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Finite-time and fixed-time sliding mode control for second-order nonlinear multiagent systems with external disturbances*

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Abstract. In this paper, the leader-following consensus of second-order nonlinear multiagent systems (SONMASs) with external disturbances is studied. Firstly, based on terminal sliding model control method, a distributed control protocol is proposed over undirected networks, which can not only suppress the external disturbances, but also make the SONMASs achieve consensus in finite time. Secondly, to make the settling time independent of the initial values of systems, we improve the protocol and ensure that the SONMASs can reach the sliding surface and achieve consensus in fixed time if the control parameters satisfy some conditions. Moreover, for general directed networks, we design a new fixed-time control protocol and prove that both the sliding mode surface and consensus for SONMASs can be reached in fixed time. Finally, several numerical simulations are given to show the effectiveness of the proposed protocols.

Keywords: multiagent systems, sliding mode control, consensus, external disturbances, finite time, fixed time.

1 Introduction

In recent years, distributed cooperative control of multiagent systems (MASs) has attracted great attentions because its wide practical applications in engineering, social science, biological science, and other fields. A particularly challenging problem in these

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fields is commonly referred to consensus. The main task on consensus of MASs is to design some feasible controllers such that all agents can achieve an agreement by sharing information among their neighbors. The consensus of MASs has been studied by many researchers since it can be used in many potential fields, such as the coordination control of UAV [10, 20], formation control of vehicles [8, 18, 26], and so on.

Depending on the number of leaders, the existing consensus results on MASs can be roughly divided into three categories: leaderless consensus [1], single leader-following consensus [11], and containment control of multiple leaders [9]. These results in [1, 9, 11] did not consider the influence of disturbances in the systems. However, in many practical applications, agents of MASs may face various disturbance signals. Therefore, the consensus of MASs with disturbances has become a hot research topic in distributed cooperative control. There are many techniques to eliminate disturbances in MASs, such as disturbance observation [6], output regulation [5], and internal model principle [13,24]. Since the sliding mode technology can achieve fast convergence rate, sliding mode control method is widely used to the control systems with disturbances. With the development of distributed cooperative control for MASs, many sliding mode control protocols [17,21,30] were proposed to solve the consensus problems of MASs.

As we all know, the convergence rate is one of the key elements in consensus of MASs. In [17, 21, 27, 30], the asymptotical consensus was considered, that means the consensus can be achieved only when the time tended to infinity. However, in practical applications, all agents should reach consensus in a finite-time interval. Hence, the finite-time $\mathcal{L}_2$ leader-follower consensus was investigated for networked Euler–Lagrange systems in [12]. In [31], the authors studied the consensus problem of MASs by using terminal sliding mode. In [15], the finite-time containment control for SONMASs with undirected networks was considered. However, the estimation of settling time depends on initial conditions of MASs in finite time consensus. To overcome this shortcoming, some fixed-time consensus and synchronization problems were considered in [7, 16, 22, 25, 32].

The above works [7, 15, 22, 25, 31, 32] show that the finite-time and fixed-time consensus of SONMASs with external disturbances were rarely considered by using sliding mode control. Thus, it is very meaningful to develop some sliding mode control protocols to study the finite-time and fixed-time consensus of SONMASs with external disturbances. Inspired by this, we propose three terminal sliding mode control protocols to study the finite-time and fixed-time consensus of SONMASs with external disturbances on undirected and directed networks, respectively. Compared with the previous works, the main contributions of the paper are at least the following three points:

(i) Compared with [17, 21, 30], a finite-time terminal sliding mode control protocol is designed, which can not only suppress the external disturbances, but also make the SONMASs achieve consensus in finite time.

(ii) In [15], although the sliding mode control protocols were proposed, the estimation of settling time depends on initial values of MASs. In this paper, we overcome this disadvantage and propose a fixed-time distributed protocol, which can suppress the external disturbances and make the SONMASs achieve consensus in fixed time.

(iii) For SONMASs with external disturbances, a novel sliding mode control protocol is carefully designed over general directed networks. Based on fixed-time stability theory, we also rigorously prove that the external disturbances can be suppressed, and the leader-following consensus can be reached in fixed time.

The remainder of this paper is organized as follows. Section 2 introduces some preliminaries including graph theory, definitions, lemmas and problem formulation. In Section 3, three consensus control protocols based on sliding mode technique are proposed, and relevant theorems are given. In Section 4, the effectiveness of the proposed control protocols is verified by numerical simulations. Finally, some conclusions are given in Section 5.

Notations. In this paper, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. I_n denotes n dimensional unit matrix. For $q = [q_1, q_2, \dots, q_N]^{\mathrm{T}}$, $\|q\|_1$, $\|q\|$, and $\|q\|_\infty$ represent the 1-norm, 2-norm, and ∞-norm of vector q, respectively. $|q|^\alpha = [|q_1|^\alpha, \dots, |q_N|^\alpha]^{\mathrm{T}}$, $\operatorname{sgn}(q) = [\operatorname{sgn}(q_1), \dots, \operatorname{sgn}(q_N)]^{\mathrm{T}}$, $\operatorname{sig}^\alpha(q) = [|q_1|^\alpha \operatorname{sgn}(q_1), \dots, |q_N|^\alpha \operatorname{sgn}(q_N)]^{\mathrm{T}}$, where $\alpha > 0$. For a matrix $A \in \mathbb{R}^{N\times N}$, let A^{T} represents its transpose, A^{-1} represents its inverse, $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ represent the maximum eigenvalue and minimum eigenvalue of A, respectively. $\mathbf{1}_N$ is the column vector with all elements of 1. The symbol $\otimes$ denotes the Kronecker product of matrices. $\operatorname{sgn}(\cdot)$ represents the sign function. $\operatorname{diag}(\cdot)$ represents the diagonal matrix.

