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Solvability of a system of integral equations in two variables in the weighted Sobolev space $W_{\omega}^{1,1}(a,b)$ using a generalized measure of noncompactness

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Abstract. In this paper, we deal with the existence of solutions for a coupled system of integral equations in the Cartesian product of weighted Sobolev spaces $\mathcal{E} = W_{\omega}^{1,1}(a,b) \times W_{\omega}^{1,1}(a,b)$. The results were achieved by equipping the space $\mathcal{E}$ with the vector-valued norms and using the measure of noncompactness constructed in [F.P. Najafabad, J.J. Nieto, H.A. Kayvanloo, Measure of noncompactness on weighted Sobolev space with an application to some nonlinear convolution type integral equations, *J. Fixed Point Theory Appl.*, 22(3), 75, 2020] to applicate the generalized Darbo's fixed point theorem [J.R. Graef, J. Henderson, and A. Ouahab, *Topological Methods for Differential Equations and Inclusions*, CRC Press, Boca Raton, FL, 2018].

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1 Introduction

Sobolev spaces $W^{m,p}(\Omega)$ ($\Omega \subseteq \mathbb{R}^N$) [7] are the classes of functions f defined a.e. on Ω with its derivatives in distributional sense $D^\alpha f$ for orders $|\alpha| \leqslant m$ in $L^p(\Omega)$. One of the most important mathematical discoveries of the XXth century was the concept of Sobolev spaces. This theory is essential in the study of nonlinear partial differential equations in modern analysis. In the early 1970s, Muckenhoupt [14] introduced the A_p class of weights, which are common in applications. Following that, many papers and books have been discussed intensively in Sobolev spaces with Muckenhoupt's weights.

Measures of noncompactness introduced by Kuratowski [12] are functions that measure the degree of noncompactness of sets in complete metric spaces. These functions play an outstanding role in fixed point theory. In 1955, Darbo presented a fixed point theorem [8] using this notion. Furthermore, several interesting papers on the solvability of various integral equations in Sobolev spaces without weights using the measures of noncompactness have been shown; see, for example, [3, 5, 11, 13].

The coupled system of integral equations describe a phenomenon in biological science, physics, electrodynamics, electromagnetic, and fluid dynamics. In the last decades, the problem of existence solutions of these equations has been taking great interest [1, 4, 16–18]. In particular, Nasiri et al. [16] gave an existing result of the following category of Volterra integral equations system:

$$\begin{aligned}
\varrho_1(\iota) &= \mathsf{A}(\iota) + f\bigl(\iota, \varrho_1(\iota), \varrho_2(\iota)\bigr) \\
&\quad + \mathsf{P}\bigl(\varrho_1(\iota), \varrho_2(\iota)\bigr) \int_0^t g\bigl(\varsigma, \varrho_1(\varsigma), \varrho_2(\varsigma)\bigr) Q\bigl(\varrho_1(\varsigma), \varrho_2(\varsigma)\bigr)\, \mathrm{d}\varsigma, \\
\varrho_2(\iota) &= \mathsf{A}(\iota) + f\bigl(\iota, \varrho_2(\iota), \varrho_1(\iota)\bigr) \\
&\quad + \mathsf{P}\bigl(\varrho_2(\iota), \varrho_1(\iota)\bigr) \int_0^t g\bigl(\varsigma, \varrho_2(\varsigma), \varrho_1(\varsigma)\bigr) Q\bigl(\varrho_2(\varsigma), \varrho_1(\varsigma)\bigr)\, \mathrm{d}\varsigma
\end{aligned} \tag{1}$$

in $\mathcal{C}([0, L], \mathcal{E}) \times \mathcal{C}([0, L], \mathcal{E})$. Here $\mathcal{E}$ is real Banach algebra, and the entries on system (1) satisfy certain conditions. The idea used here is to prove that system (1) has a coupled fixed point with the help of the measure of noncompactness defined by

$$\tilde{\mu}(\mathcal{K}_1 \times \mathcal{K}_2) = \mathcal{G}\bigl(\mu(\mathcal{K}_1), \mu(\mathcal{K}_2)\bigr),$$

where $\mathcal{K}_1, \mathcal{K}_2 \subset \mathcal{C}([0, L], \mathcal{E})$, $\mathcal{G}$ is a convex function from $\mathbb{R}_+^2$ into $\mathbb{R}_+$ satisfying $\mathcal{G}(t_1, t_2) = 0$ if and only if $t_1 = t_2 = 0$, and μ is a usual measure of noncompactness.

Another method to ensure the existence of solutions of a coupled system of integral equations is to work on some suitable generalized Banach space in the sense of Perov. In [10], the authors extended Darbo's fixed point theorem on generalized Banach spaces by replacing the set contraction factor with a matrix convergent to zero and the usual measure of noncompactness of a set $\mathcal{A}$ with a generalized (vector) measure of noncompactness

$$\mu_{\mathrm{G}}(\mathcal{A}) = \begin{pmatrix} \mu_1(\mathcal{A}) \\ \vdots \\ \mu_n(\mathcal{A}) \end{pmatrix}$$

(see Definition 8 and Theorem 1 below).

More recently, authors in [15] constructed a new measure of noncompactness on weighted Sobolev spaces $W_\omega^{m,p}(\Omega)$, where ω is A_p weight, and presented the effectiveness of this measure by studying the existence of solution of some nonlinear convolution-type integral equations using Darbo's fixed point theorem.

The organization for the rest of this manuscript is as follows: Section 2 is devoted to the presentation of definitions and some auxiliary results regarding the main objects of the monograph. In Section 3, we present existence results with the help of the so-called generalized measure of noncompactness for the following system of the integral equation (SIE):

$$\begin{aligned} \varrho_1(\iota) &= \mathsf{P}_1\bigl(\iota, \varrho_1(\iota), \varrho_2(\iota)\bigr) + \int_a^\iota \mathsf{K}_1(\iota, \varsigma)\mathsf{L}_1\bigl(\varsigma, \varrho_1(\iota), \varrho_2(\iota)\bigr)\,\mathrm{d}\varsigma, \\ \varrho_2(\iota) &= \mathsf{P}_2\bigl(\iota, \varrho_1(\iota), \varrho_2(\iota)\bigr) + \int_a^\iota \mathsf{K}_2(\iota, \varsigma)\mathsf{L}_2\bigl(\varsigma, \varrho_1(\iota), \varrho_2(\iota)\bigr)\,\mathrm{d}\varsigma, \end{aligned} \tag{2}$$

where the functions P_1, P_2, K_1, K_2, L_1, L_2 are given and verify some conditions. The functional setting of this system is the generalized Banach space $\mathcal{E} = W_\omega^{1,1}(a,b) \times W_\omega^{1,1}(a,b)$. Finally, an example is given to show the effectiveness of the obtained result.

2 Preliminaries

We recall some concepts that are necessary for this paper. So, this section deals with notations, definitions, and auxiliary results of weighted Sobolev spaces, generalized Banach spaces, generalized measures of noncompactness, and fixed point theory. Beginning with a review of the definition of weights, in particular, A_1 weights, for more details, we refer the reader to the following monographs: [9, 14].

Definition 1. (See [19].) A weight on $\mathbb{R}$ is a locally integrable function ω such that $\omega(\iota) > 0$ for a.e. $t \in \mathbb{R}$.

Definition 2. (See [19].) A weight ω is said to be an A_1 weight if there exists a positive constant A such that for every ball $B \subset \mathbb{R}$,

$$\left(\oint_B \omega(\iota)\,\mathrm{d}\iota\right) \operatorname*{ess\,sup}_{\iota \in B} \frac{1}{\omega(\iota)} \leqslant A, \quad \text{where } \oint_B \bigl|f(\iota)\bigr|\,\mathrm{d}\iota = \frac{1}{|B|}\int_B \bigl|f(\iota)\bigr|\,\mathrm{d}\iota,$$

here $|B|$ is the Lebesgue measure of the ball B. The infimum over all such constants A is called the A_1 constant of ω. We denote by A_1 the set of all A_1 weights.

Let ω be a weight, and let $(a,b) \subseteq \mathbb{R}$ be open. We define the weighted Lebesgue space $L^1_\omega((a,b))$ as the set of measurable functions f on (a,b) such that

$$\|f\|_{L^1_\omega((a,b))} = \int_a^b |f(\iota)|\omega(\iota)\,\mathrm{d}\iota < \infty.$$

Definition 3. (See [19].) Suppose that the weight ω is in A_1. Then we define the weighted Sobolev space $W^{1,1}_\omega((a,b))$ as the set of functions $f \in L^1_\omega((a,b))$ with weak derivatives $f' \in L^1_\omega((a,b))$. The weighted Sobolev space $W^{1,1}_\omega((a,b))$ is a Banach space with the norm

$$\|f\|_{W^{1,1}_\omega((a,b))} = \max\bigl\{\|f\|_{L^1_\omega((a,b))}, \|f'\|_{L^1_\omega((a,b))}\bigr\}.$$

Now, define on $\mathcal{M}_{m\times n}(\mathbb{R}_+)$ the partial order relation as follows. Let $M, N \in \mathcal{M}_{m\times n}(\mathbb{R}_+)$, $m \geqslant 1$ and $n \geqslant 1$. Put $M = (M_{i,j})_{1\leqslant j\leqslant m,\,1\leqslant i\leqslant n}$ and $N = (N_{i,j})_{1\leqslant j\leqslant m,\,1\leqslant i\leqslant n}$. Then

$$\begin{aligned} M \preccurlyeq N &\quad \text{if } N_{i,j} \geqslant M_{i,j} \text{ for all } j = 1,\dots,m,\ i = 1,\dots,n, \\ M \prec N &\quad \text{if } N_{i,j} > M_{i,j} \text{ for all } j = 1,\dots,m,\ i = 1,\dots,n. \end{aligned}$$

Let $\mathcal{A} = \prod_{i=1}^n \mathcal{A}_i$ be a bounded set of $\mathbb{R}^n$, the supremum bound (resp. the infimun bound) of $\mathcal{A}$ is the vector

$$\widehat{\sup}\{\lambda\colon\ \lambda \in \mathcal{A}\} := \begin{pmatrix} \sup\{\lambda_1\colon\ \lambda_1 \in \mathcal{A}_1\} \\ \vdots \\ \sup\{\lambda_n\colon\ \lambda_n \in \mathcal{A}_n\} \end{pmatrix}$$

$$\left(\text{resp. } \widehat{\inf}\{\lambda\colon \lambda \in \mathcal{A}\} := \begin{pmatrix} \inf\{\lambda_1\colon\ \lambda_1 \in \mathcal{A}_1\} \\ \vdots \\ \inf\{\lambda_n\colon\ \lambda_n \in \mathcal{A}_n\} \end{pmatrix}\right).$$

Definition 4. Let $\mathcal{E}$ be a vector space on $\mathsf{K} = \mathbb{R}$ or $\mathbb{C}$. By a generalized norm on $\mathcal{E}$ we mean a map

$$\|\cdot\|_{\mathrm{G}} : \mathcal{E} \to \mathbb{R}^n_+, \quad \varrho \mapsto \|\varrho\|_{\mathrm{G}} = \begin{pmatrix} \|\varrho\|_1 \\ \vdots \\ \|\varrho\|_n \end{pmatrix}$$

satisfying the following properties:

(i) For all $\varrho \in \mathcal{E}$; if $\|\varrho\|_{\mathrm{G}} = 0_{\mathbb{R}^n_+}$, then $\varrho = 0$,
(ii) $\|\lambda\varrho\|_{\mathrm{G}} = |\lambda|\|\varrho\|_{\mathrm{G}}$ for all $\varrho \in \mathcal{E}$ and $\lambda \in \mathsf{K}$, and
(iii) $\|\varrho + y\|_{\mathrm{G}} \preccurlyeq \|\varrho\|_{\mathrm{G}} + \|y\|_{\mathrm{G}}$ for all $\varrho, y \in \mathcal{E}$.

The pair $(\mathcal{E}, \|\cdot\|_{\mathrm{G}})$ is called a vector (generalized) normed space. Furthermore, $(\mathcal{E}, \|\cdot\|_{\mathrm{G}})$ is called a generalized Banach space (in short, GBS) if the vector metric space generated by its vector metric is complete.

Proposition 1. (*See* [10].) *In a GBS, in the sense of Perov, the definitions of convergence sequence, continuity, open subsets, and closed subsets are similar to those for usual Banach spaces.*

Let $(\mathcal{E}, \|\cdot\|_{\mathrm{G}})$ be a generalized Banach space. Throughout this paper and for $r = (r_1, \dots, r_n) \in \mathbb{R}_+^n$, $\varrho_0 \in \mathcal{E}$, and $i = 1, \dots, n$, we denote by

$$B(\varrho_0, r) = \{\varrho \in \mathcal{E}\colon \|\varrho_0 - \varrho\|_G \prec r\} \qquad \bigl(\text{resp. } B_i(\varrho_0, r_i) = \{\varrho \in \mathcal{E}\colon \|\varrho_0 - \varrho\|_i < r_i\}\bigr)$$

the open ball centered at ϱ_0 with radius r (resp. r_i) and by

$$\bar{B}(\varrho_0, r) = \{\varrho \in \mathcal{E}\colon \|\varrho_0 - \varrho\|_G \preccurlyeq r\} \qquad \bigl(\text{resp. } \bar{B}_i(\varrho_0, r_i) = \{\varrho \in \mathcal{E}\colon \|\varrho_0 - \varrho\|_i \leqslant r_i\}\bigr)$$

the closed ball centered at ϱ_0 with radius r (resp. r_i). If $\varrho_0 = 0$, we simply denote $B_r = B(0, r)$ and $\bar{B}_r = \bar{B}(0, r)$. Finally, we respectively denote by $\overline{\mathcal{K}}$ and $\mathrm{co}(\mathcal{K})$ the closure and the convex hull of an arbitrary subset $\mathcal{K}$ of $\mathcal{E}$.

