac{\sigma_2^2}{2}+\frac{\sigma_3^2}{2} \\
\leqslant\,& G(S)+\frac{(1-p)\beta_2(1+\alpha_2(b_2+b_4))}{\alpha_2^2}S I_2+p\beta_1(b_2+b_3)I_1-\frac{(1-p)\beta_2 S^0}{\alpha_2} \\
&-\frac{p\beta_1 S^0}{\alpha_1+S^0}+\mu_2+\mu_3+\gamma_1+\gamma_2+\frac{S^0\sigma_1^2}{2}+\frac{\sigma_2^2}{2}+\frac{\sigma_3^2}{2},
\end{aligned}
$$

where $\zeta \in (0, I_2)$ and

$$
\begin{aligned}
G(S) = &-\frac{b_1 r}{K}\left(S-S^0\right)^2+\frac{b_2 r}{K}S\left(S-S^0\right)-\frac{p\beta_1 S}{\alpha_1+S}+\frac{p\beta_1 S^0}{\alpha_1+S^0} \\
&-\frac{(1-p)\beta_2 S}{\alpha_2}+\frac{(1-p)\beta_2 S^0}{\alpha_2}.
\end{aligned}
$$

From above equation we can get

$$
G'(S) = -\frac{2b_1 r}{K}\left(S-S^0\right)+\frac{b_2 r}{K}\left(2S-S^0\right)-\frac{p\alpha_1\beta_1}{(\alpha_1+S)^2}-\frac{(1-p)\beta_2}{\alpha_2}
$$

and

$$
G''(S) = -\frac{2b_1 r}{K}+\frac{2b_2 r}{K}+\frac{2p\alpha_1\beta_1}{(\alpha_1+S)^3}.
$$

Since

$$
b_2 = \frac{K}{rS^0}\left(\frac{p\alpha_1\beta_1}{(\alpha_1+S^0)^2}+\frac{(1-p)\beta_2}{\alpha_2}\right),
$$

then we have

$$
G'(S)|_{S=S^0} = \frac{b_2 r S^0}{K}-\frac{p\alpha_1\beta_1}{(\alpha_1+S^0)^2}-\frac{(1-p)\beta_2}{\alpha_2} = 0. \tag{10}
$$

Note that

$$
b_1 > \frac{K}{rS^0}\left(\frac{p\alpha_1\beta_1}{(\alpha_1+S^0)^2}+\frac{(1-p)\beta_2}{\alpha_2}\right)+\frac{p\alpha_1\beta_1 K}{r(\alpha_1+S^0)^3},
$$

we have

$$
G''(S^0) = -\frac{2b_1 r}{K}+\frac{2b_2 r}{K}+\frac{2p\alpha_1\beta_1}{(\alpha_1+S^0)^3} < 0. \tag{11}
$$

According to (10)–(11), we get $G(S) \leqslant G(S^0) = 0$. Therefore,

$$\begin{aligned}\mathcal{L}V_1 &\leqslant -\frac{(1-p)\beta_2 S^0}{\alpha_2} - \frac{p\beta_1 S^0}{\alpha_1 + S^0} + \mu_2 + \mu_3 + \gamma_1 + \gamma_2 + \frac{S^0\sigma_1^2}{2} + \frac{\sigma_2^2}{2} + \frac{\sigma_3^2}{2} \\ &\quad + \frac{(1-p)\beta_2(1+\alpha_2(b_2+b_4))}{\alpha_2^2} S I_2 + p\beta_1(b_2+b_3)I_1 \\ &= -\mathcal{R} + \frac{(1-p)\beta_2(1+\alpha_2(b_2+b_4))}{\alpha_2^2} S I_2 + p\beta_1(b_2+b_3)I_1,\end{aligned}$$

where

$$\mathcal{R} = \frac{(1-p)\beta_2 S^0}{\alpha_2} + \frac{p\beta_1 S^0}{\alpha_1 + S^0} - \left(\mu_2 + \mu_3 + \gamma_1 + \gamma_2 + \frac{S^0\sigma_1^2}{2} + \frac{\sigma_2^2}{2} + \frac{\sigma_3^2}{2}\right) > 0.$$

