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A note on product-convolution for generalized subexponential distributions*

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Abstract. In this paper, we consider the stability property of the class of generalized subexponential distributions with respect to product-convolution. Assuming that the primary distribution is in the class of generalized subexponential distributions, we find conditions for the second distribution in order that their product-convolution belongs to the class of generalized subexponential distributions as well. The similar problem for the class of generalized subexponential positively decreasing-tailed distributions is considered.

Keywords: product-convolution, subexponential distribution, generalized subexponential distribution, positively decreasing-tailed distribution.

1 Introduction and preliminaries

This paper deals with the O-generalization of the standard class of subexponential distributions. Recall that distribution function (d.f.) F, satisfying $F(0-) = 0$, belongs to the *subexponential* class of distributions, denoted $\mathcal{S}$, if the following relation holds:

$$\lim_{x\to\infty} \frac{\overline{F^{*2}}(x)}{\overline{F}(x)} = 2, \tag{1}$$

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where $\overline{F} = 1 - F$ denotes the tail of the d.f. F, and $F^{*2} = F * F$ denotes the 2-fold convolution of d.f. F. In the general case, when d.f. F not necessarily satisfies $F(0-) = 0$, we say that F is subexponential, written as $F \in \mathcal{S}$, if $F^+ \in \mathcal{S}$, where $F^+(x) := F(x)\mathbf{1}_{[0,\infty)}(x)$. The concept of subexponentiality was introduced by Chistyakov [11]. Later on, the subexponential distributions found numerous applications in applied probability including actuarial science, financial mathematics, risk management, branching processes, queueing theory, etc.; see, for instance, [2–4, 16, 20, 29, 31, 32, 43, 47] and references therein. For the authoritative review of theoretical properties of subexponential distributions, see Section 3 of Foss et al. [18].

It is well known that class $\mathcal{S}$ represents a subset of the class of long-tailed distributions $\mathcal{L}$. Recall that a d.f. F is said to be *long-tailed*, written as $F \in \mathcal{L}$, if for any $y > 0$,

$$\lim_{x\to\infty} \frac{\overline{F}(x-y)}{\overline{F}(x)} = 1.$$

Note that for any $F \in \mathcal{L}$, there exists a function $a : [0,\infty) \to (0,\infty)$ such that, for $x \to \infty$,

(a) $a(x) \nearrow \infty$,
(b) $a(x)/x \searrow 0$,
(c) $\overline{F}(x - a(x)) \sim \overline{F}(x)$,

where and elsewhere the notations $\nearrow$ and $\searrow$ denote monotonic increase and monotonic decrease, respectively.

One can check that $F \in \mathcal{S}$ if and only if (1) holds and $F \in \mathcal{L}$.

Another important class of heavy-tailed distributions, called a class of *dominatedly varying* distributions, is defined by the following property:

$$\liminf_{x\to\infty} \frac{\overline{F}(yx)}{\overline{F}(x)} > 0.$$

This class was introduced in [17]. We denote this class of distributions by $\mathcal{D}$. Note that $\mathcal{D}$ is not contained in $\mathcal{S}$, but $\mathcal{D} \cap \mathcal{L} = \mathcal{D} \cap \mathcal{S} \subset \mathcal{S}$.

The following class, in some sense symmetric to the class $\mathcal{D}$, was introduced in [8]. A d.f. F is said *positively decreasing-tailed*, written as $F \in \mathcal{PD}$, if for any $y > 1$ (equivalently for some $y > 1$), it holds

$$\limsup_{x\to\infty} \frac{\overline{F}(yx)}{\overline{F}(x)} < 1.$$

Importantly, the class of positively decreasing-tailed distributions contains heavy-tailed, such as Pareto, as well as light-tailed distributions, such as exponential distributions or distributions with the tail

$$\overline{F}(x) \underset{x\to\infty}{\sim} c\,\mathrm{e}^{-\gamma x} x^{-\alpha}, \quad c > 0,\ \gamma > 0,\ \alpha > 1, \tag{2}$$

which belong to the class of convolution-equivalent distributions $\mathcal{S}(\gamma)$; see, e.g., [23].

More details on the class $\mathcal{PD}$, including some closure properties, can be found in [6]. Denote by $\mathcal{A} := \mathcal{PD} \cap \mathcal{S}$ the subexponential part of this class. The class $\mathcal{A}$ has some useful properties and was considered in the papers [9,22,25,26,35,40]. We note only that regularly varying (but not slowly varying) distributions are in $\mathcal{A}$.