Definition 5. A matrix $M \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ is said to be convergent to zero if

$$M^m \to 0 \quad \text{as } m \to \infty.$$

Lemma 1. *See* [20].) *Let* $M \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$. *The following affirmations are equivalent*:

(i) $M^m \to 0$ *as* $m \to \infty$.
(ii) *The matrix* $I - M$ *is invertible, and* $(I - M)^{-1} \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$.
(iii) *The spectral radius* $\rho(M)$ *is strictly less than* 1.

Definition 6. Let $(\mathcal{E}, \|\cdot\|_{\mathrm{G}})$ be a GBS, and let $\mathcal{K}$ be a subset of $\mathcal{E}$. Then $\mathcal{K}$ is said to be G-bounded if there is a vector $V \in \mathbb{R}_+^n$ such that for all $\varrho \in \mathcal{K}$, $\|\varrho\|_{\mathrm{G}} \preccurlyeq V$, and we write

$$\|\mathcal{K}\|_{\mathrm{G}} := \widehat{\sup}\{\|\varrho\|_{\mathrm{G}} : \varrho \in \mathcal{K}\} = \begin{pmatrix} \sup_{\varrho\in\mathcal{K}} \|\varrho\|_1 \\ \vdots \\ \sup_{\varrho\in\mathcal{K}} \|\varrho\|_n \end{pmatrix} \preccurlyeq V.$$

Definition 7. Let $(\mathcal{E}, \|\cdot\|_{\mathrm{G}})$ be a GBS. A subset $\mathcal{K}$ of $\mathcal{E}$ is called G-compact if every open cover of $\mathcal{K}$ has a finite subcover. $\mathcal{K}$ is said relatively G-compact if its closure is G-compact.

We denote by $\mathcal{N}_{\mathrm{G}}(\mathcal{E})$ the family of all relatively G-compact subsets of $\mathcal{E}$.

Now, we present a definition of an axiomatic measure of noncompactness for generalized Banach spaces similar to that introduced in 1980 by Banaś and Goebel [6].

Definition 8. Let $(\mathcal{E}, \|\cdot\|_{\mathrm{G}})$ be a GBS, and let $\mathcal{B}_{\mathrm{G}}(\mathcal{E})$ be the family of G-bounded subsets of $\mathcal{E}$. A map

$$\mu_{\mathrm{G}} : \mathcal{B}_{\mathrm{G}}(\mathcal{E}) \to [0, +\infty)^n, \quad \mathcal{A} \mapsto \mu_{\mathrm{G}}(\mathcal{A}) = \begin{pmatrix} \mu_1(\mathcal{A}) \\ \vdots \\ \mu_n(\mathcal{A}) \end{pmatrix}$$

is called a generalized measure of noncompactness (for short G-MNC) defined on $\mathcal{E}$ if it satisfies the following conditions:

(i) The family $\ker \mu_{\mathrm{G}}(\mathcal{E}) = \{\mathcal{A} \in \mathcal{B}_{\mathrm{G}}(\mathcal{E})\colon \mu_{\mathrm{G}}(\mathcal{A}) = 0\}$ is nonempty, and $\ker \mu_{\mathrm{G}}(\mathcal{E}) \subset \mathcal{N}_{\mathrm{G}}(\mathcal{E})$.
(ii) Monotonicity: $\mathcal{A}_1 \subseteq \mathcal{A}_2 \Rightarrow \mu_{\mathrm{G}}(\mathcal{A}_1) \preccurlyeq \mu_{\mathrm{G}}(\mathcal{A}_2)$ for all $\mathcal{A}_1, \mathcal{A}_2 \in \mathcal{B}_{\mathrm{G}}(\mathcal{E})$.
(iii) Invariance under closure and convex hull: $\mu_{\mathrm{G}}(\mathcal{A}) = \mu_{\mathrm{G}}(\overline{\mathcal{A}}) = \mu_{\mathrm{G}}(\mathrm{co}(\mathcal{A}))$ for all $\mathcal{A} \in \mathcal{B}_{\mathrm{G}}(\mathcal{E})$.
(iv) Convexity: $\mu_{\mathrm{G}}(\lambda\mathcal{A}_1 + (1-\lambda)\mathcal{A}_2) \preccurlyeq \lambda\mu_{\mathrm{G}}(\mathcal{A}_1) + (1-\lambda)\mu_{\mathrm{G}}(\mathcal{A}_2)$ for all $\mathcal{A}_1, \mathcal{A}_2 \in \mathcal{B}_{\mathrm{G}}(\mathcal{E})$ and $\lambda \in [0,1]$.
(v) Generalized Cantor intersection property: if $(\mathcal{A}_m)_{m\geqslant 1}$ is a sequence of nonempty, closed subsets of $\mathcal{E}$ such that $\mathcal{A}_1$ is G-bounded and $\mathcal{A}_1 \supseteq \mathcal{A}_2 \supseteq \cdots \supseteq \mathcal{A}_m \cdots$, and $\lim_{m\to+\infty} \mu_{\mathrm{G}}(\mathcal{A}_m) = 0_{\mathbb{R}^n_+}$, then the set $\mathcal{A}_\infty := \bigcap_{m=1}^{\infty} \mathcal{A}_m$ is nonempty and is G-compact.

Definition 9. Let $(\mathcal{E}, \|\cdot\|_{\mathrm{G}})$ be a GBS, and let μ_{G} be a G-MNC. A self-mapping $S\colon \mathcal{E} \to \mathcal{E}$ is said to be a M-set contractive with respect to μ_{G} if S maps G-bounded sets into G-bounded sets and there exists a matrix $M \in \mathcal{M}_{n\times n}(\mathbb{R}^+)$ such that

$$\mu_{\mathrm{G}}\big(S(\mathcal{A})\big) \preccurlyeq M\mu_G(\mathcal{A})$$

for every nonempty G-bounded subset $\mathcal{A}$ of $\mathcal{E}$. If the matrix M converges to zero, then we say that S satisfies the generalized Darbo condition.

Theorem 1. (*See* [10].) *Let $\mathcal{E}$ be a GBS. Then every nonempty G-bounded, closed, convex subset $\mathcal{K}$ of $\mathcal{E}$ has the fixed point property for continuous mappings satisfying the generalized Darbo condition.*

Theorem 2. (*See* [15].) *Let $\mathcal{A}$ be a bounded subset of the space $W^{1,1}_\omega((a,b))$. For $\varrho \in \mathcal{A}$ and $\varepsilon > 0$, let us denote*

$$\begin{aligned}
\mu^0(\varrho,\varepsilon) &= \sup\big\{\|\tau_h\varrho - \varrho\|_{W^{1,1}_\omega(B_T)},\ |h| < \varepsilon\big\},\\
\mu^0(\mathcal{A},\varepsilon) &= \sup\big\{\mu^0(\varrho,\varepsilon)\colon\ \varrho \in \mathcal{A}\big\},\\
\mu^0(\mathcal{A}) &= \lim_{\varepsilon\to 0}\mu^0(\mathcal{A},\varepsilon),
\end{aligned}$$

where $\tau_h\varrho(\iota) = \varrho(\iota + h)$ for $\iota, h \in \mathbb{R}$, and

$$\begin{aligned}
d_T(\mathcal{A}) &= \sup\big\{\|\varrho\|_{W^{1,1}_\omega((a,b)\setminus B_T)}\colon\ \varrho \in \mathcal{A}\big\},\\
d(\mathcal{A}) &= \lim_{T\to\infty} d_T(\mathcal{A}),\\
\mu(\mathcal{A}) &= \mu^0(\mathcal{A}) + d(\mathcal{A}).
\end{aligned}$$

The function $\mu : \mathcal{B}_{\mathrm{G}}(W^{1,1}_\omega((a,b))) \to \mathbb{R}_+$ is a measure of noncompactness on the weighted Sobolev space $W^{1,1}_\omega((a,b))$, and moreover, $\ker\mu = \mathcal{N}(W^{1,1}_\omega((a,b)))$

Proposition 2. *The space $\mathcal{E} = W^{1,1}_\omega((a,b)) \times W^{1,1}_\omega((a,b))$ define a generalized Banach space equipped with the generalized norm*

$$\|f\|_{\mathrm{G}} := \begin{pmatrix} \|f_1\|_{W^{1,1}_\omega((a,b))} \\ \|f_2\|_{W^{1,1}_\omega((a,b))} \end{pmatrix}$$

for each $f := (f_1, f_2) \in W^{1,1}_\omega((a,b)) \times W^{1,1}_\omega((a,b))$. *Furthermore, the function* $\mu_{\mathrm{G}} : \mathcal{B}_{\mathrm{G}}(W^{1,1}_\omega((a,b)) \times W^{1,1}_\omega((a,b))) \to \mathbb{R}^2_+$ *defined as*

$$\mu_{\mathrm{G}}(\mathcal{A}) = \begin{pmatrix} \mu_1(\mathcal{A}_1) \\ \mu_2(\mathcal{A}_2) \end{pmatrix} := \begin{pmatrix} \mu^0(\mathcal{A}_1) + d(\mathcal{A}_1) \\ \mu^0(\mathcal{A}_2) + d(\mathcal{A}_2) \end{pmatrix} = \mu^0_{\mathrm{G}}(\mathcal{A}) + d_{\mathrm{G}}(\mathcal{A})$$

is generalized measure of noncompactness on $\mathcal{E}$.

Definition 10. (See [2].) A function $f : \mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}$ is said to have the Carathéodory property if

(i) the function $t \to f(\iota, u)$ is measurable for any $u \in \mathbb{R}^M$,
(ii) the function $u \to f(\iota, u)$ is continuous for almost all $t \in \mathbb{R}^N$.

3 Main results

In this section, we study the existence of solutions for the system of integral equation (SIE) (2). Problem (2) will be discussed under the following assumptions:

($\mathcal{H}_1$) $\mathsf{K}_1, \mathsf{K}_2 \in \mathcal{C}^1(\overline{(a,b)^2}, \mathbb{R})$.
($\mathcal{H}_2$) The functions $\mathsf{P}_1, \mathsf{P}_2 : (a,b) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfy the Carathéodory conditions and have the continuous derivatives of order 1 with respect to the first variable, and

(a) There exists a matrix $M_1 = \left(\begin{smallmatrix} \alpha_{1,1} & \alpha_{1,2} \\ \alpha_{2,1} & \alpha_{2,2} \end{smallmatrix}\right) \in \mathcal{M}_{2\times 2}(L^\infty(a,b), \mathbb{R}_+)$ such that for each $(\iota, \varrho_1, \varrho_2), (\iota, \bar{\varrho}_1, \bar{\varrho}_2) \in (a,b) \times \mathbb{R} \times \mathbb{R}$ and for $i \in \{1,2\}$, we have

$$\begin{aligned} \left|\mathsf{P}_i(\iota, \varrho_1, \varrho_2) - \mathsf{P}_i(\iota, \bar{\varrho}_1, \bar{\varrho}_2)\right| &\leqslant \alpha_{i,1}(\iota)|\varrho_1 - \bar{\varrho}_1| + \alpha_{i,2}(\iota)|\varrho_2 - \bar{\varrho}_2|, \\ \left|\frac{\partial \mathsf{P}_i}{\partial \iota}(\iota, \varrho_1, \varrho_2) - \frac{\partial \mathsf{P}_i}{\partial \iota}(\iota, \bar{\varrho}_1, \bar{\varrho}_2)\right| &\leqslant \alpha_{i,1}(\iota)|\varrho_1 - \bar{\varrho}_1| + \alpha_{i,2}(\iota)|\varrho_2 - \bar{\varrho}_2|. \end{aligned}$$

(b) There exists a matrix $M_2 = \left(\begin{smallmatrix} H_{1,1} & H_{1,2} \\ H_{2,1} & H_{2,2} \end{smallmatrix}\right) \in \mathcal{M}_{2\times 2}(\mathbb{R}_+)$ such that for each $(\iota, \varrho_1, \varrho_2), (\iota, \bar{\varrho}_1, \bar{\varrho}_2) \in (a,b) \times \mathbb{R} \times \mathbb{R}$ and for $i \in \{1,2\}$ and $j \in \{1,2\}$, we have

$$\left|\frac{\partial \mathsf{P}_i}{\partial \varrho_j}(\iota, \varrho_1, \varrho_2) \cdot \varrho_j'(\iota) - \frac{\partial \mathsf{P}_i}{\partial \varrho_j}(\iota, \bar{\varrho}_1, \bar{\varrho}_2) \cdot \bar{\varrho}_j'(\iota)\right| \leqslant H_{i,j}\left|\varrho_j'(\iota) - \bar{\varrho}_j'(\iota)\right|.$$

(c) There exists a matrix $M_3 = \left(\begin{smallmatrix} \lambda_{1,1} & \lambda_{1,2} \\ \lambda_{2,1} & \lambda_{2,2} \end{smallmatrix}\right) \in \mathcal{M}_{2\times 2}(\mathbb{R}_+)$ such that for each $(\iota, \varrho_1, \varrho_2), (\iota, \bar{\varrho}_1, \bar{\varrho}_2) \in (a,b) \times \mathbb{R} \times \mathbb{R}$ and for $i \in \{1,2\}$ and $j \in \{1,2\}$, we have

$$\left|\frac{\partial \mathsf{P}_i}{\partial \varrho_j}(\iota, \varrho_1, \varrho_2) \cdot \varrho_j'(\iota)\right| \leqslant \lambda_{i,j}\left|\varrho_j'(\iota)\right|.$$

(d) For each $i \in \{1,2\}$ the functions $\mathsf{P}_i(\cdot, 0, 0) \in W^{1,1}_\omega(a,b)$.