Next,

$$\begin{aligned}\mathcal{L}V_2 &= (S+I_1+I_2)^{\vartheta+1}\left[rS\left(1-\frac{S}{K}\right) - (\mu_1+\delta)S - (\mu_2+\gamma_1)I_1 - (\mu_3+\gamma_2)I_2\right] \\ &\quad + \frac{\vartheta+1}{2}(S+I_1+I_2)^{\vartheta}\left(\sigma_1^2 S^2 + \sigma_2^2 I_1^2 + \sigma_3^2 I_2^2\right) \\ &\leqslant rS(S+I_1+I_2)^{\vartheta+1} - \frac{r}{K}S^{\vartheta+3} - (\mu_2+\gamma_1)I_1^{\vartheta+2} - (\mu_3+\gamma_2)I_2^{\vartheta+2} \\ &\quad + \frac{\vartheta+1}{2}(S+I_1+I_2)^{\vartheta+2}\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2\right) \\ &\leqslant -\frac{r}{2K}S^{\vartheta+3} - \frac{(\mu_2+\gamma_1)}{2}I_1^{\vartheta+2} - \frac{(\mu_3+\gamma_2)}{2}I_2^{\vartheta+2} + O.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathcal{L}V &\leqslant -M\mathcal{R} + \frac{M(1-p)\beta_2(1+\alpha_2(b_2+b_4))}{\alpha_2^2} S I_2 + Mp\beta_1(b_2+b_3)I_1 \\ &\quad - \frac{r}{2K}S^{\vartheta+3} - \frac{(\mu_2+\gamma_1)}{2}I_1^{\vartheta+2} - \frac{(\mu_3+\gamma_2)}{2}I_2^{\vartheta+2} + O.\end{aligned}$$

Construct a compact bounded subset U:

$$U = \left\{(S, I_1, I_2) \in \mathbb{R}_+^3\colon\ \epsilon \leqslant S \leqslant \frac{1}{\epsilon},\ \epsilon \leqslant I_1 \leqslant \frac{1}{\epsilon},\ \epsilon \leqslant I_2 \leqslant \frac{1}{\epsilon}\right\},$$

and $\epsilon > 0$ will be given in the later. In the set $\mathbb{R}_+^3 \setminus U$, choosing ϵ small enough such that

$$\epsilon < \frac{\alpha_2^2(\mu_3+\gamma_2)}{2M(1-p)\beta_2(1+\alpha_2(b_2+b_4))}, \tag{12}$$

$$-M\mathcal{R} + \frac{M(1-p)\beta_2(1+\alpha_2(b_2+b_4))(\vartheta+1)}{\alpha_2^2(\vartheta+2)}\epsilon + B < -1, \tag{13}$$

$$-M\mathcal{R} + Mp\beta_1(b_2+b_3)\epsilon + C \leqslant -1, \tag{14}$$

$$\epsilon < \frac{r\alpha_2^2(\vartheta+3)}{2KM(1-p)\beta_2(1+\alpha_2(b_2+b_4))}, \tag{15}$$

$$-M\mathcal{R} + \frac{M(1-p)\beta_2(1+\alpha_2(b_2+b_4))(\vartheta+1)}{\alpha_2^2(\vartheta+2)}\epsilon + D < -1, \tag{16}$$

$$-M\mathcal{R} - \frac{r}{4K\epsilon^{\vartheta+3}} + E \leqslant -1, \tag{17}$$

$$-M\mathcal{R} - \frac{\mu_2+\gamma_1}{4\epsilon^{\vartheta+3}} + F \leqslant -1, \tag{18}$$

$$-M\mathcal{R} - \frac{\mu_3+\gamma_2}{4\epsilon^{\vartheta+3}} + J \leqslant -1. \tag{19}$$

Here B, C, D, E, F and J are positive constants defined in equations (20), (21), (22), (23), (24), (25), respectively. Next, six domains are given in the following:

$$\begin{aligned}
&U_1 = \{(S,I_1,I_2) \in \mathbb{R}_+^3,\ 0<S<\epsilon\}, && U_2 = \{(S,I_1,I_2) \in \mathbb{R}_+^3,\ 0<I_1<\epsilon\},\\
&U_3 = \{(S,I_1,I_2) \in \mathbb{R}_+^3,\ 0<I_2<\epsilon\}, && U_4 = \bigl\{(S,I_1,I_2) \in \mathbb{R}_+^3,\ S>\epsilon^{-1}\bigr\},\\
&U_5 = \bigl\{(S,I_1,I_2) \in \mathbb{R}_+^3,\ I_1>\epsilon^{-1}\bigr\}, && U_6 = \bigl\{(S,I_1,I_2) \in \mathbb{R}_+^3,\ I_2>\epsilon^{-1}\bigr\}.
\end{aligned}$$

We need prove that $\mathcal{L}V(S,I_1,I_2) \leqslant -1$ on $\mathbb{R}_+^3 \backslash U$. It is clear that $\mathbb{R}_+^3 \backslash U$ is equivalent to $U_1 \cup U_2 \cup U_3 \cup U_4 \cup U_5 \cup U_6$.