The following two indices, called the upper and lower Matuszewska indices of d.f. F, respectively, were introduced in [30]:

$$J_F^+ := -\lim_{y\to\infty} \frac{\log \limsup_{x\to\infty}(\overline{F}(y\,x)/\overline{F}(x))}{\log y},$$
$$J_F^- := -\lim_{y\to\infty} \frac{\log \liminf_{x\to\infty}(\overline{F}(y\,x)/\overline{F}(x))}{\log y}.$$

Obviously, $0 \leqslant J_F^- \leqslant J_F^+ \leqslant \infty$. By definition, $F \in \mathcal{D}$ if and only if $J_F^+ < \infty$. Similarly, $F \in \mathcal{PD}$ if and only if $J_F^- > 0$. More details on the Matuszewska indices can be found in Section 2 of [8]. In particular, Proposition 2.2.1 of [8] (see also Lemma 4.1 in [34] or Lemma 3.5 in [38]) implies the following relations:

$$F \in \mathcal{D} \quad \Longrightarrow \quad x^{-\alpha} = o\big(\overline{F}(x)\big) \quad \text{for } \alpha > J_F^+;$$
$$F \in \mathcal{PD} \quad \Longrightarrow \quad \overline{F}(x) = o\big(x^{-\beta}\big) \quad \text{for } 0 < \beta < J_F^-.$$

In this paper, we consider the distribution of the product of two independent random variables (r.v.s). If X and Y are two real-valued independent r.v.s with d.f.s $F(x) = \mathbf{P}(X \leqslant x)$ and $G(x) = \mathbf{P}(Y \leqslant x)$, then d.f. of the product XY is

$$F \otimes G(x) := \mathbf{P}(XY \leqslant x) = \int\limits_{(-\infty,0)} \left(1 - F\left(\frac{x}{y}-\right)\right) \mathrm{d}G(y)$$
$$+ \int\limits_{(0,\infty)} F\left(\frac{x}{y}\right) \mathrm{d}G(y) + \big(G(0) - G(0-)\big)\mathbf{1}_{[0,\infty)}(x);$$

see, e.g., Section 1.2 of [19]. We call $F \otimes G$ a *product-convolution* of F and G. In the case of nonnegative r.v. Y, the formula above becomes

$$F \otimes G(x) = \int\limits_{(0,\infty)} F\left(\frac{x}{y}\right) \mathrm{d}G(y) + G(0)\mathbf{1}_{[0,\infty)}(x).$$

The following result on subexponentiality of the d.f. $F \otimes G$ was proved in [13]:

Theorem 1. *Assume that F is a distribution on $\mathbb{R}$ and $G(0-) = 0$, $G(0) < 1$. If $F \in \mathcal{S}$ and there exists a function $a : [0,\infty) \to [0,\infty)$ such that, for $x \to \infty$,*

(i) $a(x) \nearrow \infty$,
(ii) $\frac{a(x)}{x} \searrow 0$,
(iii) $\overline{G}(a(x)) = o(\overline{F \otimes G}(x))$,
(iv) $\overline{F}(x - a(x)) \sim \overline{F}(x)$,

then $F \otimes G \in \mathcal{S}$.

The similar result for the class $\mathcal{A}$ was proved in [36, Thm. 2.1].

Theorem 2. *Assume that F is distribution on $\mathbb{R}$ and $G(0-)=0$, $G(0)<1$. If $F\in\mathcal{A}$ and*

$$\overline{G}(yx)=o\big(\overline{F\otimes G}(x)\big) \quad \text{for any } y>0, \tag{3}$$

then $F\otimes G\in\mathcal{A}$.

Note that condition (3) is equivalent to the existence of function a satisfying conditions (i)–(iii) from Theorem 1; see Lemma 3.2 in [36]. We mention also the paper by Xu et al. [45], where necessary and sufficient conditions for the subexponentiality of the product-convolution were presented.

The aim of the present paper is to prove similar closure properties for the class of generalized subexponential distributions. In Section 2, we introduce the class of generalized subexponential distributions and formulate the main results of the paper. In Section 3, we prove the results. Some corollaries and examples are provided in Section 4.