($\mathcal{H}_3$) The functions $\mathsf{L}_1, \mathsf{L}_2 : (a,b) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfy the Carathéodory conditions and have the continuous derivatives of order 1 with respect to the first variable, and

(a) There exists a matrix $M_4 = \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix} \in \mathcal{M}_{2\times 2}(L^1((a,b),\mathbb{R}_+))$ such that for each $(s, \varrho_1, \varrho_2), (\varsigma, \bar{\varrho}_1, \bar{\varrho}_2) \in (a,b) \times \mathbb{R} \times \mathbb{R}$ and for $i \in \{1,2\}$, we have

$$\left|\mathsf{L}_i(\varsigma, \varrho_1, \varrho_2) - \mathsf{L}_i(\varsigma, \bar{\varrho}_1, \bar{\varrho}_2)\right| \leqslant \beta_{i,1}(\varsigma)|\varrho_1 - \bar{\varrho}_1| + \beta_{i,2}(\varsigma)|\varrho_2 - \bar{\varrho}_2|.$$

(b) There exists a matrix $M_5 = \begin{pmatrix} L_{1,1} & L_{1,2} \\ L_{2,1} & L_{2,2} \end{pmatrix} \in \mathcal{M}_{2\times 2}(L^1((a,b),\mathbb{R}_+)$ such that for each $(\varsigma, \varrho_1, \varrho_2), (\varsigma, \bar{\varrho}_1, \bar{\varrho}_2) \in (a,b) \times \mathbb{R} \times \mathbb{R}$ and for $i \in \{1,2\}$ and $j \in \{1,2\}$, we have

$$\left|\frac{\partial \mathsf{L}_i}{\partial \varrho_j}(\varsigma, \varrho_1, \varrho_2) \cdot \varrho_j'(\iota) - \frac{\partial \mathsf{L}_i}{\partial \varrho_j}(\varsigma, \bar{\varrho}_1, \bar{\varrho}_2) \cdot \bar{\varrho}_j(\iota)\right| \leqslant L_{i,j}(\varsigma)|\varrho_j' - \bar{\varrho}_j'|.$$

(c) There exists a matrix $M_6 = \begin{pmatrix} \eta_{1,1} & \eta_{1,2} \\ \eta_{2,1} & \eta_{2,2} \end{pmatrix} \in \mathcal{M}_{2\times 2}(L^1((a,b),\mathbb{R}_+)$ such that for each $(\varsigma, \varrho_1, \varrho_2), (\varsigma, \bar{\varrho}_1, \bar{\varrho}_2) \in (a,b) \times \mathbb{R} \times \mathbb{R}$ and for $i \in \{1,2\}$ and $j \in \{1,2\}$, we have

$$\left|\frac{\partial \mathsf{L}_i}{\partial \varrho_j}(\varsigma, \varrho_1, \varrho_2) \cdot \varrho_j'(\iota)\right| \leqslant \eta_{i,j}(\varsigma)\left|\varrho_j'(\iota)\right|.$$

(d) For each $i \in \{1,2\}$, the functions $\xi_i(\iota) := \int_a^\iota |\mathsf{L}_i(\varsigma, 0, 0)|\,\mathrm{d}\varsigma$ exist, and $\xi_i \in L^1_\omega(a,b)$.

Theorem 3. *Suppose that assumptions* ($\mathcal{H}_1$)–($\mathcal{H}_3$) *are satisfied. Then the system of integral equation* (*SIE*) (2) *has at least one solution in* $W^{1,1}_\omega((a,b)) \times W^{1,1}_\omega((a,b))$ *if there is* $r \in \mathbb{R}^2$ *such that*

$$\begin{aligned}\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} \succcurlyeq{}& \begin{pmatrix} \|\alpha_{1,1}\|_{L^\infty} + \lambda_{1,1} + K(\|\beta_{1,1}\|_{L^1} + \|\eta_{1,1}\|_{L^1}) & \|\alpha_{1,2}\|_{L^\infty} + \lambda_{1,2} + K(\|\beta_{1,2}\|_{L^1} + \|\eta_{1,2}\|_{L^1}) \\ \|\alpha_{2,1}\|_{L^\infty} + \lambda_{2,1} + K(\|\beta_{2,1}\|_{L^1} + \|\eta_{2,1}\|_{L^1}) & \|\alpha_{2,2}\|_{L^\infty} + \lambda_{2,2} + K(\|\beta_{2,2}\|_{L^1} + \|\eta_{2,2}\|_{L^1}) \end{pmatrix} \\ & \times \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} + \begin{pmatrix} \|\mathsf{P}_1(\cdot,0,0)\|_{W^{1,1}_\omega} + K\|\xi_1\|_{L^1_\omega} \\ \|\mathsf{P}_2(\cdot,0,0)\|_{W^{1,1}_\omega} + K\|\xi_2\|_{L^1_\omega} \end{pmatrix} \end{aligned} \tag{3}$$

and the matrix

$$M_* := \begin{pmatrix} \|\alpha_{1,1}\|_{L^\infty} + 2K\|\beta_{1,1}\|_{L^1} & \|\alpha_{2,1}\|_{L^\infty} + 2K\|\beta_{2,1}\|_{L^1} \\ \|\alpha_{1,2}\|_{L^\infty} + 2K\|\beta_{1,2}\|_{L^1} & \|\alpha_{2,2}\|_{L^\infty} + 2K\|\beta_{2,2}\|_{L^1} \end{pmatrix}$$

converges to zero. Here

$$K := \max_{i\in\{1,2\}}\left\{\sup_{(\iota,\varsigma)\in\overline{(a,b)}^2}\left|\mathsf{K}_i(\iota,\varsigma)\right|, \sup_{(\iota,\varsigma)\in\overline{(a,b)}^2}\left|\frac{\partial \mathsf{K}_i}{\partial \iota}(\iota,\varsigma)\right|\right\}.$$

Proof. We define the operator $\varXi : W^{1,1}_\omega(a,b) \times W^{1,1}_\omega(a,b) \to W^{1,1}_\omega(a,b) \times W^{1,1}_\omega(a,b)$ by

$$\varXi(\varrho)(\iota) = \begin{pmatrix} \varXi_1(\varrho)(\iota) \\ \varXi_2(\varrho)(\iota) \end{pmatrix} = \begin{pmatrix} \mathsf{P}_1(\iota, \varrho_1(\iota), \varrho_2(\iota)) + \int_a^\iota \mathsf{K}_1(\iota,\varsigma)\mathsf{L}_1(\varsigma, \varrho_1(\iota), \varrho_2(\iota))\,\mathrm{d}\varsigma \\ \mathsf{P}_2(\iota, \varrho_1(\iota), \varrho_2(\iota)) + \int_a^\iota \mathsf{K}_2(\iota,\varsigma)\mathsf{L}_2(\varsigma, \varrho_1(\iota), \varrho_2(\iota))\,\mathrm{d}\varsigma \end{pmatrix}.$$

The proof will be broken up into several steps.

Step 1. Our first claim is to show that the operator $\varXi$ is well defined. Looking that for each $i \in \{1,2\}$, the function $\varXi_i(\varrho)$ is measurable for any $\varrho \in W^{1,1}_\omega(a,b) \times W^{1,1}_\omega(a,b)$. Also, for any $\iota \in (a,b)$ and $i \in \{1,2\}$, we have

$$\begin{aligned} \big(\varXi_i(\varrho)\big)'(\iota) &= \frac{\partial \mathsf{P}_i}{\partial \iota}\big(\iota, \varrho_1(\iota), \varrho_2(\iota)\big) + \frac{\partial \mathsf{P}_i}{\partial \varrho_1}\big(\iota, \varrho_1(\iota), \varrho_2(\iota)\big)\cdot\varrho_1'(\iota) \\ &\quad + \frac{\partial \mathsf{P}_i}{\partial \varrho_2}\big(\iota, \varrho_1(\iota), \varrho_2(\iota)\big)\cdot\varrho_2'(\iota) + \int_a^\iota \frac{\partial \mathsf{K}_i}{\partial \iota}(\iota,\varsigma)\mathsf{L}_i\big(\varsigma, \varrho_1(\iota), \varrho_2(\iota)\big) \\ &\quad + \mathsf{K}_i(\iota,\varsigma)\bigg(\frac{\partial \mathsf{L}_i}{\partial \varrho_1}\big(\varsigma, \varrho_1(\iota), \varrho_2(\iota)\big)\cdot\varrho_1'(\iota) + \frac{\partial \mathsf{L}_i}{\partial \varrho_2}\big(\varsigma, \varrho_1(\iota), \varrho_2(\iota)\big)\cdot\varrho_2'(\iota)\bigg)\,\mathrm{d}\varsigma. \end{aligned}$$

Then $\varXi_i(\varrho)$ has a measurable derivative. Now, we shall show that $\varXi(\varrho) \in W^{1,1}_\omega(a,b) \times W^{1,1}_\omega(a,b)$ for any $\varrho \in W^{1,1}_\omega(a,b) \times W^{1,1}_\omega(a,b)$. Using our hypotheses, for arbitrarily fixed $\iota \in (a,b)$ and $i \in \{1,2\}$, we obtain

$$\begin{aligned} \big|\varXi_i(\varrho)(\iota)\big| &\leqslant \big|\mathsf{P}_i\big(\iota, \varrho_1(\iota), \varrho_2(\iota)\big) - \mathsf{P}_i(\iota,0,0) + \mathsf{P}_i(\iota,0,0)\big| \\ &\quad + \int_a^\iota \big|\mathsf{K}_i(\iota,\varsigma)\big|\big(\big|\mathsf{L}_i\big(\varsigma, \varrho_1(\iota), \varrho_2(\iota)\big) - \mathsf{L}_i(\varsigma,0,0)\big| + \big|\mathsf{L}_i(\varsigma,0,0)\big|\big)\,\mathrm{d}\varsigma \\ &\leqslant \alpha_{i,1}(\iota)\big|\varrho_1(\iota)\big| + \alpha_{i,2}(\iota)\big|\varrho_2(\iota)\big| + \big|\mathsf{P}_i(\iota,0,0)\big| \\ &\quad + K\bigg(\int_a^\iota \big|\beta_{i,1}(\varsigma)\big|\big|\varrho_1(\iota)\big| + \big|\beta_{i,2}(\varsigma)\big|\big|\varrho_2(\iota)\big|\,\mathrm{d}\varsigma + \xi_i(\iota)\bigg) \\ &\leqslant \alpha_{i,1}(\iota)\big|\varrho_1(\iota)\big| + \alpha_{i,2}(\iota)\big|\varrho_2(\iota)\big| + \big|\mathsf{P}_i(\iota,0,0)\big| \\ &\quad + K\big(\|\beta_{i,1}\|_{L^1}\big|\varrho_1(\iota)\big| + \|\beta_{i,2}\|_{L^1}\big|\varrho_2(\iota)\big| + \xi_i(\iota)\big), \end{aligned}$$

hence,

$$\begin{aligned} \int_a^\iota \big|\varXi_i(\varrho)(\iota)\big|\omega(\iota)\,\mathrm{d}\iota &\leqslant \int_a^\iota \big(\alpha_{i,1}(\iota)\big|\varrho_1(\iota)\big| + \alpha_{i,2}(\iota)\big|\varrho_2(\iota)\big| + \big|\mathsf{P}_i(\iota,0,0)\big| \\ &\qquad + K\big(\|\beta_{i,1}\|_{L^1}\big|\varrho_1(\iota)\big| + \|\beta_{i,2}\|_{L^1}\big|\varrho_2(\iota)\big|\big)\big)\omega(\iota)\,\mathrm{d}\iota \\ &\leqslant \|\alpha_{i,1}\|_{L^\infty}\|\varrho_1\|_{L^1_\omega} + \|\alpha_{i,2}\|_{L^\infty}\|\varrho_2\|_{L^1_\omega} + \big\|\mathsf{P}_i(\cdot,0,0)\big\|_{L^1_\omega} \\ &\quad + K\big(\|\beta_{i,1}\|_{L^1}\|\varrho_1\|_{L^1_\omega} + \|\beta_{i,2}\|_{L^1}\|\varrho_2\|_{L^1_\omega} + \|\xi_i\|_{L^1_\omega}\big). \end{aligned}$$