Case 1. If $(S,I_1,I_2) \in U_1$, due to

$$I_2 \leqslant \epsilon I_2 \leqslant \frac{\vartheta+1+I_2^{\vartheta+2}}{\vartheta+2}\epsilon = \frac{(\vartheta+1)\epsilon}{\vartheta+2} + \frac{\epsilon}{\vartheta+2}I_2^{\vartheta+2},$$

we get

$$\begin{aligned}
\mathcal{L}V &\leqslant -M\mathcal{R} + \frac{M(1-p)\beta_2(1+\alpha_2(b_2+b_4))(\vartheta+1)}{\alpha_2^2(\vartheta+2)}\epsilon - \frac{r}{2K}S^{\vartheta+3} - \frac{\mu_2+\gamma_1}{2}I_1^{\vartheta+2}\\
&\quad + Mb_3p\beta_1 I_1 - \left(\frac{\mu_3+\gamma_2}{2} - \frac{M(1-p)\beta_2(1+\alpha_2(b_2+b_4))\epsilon}{\alpha_2^2}\right)I_2^{\vartheta+2} + O\\
&\leqslant -M\mathcal{R} + \frac{M(1-p)\beta_2(1+\alpha_2(b_2+b_4))(\vartheta+1)}{\alpha_2^2(\vartheta+2)}\epsilon + B,
\end{aligned}$$

where

$$\begin{aligned}
B = \sup_{(S,I_1,I_2)\in\mathbb{R}_+^3} \biggl\{ &-\frac{r}{2K}S^{\vartheta+3} - \left(\frac{\mu_3+\gamma_2}{2} - \frac{M(1-p)\beta_2(1+\alpha_2(b_2+b_4))\epsilon}{\alpha_2^2}\right)I_2^{\vartheta+2}\\
&- \frac{(\mu_2+\gamma_1)}{2}I_1^{\vartheta+2} + Mp\beta_1(b_2+b_3)I_1 + O \biggr\}.
\end{aligned} \tag{20}$$

According to (9), (12) and (13), we have that $\mathcal{L}V \leqslant -1$.

Case 2. If $(S, I_1, I_2) \in U_2$,

$$\begin{aligned}\mathcal{L}V &\leqslant -M\mathcal{R} + Mp\beta_1(b_2+b_3)I_1 - \frac{r}{2K}S^{\vartheta+3} - \frac{\mu_2+\gamma_1}{2}I_1^{\vartheta+2} \\ &\quad - \frac{\mu_3+\gamma_2}{2}I_2^{\vartheta+2} + \frac{M(1-p)\beta_2(1+\alpha_2(b_2+b_4))}{\alpha_2^2}SI_2 + O \\ &\leqslant -M\mathcal{R} + Mp\beta_1(b_2+b_3)\epsilon + C,\end{aligned}$$

where

$$\begin{aligned}C = \sup_{(S,I_1,I_2)\in\mathbb{R}_+^3} \biggl\{ &-\frac{r}{2K}S^{\vartheta+3} - \frac{\mu_2+\gamma_1}{2}I_1^{\vartheta+2} - \frac{\mu_3+\gamma_2}{2}I_2^{\vartheta+2} \\ &+ \frac{M(1-p)\beta_2(1+\alpha_2(b_2+b_4))}{\alpha_2^2}SI_2 + O\biggr\}. \qquad (21)\end{aligned}$$

By (9) and (14) we obtain that $\mathcal{L}V \leqslant -1$.

Case 3. If $(S, I_1, I_2) \in U_3$, due to

$$I_2 \leqslant S\epsilon \leqslant \frac{(\vartheta+2)\epsilon}{\vartheta+3} + \frac{\epsilon}{\vartheta+3}S^{\vartheta+3},$$

we have

$$\begin{aligned}\mathcal{L}V &\leqslant -M\mathcal{R} + \frac{M(1-p)\beta_2(1+\alpha_2(b_2+b_4))(\vartheta+2)}{\alpha_2^2(\vartheta+3)}\epsilon \\ &\quad - \left(\frac{r}{2K} - \frac{M(1-p)\beta_2(1+\alpha_2(b_2+b_4))\epsilon}{\alpha_2^2(\vartheta+3)}\right)S^{\vartheta+3} \\ &\quad - \frac{\mu_2+\gamma_1}{2}I_1^{\vartheta+2} + Mp\beta_1(b_2+b_3)I_1 - \frac{\mu_3+\gamma_2}{2}I_2^{\vartheta+2} + O \\ &\leqslant -M\mathcal{R} + \frac{M(1-p)\beta_2(1+\alpha_2(b_2+b_4))(\vartheta+2)}{\alpha_2^2(\vartheta+3)}\epsilon + D,\end{aligned}$$

where

$$\begin{aligned}D = \sup_{(S,I_1,I_2)\in\mathbb{R}_+^3} \biggl\{ &-\left(\frac{r}{2K} - \frac{M(1-p)\beta_2(1+\alpha_2(b_2+b_4))\epsilon}{\alpha_2^2(\vartheta+3)}\right)S^{\vartheta+3} \\ &- \frac{\mu_2+\gamma_1}{2}I_1^{\vartheta+2} + Mp\beta_1(b_2+b_3)I_1 - \frac{\mu_3+\gamma_2}{2}I_2^{\vartheta+2} + O\biggr\}. \qquad (22)\end{aligned}$$

In view of (9), (15) and (16), we get that $\mathcal{L}V \leqslant -1$.