2 Preliminaries

2.1 Algebraic graph theory

A MAS consisting of N agents is represented by a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ where $\mathcal{V} = \{v_1, \dots, v_N\}$ is a set of nodes, $\mathcal{E}$ denotes a set of edges in which $(i, j) \in \mathcal{E}$ if there is an edge between v_i and v_j. The weighted adjacency matrix is denoted as $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$, where $a_{ij} > 0$ if $(j, i) \in \mathcal{E}$, and $a_{ij} = 0$ otherwise. The set of neighbors of agent i is denoted by $N_i = \{j \in \mathcal{V}\colon (j, i) \in \mathcal{E}\}$. $\mathcal{G}$ is an undirected graph if $a_{ij} = a_{ji}$. The graph $\mathcal{G}$ is called directed and strongly connected if there is a directed path between each pair of nodes. The graph $\mathcal{G}$ contains a directed spanning tree if there is at least one root. A node is called globally reachable if it is a root. The Laplacian matrix $L = [l_{ij}]_{N\times N}$ is defined by $l_{ij} = -a_{ij}$ for $i \neq j$, and $l_{ii} = \sum_{j\neq i}^{N} a_{ij}$. For a leader-following MAS, $b_i > 0$ if the ith agent can receive information from the leader, $b_i = 0$ otherwise. Let $B = \operatorname{diag}(b_1, \dots, b_N)$. The matrix $L + B$ is invertible if the network topology $\mathcal{G}$ has a directed spanning tree.

2.2 Definitions and lemmas

Consider the following differential equation:

$$\dot{x}(t) = f\bigl(x(t)\bigr), \qquad x(0) = x_0, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ denotes the state variable, and $f(\cdot) : \mathbb{R}^n \mapsto \mathbb{R}^n$ is a nonlinear function. Assume that the origin is the equilibrium point of system (1).

The following definitions and lemmas are given.

Definition 1. (See [19].) The origin of system (1) is said to be globally finite-time stable if for any solution $x(t)$ with $x_0 \in \mathbb{R}^n$, there is a positive number $T(x_0)$ such that $x(t) = 0$ for all $t \geqslant T(x_0)$. The positive number $T(x_0)$ is called the settling time. Moreover, if the positive number $T(x_0)$ independent of initial value x_0, the origin is said to be globally fixed-time stable.

Lemma 1. (*See* [2].) *Let $V(x(t)) : \mathbb{R}^n \to \mathbb{R}$ be a continuous, positive definite, and radially unbounded function, and $x(t) : [0, +\infty) \to \mathbb{R}^n$ is absolutely continuous on any compact interval. If there are scalars $\gamma > 0$ and $0 < k < 1$ such that $\dot{V}(x(t)) \leqslant -\gamma V(x(t))^k$, then $V(x(t)) \equiv 0$ for all $t \geqslant T$. Furthermore, the settling time can be estimated by $T = V^{1-k}(0)/\gamma(1-k)$.*

Lemma 2. (*See* [29].) *Consider the differential equation* (1). *If there is a continuous, positive definite, and radially unbounded function $V(x(t)) : \mathbb{R}^n \to \mathbb{R}$ such that any solution of* (1) *satisfies the inequality*

$$\dot{V}\big(x(t)\big) \leqslant -\big(\tau V^p\big(x(t)\big) + \phi V^q\big(x(t)\big)\big)^{\varrho}, \quad x(t) \in \mathbb{R}^n \setminus \{0\},$$

where $\tau, \phi, p, \varrho > 0$, $q \geqslant 0$, $p\varrho > 1$, $q\varrho < 1$, then the origin of system (1) *is fixed-time stable, and the settling time $T(x_0)$ is estimated by*

$$T(x_0) \leqslant \frac{1}{\phi^{\varrho}} \left(\frac{\phi}{\tau}\right)^{(1-q\varrho)/(p-q)} \left(\frac{1}{1-q\varrho} + \frac{1}{p\varrho - 1}\right).$$

Lemma 3. (*See* [29].) *Let $\xi_1, \xi_2, \ldots, \xi_N \geqslant 0$ be nonnegative numbers. Then*

$$\sum_{i=1}^{N} \xi_i^r \geqslant \left(\sum_{i=1}^{N} \xi_i\right)^r, \quad 0 < r \leqslant 1,$$
$$\sum_{i=1}^{N} \xi_i^r \geqslant N^{1-r} \left(\sum_{i=1}^{N} \xi_i\right)^r, \quad 1 < r \leqslant \infty.$$

2.3 Problem formulation

Consider a SONMAS consisting of N followers and a leader. The dynamics of the ith ($i = 1, 2, \ldots, N$) follower is described by

$$\begin{aligned} \dot{x}_i(t) &= v_i(t), \\ \dot{v}_i(t) &= f\big(x_i(t), v_i(t)\big) + u_i(t) + w_i(t), \end{aligned} \tag{2}$$

where $x_i(t) \in \mathbb{R}^n$ and $v_i(t) \in \mathbb{R}^n$ denote the position and velocity, respectively. $w_i(t) \in \mathbb{R}^n$ is the external disturbance, $u_i(t) \in \mathbb{R}^n$ is the control input, and $f(x_i(t), v_i(t))$: $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}^n$ is a nonlinear function, which represents the inherent dynamics. In addition, we assume that the external disturbance is bounded, which satisfies $\|w_i(t)\|_\infty \leqslant b < \infty$ for $b > 0$.

The dynamics of the leader is described by

$$\begin{aligned} \dot{x}_0(t) &= v_0(t), \\ \dot{v}_0(t) &= f\big(x_0(t), v_0(t)\big) + w_0(t), \end{aligned} \tag{3}$$

where $x_0(t) \in \mathbb{R}^n$ and $v_0(t) \in \mathbb{R}^n$ are the position and velocity of the leader, respectively. $f(x_0(t), v_0(t))$ is a nonlinear function, which represents the inherent dynamics. For convenience, we write $f(x_0(t), v_0(t))$ as $f_0(t)$. In addition, we also assume that the external disturbance is bounded, which satisfies $\|w_0(t)\|_\infty \leqslant c < \infty$ for $c > 0$.