2 Product-convolution properties for generalized subexponential distributions

Klüppelberg [24] proposed the following natural generalization of the class of subexponential distributions. A d.f. F is said to be *generalized subexponential* (or O-subexponential), denoted by $F\in\mathcal{OS}$, if

$$\limsup_{x\to\infty}\frac{\overline{F*F}(x)}{\overline{F}(x)}<\infty.$$

Later, this class was studied in the papers [5, 10, 28, 33, 42, 44].

As in the case of class $\mathcal{OS}$, one can introduce the O-version of class $\mathcal{L}$ as follows (see [33]): a distribution F on $\mathbb{R}$ is said to belong to the class $\mathcal{OL}$ of *generalized long-tailed distributions* if for any (or some) $y>0$,

$$\limsup_{x\to\infty}\frac{\overline{F}(x-y)}{\overline{F}(x)}<\infty.$$

Similarly to inclusion $\mathcal{S}\subset\mathcal{L}$, it holds that $\mathcal{OS}\subset\mathcal{OL}$ (see, e.g., Proposition 2.1(ii) of [33]). Examples of d.f.s $F\in\mathcal{OL}\setminus\mathcal{OS}$ can be found in [14, 46], some useful characterizations of class $\mathcal{OL}$ are given in [1].

The class of generalized subexponential distributions forms a wide class of distributions. Both $\mathcal{OS}$ and $\mathcal{OL}$ admit some light-tailed distributions. Presented definitions imply that:

$$\mathcal{A}\subset\mathcal{S}\subset\mathcal{L}\cap\mathcal{OS}, \qquad \mathcal{D}\subset\mathcal{OS}.$$

Lin and Wang [28], Wang et al. [41] provided examples showing that $\mathcal{L}\cap\mathcal{OS}\setminus\mathcal{S}$ is nonempty.

Next, we formulate the main results of the paper. Theorem 3 is analogous to Theorem 1. Theorem 4 is analogous to Theorem 2.

Theorem 3. *Assume that F is distribution on $\mathbb{R}$ and $G(0-) = 0$, $G(0) < 1$. If $F \in \mathcal{OS}$ and*

$$\sup_{y>0} \limsup_{x\to\infty} \frac{\overline{G}(yx)}{\overline{F \otimes G}(x)} < \infty, \tag{4}$$

then $F \otimes G \in \mathcal{OS}$.

As in the case of condition (3), we show in Lemma 2 below that (4) can be equivalently characterized in terms of corresponding increasing function $a(x)$. Some examples will be constructed in Section 4 to illustrate condition (4). In relation to the results of [14], we have that if $F \in \mathcal{L} \cap \mathcal{OS}$, $G \in \mathcal{L}$ satisfies (4), then $F \otimes G \in \mathcal{L} \cap \mathcal{OS}$. We also remind that the class $\mathcal{L}$ is closed under product convolution; see Tang [37].

Now introduce the generalization of class $\mathcal{A}$:

$$\mathcal{OA} := \mathcal{PD} \cap \mathcal{OS}.$$

Theorem 4. *Assume that F is distribution on $\mathbb{R}$ and $G(0-) = 0$, $G(0) < 1$. If $F \in \mathcal{OA}$ and condition* (3) *holds, then $F \otimes G \in \mathcal{OA}$.*

Note that both theorems are rather substantial extensions of Theorems 1 and 2, as they allow to consider such (heavy-tailed) subclass of $\mathcal{OS}$ as $\mathcal{D}$, as well as some light-tailed distributions (see examples in Section 4).

3 Proof of Theorems 3 and 4

In order to prove the theorems, we need two auxiliary lemmas. The first lemma deals with the bound of probability $\mathbf{P}(qX+rY > x)$, $0 < r \leqslant q$, in the case where d.f. of X belongs to class $\mathcal{OS}$. A similar result in the case of the class $\mathcal{S}$ was proved in [39, Lemma 5.1].