Also,

$$\begin{aligned}
&\big|\big(\varXi_i(\varrho)\big)'(\iota)\big| \\
&\quad\leqslant \left|\frac{\partial \mathsf{P}_i}{\partial\iota}\big(\iota,\varrho_1(\iota),\varrho_2(\iota)\big) - \frac{\partial \mathsf{P}_i}{\partial\iota}(\iota,0,0) + \frac{\partial \mathsf{P}_i}{\partial\iota}(\iota,0,0)\right| \\
&\qquad + \left|\frac{\partial \mathsf{P}_i}{\partial\varrho_1}\big(\iota,\varrho_1(\iota),\varrho_2(\iota)\big)\cdot\varrho_1'(\iota)\right| + \left|\frac{\partial \mathsf{P}_i}{\partial\varrho_2}\big(\iota,\varrho_1(\iota),\varrho_2(\iota)\big)\cdot\varrho_2'(\iota)\right| \\
&\qquad + \int_a^\iota \left|\frac{\partial \mathsf{K}_i}{\partial\iota}(\iota,\varsigma)\right|\big(\big|\mathsf{L}_i\big(\varsigma,\varrho_1(\iota),\varrho_2(\iota)\big) - \mathsf{L}_i(\varsigma,0,0)\big| + \big|\mathsf{L}_i(\varsigma,0,0)\big|\big) \\
&\qquad + \big|\mathsf{K}_i(\iota,\varsigma)\big|\left(\left|\frac{\partial \mathsf{L}_i}{\partial\varrho_1}\big(\varsigma,\varrho_1(\iota),\varrho_2(\iota)\big)\cdot\varrho_1'(\iota)\right| + \left|\frac{\partial \mathsf{L}_i}{\partial\varrho_2}\big(\varsigma,\varrho_1(\iota),\varrho_2(\iota)\big)\cdot\varrho_2'(\iota)\right|\right)\mathrm{d}\varsigma \\
&\quad\leqslant \alpha_{i,1}(\iota)\big|\varrho_1(\iota)\big| + \alpha_{i,2}(\iota)\big|\varrho_2(\iota)\big| + \left|\frac{\partial \mathsf{P}_i}{\partial\iota}(\iota,0,0)\right| + \lambda_{i,1}\big|\varrho_1'(\iota)\big| + \lambda_{i,2}\big|\varrho_2'(\iota)\big| \\
&\qquad + K\int_a^\iota \beta_{i,1}(\varsigma)\big|\varrho_1(\iota)\big| + \beta_{i,2}(\varsigma)\big|\varrho_2(\iota)\big| + \big|\mathsf{L}_i(\varsigma,0,0)\big|\,\mathrm{d}\varsigma \\
&\qquad + \eta_{i,1}(\varsigma)\big|\varrho_1'(\iota)\big| + \eta_{i,2}(\varsigma)\big|\varrho_2'(\iota)\big| \\
&\quad\leqslant \alpha_{i,1}(\iota)\big|\varrho_1(\iota)\big| + \alpha_{i,2}(\iota)\big|\varrho_2(\iota)\big| + \left|\frac{\partial \mathsf{P}_i}{\partial\iota}(\iota,0,0)\right| + \lambda_{i,1}\big|\varrho_1'(\iota)\big| + \lambda_{i,2}\varrho_2'(\iota)\big| \\
&\qquad + K\big(\|\beta_{i,1}\|_{L^1}\big|\varrho_1(\iota)\big| + \|\beta_{i,2}\|_{L^1}\big|\varrho_2(\iota)\big| + \xi_i(\iota) \\
&\qquad + \|\eta_{i,1}\|_{L^1}\big|\varrho_1'(\iota)\big| + \|\eta_{i,2}\|_{L^1}\big|\varrho_2'(\iota)\big|\big),
\end{aligned}$$

then

$$\begin{aligned}
\big\|\big(\varXi_i(\varrho)\big)'\big\|_{L^1_\omega} &\leqslant \|\alpha_{i,1}\|_{L^\infty}\|\varrho_1\|_{L^1_\omega} + \|\alpha_{i,2}\|_{L^\infty}\|\varrho_2\|_{L^1_\omega} \\
&\quad + \left\|\frac{\partial \mathsf{P}_i}{\partial\iota}(\cdot,0,0)\right\|_{L^1_\omega} + \lambda_{i,1}\|\varrho_1'\|_{L^1_\omega} + \lambda_{i,2}\|\varrho_2'\|_{L^1_\omega} \\
&\quad + K\big(\|\beta_{i,1}\|_{L^1}\|\varrho_1\|_{L^1_\omega} + \|\beta_{i,2}\|_{L^1}\|\varrho_2\|_{L^1_\omega} \\
&\quad + \|\eta_{i,1}\|_{L^1}\big|\|\varrho_1'\|_{L^1_\omega} + \|\eta_{i,2}\|_{L^1}\big|\|\varrho_2'\|_{L^1_\omega} + \|\xi_i\|_{L^1_\omega}\big).
\end{aligned}$$

Thus,

$$\begin{aligned}
\big\|\varXi_i(\varrho)\big\|_{W^{1,1}_\omega} &\leqslant \|\alpha_{i,1}\|_{L^\infty}\|\varrho_1\|_{W^{1,1}_\omega} + \|\alpha_{i,2}\|_{L^\infty}\|\varrho_2\|_{W^{1,1}_\omega} \\
&\quad + \big\|\mathsf{P}_i(\cdot,0,0)\big\|_{W^{1,1}_\omega} + \lambda_{i,1}\|\varrho_1'\|_{L^1_\omega} + \lambda_{i,2}\|\varrho_2'\|_{L^1_\omega} \\
&\quad + K\big(\|\beta_{i,1}\|_{L^1}\|\varrho_1\|_{L^1_\omega} + \|\beta_{i,2}\|_{L^1}\|\varrho_2\|_{L^1_\omega} \\
&\quad + \|\eta_{i,1}\|_{L^1}\|\varrho_1'\|_{L^1_\omega} + \|\eta_{i,2}\|_{L^1}\|\varrho_2'\|_{L^1_\omega} + \|\xi_i\|_{L^1_\omega}\big) \\
&< \infty,
\end{aligned}$$

this means that the operator $\varXi$ maps $W^{1,1}_\omega(a,b)\times W^{1,1}_\omega(a,b)$ into $W^{1,1}_\omega(a,b)\times W^{1,1}_\omega(a,b)$.

Keeping in the mind that the vector r fulfills inequality (3), thus for all $\varrho \in \bar{B}_r$,

$$\begin{aligned}
&\|\Xi(\varrho)\|_{\mathrm{G}} \\
&\quad\preccurlyeq \begin{pmatrix} \|\alpha_{1,1}\|_{L^\infty}+\lambda_{1,1}+K(\|\beta_{1,1}\|_{L^1}+\|\eta_{1,1}\|_{L^1}) & \|\alpha_{1,2}\|_{L^\infty}+\lambda_{1,2}+K(\|\beta_{1,2}\|_{L^1}+\|\eta_{1,2}\|_{L^1}) \\ \|\alpha_{2,1}\|_{L^\infty}+\lambda_{2,1}+K(\|\beta_{2,1}\|_{L^1}+\|\eta_{2,1}\|_{L^1}) & \|\alpha_{2,2}\|_{L^\infty}+\lambda_{2,2}+K(\|\beta_{2,2}\|_{L^1}+\|\eta_{2,2}\|_{L^1}) \end{pmatrix} \\
&\qquad \times \begin{pmatrix} \|\varrho_1\|_{W_\omega^{1,1}} \\ \|\varrho_2\|_{W_\omega^{1,1}} \end{pmatrix} + \begin{pmatrix} \|\mathsf{P}_1(\cdot,0,0)\|_{W_\omega^{1,1}} + K\|\xi_1\|_{L_\omega^1} \\ \|\mathsf{P}_2(\cdot,0,0)\|_{W_\omega^{1,1}} + K\|\xi_2\|_{L_\omega^1} \end{pmatrix} \\
&\quad\preccurlyeq \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}.
\end{aligned} \tag{4}$$

Due to (4), we derive that Ξ is a mapping from $\bar{B}_r$ into $\bar{B}_r$.

Step 2. Our claim here is to proof the continuity of Ξ. To this end, let $(\varrho_n)_{n\in\mathbb{N}} := (\varrho_{1_n}, \varrho_{2_n})_{n\in\mathbb{N}}$ be a convergence sequence to some $x := (\varrho_1, \varrho_2)$ in $W_\omega^{1,1}(a,b) \times W_\omega^{1,1}(a,b)$. Then for each $i \in \{1,2\}$,

$$\begin{aligned}
&\big\|\Xi_i(\varrho_n) - \Xi_i(\varrho)\big\|_{L_\omega^1} \\
&\quad\leqslant \int_a^b \big|\mathsf{P}_i\big(\iota, \varrho_{1_n}(\iota), \varrho_{2_n}(\iota)\big) - \mathsf{P}_i\big(\iota, \varrho_1(\iota), \varrho_2(\iota)\big)\big|\omega(\iota)\,\mathrm{d}\iota \\
&\qquad + \int_a^b \int_a^\iota \big|\mathsf{K}_i(\iota,\varsigma)\big|\big|\mathsf{L}_i\big(\varsigma, \varrho_{1_n}(\iota), \varrho_{2_n}(\iota)\big) - \mathsf{L}_i\big(\varsigma, \varrho_1(\iota), \varrho_2(\iota)\big)\big|\,\mathrm{d}\varsigma\omega(\iota)\,\mathrm{d}\iota \\
&\quad\leqslant \int_a^b \big(a_{i,1}(\iota)\big|\varrho_{1_n}(\iota) - \varrho_1(\iota)\big| + \alpha_{i,2}(\iota)\big|\varrho_{2_n}(\iota) - \varrho_2(\iota)\big|\big)\omega(\iota)\,\mathrm{d}\iota \\
&\qquad + K\int_a^b \int_a^\iota \big|\beta_{i,1}(\varsigma)\big|\big|\varrho_{1_n}(\iota) - \varrho_1(\iota)\big| + \big|\beta_{i,2}(\varsigma)\big|\big|\varrho_{2_n}(\iota) - \varrho_2(\iota)\big|\,\mathrm{d}\varsigma\omega(\iota)\,\mathrm{d}\iota \\
&\quad\leqslant \|\alpha_{i,1}\|_{L^\infty}\|\varrho_{1_n} - \varrho_1\|_{L_\omega^1} + \|\alpha_{i,2}\|_{L^\infty}\|\varrho_{2_n} - \varrho_2\|_{L_\omega^1} \\
&\qquad + K\Bigg(\int_a^b \big|\beta_{i,1}(\varsigma)\big|\,\mathrm{d}\varsigma \int_a^b \big|\varrho_{1_n}(\iota) - \varrho_1(\iota)\big|\omega(\iota)\,\mathrm{d}\iota \\
&\qquad + \int_a^b \big|\beta_{i,2}(\varsigma)\big|\,\mathrm{d}\varsigma \int_a^b \big|\varrho_{2_n}(\iota) - \varrho_2(\iota)\big|\omega(\iota)\,\mathrm{d}\iota\Bigg) \\
&\quad\leqslant \big(\|\alpha_{i,1}\|_{L^\infty} + K\|\beta_{i,1}\|_{L^1}\big)\|\varrho_{1_n} - \varrho_1\|_{W_\omega^{1,1}} \\
&\qquad + \big(\|\alpha_{i,2}\|_{L^\infty} + K\|\alpha_{i,2}\|_{L^1}\big)\|\varrho_{2_n} - \varrho_2\|_{W_\omega^{1,1}} \\
&\quad\to 0 \quad \text{as } n \to +\infty.
\end{aligned}$$