Definition 2. For the SONMASs (2)–(3), the fixed-time leader-following consensus is achieved for any initial conditions if there exists a positive constant $\mathcal{T}$ such that

$$\begin{aligned} \lim_{t\to\mathcal{T}} \big\|x_i(t) - x_0(t)\big\| &= 0, & \lim_{t\to\mathcal{T}} \big\|v_i(t) - v_0(t)\big\| &= 0, \\ \big\|x_i(t) - x_0(t)\big\| &\equiv 0, & \big\|v_i(t) - v_0(t)\big\| &\equiv 0, \end{aligned}$$

for all $t \geqslant \mathcal{T}$, $i = 1, 2, \ldots, N$, where $\mathcal{T} > 0$ is called the settling time.

Before moving on, the following assumptions are presented.

Assumption 1. For the nonlinear function $f(\cdot)$, there are two nonnegative constants l_1 and l_2 such that

$$\big\|f\big(x(t), z(t)\big) - f\big(y(t), \varsigma(t)\big)\big\| \leqslant l_1\big\|x(t) - y(t)\big\| + l_2\big\|z(t) - \varsigma(t)\big\|, \tag{4}$$

where $x(t), y(t), z(t), \varsigma(t) \in \mathbb{R}^n$.

Remark 1. Assumption 1 is very mild and usually used in SONMASs [17,21,28], which is so-called QUAD condition on vector fields [3]. This condition can be satisfied for all linear and piecewise-linear continuous functions. In addition, the condition is also satisfied if $\partial f_i/\partial x_j$ $(i, j = 1, 2, \ldots, n)$ are uniformly bounded, which includes many well-known systems.

Assumption 2. The leader is globally reachable and the communication topology of followers is undirected.

Assumption 3. The leader is globally reachable and the communication topology of followers is directed.

3 Main results

In this section, three different control protocols are designed for SONMASs described by (2) and (3). To achieve consensus between leader and followers, we need to design the following error variables.

Define the consensus errors as follows:

$$\begin{aligned} e_{x_i}(t) &= \sum_{j=1}^{N} a_{ij}\big(x_i(t) - x_j(t)\big) + b_i\big(x_i(t) - x_0(t)\big), \quad i = 1, 2, \ldots, N, \\ e_{v_i}(t) &= \sum_{j=1}^{N} a_{ij}\big(v_i(t) - v_j(t)\big) + b_i\big(v_i(t) - v_0(t)\big), \quad i = 1, 2, \ldots, N. \end{aligned} \tag{5}$$

Let $e_x(t) = [e_{x_1}^{\mathrm{T}}(t), e_{x_2}^{\mathrm{T}}(t), \ldots, e_{x_N}^{\mathrm{T}}(t)]^{\mathrm{T}}$, $e_v(t) = [e_{v_1}^{\mathrm{T}}(t), e_{v_2}^{\mathrm{T}}(t), \ldots, e_{v_N}^{\mathrm{T}}(t)]^{\mathrm{T}}$, then (5) can be rewritten as

$$e_x(t) = \big((L + B) \otimes I_n\big)\tilde{x}(t), \qquad e_v(t) = \big((L + B) \otimes I_n\big)\tilde{v}(t), \tag{6}$$

where $\tilde{x}(t) = x(t) - \mathbf{1}_N \otimes x_0(t)$, $\tilde{v}(t) = v(t) - \mathbf{1}_N \otimes v_0(t)$, $x(t) = [x_1^{\mathrm{T}}(t), x_2^{\mathrm{T}}(t), \ldots, x_N^{\mathrm{T}}(t)]^{\mathrm{T}}$, and $v(t) = [v_1^{\mathrm{T}}(t), v_2^{\mathrm{T}}(t), \ldots, v_N^{\mathrm{T}}(t)]^{\mathrm{T}}$.

Combining (2), (3), and (6), it yields

$$\begin{aligned} \dot{e}_x(t) &= e_v(t), \\ \dot{e}_v(t) &= \big((L + B) \otimes I_n\big)\big[F\big(x(t), v(t)\big) - \big(\mathbf{1}_N \otimes f_0(t)\big) \\ &\quad + u(t) + w(t) - \big(\mathbf{1}_N \otimes w_0(t)\big)\big], \end{aligned} \tag{7}$$

where $F(x(t), v(t)) = [f^{\mathrm{T}}(x_1(t), v_1(t)), f^{\mathrm{T}}(x_2(t), v_2(t)), \ldots, f^{\mathrm{T}}(x_N(t), v_N(t))]^{\mathrm{T}}$, $w(t) = [w_1^{\mathrm{T}}(t), w_2^{\mathrm{T}}(t), \ldots, w_N^{\mathrm{T}}(t)]^{\mathrm{T}}$, and $u(t) = [u_1^{\mathrm{T}}(t), u_2^{\mathrm{T}}(t), \ldots, u_N^{\mathrm{T}}(t)]^{\mathrm{T}}$.

3.1 Finite-time consensus with undirected networks

We choose the sliding mode manifold as follows:

$$\sigma_i(t) = e_{v_i}(t) + \operatorname{sig}^{\alpha}\big(e_{x_i}(t)\big) + \operatorname{sig}^{\beta}\big(e_{x_i}(t)\big), \quad i = 1, 2, \ldots, N, \tag{8}$$

where $0 < \alpha < 1$ and $\beta > 1$. The sliding mode manifold (8) can be rewritten in the following comport form:

$$\sigma(t) = e_v(t) + \operatorname{sig}^{\alpha}\big(e_x(t)\big) + \operatorname{sig}^{\beta}\big(e_x(t)\big),$$

where $\sigma(t) = [\sigma_1^{\mathrm{T}}(t), \ldots, \sigma_N^{\mathrm{T}}(t)]^{\mathrm{T}}$, $\operatorname{sig}^{\alpha}(e_x(t)) = [(\operatorname{sig}^{\alpha}(e_{x_1}(t)))^{\mathrm{T}}, (\operatorname{sig}^{\alpha}(e_{x_2}(t)))^{\mathrm{T}}, \ldots, (\operatorname{sig}^{\alpha}(e_{x_N}(t)))^{\mathrm{T}}]^{\mathrm{T}}$.