Lemma 1. *Suppose that X and Y are two independent r.v.s with corresponding distributions F and G. If $F \in \mathcal{OS}$ and $\overline{G}(x) = O(\overline{F}(x))$, then*

$$\sup_{x>0} \sup_{\{r,q:\, 0<r\leqslant q\}} \frac{\mathbf{P}(qX + rY > x)}{\overline{F}(x/q) + \overline{G}(x/r)} < \infty. \tag{5}$$

Proof. For $0 < r \leqslant q$ and $x > 0$, we have

$$\begin{aligned} \mathbf{P}(qX + rY > x) &= \biggl(\int\limits_{(-\infty,0]} + \int\limits_{(0,\, x/q]} + \int\limits_{(x/q,\, \infty)} \biggr) \overline{G}\biggl(\frac{x - qy}{r}\biggr) \, \mathrm{d}F(y) \\ &=: J_1 + J_2 + J_3. \end{aligned} \tag{6}$$

Obviously,

$$\frac{J_1}{\overline{G}(x/r)} = \int\limits_{(-\infty,0]} \frac{\overline{G}((x - qy)/r)}{\overline{G}(x/r)} \, \mathrm{d}F(y) \leqslant F(0). \tag{7}$$

To estimate J_2, observe that

$$\begin{aligned}\frac{J_2}{\overline{F}(x/q)} &\leqslant \frac{1}{\overline{F}(x/q)} \int\limits_{(0,\,x/q]} \overline{G}\left(\frac{x}{q}-y\right) \mathrm{d}F(y) \\ &\leqslant \sup_{z\geqslant 0} \frac{\overline{G}(z)}{\overline{F}(z)} \frac{1}{\overline{F}(x/q)} \int\limits_{(0,\,x/q]} \overline{F}\left(\frac{x}{q}-y\right) \mathrm{d}F(y) \\ &= \sup_{z\geqslant 0} \frac{\overline{G}(z)}{\overline{F}(z)} \frac{1}{\overline{F}(x/q)} \mathbf{P}\left(X_1+X_2>\frac{x}{q},\ 0<X_2\leqslant\frac{x}{q}\right),\end{aligned}$$

where X_1, X_2 are independent copies of X. Hence, by $F \in \mathcal{OS}$ and $\overline{G}(x) = O(\overline{F}(x))$,

$$\frac{J_2}{\overline{F}(x/q)} \leqslant \sup_{z\geqslant 0} \frac{\overline{G}(z)}{\overline{F}(z)} \sup_{u\in\mathbb{R}} \frac{\overline{F*F}(u)}{\overline{F}(u)} \leqslant C \tag{8}$$

for some positive constant C. Obviously,

$$J_3 \leqslant \overline{F}\left(\frac{x}{q}\right). \tag{9}$$

The estimates (7)–(9) and (6) imply that

$$\begin{aligned}\sup_{0<r\leqslant q} \frac{\mathbf{P}(qX+rY>x)}{\overline{F}(x/q)+\overline{G}(x/r)} &\leqslant \sup_{0<r\leqslant q}\left(\frac{J_1}{\overline{G}(x/r)}+\frac{J_2}{\overline{F}(x/q)}+\frac{J_3}{\overline{F}(x/q)}\right) \\ &\leqslant F(0)+C+1,\end{aligned}$$

that is the estimate (5). Lemma is proved. □

The following lemma is similar to Lemma 3.2 in [36].

Lemma 2. *Assume that F is distribution on $\mathbb{R}$ and $G(0-)=0$, $G(0)<1$. Then* (4) *holds if and only if there exists a positive function $a(x)$ satisfying conditions*

(i) $a(x) \nearrow \infty$,
(ii) $a(x)/x \searrow 0$,
(iii) $\overline{G}(a(x)) = O(\overline{F\otimes G}(x))$.

Proof. We begin with the sufficiency. Since

$$\overline{G}\big(a(x)\big) = O\big(\overline{F\otimes G}(x)\big),$$

it follows that there is finite constant M such that

$$\limsup_{x\to\infty} \frac{\overline{G}(a(x))}{\overline{F\otimes G}(x)} = M.$$

Thus,

$$\overline{G}\big(a(x)\big) \leqslant (M+1)\overline{F\otimes G}(x), \quad x \geqslant x_M,$$

for some positive x_M. The property $a(x)/x \searrow 0$ implies that for any $y > 0$, there exists large enough x_y such that $a(x) \leqslant yx$ for any $x > x_y$. Hence, for any $x > x_M \vee x_y$, inequality $\overline{G}(yx)) \leqslant \overline{G}(a(x))$ holds, so that

$$\limsup_{x\to\infty} \frac{\overline{G}(yx)}{\overline{F\otimes G}(x)} \leqslant \limsup_{x\to\infty} \frac{\overline{G}(a(x))}{\overline{F\otimes G}(x)} \leqslant M+1,$$

which implies condition (4).