On the other hand, we have for each $i \in \{1,2\}$,

$$
\begin{aligned}
&\big\|\big(\Xi_i(\varrho_n)\big)' - \big(\Xi_i(\varrho)\big)'\big\|_{L^1_\omega} \\
&\quad\leqslant \int_a^b \left|\frac{\partial \mathsf{P}_i}{\partial \iota}\big(\iota, \varrho_{1_n}(\iota), \varrho_{2_n}(\iota)\big) - \frac{\partial \mathsf{P}_i}{\partial \iota}\big(\iota, \varrho_1(\iota), \varrho_2(\iota)\big)\right|\omega(\iota)\,\mathrm{d}\iota \\
&\qquad+ \int_a^b \left|\frac{\partial \mathsf{P}_i}{\partial \varrho_1}\big(\iota, \varrho_{1_n}(\iota), \varrho_{2_n}(\iota)\big)\cdot\varrho_{1_n}'(\iota) - \frac{\partial \mathsf{P}_i}{\partial \varrho_1}\big(\iota, \varrho_1(\iota), \varrho_2(\iota)\big)\cdot\varrho_1'(\iota)\right|\omega(\iota)\,\mathrm{d}\iota \\
&\qquad+ \int_a^b \left|\frac{\partial \mathsf{P}_i}{\partial \varrho_2}\big(\iota, \varrho_{1_n}(\iota), \varrho_{2_n}(\iota)\big)\cdot\varrho_{2_n}'(\iota) - \frac{\partial \mathsf{P}_i}{\partial \varrho_2}\big(\iota, \varrho_1(\iota), \varrho_2(\iota)\big)\cdot\varrho_2'(\iota)\right|\omega(\iota)\,\mathrm{d}\iota \\
&\qquad+ \int_a^b \int_a^\iota \left|\frac{\partial \mathsf{K}_i}{\partial \iota}(\iota,\varsigma)\right|\big|\mathsf{L}_i\big(\varsigma, \varrho_{1_n}(\iota), \varrho_{2_n}(\iota)\big) - \mathsf{L}_i\big(\varsigma, \varrho_1(\iota), \varrho_2(\iota)\big)\big|\,\mathrm{d}\varsigma\,\omega(\iota)\,\mathrm{d}\iota \\
&\qquad+ \int_a^b \int_a^\iota \big|\mathsf{K}_i(\iota,\varsigma)\big|\bigg|\frac{\partial \mathsf{L}_i}{\partial \varrho_1}\big(\varsigma, \varrho_{1_n}(\iota), \varrho_{2_n}(\iota)\big)\cdot\varrho_{1_n}'(\iota) \\
&\qquad\qquad - \frac{\partial \mathsf{L}_i}{\partial \varrho_1}\big(\varsigma, \varrho_1(\iota), \varrho_2(\iota)\big)\cdot\varrho_1'(\iota)\bigg|\,\mathrm{d}\varsigma\,\omega(\iota)\,\mathrm{d}\iota \\
&\qquad+ \int_a^b \int_a^\iota \big|\mathsf{K}_i(\iota,\varsigma)\big|\bigg|\frac{\partial \mathsf{L}_i}{\partial \varrho_2}\big(\varsigma, \varrho_{1_n}(\iota), \varrho_{2_n}(\iota)\big)\cdot\varrho_{2_n}'(\iota) \\
&\qquad\qquad - \frac{\partial \mathsf{L}_i}{\partial \varrho_2}\big(\varsigma, \varrho_1(\iota), \varrho_2(\iota)\big)\cdot\varrho_2'(\iota)\bigg|\,\mathrm{d}\varsigma\,\omega(\iota)\,\mathrm{d}\iota \\
&\quad\leqslant \int_a^b \big(\alpha_{i,1}(\iota)\big|\varrho_{1_n}(\iota) - \varrho_1(\iota)\big| + \alpha_{i,2}(\iota)\big|\varrho_{2_n}(\iota) - \varrho_2(\iota)\big|\big)\omega(\iota)\,\mathrm{d}\iota \\
&\qquad+ \int_a^b \big(H_{i,1}\big|\varrho_{1_n}'(\iota) - \varrho_1'(\iota)\big| + H_{i,2}\big|\varrho_{2_n}'(\iota) - \varrho_2'(\iota)\big|\big)\omega(\iota)\,\mathrm{d}\iota \\
&\qquad+ K\int_a^b \left(\int_a^\iota \big|\beta_{i,1}(\varsigma)\big|\big|\varrho_{1_n}(\iota) - \varrho_1(\iota)\big| + \big|\beta_{i,2}(\varsigma)\big|\big|\varrho_{2_n}(\iota) - \varrho_2(\iota)\big|\,\mathrm{d}\varsigma\right)\omega(\iota)\,\mathrm{d}\iota \\
&\qquad+ K\int_a^b \left(\int_a^\iota L_{i,1}(\varsigma)\big|\varrho_{1_n}'(\iota) - \varrho_1'(\iota)\big| + L_{i,2}(\varsigma)\big|\varrho_{2_n}'(\iota) - \varrho_2'(\iota)\big|\,\mathrm{d}\varsigma\right)\omega(\iota)\,\mathrm{d}\iota \\
&\quad\leqslant \big(\|\alpha_{i,1}\|_{L^\infty} + H_{i,1} + K(\|\beta_{i,1}\|_{L^1} + \|L_{i,1}\|_{L^1})\big)\|\varrho_{1_n} - \varrho_1\|_{W^{1,1}_\omega} \\
&\qquad+ \big(\|\alpha_{i,2}\|_{L^\infty} + H_{i,2} + K\big(\|\beta_{i,1}\|_{L^1} + \|L_{i,2}\|_{L^1}\big)\big)\|\varrho_{2_n} - \varrho_2\|_{W^{1,1}_\omega} \\
&\quad\to 0 \quad \text{as } n \to +\infty.
\end{aligned}
$$

Step 3. The operator Ξ is G-set contractive with respect to μ_{G}. Indeed, let $\mathcal{A}_i$ be a nonempty and bounded subset of B_{r_i}, and let $\iota, h \in B_T$ be such that $|h| < \varepsilon$ and $\varrho_i \in \mathcal{A}_i$, and by applying the same procedure of the previous step we get

$$\begin{aligned}
&\int_a^b \big|\big(\Xi_i(\varrho)\big)'(\iota+h) - \big(\Xi_i(\varrho)\big)'(\iota)\big|\omega(\iota)\,\mathrm{d}\iota \\
&\quad\leqslant \int_a^b \big|\big(\Xi_i(\varrho)\big)'(\iota+h) - \big(\Xi_i\big(\varrho(\iota+h)\big)\big)'(\iota+h)\big|\omega(\iota)\,\mathrm{d}\iota \\
&\qquad + \int_a^b \big|\big(\Xi_i\big(\varrho(\iota+h)\big)\big)'(\iota+h) - \big(\Xi_i(\varrho)\big)'(\iota)\big|\omega(\iota)\,\mathrm{d}\iota \\
&\quad\leqslant \big(\|\alpha_{i,1}\|_{L^\infty} + H_{i,1} + K\big(\|\beta_{i,1}\|_{L^1} + \|L_{i,1}\|_{L^1}\big)\big)\|\tau_h\varrho_1 - \varrho_1\|_{W_\omega^{1,1}} \\
&\qquad + \big(\|\alpha_{i,2}\|_{L^\infty} + H_{i,2} + K\big(\|\beta_{i,2}\|_{L^1} + \|L_{i,2}\|_{L^1}\big)\big)\|\tau_h\varrho_2 - \varrho_2\|_{W_\omega^{1,1}} \\
&\qquad + \int_a^b \left|\frac{\partial \mathsf{P}_i}{\partial\iota}\big(\iota+h,\ \varrho_1(t+h),\ \varrho_2(t+h)\big) - \frac{\partial \mathsf{P}_i}{\partial\iota}\big(\iota,\varrho_1(\iota),\varrho_2(\iota)\big)\right|\omega(\iota)\,\mathrm{d}\iota \\
&\qquad + \int_a^b \left|\frac{\partial \mathsf{P}_i}{\partial\varrho_1}\big(\iota+h,\ \varrho_1(t+h),\ \varrho_2(t+h)\big)\cdot\varrho_1'(\iota+h)\right. \\
&\qquad\qquad \left. - \frac{\partial \mathsf{P}_i}{\partial\varrho_1}\big(\iota,\varrho_1(\iota),\varrho_2(\iota)\big)\cdot\varrho_1'(\iota)\right|\omega(\iota)\,\mathrm{d}\iota \\
&\qquad + \int_a^b \left|\frac{\partial \mathsf{P}_i}{\partial\varrho_2}\big(\iota+h,\ \varrho_1(t+h),\varrho_2(t+h)\big)\cdot\varrho_2'(\iota+h)\right. \\
&\qquad\qquad \left. - \frac{\partial \mathsf{P}_i}{\partial\varrho_2}\big(\iota,\varrho_1(\iota),\varrho_2(\iota)\big)\cdot\varrho_2'(\iota)\right|\omega(\iota)\,\mathrm{d}\iota \\
&\qquad + \int_a^b\int_a^\iota \left|\frac{\partial \mathsf{K}_i}{\partial\iota}(\iota+h,\,\varsigma)\right|\big|\mathsf{L}_i(\varsigma,\varrho_1(t+h),\tau_h\varrho_2) - \mathsf{L}_i\big(\varsigma,\varrho_1(\iota),\varrho_2(\iota)\big)\big|\,\mathrm{d}\varsigma\,\omega(\iota)\,\mathrm{d}\iota \\
&\qquad + \int_a^b\int_a^\iota \left|\frac{\partial \mathsf{K}_i}{\partial\iota}(\iota+h,\,\varsigma) - \frac{\partial \mathsf{K}_i}{\partial\iota}(\iota,\varsigma)\right|\big|\mathsf{L}_i\big(\varsigma,\varrho_1(\iota),\varrho_2(\iota)\big)\big|\,\mathrm{d}\varsigma\,\omega(\iota)\,\mathrm{d}\iota \\
&\qquad + \int_a^b\int_\iota^{t+h} \left|\frac{\partial \mathsf{K}_i}{\partial\iota}(\iota+h,\,\varsigma)\mathsf{L}_i\big(\varsigma,\varrho_1(t+h),\varrho_2(t+h)\big)\right|\mathrm{d}\varsigma\,\omega(\iota)\,\mathrm{d}\iota \\
&\qquad + \int_a^b\int_a^\iota \big|\mathsf{K}_i(\iota+h,\,\varsigma)\big|\left|\frac{\partial \mathsf{L}_i}{\partial\varrho_1}\big(\varsigma,\varrho_1(t+h),\varrho_2(t+h)\big)\cdot\varrho_1'(\iota+h)\right. \\
&\qquad\qquad \left. - \frac{\partial \mathsf{L}_i}{\partial\varrho_1}\big(\varsigma,\varrho_1(t),\varrho_2(\iota)\big)\cdot\varrho_1'(\iota)\right|\mathrm{d}\varsigma\,\omega(\iota)\,\mathrm{d}\iota
\end{aligned}$$

$$
\begin{aligned}
&+\int_a^b\int_a^\iota \big|\mathsf{K}_i(\iota,\varsigma)-\mathsf{K}_i(\iota+h,\,\varsigma)\big|\left|\frac{\partial\mathsf{L}_i}{\partial\varrho_1}\big(\varsigma,\varrho_1(\iota),\varrho_2(\iota)\big)\cdot\varrho_1'(\iota)\right|\mathrm{d}\varsigma\omega(\iota)\,\mathrm{d}\iota\\
&+\int_a^b\int_t^{t+h}\left|\mathsf{K}_i(\iota+h,\,\varsigma)\frac{\partial\mathsf{L}_i}{\partial\varrho_1}\big(\varsigma,\varrho_1(t+h),\varrho_2(t+h)\big)\cdot\varrho_1'(\iota+h)\right|\mathrm{d}\varsigma\omega(\iota)\,\mathrm{d}\iota\\
&+\int_a^b\int_a^\iota \big|\mathsf{K}_i(\iota+h,\,\varsigma)\big|\left|\frac{\partial\mathsf{L}_i}{\partial\varrho_2}\big(\varsigma,\varrho_1(t+h),\varrho_2(t+h)\big)\cdot\varrho_2'(\iota+h)\right.\\
&\qquad\left.-\frac{\partial\mathsf{L}_i}{\partial\varrho_2}\big(\varsigma,\varrho_1(t),\varrho_2(\iota)\big)\cdot\varrho_2'(\iota)\right|\mathrm{d}\varsigma\omega(\iota)\,\mathrm{d}\iota\\
&+\int_a^b\int_a^\iota \big|\mathsf{K}_i(\iota,\varsigma)-\mathsf{K}_i(\iota+h,\varsigma)\big|\left|\frac{\partial\mathsf{L}_i}{\partial\varrho_2}\big(\varsigma,\varrho_1(\iota),\varrho_2(\iota)\big)\cdot\varrho_2'(\iota)\right|\mathrm{d}\varsigma\omega(\iota)\,\mathrm{d}\iota\\
&+\int_a^b\int_t^{t+h}\left|\mathsf{K}_i(\iota+h,\,\varsigma)\frac{\partial\mathsf{L}_i}{\partial\varrho_2}\big(\varsigma,\varrho_1(t+h),\varrho_2(t+h)\big)\cdot\varrho_2'(\iota+h)\right|\mathrm{d}\varsigma\omega(\iota)\,\mathrm{d}\iota,
\end{aligned}
$$

then

$$
\begin{aligned}
&\int_a^b \big|\big(\Xi_i(\varrho)\big)'(\iota+h)-\big(\Xi_i(\varrho)\big)'(\iota)\big|\omega(\iota)\,\mathrm{d}\iota\\
&\quad\leqslant \big(\|\alpha_{i,1}\|_{L^\infty}+2H_{i,1}+2K\big(\|\beta_{i,1}\|_{L^1}+\|L_{i,1}\|_{L^1}\big)\big)\|\tau_h\varrho_1-\varrho_1\|_{W^{1,1}_\omega}\\
&\qquad+\left\|\tau_h\left(\frac{\partial\mathsf{P}_i}{\partial\iota}\right)-\frac{\partial\mathsf{P}_i}{\partial\iota}\right\|_{L^1_\omega}\\
&\qquad+\big(\|\alpha_{i,2}\|_{L^\infty}+2H_{i,2}+2K\big(\|\beta_{i,2}\|_{L^1}+\|L_{i,2}\|_{L^1}\big)\big)\|\tau_h\varrho_2-\varrho_2\|_{W^{1,1}_\omega}\\
&\qquad+\sup_{(\iota,\varsigma)\in\overline{(a,b)}^2}\left|\frac{\partial\mathsf{K}_i}{\partial\iota}(\iota+h,\,\varsigma)-\frac{\partial\mathsf{K}_i}{\partial\iota}(\iota,\varsigma)\right|\int_a^b\int_a^\iota\big|\mathsf{L}_i\big(\varsigma,\varrho_1(\iota),\varrho_2(\iota)\big)\big|\,\mathrm{d}\varsigma\omega(\iota)\,\mathrm{d}\iota\\
&\qquad+\sup_{(\iota,\varsigma)\in\overline{(a,b)}^2}\big|\mathsf{K}_i(\iota,\varsigma)-\mathsf{K}_i(\iota+h,\,\varsigma)\big|\\
&\qquad\times\left(\int_a^b\int_a^\iota\left|\frac{\partial\mathsf{L}_i}{\partial\varrho_1}\big(\varsigma,\varrho_1(\iota),\varrho_2(\iota)\big)\cdot\varrho_1'(\iota)\right|+\left|\frac{\partial\mathsf{L}_i}{\partial\varrho_2}\big(\varsigma,\varrho_1(\iota),\varrho_2(\iota)\big)\cdot\varrho_2'(\iota)\right|\mathrm{d}\varsigma\omega(\iota)\,\mathrm{d}\iota\right)\\
&\qquad+K\int_a^b\int_t^{t+h}\big|\mathsf{L}_i\big(\varsigma,\varrho_1(t+h),\varrho_2(t+h)\big)\big|+\left|\frac{\partial\mathsf{L}_i}{\partial\varrho_1}\big(\varsigma,\varrho_1(t+h),\tau_h\varrho_2\big)\cdot\varrho_1'(\iota+h)\right|\\
&\qquad+\left|\frac{\partial\mathsf{L}_i}{\partial\varrho_2}\big(\varsigma,\varrho_1(t+h),\varrho_2(t+h)\big)\cdot\varrho_2'(\iota+h)\right|\mathrm{d}\varsigma\omega(\iota)\,\mathrm{d}\iota,
\end{aligned}
$$