In order to guarantee the closed-loop system reaching the sliding mode manifold in finite time, the following control protocol is proposed for the ith agent:

$$\begin{aligned} u_i(t) = {} & -\big(k_1\big\|e_x(t)\big\| + k_2\big\|e_v(t)\big\| + k_3 + k_4\big\|\operatorname{diag}\big(e_v(t)\big)\big|e_x(t)\big|^{\alpha-1}\big\| \\ & + k_5\big\|\operatorname{diag}\big(e_v(t)\big)\big|e_x(t)\big|^{\beta-1}\big\|\big)\operatorname{sgn}\big(\sigma_i(t)\big), \quad i = 1, 2, \ldots, N, \end{aligned} \tag{9}$$

where k_1, k_2, k_3, k_4, and k_5 are positive constants to be determined.

Similarly, controller (9) can be rewritten in the following comport form:

$$u(t) = -\big(k_1\|e_x(t)\| + k_2\|e_v(t)\| + k_3 + k_4\|\mathrm{diag}\big(e_v(t)\big)\big|e_x(t)\big|^{\alpha-1}\| \\ + k_5\|\mathrm{diag}\big(e_v(t)\big)\big|e_x(t)\big|^{\beta-1}\|\big)\,\mathrm{sgn}(\sigma(t)). \tag{10}$$

According to (7) and (10), we obtain

$$\begin{aligned} \dot e_x(t) &= e_v(t),\\ \dot e_v(t) &= \big((L+B)\otimes I_n\big)\big[F\big(x(t),v(t)\big) - \big(\mathbf{1}_N\otimes f_0(t)\big) + w(t) - \big(\mathbf{1}_N\otimes w_0(t)\big)\\ &\quad - \big(k_1\|e_x(t)\| + k_2\|e_v(t)\| + k_3 + k_4\|\mathrm{diag}\big(e_v(t)\big)\big|e_x(t)\big|^{\alpha-1}\|\\ &\quad + k_5\|\mathrm{diag}\big(e_v(t)\big)\big|e_x(t)\big|^{\beta-1}\|\big)\,\mathrm{sgn}\big(\sigma(t)\big)\big]. \end{aligned} \tag{11}$$

Theorem 1. *Suppose that Assumptions* 1 *and* 2 *hold. The finite-time leader-following consensus can be achieved for SONMASs* (2)–(3) *under the protocol* (10) *if the following inequalities are satisfied*:

$$\begin{aligned} &k_1 \geqslant l_1\lambda_{\max}\big((L+B)^{-1}\big), \quad k_2 \geqslant l_2\lambda_{\max}\big((L+B)^{-1}\big), \quad k_3 > b+c,\\ &k_4 \geqslant \alpha\lambda_{\max}\big((L+B)^{-1}\big), \quad k_5 \geqslant \beta\lambda_{\max}\big((L+B)^{-1}\big). \end{aligned} \tag{12}$$

Proof. Consider the following Lyapunov function candidate:

$$V(t) = \frac{1}{2}\sigma^{\mathrm T}(t)\big((L+B)^{-1}\otimes I_n\big)\sigma(t).$$

The time derivative of $V(t)$ along system (11) can be calculated as

$$\begin{aligned} \dot V(t) = {}& \sigma^{\mathrm T}(t)\big[F\big(x(t),v(t)\big) - F\big(x_0(t),v_0(t)\big) + w(t) - \big(\mathbf{1}_N\otimes w_0(t)\big)\\ & - \big(k_1\|e_x(t)\| + k_3 + k_2\|e_v(t)\| + k_4\|\mathrm{diag}\big(e_v(t)\big)\big|e_x(t)\big|^{\alpha-1}\|\\ & + k_5\|\mathrm{diag}\big(e_v(t)\big)\big|e_x(t)\big|^{\beta-1}\|\big)\,\mathrm{sgn}\big(\sigma(t)\big)\big]\\ & + \alpha\sigma^{\mathrm T}(t)\big((L+B)^{-1}\otimes I_n\big)\,\mathrm{diag}\big(e_v(t)\big)\big|e_x(t)\big|^{\alpha-1}\\ & + \beta\sigma^{\mathrm T}(t)\big((L+B)^{-1}\otimes I_n\big)\,\mathrm{diag}\big(e_v(t)\big)\big|e_x(t)\big|^{\beta-1}, \end{aligned} \tag{13}$$

where $F(x_0(t),v_0(t)) = \mathbf{1}_N\otimes f_0(t)$.

Combining (4) and (6), it is easy to verify that

$$\begin{aligned} &\sigma^{\mathrm T}\big[F\big(x(t),y(t)\big) - F\big(x_0(t),v_0(t)\big)\big]\\ &\quad\leqslant l_1\lambda_{\max}\big((L+B)^{-1}\big)\big\|e_x(t)\big\|\big\|\sigma(t)\big\| + l_2\lambda_{\max}\big((L+B)^{-1}\big)\big\|e_v(t)\big\|\big\|\sigma(t)\big\|. \end{aligned}$$

Furthermore, we have

$$\big\|\sigma^{\mathrm T}(t)\big(w(t) - \big(\mathbf{1}_N\otimes w_0(t)\big)\big)\big\|_\infty \leqslant \big\|\sigma(t)\big\|_1(b+c).$$