Let us prove the necessity. Condition (4) implies

$$\sup_{n\in\mathbb{N}} \limsup_{x\to\infty} \frac{\overline{G}(x/n)}{\overline{F\otimes G}(x)} \leqslant M^*$$

for some positive constant $M^* < \infty$. Hence,

$$\limsup_{x\to\infty} \frac{\overline{G}(x/n)}{\overline{F\otimes G}(x)} \leqslant M^*$$

for an arbitrary $n \in \mathbb{N}$. Now we can construct the sequence $x_1, x_2, \dots$ such that $x_1 = 1$, $x_n > 2x_{n-1}$ ($n \geqslant 2$), and

$$\frac{\overline{G}(x/n)}{\overline{F\otimes G}(x)} \leqslant M^* + 1, \quad x \geqslant x_n. \tag{10}$$

Denote $x_n^* = (n/(n-1))x_n$ and define the function $a : [0,\infty) \to (0,\infty)$:

$$a(x) = x_2 \mathbf{1}_{[0,x_2)}(x) + \sum_{n=2}^{\infty} \left(\frac{x_n}{n-1} \mathbf{1}_{[x_n,x_n^*)}(x) + \frac{x}{n} \mathbf{1}_{[x_n^*,x_{n+1})}(x) \right).$$

Function $a(x)$ is increasing to infinity. Also, for $x \in [x_n, x_{n+1})$, $n \geqslant 2$, we have

$$\frac{a(x)}{x} = \begin{cases} \frac{x_n}{(n-1)x} \leqslant \frac{1}{n-1} & \text{if } x \in [x_n, x_n^*), \\ \frac{1}{n} & \text{if } x \in [x_n^*, x_{n+1}). \end{cases}$$

Hence, $a(x)/x$ tends to zero for $x \to \infty$.

Further, for $x \in [x_n, x_n^*)$, $n \geqslant 2$, we have

$$\frac{\overline{G}(a(x))}{\overline{F\otimes G}(x)} = \frac{\overline{G}(x_n/(n-1))}{\overline{F\otimes G}(x)} \leqslant \frac{\overline{G}(x_n^*/n)}{\overline{F\otimes G}(x_n^*)} \leqslant M^* + 1$$

due to (10) because $x_n^* > x_n$. Similarly, for $x \in [x_n^*, x_{n+1})$, $n \geqslant 2$, by (10) we have

$$\frac{\overline{G}(a(x))}{\overline{F\otimes G}(x)} = \frac{\overline{G}(x/n)}{\overline{F\otimes G}(x)} \leqslant M^* + 1,$$

and we obtain that for all $x \geqslant x_2$,

$$\frac{\overline{G}(a(x))}{\overline{F \otimes G}(x)} \leqslant M^* + 1,$$

implying $\overline{G}(a(x)) = O(\overline{F \otimes G}(x))$. □

Proof of Theorem 3. Introduce r.v.s X_1, X_2, Y_1, Y_2 such that $\{X, X_1, X_2, Y, Y_1, Y_2\}$ are independent with X_1, X_2 identically distributed to X, having d.f. F, and Y_1, Y_2 identically distributed to Y, having d.f. G. According to Lemma 2, there exists a function $a(x)$ satisfying conditions (i)–(iii) of the lemma. Then for this function $a(x)$ and for $x > 0$, we have

$$\begin{aligned}\mathbf{P}(X_1Y_1 + X_2Y_2 > x) &= \mathbf{P}\big(X_1Y_1 + X_2Y_2 > x,\ \{Y_1 > a(x)\} \cup \{Y_2 > a(x)\}\big) \\ &\quad + \mathbf{P}\big(X_1Y_1 + X_2Y_2 > x,\ Y_2 \leqslant Y_1 \leqslant a(x)\big) \\ &\quad + \mathbf{P}\big(X_1Y_1 + X_2Y_2 > x,\ Y_1 < Y_2 \leqslant a(x)\big) \\ &=: K_1 + K_2 + K_3. \end{aligned} \tag{11}$$