it follows that

$$\begin{aligned}
&\big\|\tau_h(\Xi_i\varrho)' - (\Xi_i\varrho)'\big\|_{L^1_\omega} \\
&\quad\leqslant \big(\|\alpha_{i,1}\|_{L^\infty} + 2H_{i,1} + 2K\big(\|\beta_{i,1}\|_{L^1} + \|L_{i,1}\|_{L^1}\big)\big)\|\tau_h\varrho_1 - \varrho_1\|_{W^{1,1}_\omega} \\
&\qquad + \big\|\tau_h(\mathsf{P}_i) - \mathsf{P}_i\big\|_{W^{1,1}_\omega} \\
&\qquad + \big(\|\alpha_{i,2}\|_{L^\infty} + 2H_{i,2} + 2K\big(\|\beta_{i,2}\|_{L^1} + \|L_{i,2}\|_{L^1}\big)\big)\|\tau_h\varrho_2 - \varrho_2\|_{W^{1,1}_\omega} \\
&\qquad + \sup_{(\iota,\varsigma)\in\overline{(a,b)}^2}\left|\frac{\partial\mathsf{K}_i}{\partial\iota}(\iota+h,\varsigma) - \frac{\partial\mathsf{K}_i}{\partial\iota}(\iota,\varsigma)\right| \\
&\qquad \times \big(\|\beta_{i,1}\|_{L^1}\|\varrho_1\|_{L^1_\omega} + \|\beta_{i,2}\|_{L^1}\|\varrho_2\|_{L^1_\omega} + \|\xi_i\|_{L^1_\omega}\big) \\
&\qquad + \sup_{(\iota,\varsigma)\in\overline{(a,b)}^2}\big|\mathsf{K}_i(\iota,\varsigma) - \mathsf{K}_i(\iota+h,\varsigma)\big|\big(\|\eta_{i,1}\|_{L^1}\big|\|\varrho_1'\|_{L^1_\omega} + \|\eta_{i,2}\|_{L^1}\big|\|\varrho_2'\|_{L^1_\omega}\big) \\
&\qquad + K\big(\|\beta_{i,1}\|_{L^1(b-a,\,b-a+h)}\|\varrho_1\|_{L^1_\omega} + \|\beta_{i,2}\|_{L^1(b-a,\,b-a+h)}\|\varrho_2\|_{L^1_\omega} + \|\tau_h\xi_i - \xi_i\|_{L^1_\omega} \\
&\qquad + \|\eta_{i,1}\|_{L^1(b-a,\,b-a+h)}\big|\|\tau_h\varrho_1'\|_{L^1_\omega} + \|\eta_{i,2}\|_{L^1(b-a,b-a+h)}\big|\|\tau_h\varrho_2'\|_{L^1_\omega}\big).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\mu_i^0\big(\big(\Xi_i(\varrho)\big),\varepsilon\big) \\
&\quad\leqslant \big(\|\alpha_{i,1}\|_{L^\infty} + 2H_{i,1} + 2K\big(\|\beta_{i,1}\|_{L^1} + \|L_{i,1}\|_{L^1}\big)\big)\mu_1^0(\varrho_1,\varepsilon) + \mu_i^0(\mathsf{P}_i,\varepsilon) \\
&\qquad + \big(\|\alpha_{i,2}\|_{L^\infty} + 2H_{i,2} + 2K\big(\|\beta_{i,1}\|_{L^1} + \|L_{i,2}\|_{L^1}\big)\big)\mu_2^0(\varrho_2,\varepsilon) \\
&\qquad + \sup_{(\iota,\varsigma)\in\overline{(a,b)}^2}\left|\frac{\partial\mathsf{K}_i}{\partial\iota}(\iota+h,\varsigma) - \frac{\partial\mathsf{K}_i}{\partial\iota}(\iota,\varsigma)\right| \\
&\qquad \times \big(\|\beta_{i,1}\|_{L^1}\|\varrho_1\|_{L^1_\omega} + \|\beta_{i,2}\|_{L^1}\|\varrho_2\|_{L^1_\omega} + \|\xi_i\|_{L^1_\omega}\big) \\
&\qquad + \sup_{(\iota,\varsigma)\in\overline{(a,b)}^2}\big|\mathsf{K}_i(\iota,\varsigma) - \mathsf{K}_i(\iota+h,\varsigma)\big|\big(\|\eta_{i,1}\|_{L^1}|\|\varrho_1'\|_{L^1_\omega} + \|\eta_{i,2}\|_{L^1}|\|\varrho_2'\|_{L^1_\omega}\big) \\
&\qquad + K\|\beta_{i,1}\|_{L^1(b-a,\,b-a+h)}\|\varrho_1\|_{L^1_\omega} + \|\beta_{i,2}\|_{L^1(b-a,b-a+h)}\|\varrho_2\|_{L^1_\omega} \\
&\qquad + \|\tau_h\xi_i - \xi_i\|_{L^1_\omega} \\
&\qquad + \|\eta_{i,1}\|_{L^1(b-a,\,b-a+h)}\big|\|\tau_h\varrho_1'\|_{L^1_\omega} + \|\eta_{i,2}\|_{L^1(b-a,\,b-a+h)}\big|\|\tau_h\varrho_2'\|_{L^1_\omega}.
\end{aligned}$$

Since for each $i \in \{1,2\}$, $\{\mathsf{P}_i\}$, $\{\mathsf{K}_i\}$ are compact in $W^{1,1}_\omega(a,b)$ and $\{\xi_i\}$ is compact in $L^1_\omega(a,b)$, we have $\mu_i^0(\mathsf{P}_i,\varepsilon) \to 0$, $\|\tau_h(\xi_i) - \xi_i\|_{L^1_\omega} \to 0$ as $\varepsilon \to 0$. Then we obtain

$$\begin{aligned}
&\mu^0_{\mathrm{G}}(\Xi\mathcal{A}) \\
&\quad\preccurlyeq \begin{pmatrix} \|\alpha_{1,1}\|_{L^\infty}+2H_{1,1}+2K(\|\beta_{1,1}\|_{L^1}+\|L_{1,1}\|_{L^1}) & \|\alpha_{2,1}\|_{L^\infty}+2H_{2,1}+2K(\|\beta_{2,1}\|_{L^1}+\|L_{2,1}\|_{L^1}) \\ \|\alpha_{1,2}\|_{L^\infty}+2H_{1,2}+2K(\|\beta_{1,2}\|_{L^1}+\|L_{1,2}\|_{L^1}) & \|\alpha_{2,2}\|_{L^\infty}+2H_{2,2}+2K(\|\beta_{2,2}\|_{L^1}+\|L_{2,2}\|_{L^1}) \end{pmatrix} \\
&\qquad \times \mu^0_{\mathrm{G}}(\mathcal{A}).
\end{aligned}$$

Next, let us fix an arbitrary number $T > 0$. Then, taking into account our hypotheses, for an arbitrary function $\varrho \in \mathcal{A}$, we have

$$\begin{aligned}
&\int\limits_{(a,b)\setminus B_T} \left|\left(\Xi_i(\varrho)\right)'(\iota)\right|\omega(\iota)\,\mathrm{d}\iota \\
&\quad\leqslant \|\alpha_{i,1}\|_{L^\infty((a,b)\setminus B_T)}\|\varrho_1\|_{L^1_\omega((a,b)\setminus B_T)} + \|\alpha_{i,2}\|_{L^\infty((a,b)\setminus B_T)}\|\varrho_2\|_{L^1_\omega((a,b)\setminus B_T)} \\
&\qquad + \left\|\frac{\partial \mathsf{P}_i}{\partial\iota}(\cdot,0,0)\right\|_{L^1_\omega((a,b)\setminus B_T)} + \lambda_{i,1}\|\varrho_1'\|_{L^1_\omega((a,b)\setminus B_T)} + \lambda_{i,2}\|\varrho_2'\|_{L^1_\omega((a,b)\setminus B_T)} \\
&\qquad + K\big(\|\beta_{i,1}\|_{L^1}\|\varrho_1\|_{L^1_\omega((a,b)\setminus B_T)} + \|\beta_{i,2}\|_{L^1}\|\varrho_2\|_{L^1_\omega((a,b)\setminus B_T)} \\
&\qquad + \|\eta_{i,1}\|_{L^1}\|\varrho_1'\|_{L^1_\omega((a,b)\setminus B_T)} + \|\eta_{i,2}\|_{L^1}\|\varrho_2'\|_{L^1_\omega((a,b)\setminus B_T)} + \|\xi_i\|_{L^1_\omega((a,b)\setminus B_T)}\big) \\
&\quad\leqslant \big(\|\alpha_{i,1}\|_{L^\infty((a,b)\setminus B_T)} + \lambda_{i,1} + K\|\beta_{i,1}\|_{L^1} + K\|\eta_{i,1}\|_{L^1}\big)\|\varrho_1\|_{W^{1,1}_\omega((a,b)\setminus B_T)} \\
&\qquad + \big(\|\alpha_{i,2}\|_{L^\infty((a,b)\setminus B_T)} + \lambda_{i,2} + K\|\beta_{i,2}\|_{L^1} + K\|\eta_{i,2}\|_{L^1}\big)\|\varrho_2\|_{W^{1,1}_\omega((a,b)\setminus B_T)} \\
&\qquad + \big\|\mathsf{P}_i(\cdot,0,0)\big\|_{W^{1,1}_\omega((a,b)\setminus B_T)} + K\|\xi_i\|_{L^1_\omega((a,b)\setminus B_T)}.
\end{aligned}$$

But for each $i \in \{1,2\}$, $\|\mathsf{P}_i(\cdot,0,0)\|_{W^{1,1}_\omega((a,b)\setminus B_T)} \to 0$, $\|\alpha_{i,1}\|_{L^\infty((a,b)\setminus B_T)} \to 0$, $\|\alpha_{i,2}\|_{L^\infty((a,b)\setminus B_T)} \to 0$, $\|\xi_i\|_{L^1_\omega((a,b)\setminus B_T)} \to 0$ when $T \to +\infty$, hence,

$$d_{\mathrm{G}}(\Xi\mathcal{A}) \preccurlyeq \begin{pmatrix} \lambda_{1,1}+K(\|\beta_{1,1}\|_{L^1}+\|\eta_{1,1}\|_{L^1}) & \lambda_{1,2}+K(\|\beta_{1,2}\|_{L^1}+\|\eta_{1,2}\|_{L^1}) \\ \lambda_{2,1}+K(\|\beta_{2,1}\|_{L^1}+\|\eta_{2,1}\|_{L^1}) & \lambda_{2,2}+K(\|\beta_{2,2}\|_{L^1}+\|\eta_{2,2}\|_{L^1}) \end{pmatrix} d_{\mathrm{G}}(\mathcal{A}),$$

then

$$\begin{aligned}
\mu_{\mathrm{G}}\big(\Xi(\mathcal{A})\big) \preccurlyeq \big(&\|\alpha_{i,j}\|_{L^\infty} + 2H_{i,j} + 2K\big(\|\beta_{i,j}\|_{L^1} + \|L_{i,j}\|_{L^1}\big) \\
&+ \lambda_{i,j} + K\|\eta_{i,j}\|_{L^1}\big)_{1\leqslant i,j\leqslant 2}\mu_{\mathrm{G}}(\mathcal{A}). \qquad (5)
\end{aligned}$$