Based on (12), the following inequalities hold:

$$
\begin{aligned}
&-k_3\sigma^{\mathrm{T}}(t)\operatorname{sgn}\bigl(\sigma(t)\bigr)+\sigma^{\mathrm{T}}(t)\bigl(w(t)-\bigl(\mathbf{1}_N\otimes w_0(t)\bigr)\bigr)\\
&\quad\leqslant -k_3\bigl\|\sigma(t)\bigr\|_1+\bigl\|\sigma(t)\bigr\|_1\bigl\|w(t)-\bigl(\mathbf{1}_N\otimes w_0(t)\bigr)\bigr\|_\infty\\
&\quad\leqslant -\bigl(k_3-(b+c)\bigr)\bigl\|\sigma(t)\bigr\|_1\leqslant 0,\\
&\alpha\sigma^{\mathrm{T}}(t)\bigl((L+B)^{-1}\otimes I_n\bigr)\operatorname{diag}\bigl(e_v(t)\bigr)\bigl|e_x(t)\bigr|^{\alpha-1}\\
&\quad\leqslant \alpha\lambda_{\max}\bigl((L+B)^{-1}\bigr)\bigl\|\sigma(t)\bigr\|\bigl\|\operatorname{diag}\bigl(e_v(t)\bigr)\bigl|e_x(t)\bigr|^{\alpha-1}\bigr\|,\\
&-\sigma^{\mathrm{T}}(t)k_4\bigl\|\operatorname{diag}\bigl(e_v(t)\bigr)\bigl|e_x(t)\bigr|^{\alpha-1}\bigr\|\operatorname{sgn}\bigl(\sigma(t)\bigr)\\
&\qquad+\alpha\sigma^{\mathrm{T}}(t)\lambda_{\max}\bigl((L+B)^{-1}\bigr)\operatorname{diag}\bigl(e_v(t)\bigr)\bigl|e_x(t)\bigr|^{\alpha-1}\\
&\quad\leqslant -k_4\bigl\|\sigma(t)\bigr\|_1\bigl\|\operatorname{diag}\bigl(e_v(t)\bigr)\bigl|e_x(t)\bigr|^{\alpha-1}\bigr\|\\
&\qquad+\alpha\lambda_{\max}\bigl((L+B)^{-1}\bigr)\bigl\|\sigma(t)\bigr\|_1\bigl\|\operatorname{diag}\bigl(e_v(t)\bigr)\bigl|e_x(t)\bigr|^{\alpha-1}\bigr\|,\\
&\beta\sigma^{\mathrm{T}}(t)\bigl((L+B)^{-1}\otimes I_n\bigr)\operatorname{diag}\bigl(e_v(t)\bigr)\bigl|e_x(t)\bigr|^{\beta-1}\\
&\quad\leqslant \beta\lambda_{\max}\bigl((L+B)^{-1}\bigr)\bigl\|\sigma(t)\bigr\|\bigl\|\operatorname{diag}\bigl(e_v(t)\bigr)\bigl|e_x(t)\bigr|^{\beta-1}\bigr\|,\\
&-\sigma^{\mathrm{T}}(t)k_5\bigl\|\operatorname{diag}\bigl(e_v(t)\bigr)\bigl|e_x(t)\bigr|^{\beta-1}\bigr\|\operatorname{sgn}\bigl(\sigma(t)\bigr)\\
&\qquad+\beta\sigma^{\mathrm{T}}(t)\lambda_{\max}\bigl((L+B)^{-1}\bigr)\operatorname{diag}\bigl(e_v(t)\bigr)\bigl|e_x(t)\bigr|^{\beta-1}\\
&\quad\leqslant -k_5\bigl\|\sigma(t)\bigr\|_1\bigl\|\operatorname{diag}\bigl(e_v(t)\bigr)\bigl|e_x(t)\bigr|^{\beta-1}\bigr\|\\
&\qquad+\beta\lambda_{\max}\bigl((L+B)^{-1}\bigr)\bigl\|\sigma(t)\bigr\|_1\bigl\|\operatorname{diag}\bigl(e_v(t)\bigr)\bigl|e_x(t)\bigr|^{\beta-1}\bigr\|.
\end{aligned}
$$

Therefore, equality (13) can be written as

$$
\begin{aligned}
\dot V(t)\leqslant{}& l_1\lambda_{\max}\bigl((L+B)^{-1}\bigr)\bigl\|e_x(t)\bigr\|\bigl\|\sigma(t)\bigr\|+l_2\lambda_{\max}\bigl((L+B)^{-1}\bigr)\bigl\|e_v(t)\bigr\|\bigl\|\sigma(t)\bigr\|\\
&-k_1\bigl\|\sigma(t)\bigr\|\bigl\|e_x(t)\bigr\|-k_2\bigl\|\sigma(t)\bigr\|\bigl\|e_v(t)\bigr\|-\bigl(k_3-(b+c)\bigr)\bigl\|\sigma(t)\bigr\|_1\\
&-k_4\bigl\|\sigma(t)\bigr\|_1\bigl\|\operatorname{diag}\bigl(e_v(t)\bigr)\bigl|e_x(t)\bigr|^{\alpha-1}\bigr\|+\alpha\lambda_{\max}\bigl((L+B)^{-1}\bigr)\bigl\|\sigma(t)\bigr\|_1\\
&\times\bigl\|\operatorname{diag}\bigl(e_v(t)\bigr)\bigl|e_x(t)\bigr|^{\alpha-1}\bigr\|-k_5\bigl\|\sigma(t)\bigr\|_1\bigl\|\operatorname{diag}\bigl(e_v(t)\bigr)\bigl|e_x(t)\bigr|^{\beta-1}\bigr\|\\
&+\beta\lambda_{\max}\bigl((L+B)^{-1}\bigr)\bigl\|\sigma(t)\bigr\|_1\bigl\|\operatorname{diag}\bigl(e_v(t)\bigr)\bigl|e_x(t)\bigr|^{\beta-1}\bigr\|\\
\leqslant{}&-\bigl(k_3-(b+c)\bigr)\bigl\|\sigma(t)\bigr\|_1\\
\leqslant{}&-\bigl(k_3-(b+c)\bigr)\sqrt{2\lambda_{\min}(L+B)}V^{1/2}(t).
\end{aligned}
$$

By Lemma 1, it can be concluded that the closed-loop system will reach the sliding mode surface $\sigma(t)=0$ in finite time. The settling time can be estimated as

$$
T_1=\frac{\sqrt{2}V^{1/2}(0)}{(k_3-(b+c))\sqrt{\lambda_{\min}(L+B)}},
$$

where $V(0)$ is the initial value of $V(t)$.