Clearly, $K_1 \leqslant 2\overline{G}(a(x))$. For the second term, using Lemma 1, we obtain

$$\begin{aligned} K_2 &= \mathbf{P}\big(X_1Y_1 > x,\ 0 < Y_1 \leqslant a(x)\big)\mathbf{P}(Y_2 = 0) \\ &\quad + \int\limits_{(0,a(x)]} \bigg(\int\limits_{[y_2,a(x)]} \mathbf{P}(X_1y_1 + X_2y_2 > x)\,\mathrm{d}G(y_1)\bigg)\mathrm{d}G(y_2) \\ &\leqslant \mathbf{P}(X_1Y_1 > x) + \bigg[\sup_{x>0}\ \sup_{\{y_1,y_2:\,0<y_2\leqslant y_1\}} \frac{\mathbf{P}(X_1y_1 + X_2y_2 > x)}{\mathbf{P}(X_1y_1 > x) + \mathbf{P}(X_2y_2 > x)}\bigg] \\ &\quad \times \int\limits_{(0,a(x)]} \bigg(\int\limits_{[y_2,a(x)]} \big(\mathbf{P}(X_1y_1 > x) + \mathbf{P}(X_2y_2 > x)\big)\,\mathrm{d}G(y_1)\bigg)\mathrm{d}G(y_2) \\ &= \mathbf{P}(X_1Y_1 > x) + C\big(\mathbf{P}\big(X_1Y_1 > x,\ 0 < Y_2 \leqslant Y_1 \leqslant a(x)\big) \\ &\quad + \mathbf{P}\big(X_2Y_2 > x,\ 0 < Y_2 \leqslant Y_1 \leqslant a(x)\big)\big) \\ &\leqslant (1 + 2C)\,\mathbf{P}(XY > x) \end{aligned}$$

with finite positive constant C. By symmetry, $K_3 \leqslant (1 + 2C)\mathbf{P}(XY > x)$.

Substituting the obtained estimates for K_1, K_2, K_3 to (11), we get

$$\mathbf{P}(X_1Y_1 + X_2Y_2 > x) \leqslant 2\overline{G}\big(a(x)\big) + 2(1 + 2C)\mathbf{P}(XY > x) \tag{12}$$

for $x > 0$. Now applying property (iii) of function $a(x)$, we get from (12) that

$$\limsup_{x\to\infty} \frac{\mathbf{P}(X_1Y_1 + X_2Y_2 > x)}{\mathbf{P}(XY > x)} < \infty,$$

i.e., $F \otimes G \in \mathcal{OS}$. Theorem is proved. □

Proof of Theorem 4. According to Theorem 3 and Lemma 2, it holds that $F \otimes G \in \mathcal{OS}$. It suffices to prove that $F \otimes G \in \mathcal{PD}$. Since $F \in \mathcal{PD}$, it holds that

$$\limsup_{x\to\infty} \frac{\overline{F}(y^* x)}{\overline{F}(x)} < 1$$

for some $y^* > 1$. For this constant y^* and function $a(x)$ defined in (i)–(iii) of Theorem 2, along the lines of considerations on p. 238 of [36], we get

$$\begin{aligned}
\limsup_{x\to\infty} \frac{\overline{F\otimes G}(y^* x)}{\overline{F\otimes G}(x)} &= \limsup_{x\to\infty} \frac{(\int_{(0,a(x)]} + \int_{(a(x),\infty)})\overline{F}(y^* x/z)\,\mathrm{d}G(z)}{\overline{F\otimes G}(x)} \\
&\leqslant \limsup_{x\to\infty} \sup_{0<z\leqslant a(x)} \frac{\overline{F}(y^* x/z)}{\overline{F}(x/z)} + \limsup_{x\to\infty} \frac{\overline{G}(a(x))}{\overline{F\otimes G}(x)} \\
&= \limsup_{x\to\infty} \frac{\overline{F}(y^* x)}{\overline{F}(x)} + \limsup_{x\to\infty} \frac{\overline{G}(a(x))}{\overline{F\otimes G}(x)}.
\end{aligned}$$

This estimate and conditions of the theorem imply that

$$\limsup_{x\to\infty} \frac{\overline{F\otimes G}(y^* x)}{\overline{F\otimes G}(x)} < 1,$$

i.e., $F \otimes G \in \mathcal{PD}$. Theorem is proved. □

4 Corollaries and examples

In this section, we present some corollaries and examples to illustrate the main results of the paper. In the corollaries below, we provide several simple criteria for the validity of conditions of Theorems 3 and 4.