Case 4. If $(S, I_1, I_2) \in U_4$,

$$\begin{aligned}\mathcal{L}V &\leqslant -M\mathcal{R} - \frac{r}{4K}S^{\vartheta+3} - \frac{r}{4K}S^{\vartheta+3} - \frac{\mu_2+\gamma_1}{2}I_1^{\vartheta+2} - \frac{\mu_3+\gamma_2}{2}I_2^{\vartheta+2} \\ &\quad + \frac{M(1-p)\beta_2(1+\alpha_2(b_2+b_4))}{\alpha_2^2}SI_2 + Mp\beta_1(b_2+b_3)I_1 + O \\ &\leqslant -M\mathcal{R} - \frac{r}{4K\epsilon^{\vartheta+3}} + E,\end{aligned}$$

where

$$E = \sup_{(S,I_1,I_2)\in\mathbb{R}^3_+}\biggl\{-\frac{r}{4K}S^{\vartheta+3} - \frac{\mu_2+\gamma_1}{2}I_1^{\vartheta+2} - \frac{\mu_3+\gamma_2}{2}I_2^{\vartheta+2} + \frac{M(1-p)\beta_2(1+\alpha_2(b_2+b_4))}{\alpha_2^2}SI_2 + Mp\beta_1(b_2+b_3)I_1 + O\biggr\}. \tag{23}$$

Together with (17), we have that $\mathcal{L}V \leqslant -1$.

Case 5. If $(S, I_1, I_2) \in U_5$,

$$\begin{aligned}\mathcal{L}V &\leqslant -M\mathcal{R} - \frac{\mu_2+\gamma_1}{4}I_1^{\vartheta+2} - \frac{r}{2K}S^{\vartheta+3} - \frac{\mu_2+\gamma_1}{4}I_1^{\vartheta+2} - \frac{\mu_3+\gamma_2}{2}I_2^{\vartheta+2} \\ &\quad + \frac{M(1-p)\beta_2(1+\alpha_2(b_2+b_4))}{\alpha_2^2}SI_2 + Mp\beta_1(b_2+b_3)I_1 + O \\ &\leqslant -M\mathcal{R} - \frac{\mu_2+\gamma_1}{4\epsilon^{\vartheta+3}} + F,\end{aligned}$$

where

$$F = \sup_{(S,I_1,I_2)\in\mathbb{R}^3_+}\biggl\{-\frac{r}{2K}S^{\vartheta+3} - \frac{\mu_2+\gamma_1}{4}I_1^{\vartheta+2} - \frac{\mu_3+\gamma_2}{2}I_2^{\vartheta+2} + \frac{M(1-p)\beta_2(1+\alpha_2(b_2+b_4))}{\alpha_2^2}SI_2 + Mp\beta_1(b_2+b_3)I_1 + O\biggr\}. \tag{24}$$

By (18) we get that $\mathcal{L}V \leqslant -1$.

Case 6. If $(S, I_1, I_2) \in U_6$,

$$\begin{aligned}\mathcal{L}V &\leqslant -M\mathcal{R} - \frac{\mu_3+\gamma_2}{4}I_2^{\vartheta+2} - \frac{r}{2K}S^{\vartheta+3} - \frac{\mu_2+\gamma_1}{2}I_1^{\vartheta+2} - \frac{\mu_3+\gamma_2}{4}I_2^{\vartheta+2} \\ &\quad + \frac{M(1-p)\beta_2(1+\alpha_2(b_2+b_4))}{\alpha_2^2}SI_2 + Mp\beta_1(b_2+b_3)I_1 + O \\ &\leqslant -M\mathcal{R} - \frac{\mu_3+\gamma_2}{4\epsilon^{\vartheta+3}} + J,\end{aligned}$$

where

$$J = \sup_{(S,I_1,I_2)\in\mathbb{R}^3_+}\biggl\{-\frac{r}{2K}S^{\vartheta+3} - \frac{\mu_2+\gamma_1}{2}I_1^{\vartheta+2} - \frac{\mu_3+\gamma_2}{4}I_2^{\vartheta+2} + \frac{M(1-p)\beta_2(1+\alpha_2(b_2+b_4))}{\alpha_2^2}SI_2 + Mp\beta_1(b_2+b_3)I_1 + O\biggr\}. \tag{25}$$

From (19) we derive that $\mathcal{L}V \leqslant -1$ for all $(S, I_1, I_2) \in U_6$.

Clearly, ϵ is small enough such that

$$\mathcal{L}V(S, I_1, I_2) \leqslant -1, \quad (S, I_1, I_2) \in \mathbb{R}^3_+ \setminus U. \qquad \square$$

6 Numerical simulations

We construct the following equivalent model to facilitate computer simulation:

$$\begin{aligned} S(j+1) &= \biggl[rS(j)\biggl(1-\frac{S(j)}{K}\biggr) - \frac{p\beta_1 S(j)I_1(j)}{\alpha_1 + S(j)} - \frac{(1-p)\beta_2 S(j)I_2(j)}{\alpha_2 + I_2(j)} \\ &\quad - \mu_1 S(j) - \delta S(j)\biggr]\Delta t + \sigma_1 S(j)\sqrt{\Delta t}\xi, \\ I_1(j+1) &= \biggl[\frac{p\beta_1 S(j)I_1(j)}{\alpha_1 + S(j)} - (\mu_2+\gamma_1)I_1(j)\biggr]\Delta t + \sigma_2 I_1(j)\sqrt{\Delta t}\zeta, \\ I_2(j+1) &= \biggl[\frac{(1-p)\beta_2 S(j)I_2(j)}{\alpha_2 + I_2(j)} - (\mu_3+\gamma_2)I_2(j)\biggr]\Delta t + \sigma_3 I_2(j)\sqrt{\Delta t}\eta, \end{aligned}$$

where ξ, ζ, η are independent random variables, Δt is the time taken divided by the step size.