For $t \geqslant T_1$, the sliding mode surface $\sigma(t) = 0$, that is, $e_v(t) + \operatorname{sig}^{\alpha}(e_x(t)) + \operatorname{sig}^{\beta}(e_x(t)) = 0$, and we have $e_v(t) = -\operatorname{sig}^{\alpha}(e_x(t)) - \operatorname{sig}^{\beta}(e_x(t))$. Since $\dot{e}_x(t) = e_v(t)$, so $\dot{e}_x(t) = -\operatorname{sig}^{\alpha}(e_x(t)) - \operatorname{sig}^{\beta}(e_x(t))$.

Define the Lyapunov function $V_1(t) = {e_x}^{\mathrm{T}}(t)e_x(t)/2$. Then taking the time derivative of $V_1(t)$, we obtain

$$\begin{aligned}\dot{V}_1(t) &= e_x^{\mathrm{T}}(t)e_v(t) = -\sum_{i=1}^{N}\sum_{j=1}^{n}\big(\big(e_{x_{ij}}(t)\big)^2\big)^{(\alpha+1)/2} - \sum_{i=1}^{N}\sum_{j=1}^{n}\big(\big(e_{x_{ij}}(t)\big)^2\big)^{(\beta+1)/2} \\ &\leqslant -\big(\big\|e_x(t)\big\|^2\big)^{(\alpha+1)/2} - (Nn)^{(1-\beta)/2}\big(\big\|e_x(t)\big\|^2\big)^{(\beta+1)/2} \\ &= -\big(2V_1(t)\big)^{(\alpha+1)/2} - (Nn)^{(1-\beta)/2}\big(2V_1(t)\big)^{(\beta+1)/2}.\end{aligned}$$

By using Lemma 2, it is shown that $V_1(t)$ converges to 0 in fixed time, which means $e_x(t) \to 0$ and $e_v(t) \to 0$ in fixed time. The estimation of settling time is

$$T \leqslant T_1 + \frac{1}{2^{(\alpha+1)/2}}\left[\frac{1}{(Nn)^{(1-\beta)/2}2^{(\beta-\alpha)/2}}\right]^{(1-\alpha)/(\beta-\alpha)}\left(\frac{2}{1-\alpha} + \frac{2}{\beta-1}\right). \qquad \square$$

Remark 2. In existing works [17, 21, 30], although the sliding mode control technique was applied in the consensus of MASs, the main results focused on asymptotical convergence. In addition, similar to [14], the traditional variable structure control adopts linear sliding mode, and the tracking error will converge in infinite time. In order to overcome this shortcoming, a new terminal sliding mode control protocol, which both can suppress the external disturbances and make all agents achieve consensus in finite time, is proposed in this paper.

3.2 Fixed-time consensus with undirected networks

In this section, in order to solve the drawback that the settling time depends on initial value of the system, we improve the proposed protocol (10) and prove that both the sliding surface and the consensus of SONMASs can be achieved in fixed time.

The following control protocol is proposed:

$$\begin{aligned}u(t) = &-\big(k_1\big\|e_x(t)\big\| + k_2\big\|e_v(t)\big\| + k_3 + k_4\big\|\operatorname{diag}\big(e_v(t)\big)\big|e_x(t)\big|^{\alpha-1}\big\| \\ &+ k_5\big\|\operatorname{diag}\big(e_v(t)\big)\big|e_x(t)\big|^{\beta-1}\big\|\big)\operatorname{sgn}\big(\sigma(t)\big) - \theta\operatorname{sig}^h\big(\sigma(t)\big),\end{aligned} \tag{14}$$

where $0 < \alpha < 1$, $\beta > 1$, $h > 1$, θ, k_1, k_2, k_3, k_4, and k_5 are positive constants to be determined.

According to Eqs. (7) and (14), we have

$$\begin{aligned}\dot{e}_x(t) &= e_v(t), \\ \dot{e}_v(t) &= \big((L+B)\otimes I_n\big)\big[F\big(x(t), v(t)\big) - \big(\mathbf{1}_N\otimes f_0(t)\big) + w(t) - \big(\mathbf{1}_N\otimes w_0(t)\big) \\ &\quad - \big(k_1\big\|e_x(t)\big\| + k_2\big\|e_v(t)\big\| + k_3 + k_4\big\|\operatorname{diag}\big(e_v(t)\big)\big|e_x(t)\big|^{\alpha-1}\big\| \\ &\quad + k_5\big\|\operatorname{diag}\big(e_v(t)\big)\big|e_x(t)\big|^{\beta-1}\big\|\big)\operatorname{sgn}\big(\sigma(t)\big) - \theta\operatorname{sig}^h\big(\sigma(t)\big)\big].\end{aligned} \tag{15}$$

Theorem 2. *Suppose that Assumptions* 1 *and* 2 *hold. The fixed-time leader-following consensus can be achieved for SONMASs* (2)–(3) *under protocol* (14) *if the following inequalities are satisfied*:

$$\begin{aligned} &k_1 \geqslant l_1\lambda_{\max}\big((L+B)^{-1}\big), \quad k_2 \geqslant l_2\lambda_{\max}\big((L+B)^{-1}\big), \quad k_3 \geqslant b+c+1, \\ &k_4 \geqslant \alpha\lambda_{\max}\big((L+B)^{-1}\big), \quad k_5 \geqslant \beta\lambda_{\max}\big((L+B)^{-1}\big). \end{aligned} \tag{16}$$

Proof. Consider the following Lyapunov function candidate:

$$V(t) = \frac{1}{2}\sigma^{\mathrm{T}}(t)\big((L+B)^{-1}\otimes I_n\big)\sigma(t).$$