Corollary 1. *If $F \in \mathcal{OS}$, G is d.f. of nonnegative, nondegenerate at zero, and bounded r.v., then $F \otimes G \in \mathcal{OS}$.*

Example 1. Using the distribution in (2) as a benchmark, one can construct (light-tailed) distributions from $\mathcal{OS}$ as in [14]. For example, distribution F with the tail

$$\overline{F}(x) = \mathbf{1}_{(-\infty,0)}(x) + \frac{\mathrm{e}^{-x}}{(1+x)^3}\left(1 + \frac{\sin x}{a}\right)\mathbf{1}_{[0,\infty)}(x),$$

where $a > 2$, is in $\mathcal{OS}$ and, moreover, is light-tailed; see Example 2 in [14]. Further, by Corollary 1, any product-convolution of F with G having bounded support is still in $\mathcal{OS}$. In this way, one construct a rich enough class of distributions in $\mathcal{OS}$. If, for instance, G is discrete uniform distribution at the points $1, \dots, N$ with masses $1/N$, then

$$\overline{F\otimes G}(x) = \mathbf{1}_{(-\infty,0)}(x) + \frac{1}{N}\sum_{k=1}^{N} \frac{\mathrm{e}^{-x/k}}{(1+x/k)^3}\left(1 + \frac{\sin(x/k)}{a}\right)\mathbf{1}_{[0,\infty)}(x)$$

and $F \otimes G \in \mathcal{OS}$.

Assume now that G has unbounded support. The next corollary is an O-version of Corollary 2.1 in [36].

Corollary 2. *Suppose F is distribution on $\mathbb{R}$, $G(0-) = 0$ and G has unbounded support.*

(i) *If $F \in \mathcal{OS}$ ($F \in \mathcal{OA}$) and $\overline{G}(y^*x) = o(\overline{F}(x))$ for some $y^* > 0$, then $F \otimes G \in \mathcal{OS}$ ($F \otimes G \in \mathcal{OA}$, respectively).*

(ii) *If $F \in \mathcal{OS}$ ($F \in \mathcal{OA}$) and $\overline{G}(y^*x) = o(\overline{G}(x))$ for some $y^* > 1$, then $F \otimes G \in \mathcal{OS}$ ($F \otimes G \in \mathcal{OA}$, respectively).*

(iii) *If $F \in \mathcal{OS}$ ($F \in \mathcal{OA}$), $J_G^- > 0$ and $\int_{[0,\infty)} x^p \, \mathrm{d}F(x) = \infty$ for some $p \in (0, J_G^-)$, then $F \otimes G \in \mathcal{OS}$ ($F \otimes G \in \mathcal{OA}$, respectively).*

Proof. (i) For any $y > 0$, $x > 0$ and for given $y^* > 0$, we have

$$\overline{F \otimes G}(x) \geqslant \overline{G}\left(\frac{y^*}{y}\right)\overline{F}\left(\frac{yx}{y^*}\right).$$

Hence,

$$\limsup_{x\to\infty} \frac{\overline{G}(yx)}{\overline{F \otimes G}(x)} \leqslant \frac{1}{\overline{G}(y/y^*)} \limsup_{x\to\infty} \frac{\overline{G}(yx)}{\overline{F}(yx/y^*)} = 0,$$

and the assertion follows from Theorems 3 and 4, respectively.

(ii) For any $y > 0$, $x > 0$, we have $\overline{F \otimes G}(x) \geqslant \overline{F}(y^*/y)\overline{G}(yx/y^*)$. Therefore, the statement follows from

$$\limsup_{x\to\infty} \frac{\overline{G}(yx)}{\overline{F \otimes G}(x)} \leqslant \frac{1}{\overline{F}(y^*/y)} \limsup_{x\to\infty} \frac{\overline{G}(yx)}{\overline{G}(yx/y^*)} = 0$$

and Theorems 3, 4.

(iii) follows from Corollary 2.1(3) in [36]. □

Example 2. Let F be the generalized Peter and Paul distribution $\mathcal{PP}_{\{a,b\}}$ with parameters $b > 1$, $a > 0$:

$$\begin{aligned}\overline{F}(x) &= \mathbf{1}_{(-\infty,0)}(x) + \left(b^a - 1\right)\mathbf{1}_{[0,\infty)}(x) \sum_{k\in\mathbb{N}:\, b^k > x} b^{-ka} \\ &= \mathbf{1}_{(-\infty,b)}(x) + b^{-a\lfloor\log_b x\rfloor}\mathbf{1}_{[b,\infty)}(x).\end{aligned}$$

See, for instance, [7, 15, 21, 27].