In system (2), let $K = 4$, $r = 1.5$, $\mu_1 = 0.1$, $\beta_1 = 0.5$, $\mu_2 = 0.1$, $\beta_2 = 0.46$, $\mu_3 = 0.07$, $\alpha_1 = 0.3$, $\alpha_2 = 0.6$, $\delta = 0.4$, $\gamma_1 = 0.1$, $\gamma_2 = 0.1$, $p = 0.8$, $\sigma_1 = 0.15$. With the changes of σ_2 and σ_3, the diseases I_1 and I_2 will be extinct or persistent.

In each figure below, every figure has two subfigures. The first subfigure represents the development trend of $S(t)$, $I_1(t)$, $I_2(t)$, respectively. The second subfigure is the probability density of $S(t)$, $I_1(t)$, $I_2(t)$. From Theorem 3 we know that the diseases $I_i(t)$ $(i = 1, 2)$ will be extinct when $\mathfrak{R}_i < 1$ $(i = 1, 2)$. When $\mathfrak{R}_i > 1$ $(i = 1, 2)$, the diseases will be persistent.

In our simulations, we only consider the influence with the changes in white noise on the disease. When the values of white noises are large than a certain value, the disease will be extinct. When the white noise is less than a certain value, the disease will be persistent. The figures are consistent with the theorem in our paper.

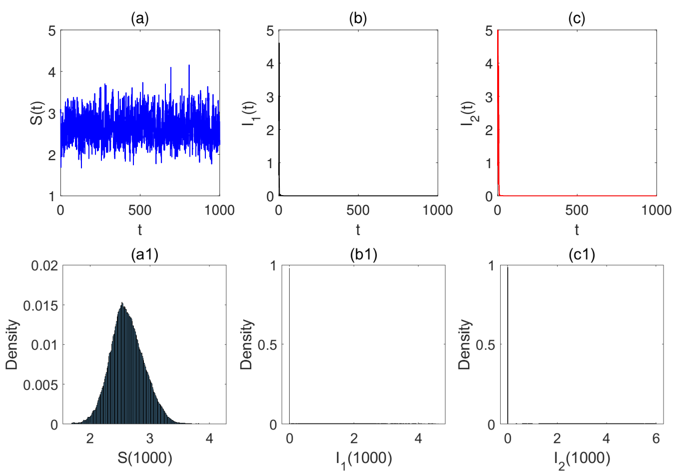

Figure 1. Extinction of $I_1(t)$ and $I_2(t)$.

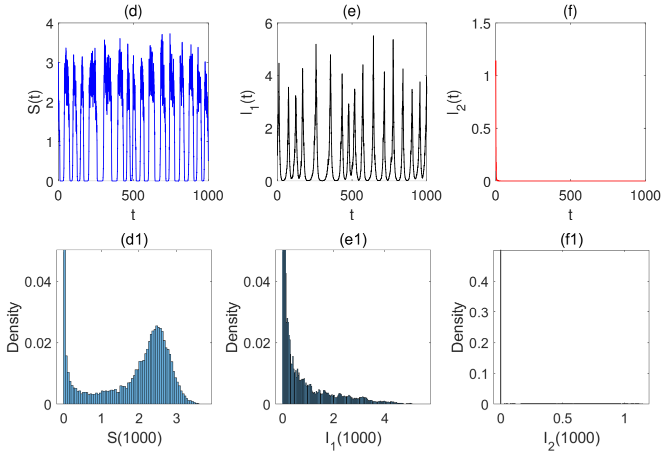

Figure 2. $I_1(t)$ is persistent and $I_2(t)$ is extinct.

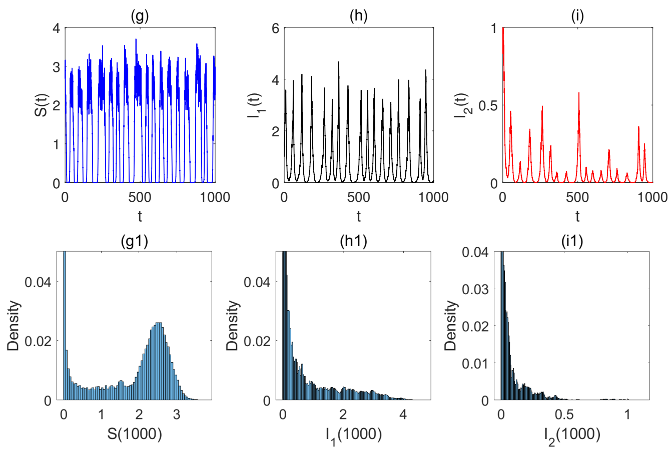

Figure 3. Persistence of $I_1(t)$ and $I_2(t)$.