The time derivative of $V(t)$ along system (15) can be calculated as

$$\begin{aligned} \dot{V}(t) = {} & \sigma^{\mathrm{T}}(t)\big[F\big(x(t),v(t)\big) - F\big(x_0(t),v_0(t)\big) + w(t) - \big(\mathbf{1}_N\otimes w_0(t)\big) \\ & - \big(k_1\|e_x(t)\| + k_3 + k_2\|e_v(t)\| + k_4\big\|\mathrm{diag}\big(e_v(t)\big)\big|e_x(t)\big|^{\alpha-1}\big\| \\ & + k_5\big\|\mathrm{diag}\big(e_v(t)\big)\big|e_x(t)\big|^{\beta-1}\big\|\big)\,\mathrm{sgn}\big(\sigma(t)\big) - \theta\,\mathrm{sig}^h\big(\sigma(t)\big)\big] \\ & + \alpha\sigma^{\mathrm{T}}(t)\big((L+B)^{-1}\otimes I_n\big)\,\mathrm{diag}\big(e_v(t)\big)\big|e_x(t)\big|^{\alpha-1} \\ & + \beta\sigma^{\mathrm{T}}(t)\big((L+B)^{-1}\otimes I_n\big)\,\mathrm{diag}\big(e_v(t)\big)\big|e_x(t)\big|^{\beta-1}. \end{aligned} \tag{17}$$

Based on Lemma 3, we have

$$\begin{aligned} -\sum_{i=1}^{N}\sum_{j=1}^{n}|\sigma_{ij}(t)|^{h+1} &= -\sum_{i=1}^{N}\sum_{j=1}^{n}(|\sigma_{ij}(t)|^2)^{(h+1)/2} \\ &\leqslant -(Nn)^{(1-h)/2}\left(\sum_{i=1}^{N}\sum_{j=1}^{n}|\sigma_{ij}(t)|^2\right)^{(h+1)/2}. \end{aligned}$$

Similar to analysis of Theorem 1, combining with (16), inequality (17) can be written as

$$\begin{aligned} \dot{V}(t) &\leqslant -k_3\big\|\sigma(t)\big\|_1 - \theta(Nn)^{(1-h)/2}\left(\sum_{i=1}^{N}\sum_{j=1}^{n}\big|\sigma_{ij}(t)\big|^2\right)^{(h+1)/2} \\ &\leqslant -k_3\big(2\lambda_{\min}(L+B)V(t)\big)^{1/2} \\ &\quad - \theta(Nn)^{(1-h)/2}\big(2\lambda_{\min}(L+B)V(t)\big)^{(h+1)/2}. \end{aligned}$$

Using Lemma 2, we can conclude that the closed-loop system will reach the sliding mode surface $\sigma(t)=0$ in fixed time. Moreover, the settling time can be estimated as

$$T_2 \leqslant \frac{1}{k_3\sqrt{2}\sqrt{\lambda_{\min}(L+B)}}\left[\frac{k_3\sqrt{2}}{\theta(Nn)^{(1-h)/2}2^{(h+1)/2}\big(\lambda_{\min}(L+B)^{h/2}\big)}\right]^{1/h}\left(2+\frac{2}{h-1}\right).$$

Therefore, for $t \geqslant T_2$, we have $e_v(t) = -\operatorname{sig}^\alpha(e_x(t)) - \operatorname{sig}^\beta(e_x(t))$. Since $\dot{e}_x(t) = e_v(t)$, then we can obtain that $\dot{e}_x(t) = -\operatorname{sig}^\alpha(e_x(t)) - \operatorname{sig}^\beta(e_x(t))$.

Define the Lyapunov function $V_1(t) = e_x^{\mathrm{T}}(t)e_x(t)/2$. Then taking the time derivative of $V_1(t)$ yields

$$\begin{aligned}\dot{V}_1(t) &= e_x^{\mathrm{T}}(t)e_v(t)\\ &= -\sum_{i=1}^{N}\sum_{j=1}^{n}\bigl(\bigl(e_{x_{ij}}(t)\bigr)^2\bigr)^{(\alpha+1)/2} - \sum_{i=1}^{N}\sum_{j=1}^{n}\bigl(\bigl(e_{x_{ij}}(t)\bigr)^2\bigr)^{(\beta+1)/2}\\ &\leqslant -\bigl(2V_1(t)\bigr)^{(\alpha+1)/2} - (Nn)^{(1-\beta)/2}\bigl(2V_1(t)\bigr)^{(\beta+1)/2}.\end{aligned}$$

According to Lemma 2, we can conclude that $V_1(t)$ converges to 0 in fixed time, which means that $e_x(t) \to 0$ and $e_v(t) \to 0$ in fixed time. The estimation of settling time is

$$T \leqslant T_2 + \frac{1}{2^{(\alpha+1)/2}}\left(\frac{1}{(Nn)^{(1-\beta)/2}2^{(\beta-\alpha)/2}}\right)^{(1-\alpha)/(\beta-\alpha)}\left(\frac{2}{1-\alpha} + \frac{2}{\beta-1}\right). \qquad \square$$

Remark 3. In control protocol (9), the sign function is applied to make the systems reach the sliding mode surface in finite time. However, in protocol (14), we make an improvement by adding a term $-\theta \operatorname{sig}(\sigma(t))^h$ to improve the convergence time. In theoretical analysis, we also prove the fixed-time reachability of sliding mode surface. Moreover, the improved protocol (14) can make all agents reach consensus in fixed time.

3.3 Fixed-time consensus with directed networks

The consensus of SONMASs with undirected networks is concerned in previous sections. In some practical, the communication among agents may be directed. Hence, the fixed-time consensus of SONMASs with directed networks will be discussed in this section.