It is easy to check that $F \in \mathcal{OA}$. Indeed, $F \in \mathcal{D}$ with $J_F^+ = J_F^- = a$. Also, for $x > b$,

$$\frac{\overline{F}(yx)}{\overline{F}(x)} = \frac{b^{a\lfloor\log_b x\rfloor}}{b^{a\lfloor\log_b x + \log_b y\rfloor}} \leqslant \left(\frac{b}{y}\right)^a < 1 \quad \text{if } y > b > 1.$$

Hence, $F \in \mathcal{D} \cap \mathcal{PD} \subset \mathcal{OA}$. As $\overline{F}(x) \geqslant x^{-a}$ for $x > b$, by Corollary 2(i), $F \otimes G \in \mathcal{OA}$ for any d.f. G satisfying $\overline{G}(x) = o(x^{-a})$.

In the corollaries and examples above, condition (4) was implied by

$$\limsup_{x\to\infty}\frac{\overline{G}(yx)}{\overline{F\otimes G}(x)}=0\quad\text{for any } y>0.$$

Next corollary considers the case where

$$\limsup_{x\to\infty}\frac{\overline{G}(yx)}{\overline{F\otimes G}(x)}\leqslant C<\infty$$

uniformly in $y>0$. Recall that d.f. G on $\mathbb{R}_+$ is *slowly varying*, denoted $G\in\mathcal{R}_0$, if $\lim_{x\to\infty}\overline{G}(yx)/\overline{G}(x)=1$ for any $y>0$.

Corollary 3. *If $F\in\mathcal{OS}$ and $G\in\mathcal{R}_0$, then $F\otimes G\in\mathcal{OS}$.*

Proof. Take $\kappa>0$ such that $\overline{F}(\kappa)>0$. For this κ and for any $x>0$, $y>0$, we have

$$\frac{\overline{G}(yx)}{\overline{F\otimes G}(x)}\leqslant\frac{\overline{G}\big(y\kappa(x/\kappa)\big)}{\overline{G}(x/\kappa)\overline{F}(\kappa)}.$$

Thus, as $G\in\mathcal{R}_0$, we obtain that for any $y>0$,

$$\limsup_{x\to\infty}\frac{\overline{G}(yx)}{\overline{F\otimes G}(x)}\leqslant\frac{1}{\overline{F}(\kappa)},$$

and the assertion of the corollary follows from Theorem 3. □

Example 3. Let us consider d.f. F with tail satisfying asymptotic relation (2) It is well known (see, e.g., [12]) that such d.f. F belongs to $\mathcal{OS}$ (and $\mathcal{PD}$). Thus, by Corollary 3, $F\otimes G\in\mathcal{OS}$ for any slowly varying d.f. G. Take, for example,

$$\begin{aligned}\overline{F}(x)&=\mathbf{1}_{(-\infty,1)}(x)+\mathrm{e}^{1-x}x^{-2}\mathbf{1}_{[1,\infty)}(x),\\ \overline{G}(x)&=\mathbf{1}_{(-\infty,1)}(x)+\frac{1}{1+\lfloor\log x\rfloor}\mathbf{1}_{[1,\infty)}(x).\end{aligned}$$

Distribution G has mass points e^k with $G(\{\mathrm{e}^k\}):=\overline{G}(\mathrm{e}^{k-1})-\overline{G}(\mathrm{e}^k)=1/(k(k+1))$, $k=1,2,\dots$. Thus,

$$\begin{aligned}\overline{F\otimes G}(x)&=\sum_{k=1}^{\infty}\overline{F}\left(\frac{x}{\mathrm{e}^k}\right)G\big(\{\mathrm{e}^k\}\big)\\ &=\mathbf{1}_{(-\infty,\mathrm{e})}(x)+\left(\frac{1}{x^2}\sum_{k=1}^{\lfloor\log x\rfloor}\frac{\mathrm{e}^{2k+1-x\mathrm{e}^{-k}}}{k(k+1)}+\frac{1}{1+\lfloor\log x\rfloor}\right)\mathbf{1}_{[\mathrm{e},\infty)}(x)\end{aligned}$$

and $F\otimes G\in\mathcal{OS}$.

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