In Fig. 1, we let $\sigma_2 = 0.8$ and $\sigma_3 = 0.75$. By calculating we obtain that $\mathfrak{R}_1 = 0.6899 < 1$ and $\mathfrak{R}_2 = 0.7334 < 1$, then the conditions of Theorem 3 hold. So $I_1(t)$ and $I_2(t)$ are extinct.

In Fig. 2, we let $\sigma_2 = 0.1$ and $\sigma_3 = 0.7$. By calculating we gain that $\mathfrak{R}_1 = 1.75 > 1$ and $\mathfrak{R}_2 = 0.785 < 1$, which satisfy condition (i) of Theorem 4. We can obtain that $I_1(t)$ is persistent (see Fig. 2(e)) and $I_2(t)$ is extinct (see Fig. 2(f)).

In Fig. 3, we let $\sigma_2 = 0.1$ and $\sigma_3 = 0.1$. By calculating we get that $\mathfrak{R}_1 = 1.75 > 1$ and $\mathfrak{R}_2 = 1.4701 > 1$, then condition (iii) of Theorem 4 holds. So $I_1(t)$ and $I_2(t)$ are persistent.

In Fig. 4, we simulate the influence of different noise intensities on system (2). It is found that as the intensity of white noise increases, the number of infections will decrease.

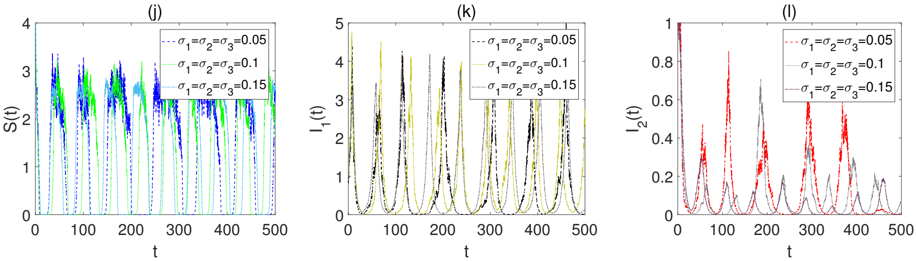

Figure 4. The influence of different noise intensities on the system (2).

In addition, as time goes on, it shows periodic outbreaks and the duration of the outbreak is shortened.

Diseases are always affected by various noises in the environment, then the changes of environmental noise can lead to changes in diseases. According to Theorems 3 and 4, the conditions of extinction and persistence in mean about system (2) have been established. These theorems are in fact a development of the papers by Cai [2], Liu [16] and Meng [21]. Furthermore, we used a new stochastic method to investigate the extinction and persistence, which is different from the previous works [2, 6, 7, 16, 18, 21]. The results that obtained in the present work can be applied to stochastic model of proportional disturbance. The obtained theory is a positive and effective guidance for cross infection. Many diseases, such as diphtheria, typhoid and influenza, are cured, the susceptible class can have permanent immunity. This feature can be well reflected in this model.

The model can introduce telephone noises such as continuous time Markov chain [5, 17, 27]. We only study one susceptible person, and we can study multiple susceptible persons. We also can investigate a susceptible person infected with more than three diseases [32]. We can explore the periodic solution of the epidemic model [23]. The impact of white noise on not only mortality rate but also infection rates also will be considered. The methods also can be used in stochastic food chain models [26, 31]. In our future work, we will solve these problems.

7 Conclusion

This paper provides a modeling framework based on stochastic differential equations to explore the long-term dynamics of epidemic cross-infection with SIR epidemiological laws. Since the interaction between the disease and the environment is full of stochasticity, it is of great practical significance to explore the mechanism of environmental stochasticity on the dynamics of infection. For this reason, we assumed that each component of the population is subject to environmental stochasticity, which is positively correlated with the density of each component of the population. In addition, in view of the fact that immunization is widely used in the control of epidemic diseases and has achieved miraculous achievements repeatedly, we also considered the impact of immunization on the spread of diseases. With the help of stochastic analysis tools and auxiliary systems, we studied the properties of the global positive solution of the proposed system. Furthermore, we provided our main theoretical results including the extinction, persistence and the existence of a unique stationary distribution of the proposed model. The results show that: (i) When $\mathfrak{R}_j < 1$, $j = 1, 2$, both types of diseases will be extinct with probability one. Since the intensities of environmental stochasticities σ_i, $i = 2, 3$, are negatively correlated with the conditions for extinction $\mathfrak{R}_j$, $j = 1, 2$, respectively, which indicates that environmental stochasticity is not conducive to the survival of the diseases. (ii) When $\mathfrak{R}_1 > 1$ and $\mathfrak{R}_2 < 1$, $I_1(t)$ will be persistent in the mean, while $I_2(t)$ will be extinct; when $\mathfrak{R}_1 < 1$ and $\mathfrak{R}_2 > 1$, $I_2(t)$ will be persistent in the mean, while $I_1(t)$ will be extinct; when $\mathfrak{R}_j > 1$, $j = 1, 2$, both $I_1(t)$ and $I_2(t)$ will be persistent in the mean. Similar to the item (ii), this result suggests that small stochasticity is beneficial to the survival of the diseases. (iii) When $R_1 > 1$ and the parameters meet some other constraints (see Theorem 5 for detail), the stochastic system has a unique ergodic stationary distribution. Ergodicity means that the statistical properties of the stochastic system will not change over time, which allows us to estimate the contour of the stationary distribution by simulating a trajectory of the solution. In this scenario, the results also indicate that the small stochasticity is necessary for the existence of the stationary distribution, i.e., small stochasticity contributes to the survival of the diseases.