By the same way we find for $i \in \{1,2\}$,

$$\begin{aligned}
&\int_a^b \left|\big(\Xi_i(\varrho)\big)(\iota+h) - \big(\Xi_i(\varrho)\big)(\iota)\right|\omega(\iota)\,\mathrm{d}\iota \\
&\quad\leqslant \int_a^b \left|\big(\Xi_i(\varrho)\big)(\iota+h) - \big(\Xi_i\big(\varrho(\iota+h)\big)\big)(\iota+h)\right|\omega(\iota)\,\mathrm{d}\iota \\
&\qquad + \int_a^b \left|\big(\Xi_i\big(\varrho(\iota+h)\big)\big)(\iota+h) - \big(\Xi_i(\varrho)\big)(\iota)\right|\omega(\iota)\,\mathrm{d}\iota \\
&\quad\leqslant \big(\|\alpha_{i,1}\|_{L^\infty}+K\|\beta_{i,1}\|_{L^1}\big)\|\tau_h\varrho_1-\varrho_1\|_{W^{1,1}_\omega}+\big(\|\alpha_{i,2}\|_{L^\infty}+K\|\alpha_{i,2}\|_{L^1}\big)\|\tau_h\varrho_2-\varrho_2\|_{W^{1,1}_\omega} \\
&\qquad + \int_a^b \left|\mathsf{P}_i\big(\iota+h,\ \varrho_1(\iota+h),\tau_h\varrho_2(\iota+h)\big) - \mathsf{P}_i\big(\iota,\varrho_1(\iota),\varrho_2(\iota)\big)\right|\omega(\iota)\,\mathrm{d}\iota
\end{aligned}$$

$$
\begin{aligned}
&+\int_a^b\int_a^\iota \big|\mathsf{K}_i(\iota+h,\varsigma)\big|\big|\mathsf{L}_i(\varsigma,\varrho_1(t+h),\varrho_2(t+h)\big)-\mathsf{L}_i\big(\varsigma,\varrho_1(\iota),\varrho_2(t)\big)\big|\,\mathrm{d}\varsigma\omega(\iota)\,\mathrm{d}\iota\\
&+\int_a^b\int_a^\iota \big|\mathsf{K}_i(\iota+h,\varsigma)-\mathsf{K}_i(\iota,\varsigma)\big|\big|\mathsf{L}_i\big(\varsigma,\varrho_1(\iota),\varrho_2(\iota)\big)\big|\,\mathrm{d}\varsigma\omega(\iota)\,\mathrm{d}\iota\\
&+\int_a^b\int_t^{t+h} \big|\mathsf{K}_i(\iota,\varsigma)\mathsf{L}_i\big(\varsigma,\varrho_1(\iota),\varrho_2(\iota)\big)\big|\,\mathrm{d}\varsigma\omega(\iota)\,\mathrm{d}\iota\\
\leqslant{}& \big(\|\alpha_{i,1}\|_{L^\infty}+2K\|\beta_{i,1}\|_{L^1}\big)\mu_1^0(\varrho_1,\varepsilon)+\big(\|\alpha_{i,2}\|_{L^\infty}+2K\|\alpha_{i,2}\|_{L^1}\big)\mu_2^0(\varrho_2,\varepsilon)\\
&+\mu_i^0(\mathsf{P}_i,\varepsilon)+\sup_{(\iota,\varsigma)\in\overline{(a,b)}^2}\big|\mathsf{K}_i(\iota,\varsigma)-\mathsf{K}_i(\iota+h,\varsigma)\big|\\
&\times\big(\|\beta_{i,1}\|_{L^1}\|\varrho_1\|_{L^1_\omega}+\|\beta_{i,2}\|_{L^1}\|\varrho_2\|_{L^1_\omega}\big)\\
&+K\big(\|\beta_{i,1}\|_{L^1(b-a,\,b-a+h)}\|\varrho_1\|_{L^1_\omega}+\|\beta_{i,2}\|_{L^1(b-a,\,b-a+h)}\|\varrho_2\|_{L^1_\omega}+\|\tau_h\xi_i-\xi_i\|_{L^1_\omega}\big),
\end{aligned}
$$

hence,

$$
\mu_{\mathrm{G}}^0(\varXi\mathcal{A})\preccurlyeq\begin{pmatrix}\|\alpha_{1,1}\|_{L^\infty}+2K\|\beta_{1,1}\|_{L^1} & \|\alpha_{2,1}\|_{L^\infty}+2K\|\beta_{2,1}\|_{L^1}\\ \|\alpha_{1,2}\|_{L^\infty}+2K\|\beta_{1,2}\|_{L^1} & \|\alpha_{2,2}\|_{L^\infty}+2K\|\beta_{2,2}\|_{L^1}\end{pmatrix}\mu_{\mathrm{G}}^0(\mathcal{A}),
$$

$$
\begin{aligned}
&\int_{(a,b)\setminus B_T}\big|\big(\varXi_i(\varrho)\big)(\iota)\big|\omega(\iota)\,\mathrm{d}\iota\\
&\quad\leqslant\big(\|\alpha_{i,1}\|_{L^\infty((a,b)\setminus B_T)}+K\|\beta_{i,1}\|_{L^1}\big)\|\varrho_1\|_{L^1_\omega((a,b)\setminus B_T)}+\big\|\mathsf{P}_i(\cdot,0,0)\big\|_{L^1_\omega((a,b)\setminus B_T)}\\
&\qquad+\big(\|\alpha_{i,2}\|_{L^\infty((a,b)\setminus B_T)}+K\|\beta_{i,2}\|_{L^1}\big)\|\varrho_2\|_{L^1_\omega((a,b)\setminus B_T)}+\|\xi_i\|_{L^1_\omega((a,b)\setminus B_T)},
\end{aligned}
$$

$$
d_{\mathrm{G}}(\varXi\mathcal{A})\preccurlyeq K\begin{pmatrix}\|\beta_{1,1}\|_{L^1} & \|\beta_{1,12}\|_{L^1}\\ \|\beta_{2,1}\|_{L^1} & \|\beta_{2,2}\|_{L^1}\end{pmatrix}d_{\mathrm{G}}(\mathcal{A}).
$$

So

$$
\mu_{\mathrm{G}}(\varXi\mathcal{A})\preccurlyeq\begin{pmatrix}\|\alpha_{1,1}\|_{L^\infty}+2K\|\beta_{1,1}\|_{L^1} & \|\alpha_{1,2}\|_{L^\infty}+2K\|\beta_{1,2}\|_{L^1}\\ \|\alpha_{2,1}\|_{L^\infty}+2K\|\beta_{2,1}\|_{L^1} & \|\alpha_{2,2}\|_{L^\infty}+2K\|\beta_{2,2}\|_{L^1}\end{pmatrix}\mu_{\mathrm{G}}(\mathcal{A}). \tag{6}
$$

Now, by combining (5), (6) we find

$$
\mu_{\mathrm{G}}\big(\varXi(\mathcal{A})\big)\preccurlyeq M_*\mu_{\mathrm{G}}(\mathcal{A}).
$$

Therefore, by the generalized Darbo fixed point Theorem 1 system (2) has at least one solution in $\bar{B}_r\subset W^{1,1}_\omega(a,b)\times W^{1,1}_\omega(a,b)$. □

Example 1. Consider the following coupled functional integral equation:

$$\begin{aligned}\varrho_1(\iota) &= \frac{\cos(\mathrm{e}^{-8}\varrho_1(\iota) + \mathrm{e}^{-3}\varrho_2(\iota))}{|\iota| + 5} \\ &\quad + \int_{-1}^{\iota} \mathrm{e}^{-4} \ln\bigl((\varsigma + t)^2 + 1\bigr) \arctan\bigl(\mathrm{e}^{\varrho_1(\iota)/20 + \varrho_2(\iota)/10 + \varsigma^2}\bigr)\, \mathrm{d}\varsigma, \\ \varrho_2(\iota) &= \frac{\sin(\mathrm{e}^{-9}\varrho_1(\iota) + \mathrm{e}^{-5}\varrho_2(\iota))}{|\iota| + 3} \\ &\quad + \int_{-1}^{\iota} \arctan\frac{t + s}{2} \ln\biggl(\cosh\biggl(\frac{\varrho_1(\iota)}{\varsigma^2 + 16} + \frac{\varrho_2(\iota)}{|\varsigma| + 8}\biggr)\biggr)\, \mathrm{d}\varsigma.\end{aligned} \tag{7}$$

Then

$$\mathsf{P}_1\bigl(\iota, \varrho_1(\iota), \varrho_2(\iota)\bigr) = \frac{\cos(\mathrm{e}^{-8}\varrho_1(\iota) + \mathrm{e}^{-3}\varrho_2(\iota))}{|\iota| + 5}, \qquad \mathsf{K}_1(\iota, \varsigma) = \mathrm{e}^{-4} \ln\bigl((\varsigma + t)^2 + 1\bigr),$$

$$\mathsf{P}_2(\iota, \varrho_1(\iota), \varrho_2(\iota)) = \frac{\sin(\mathrm{e}^{-9}\varrho_1(\iota) + \mathrm{e}^{-5}\varrho_2(\iota))}{|\iota| + 3}, \qquad \mathsf{K}_2(\iota, \varsigma) = \arctan\frac{t + s}{2},$$

$$\begin{aligned}\mathsf{L}_1\bigl(\varsigma, \varrho_1(\iota), \varrho_2(\iota)\bigr) &= \arctan\bigl(\mathrm{e}^{\varrho_1(\iota)/20 + \varrho_2(\iota)/10 + \varsigma^2}\bigr), \\ \mathsf{L}_2\bigl(\varsigma, \varrho_1(\iota), \varrho_2(\iota)\bigr) &= \ln\biggl(\cosh\biggl(\frac{\varrho_1(\iota)}{\varsigma^2 + 16} + \frac{\varrho_2(\iota)}{|\varsigma| + 8}\biggr)\biggr),\end{aligned}$$

and we have

$$\begin{aligned}\frac{\partial \mathsf{P}_1}{\partial \iota}(\iota, \varrho_1, \varrho_2) &= \operatorname{sgn}(\iota)\frac{\cos(\mathrm{e}^{-8}\varrho_1 + \mathrm{e}^{-3}\varrho_2)}{(|\iota| + 5)^2}, \\ \frac{\partial \mathsf{P}_2}{\partial \iota}(\iota, \varrho_1, \varrho_2) &= \operatorname{sgn}(\iota)\frac{\sin(\mathrm{e}^{-9}\varrho_1 + \mathrm{e}^{-5}\varrho_2)}{(|\iota| + 3)^2},\end{aligned}$$

and we simply check that

$$\begin{aligned}\bigl|\mathsf{P}_1(\iota, \varrho_1, \varrho_2) - \mathsf{P}_1(\iota, \bar{\varrho}_1, \bar{\varrho}_2)\bigr| &\leqslant \frac{\mathrm{e}^{-8}}{|\iota| + 5}|\varrho_1 - \bar{\varrho}_1| + \frac{\mathrm{e}^{-3}}{|\iota| + 5}|\varrho_2 - \bar{\varrho}_2|, \\ \bigl|\mathsf{P}_2(\iota, \varrho_2, \varrho_2) - \mathsf{P}_1(\iota, \bar{\varrho}_1, \bar{\varrho}_2)\bigr| &\leqslant \frac{\mathrm{e}^{-9}}{|\iota| + 3}|\varrho_1 - \bar{\varrho}_1| + \frac{\mathrm{e}^{-5}}{|\iota| + 3}|\varrho_2 - \bar{\varrho}_2|\end{aligned}$$

and

$$\begin{aligned}\biggl|\frac{\partial \mathsf{P}_1}{\partial \iota}(\iota, \varrho_1, \varrho_2) - \frac{\partial \mathsf{P}_1}{\partial \iota}(\iota, \bar{\varrho}_1, \bar{\varrho}_2)\biggr| &\leqslant \frac{\mathrm{e}^{-8}}{(|\iota| + 5)^2}|\varrho_1 - \bar{\varrho}_1| + \frac{\mathrm{e}^{-3}}{(|\iota| + 5)^2}|\varrho_2 - \bar{\varrho}_2|, \\ \biggl|\frac{\partial \mathsf{P}_2}{\partial \iota}(\iota, \varrho_1, \varrho_2) - \frac{\partial \mathsf{P}_2}{\partial \iota}(\iota, \bar{\varrho}_1, \bar{\varrho}_2)\biggr| &\leqslant \frac{\mathrm{e}^{-9}}{(|\iota| + 3)^2}|\varrho_1 - \bar{\varrho}_1| + \frac{\mathrm{e}^{-5}}{(|\iota| + 3)^2}|\varrho_2 - \bar{\varrho}_2|.\end{aligned}$$

Thus,

$$M_1 = \begin{pmatrix} \frac{e^{-8}}{|\iota|+5} & \frac{e^{-3}}{|\iota|+5} \\ \frac{e^{-9}}{|\iota|+3} & \frac{e^{-5}}{|\iota|+3} \end{pmatrix} \in \mathcal{M}_{2\times 2}(L^\infty(a,b), \mathbb{R}_+)$$

with

$$\begin{pmatrix} \|\alpha_{1,1}\|_{L^\infty} & \|\alpha_{1,2}\|_{L^\infty} \\ \|\alpha_{2,1}\|_{L^\infty} & \|\alpha_{2,2}\|_{L^\infty} \end{pmatrix} = \begin{pmatrix} \frac{e^{-8}}{5} & \frac{e^{-3}}{5} \\ \frac{e^{-9}}{3} & \frac{e^{-5}}{3} \end{pmatrix}.$$

Furthermore,

$$\begin{aligned}
\frac{\partial \mathsf{P}_1}{\partial \varrho_1}(\iota, \varrho_1, \varrho_2) &= -e^{-8} \frac{\sin(e^{-8}\varrho_1 + e^{-3}\varrho_2)}{(|\iota|+5)^2}, \\
\frac{\partial \mathsf{P}_1}{\partial \varrho_2}(\iota, \varrho_1, \varrho_2) &= -e^{-3} \frac{\sin(e^{-8}\varrho_1 + e^{-3}\varrho_2)}{(|\iota|+5)^2}, \\
\frac{\partial \mathsf{P}_2}{\partial \varrho_1}(\iota, \varrho_1, \varrho_2) &= e^{-9} \frac{\sin(e^{-9}\varrho_1 + e^{-5}\varrho_2)}{(|\iota|+3)^2}, \\
\frac{\partial \mathsf{P}_2}{\partial \varrho_2}(\iota, \varrho_1, \varrho_2) &= e^{-5} \frac{\sin(e^{-9}\varrho_1 + e^{-5}\varrho_2)}{(|\iota|+3)^2},
\end{aligned}$$

then we can verify easily that

$$\begin{aligned}
\left| \frac{\partial \mathsf{P}_1}{\partial \varrho_1}(\iota, \varrho_1, \varrho_2)\varrho_1'(\iota) - \frac{\partial \mathsf{P}_1}{\partial \varrho_1}(\iota, \bar{\varrho}_1, \bar{\varrho}_2)\bar{\varrho}_1'(\iota) \right| &\leqslant \frac{2e^{-8}}{5} \left|\varrho_1'(\iota) - \bar{\varrho}_1'(\iota)\right|, \\
\left| \frac{\partial \mathsf{P}_1}{\partial \varrho_2}(\iota, \varrho_1, \varrho_2)\varrho_2'(\iota) - \frac{\partial \mathsf{P}_1}{\partial \varrho_2}(\iota, \bar{\varrho}_1, \bar{\varrho}_2)\bar{\varrho}_2'(\iota) \right| &\leqslant \frac{2e^{-3}}{5} \left|\varrho_2'(\iota) - \bar{\varrho}_2'(\iota)\right|, \\
\left| \frac{\partial \mathsf{P}_2}{\partial \varrho_1}(\iota, \varrho_1, \varrho_2)\varrho_1'(\iota) - \frac{\partial \mathsf{P}_2}{\partial \varrho_1}(\iota, \bar{\varrho}_1, \bar{\varrho}_2)\bar{\varrho}_1'(\iota) \right| &\leqslant \frac{2e^{-9}}{3} \left|\varrho_1'(\iota) - \bar{\varrho}_1'(\iota)\right|, \\
\left| \frac{\partial \mathsf{P}_2}{\partial \varrho_2}(\iota, \varrho_1, \varrho_2)\varrho_2'(\iota) - \frac{\partial \mathsf{P}_2}{\partial \varrho_1}(\iota, \bar{\varrho}_1, \bar{\varrho}_2)\bar{\varrho}_2'(\iota) \right| &\leqslant \frac{2e^{-5}}{3} \left|\varrho_2'(\iota) - \bar{\varrho}_2'(\iota)\right|
\end{aligned}$$

and

$$\begin{aligned}
\left| \frac{\partial \mathsf{P}_1}{\partial \varrho_1}(\iota, \varrho_1, \varrho_2)\varrho_1'(\iota) \right| &\leqslant \frac{e^{-8}}{5} \left|\varrho_1'(\iota)\right|, &\qquad \left| \frac{\partial \mathsf{P}_1}{\partial \varrho_2}(\iota, \varrho_1, \varrho_2)\varrho_2'(\iota) \right| &\leqslant \frac{e^{-3}}{5} \left|\varrho_2'(\iota)\right|, \\
\left| \frac{\partial \mathsf{P}_2}{\partial \varrho_1}(\iota, \varrho_1, \varrho_2)\varrho_1'(\iota) \right| &\leqslant \frac{e^{-9}}{3} \left|\varrho_1'(\iota)\right|, &\qquad \left| \frac{\partial \mathsf{P}_2}{\partial \varrho_2}(\iota, \varrho_1, \varrho_2)\varrho_2'(\iota) \right| &\leqslant \frac{e^{-5}}{3} \left|\varrho_2'(\iota)\right|.
\end{aligned}$$

Hence,

$$M_2 = \begin{pmatrix} \frac{2e^{-8}}{5} & \frac{2e^{-3}}{5} \\ \frac{2e^{-9}}{3} & \frac{2e^{-5}}{3} \end{pmatrix}, \qquad M_3 = \begin{pmatrix} \frac{e^{-8}}{5} & \frac{e^{-3}}{5} \\ \frac{e^{-9}}{3} & \frac{e^{-5}}{3} \end{pmatrix}.$$

It is easy to see that for each $i \in \{1,2\}$, $\mathsf{P}_i(\cdot,0,0)$ satisfies assumption $(\mathcal{H}_2)$(d). Since we have $\mathsf{P}_1(\iota,0,0) = 1/(|\iota|+5)$, $\partial \mathsf{P}_1/\partial\iota(\iota,0,0) = \operatorname{sgn}(\iota)(1/(|\iota|+5)^2)$, $\mathsf{P}_2(\iota,0,0) = \partial \mathsf{P}_2/\partial\iota(\iota,0,0) \equiv 0$. Then we obtain

$$\left\|\mathsf{P}_1(\iota,0,0)\right\|_{W^{1,p}_\omega((-1,1))} \leqslant \frac{e^1 - e^{-1}}{5}, \qquad \left\|\mathsf{P}_2(\iota,0,0)\right\|_{W^{1,p}_\omega((-1,1))} = 0.$$

By the same way we get

$$\begin{aligned}&\bigl|\mathsf{L}_1(\varsigma,\varrho_1,\varrho_2)-\mathsf{L}_1(\varsigma,\bar\varrho_1,\bar\varrho_2)\bigr|\\&\quad\leqslant\frac{1}{40}\operatorname{sech}\left(\frac{\varrho_1}{20}+\frac{\varrho_2}{10}+\varsigma^2\right)|\varrho_1-\bar\varrho_1|+\frac{1}{20}\operatorname{sech}\left(\frac{\varrho_1(\iota)}{20}+\frac{\varrho_2(\iota)}{10}+\varsigma^2\right)|\varrho_2-\bar\varrho_2|,\\&\bigl|\mathsf{L}_2(\varsigma,\varrho_2,\varrho_2)-\mathsf{L}_1(\varsigma,\bar\varrho_1,\bar\varrho_2)\bigr|\\&\quad\leqslant\frac{1}{\varsigma^2+16}\operatorname{th}\left(\frac{\varrho_1}{\varsigma^2+16}+\frac{\varrho_2}{|\varsigma|+8}\right)|\varrho_1-\bar\varrho_1|+\frac{1}{|\varsigma|+8}\operatorname{th}\left(\frac{\varrho_1}{\varsigma^2+16}+\frac{\varrho_2}{|\varsigma|+8}\right)|\varrho_2-\bar\varrho_2|,\end{aligned}$$

$$\begin{pmatrix}\|\beta_{1,1}\|_{L^1} & \|\beta_{1,2}\|_{L^1}\\ \|\beta_{2,1}\|_{L^1} & \|\beta_{2,2}\|_{L^1}\end{pmatrix}=\begin{pmatrix}\frac{1}{20} & \frac{1}{10}\\ \frac{1}{8} & \frac{1}{4}\end{pmatrix},$$

and

$$\begin{pmatrix}\|L_{1,1}\|_{L^1} & \|L_{1,2}\|_{L^1}\\ \|L_{2,1}\|_{L^1} & \|L_{2,2}\|_{L^1}\end{pmatrix}=\begin{pmatrix}\|\eta_{1,1}\|_{L^1} & \|\eta_{1,2}\|_{L^1}\\ \|\eta_{2,1}\|_{L^1} & \|\eta_{2,2}\|_{L^1}\end{pmatrix}=\begin{pmatrix}\frac{1}{10} & \frac{1}{5}\\ \frac{1}{4} & \frac{1}{2}\end{pmatrix}.$$

Furthermore, condition $(\mathcal{H}_2)$(d) can be easily verified. Moreover, $K=\pi/4$. Finally, the matrix

$$M_*=\begin{pmatrix}\frac{\mathrm{e}^{-8}}{5}+\frac{\pi}{40} & \frac{\mathrm{e}^{-3}}{5}+\frac{\pi}{20}\\ \frac{\mathrm{e}^{-9}}{3}+\frac{\pi}{16} & \frac{\mathrm{e}^{-5}}{3}+\frac{\pi}{8}\end{pmatrix}$$

has two eigenvalues: $|\sigma_1|\approx 0.0037<1$, $|\sigma_2|\approx 0.4772<1$. Therefore, M_* converges to zero. All the conditions in Theorem 3 are satisfied, so system (7) has at least one solution in the space $W^{1,1}_\omega((-1,1))\times W^{1,1}_\omega((-1,1))$, where $\omega(\iota)=\mathrm{e}^{-\iota}$.

References

1. A. Aghajani, R. Allahyari, M. Mursaleen, A generalization of Darbo's theorem with application to the solvability of systems of integral equations, *J. Comput. Appl. Math.*, **260**:68–77, 2014, https://doi.org/10.1016/j.cam.2013.09.039.
2. A. Aghajani, D. O'Regan, A.S. Haghighi, Measure of noncompactness on $L^p(\mathbb{R}^N)$ and applications, *CUBO*, **17**(1):85–97, 2015, https://revistas.ufro.cl/ojs/index.php/cubo/article/view/1153.
3. R. Allahyari, R. Arab, A.S. Haghigh, Measures of noncompactness in a Sobolev space and integro-differential equations, *Bull. Aust. Math. Soc.*, **94**(3):497–506, 2016, https://doi.org/10.1017/S0004972716000320.
4. A.B. Amar, A. Jeribi, B. Krichen, Fixed point theorems for block operator matrix and an application to a structured problem under boundary conditions of Rotenberg's model type, *Math. Slovaca*, **64**(1):155–174, 2014, https://doi.org/10.2478/s12175-013-0193-3.
5. H. Amiri, M. Khanehgir, R. Allahyari, A generalization of Darbo's theorem with application to the solvability of systems of integral-differential equations in Sobolev spaces, *Int. J. Nonlinear Anal. Appl.*, **12**(1):287–300, 2021, https://doi.org/10.22075/IJNAA.2021.4784.

6. J. Banaś, K. Goebel, *Measures of Noncompactness in Banach Spaces*, Lect. Notes Pure Appl. Math. Vol. 60, Marcel Dekker, New York, 1980.
7. H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer, New York, 2010, https://doi.org/10.1007/978-0-387-70914-7.
8. G. Darbo, Punti uniti in trasformazioni a codominio non compatto, *Rend. Sem. Mat. Univ. Padova*, **24**:84–92, 1955, http://www.numdam.org/item/RSMUP_1955__24__84_0.
9. J. García-Cuerva, J.L.R. de Francia, *Weighted Norm Inequalities and Related Topics*, Elsevier, Amsterdam, 2011, https://blackwells.co.uk/bookshop/product/Weighted-Norm-Inequalities-and-Related-Topics-by-Garcia-Cuerva-J/9780080872278.
10. J.R. Graef, J. Henderson, A. Ouahab, *Topological Methods for Differential Equations and Inclusions*, CRC Press, Boca Raton, FL, 2018, https://doi.org/10.1201/9780429446740.
11. M. Khanehgir, R. Allahyari, N. Gholamian, Construction of a measure of noncompactness in Sobolev spaces with an application to functional integral-differential equations, *Math. Sci.*, **12**(1):17–24, 2018, https://doi.org/10.1007/s40096-017-0240-2.
12. C. Kuratowski, Sur les espaces complets, *Fundam. Math.*, **15**(1):301–309, 1930, http://eudml.org/doc/212357.
13. M.H.Z. Moghaddam, R. Allahyari, M.E. Omidvar, A.S. Haghighi, Existence of solutions for some classes of integro-differential equations in the Sobolev space $W^{n,p}(\mathbb{R}^+)$, *J. Fixed Point Theory Appl.*, **20**(3):99, 2018, https://doi.org/10.1007/s11784-018-0574-x.
14. B. Muckenhoupt, Weighted norm inequalities for the hardy maximal function, *Trans. Am. Math. Soc.*, **165**:207–226, 1972, https://doi.org/10.2307/1995882.
15. F.P. Najafabad, J.J. Nieto, H.A. Kayvanloo, Measure of noncompactness on weighted Sobolev space with an application to some nonlinear convolution type integral equations, *J. Fixed Point Theory Appl.*, **22**(3):75, 2020, https://doi.org/10.1007/s11784-020-00809-1.
16. H. Nasiri, J. Roshan, M. Mursaleen, Solvability of system of Volterra integral equations via measure of noncompactness, *Comput. Appl. Math.*, **40**(5):166, 2021, https://doi.org/10.1007/s40314-021-01552-0.
17. J.J. Nieto, A. Ouahab, R. Rodríguez-López, Fixed point theorems in generalized Banach algebras and applications, *Fixed Point Theory*, **19**(2):707–732, 2018, https://doi.org/10.24193/fpt-ro.2018.2.54.
18. A. Ouahab, Some Perov's and Krasnosel'skii type fixed point results and application, *Commun. Appl. Anal.*, **19**(4):623–642, 2015, http://www.acadsol.eu/en/articles/19/4/12.pdf.
19. B.O. Turesson, *Nonlinear Potential Theory and Weighted Sobolev Spaces*, Lect. Notes Math. Vol. 1736, Springer, Berlin, Heidelberg, 2000, https://doi.org/10.1007/BFb0103908.
20. R.S. Varga, *Matrix Iterative Analysis*, Springer, Berlin, Heidelberg, 2000, https://doi.org/doi.org/10.1007/978-3-642-05156-2.