This work is just our preliminary exploration of how stochasticity affects the dynamics of disease transmission. In order to have a more comprehensive understanding of the interaction between environmental stochasticity and the diseases, the following explorations are needed: (i) The type of stochasticity considered in this article is white noise, and in complex actual environments, there are other types of stochasticity such as telegraph noise [5, 17]. It has practical significance to investigate how these different types of noise synergistically affect the spread of diseases. (ii) This article only considers the situation of one type of susceptibles. Next, we can explore more complex scenarios such as by including more than three types of diseases [32]. (iii) This article assumes that stochasticity is positively correlated with the density of each component. Next, we can also consider the situation where stochasticity mainly perturbs the infection rate, which will induce a different stochastic system with degenerate diffusion terms, and its theoretical analysis is also more challenging.

References

1. S. Cai, Y. Cai, X. Mao, A stochastic differential equation SIS epidemic model with two independent Brownian motions, *J. Math. Anal. Appl.*, **474**(2):1536–1550, 2019, `https://doi.org/10.1016/j.jmaa.2019.02.039`.

2. Y. Cai, Y. Kang, W. Wang, A stochastic SIRS epidemic model with nonlinear incidence rate, *Appl. Math. Comput.*, **305**:221–240, 2017, `https://doi.org/10.1016/j.amc.2017.02.003`.

3. V. Capasso, G. Serio, A generalization of the Kermack-McKendrick deterministic epidemic model, *Math. Biosci.*, **42**(1-2):43–61, 1978, `https://doi.org/10.1016/0025-5564(78)90006-8`.

4. Y. Chen, W. Zhao, Dynamical analysis of a stochastic SIRS epidemic model with saturating contact rate, *Math. Biosci. Eng.*, **17**(5):5925–5943, 2020, `https://doi.org/10.3934/mbe.2020316`.
5. Y. Deng, M. Liu, Analysis of a stochastic tumor-immune model with regime switching and impulsive perturbations, *Appl. Math. Modelling*, **78**:482–504, 2020, `https://doi.org/10.1016/j.apm.2019.10.010`.
6. N. Dieu, D. Nguyen, N. Du, G. Yin, Classification of asymptotic behavior in a stochastic SIR model, *SIAM J. Appl. Dyn. Syst.*, **15**(2):1062–1084, 2016, `https://doi.org/10.1137/15M1043315`.
7. J. Ghosh, P. Majumdar, U. Ghosh, Qualitative analysis and optimal control of an SIR model with logistic growth, non-monotonic incidence and saturated treatment, *Math. Model. Nat. Phenom.*, **16**:13, 2021, `https://doi.org/10.1051/mmnp/2021004`.
8. A. Gray, D. Greenhalgh, L. Hu, X. Mao, J. Pan, A stochastic differential equation SIS epidemic model, *SIAM J. Appl. Math.*, **71**(3):876–902, 2011, `https://doi.org/10.1137/10081856X`.
9. P. Han, Z. Chang, X. Meng, Asymptotic dynamics of a stochastic sir epidemic system affected by mixed nonlinear incidence rates, *Complexity*, **2020**, 2020, `https://doi.org/10.1155/2020/8596371`.
10. H.W. Hethcote, Qualitative analyses of communicable disease models, *Math. Biosci.*, **28**(3-4):335–356, 1976, `https://doi.org/10.1016/0025-5564(76)90132-2`.
11. W.O. Kermazk, A.G. McKendrick, A contribution to the mathematical theory of epidemic, *Proc. R. Soc. London A.*, **115**(772):700–721, 1927, `https://doi.org/10.1098/rspa.1927.0118`.
12. R. Khasminskii, *Stochastic Stability of Differential Equations*, Springer, Heidelberg, 1980, `https://doi.org/10.1007/978-3-642-23280-0`.
13. G. Lan, Y. Huang, C. Wei, S. Zhang, A stochastic SIS epidemic model with saturating contact rate, *Physica A*, **529**:121504, 2019, `https://doi.org/10.1016/j.physa.2019.121504`.
14. M. Liu, K. Wang, Dynamics of a two-prey one-predator system in random environments, *J. Nonlinear Sci.*, **23**(5):751–775, 2013, `https://doi.org/10.1007/s00332-013-9167-4`.
15. Q. Liu, D. Jiang, T. Hayat, B. Ahmad, Analysis of a delayed vaccinated SIR epidemic model with temporary immunity and Lévy jumps, *Nonlinear Anal., Hybrid Syst.*, **27**:29–43, 2018, `https://doi.org/10.1016/j.nahs.2017.08.002`.
16. Q. Liu, D. Jiang, T. Hayat, A. Alsaedi, Threshold behavior in a stochastic delayed SIS epidemic model with vaccination and double diseases, *J. Franklin Inst.*, **356**(13):7466–7485, 2019, `https://doi.org/10.1016/j.jfranklin.2018.11.055`.
17. Q. Liu, D. Jiang, N. Shi, Threshold behavior in a stochastic SIQR epidemic model with standard incidence and regime switching, *Appl. Math. Comput.*, **316**:310–325, 2018, `https://doi.org/10.1016/j.amc.2017.08.042`.
18. Z. Liu, Dynamics of positive solutions to SIR and SEIR epidemic models with saturated incidence rates, *Nonlinear Anal., Real World Appl.*, **14**(3):1286–1299, 2013, `https://doi.org/10.1016/j.nonrwa.2012.09.016`.

19. X. Mao, *Stochastic Differential Equations and Their Applications*, Woodhead, Philadelphia, 2007.

20. X. Mao, G. Marion, E. Renshaw, Environmental Brownian noise suppresses explosions in population dynamics, *Stochastic Processes Appl.*, **97**(1):95–110, 2002, `https://doi.org/10.1016/S0304-4149(01)00126-0`.

21. X. Meng, S. Zhao, T. Feng, T. Zhang, Dynamics of a novel nonlinear stochastic SIS epidemic model with double epidemic hypothesis, *J. Math. Anal. Appl.*, **433**(1):227–242, 2016, `https://doi.org/10.1016/j.jmaa.2015.07.056`.

22. S. Pasquali, The stochastic logistic equation: stationary solutions and their stability, *Rend. Semin. Mat. Univ. Padova*, **106**:165–183, 2001.

23. H. Qi, X. Leng, X. Meng, T. Zhang, Periodic solution and ergodic stationary distribution of SEIS dynamical systems with active and latent patients, *Qual. Theory Dyn. Syst.*, **18**(2):347–369, 2019, `https://doi.org/10.1007/s12346-018-0289-9`.

24. Z. Teng, L. Wang, Persistence and extinction for a class of stochastic SIS epidemic models with nonlinear incidence rate, *Physica A*, **451**:507–518, 2016, `https://doi.org/10.1016/j.physa.2016.01.084`.

25. C. Xu, S. Yuan, T. Zhang, Average break-even concentration in a simple chemostat model with telegraph noise, *Nonlinear Anal., Hybrid Syst.*, **29**:373–382, 2018, `https://doi.org/10.1016/j.nahs.2018.03.007`.

26. X. Yu, S. Yuan, T. Zhang, Asymptotic properties of stochastic nutrient-plankton food chain models with nutrient recycling, *Nonlinear Anal., Hybrid Syst.*, **34**:209–225, 2019, `https://doi.org/10.1016/j.nahs.2019.06.005`.

27. T. Zhang, F. Deng, Y. Sun, P. Shi, Fault estimation and fault-tolerant control for linear discrete time-varying stochastic systems, *Sci. China, Inf. Sci.*, **64**(10):1–16, 2021, `https://doi.org/10.1007/s11432-021-3280-4`.

28. X. Zhang, X. Zhang, The threshold of a deterministic and a stochastic SIQS epidemic model with varying total population size, *Appl. Math. Modelling*, **91**:749–767, 2021, `https://doi.org/10.1016/j.apm.2020.09.050`.

29. Y. Zhao, D. Jiang, The threshold of a stochastic SIS epidemic model with vaccination, *Appl. Math. Comput.*, **243**:718–727, 2014, `https://doi.org/10.1016/j.amc.2014.05.124`.

30. B. Zhou, D. Jiang, Y. Dai, T. Hayatand A. Alsaedi, Stationary distribution and probability density function of a stochastic SIRSI epidemic model with saturation incidence rate and logistic growth, *Chaos Solitons Fractals*, **142**:110519, 2021, `https://doi.org/10.1016/j.chaos.2020.110601`.

31. C. Zhu, G. Yin, Asymptotic properties of hybrid diffusion systems, *SIAM J. Control Optim.*, **46**(4):1155–1179, 2007, `https://doi.org/10.1137/060649343`.

32. F. Zhu, X. Meng, T. Zhang, Optimal harvesting of a competitive n-species stochastic model with delayed diffusions, *Math. Biosci. Eng.*, **16**(3):1554–1574, 2019, `https://doi.org/10.3934/mbe.2019074`.