We choose the sliding mode manifold as follows:

$$\bar{\sigma}(t) = e_v(t) + \mu \operatorname{sig}^\alpha\bigl(e_x(t)\bigr) + \mu \operatorname{sig}^\beta\bigl(e_x(t)\bigr),$$

where $0 < \alpha < 1$, $1 < \beta$, μ is a positive constant, $\bar{\sigma}(t) = [\bar{\sigma}_1^{\mathrm{T}}(t), \bar{\sigma}_2^{\mathrm{T}}(t), \dots, \bar{\sigma}_N^{\mathrm{T}}(t)]^{\mathrm{T}}$, $\operatorname{sig}^\alpha(e_x(t)) = [(\operatorname{sig}^\alpha(e_{x_1}(t)))^{\mathrm{T}}, \dots, (\operatorname{sig}^\alpha(e_{x_N}(t)))^{\mathrm{T}}]^{\mathrm{T}}$.

The following control protocol is proposed:

$$\begin{aligned}u(t) &= -(k_1\|e_x(t)\| + k_2\|e_v(t)\| + k_3)\operatorname{sgn}\bigl(\bar{\sigma}(t)\bigr)\\ &\quad - k_4u_a(t) - k_5u_b(t) - \eta u_c(t),\\ u_a(t) &= \bigl\|\operatorname{diag}\bigl(e_v(t)\bigr)\bigl|e_x(t)\bigr|^{\alpha-1}\bigr\|\bigl((L+B)^{-1}\otimes I_n\bigr)\operatorname{sgn}\bigl(\bar{\sigma}(t)\bigr),\\ u_b(t) &= \bigl\|\operatorname{diag}\bigl(e_v(t)\bigr)\bigl|e_x(t)\bigr|^{\beta-1}\bigr\|\bigl((L+B)^{-1}\otimes I_n\bigr)\operatorname{sgn}\bigl(\bar{\sigma}(t)\bigr),\\ u_c(t) &= \bigl((L+B)^{-1}\otimes I_n\bigr)\operatorname{sig}^h\bigl(\bar{\sigma}(t)\bigr),\end{aligned} \tag{18}$$

where η, k_1, k_2, k_3, k_4, and k_5 are positive constants to be determined, $h > 1$, $u_a(t) = [u_{a_1}^{\mathrm{T}}(t), u_{a_2}^{\mathrm{T}}(t), \dots, u_{a_N}^{\mathrm{T}}(t)]^{\mathrm{T}}$, $u_b(t) = [u_{b_1}^{\mathrm{T}}(t), u_{b_2}^{\mathrm{T}}(t), \dots, u_{b_N}^{\mathrm{T}}(t)]^{\mathrm{T}}$, and $u_c(t) = [u_{c_1}^{\mathrm{T}}(t), u_{c_2}^{\mathrm{T}}(t), \dots, u_{c_N}^{\mathrm{T}}(t)]^{\mathrm{T}}$.

According to Eq. (7) and (18), we obtain

$$\begin{aligned}
\dot{e}_x(t) &= e_v(t),\\
\dot{e}_v(t) &= \bigl((L+B)\otimes I_n\bigr)\bigl[F\bigl(x(t),v(t)\bigr) - \bigl(\mathbf{1}_N\otimes f_0(t)\bigr) + w(t)\\
&\quad - \bigl(\mathbf{1}_N\otimes w_0(t)\bigr) - \bigl(k_1\|e_x(t)\| + k_2\|e_v(t)\| + k_3\bigr)\operatorname{sgn}\bigl(\bar{\sigma}(t)\bigr)\\
&\quad - k_4u_a(t) - k_5u_b(t) - \eta u_c(t)\bigr].
\end{aligned}\tag{19}$$

Theorem 3. *Suppose that Assumptions* 1 *and* 3 *hold. The fixed-time leader-following consensus can be achieved for SONMASs* (2)–(3) *under protocol* (18) *if the following inequalities are satisfied*:

$$\begin{aligned}
&k_1 \geqslant l_1\|(L+B)^{-1}\|, \quad k_2 \geqslant l_2\|(L+B)^{-1}\|,\\
&k_3 \geqslant b+c+1, \quad k_4 \geqslant \mu, \quad k_5 \geqslant \mu.
\end{aligned}\tag{20}$$

Proof. Consider the following Lyapunov function candidate:

$$V_2(t) = \frac{1}{2}\bar{\sigma}^{\mathrm{T}}(t)\bar{\sigma}(t).\tag{21}$$

The time derivative of $V_2(t)$ along system (19) can be calculated as

$$\begin{aligned}
\dot{V}_2(t) &= \bar{\sigma}^{\mathrm{T}}(t)\bigl[\bigl((L+B)\otimes I_n\bigr)\bigl(F\bigl(x(t),v(t)\bigr) - F\bigl(x_0(t),v_0(t)\bigr)\\
&\quad + w(t) - \bigl(\mathbf{1}_N\otimes w_0(t)\bigr) - \bigl(k_1\|e_x(t)\| + k_2\|e_v(t)\| + k_3\bigr)\operatorname{sgn}\bigl(\bar{\sigma}(t)\bigr)\\
&\quad - k_4u_a(t) - k_5u_b(t) - \eta u_c(t)\bigr)\bigr] + \mu\bar{\sigma}^{\mathrm{T}}(t)\operatorname{diag}\bigl(e_v(t)\bigr)\bigl|e_x(t)\bigr|^{\alpha-1}\\
&\quad + \mu\bar{\sigma}^{\mathrm{T}}(t)\operatorname{diag}\bigl(e_v(t)\bigr)\bigl|e_x(t)\bigr|^{\beta-1}.
\end{aligned}$$

Similar to analysis of Theorem 2, based on (20), it yields

$$\begin{aligned}
\dot{V}_2(t) &\leqslant -k_3\|\bar{\sigma}(t)\|_1 - \eta(Nn)^{(1-h)/2}\Biggl(\sum_{i=1}^{N}\sum_{j=1}^{n}\bigl|\bar{\sigma}_{ij}(t)\bigr|^2\Biggr)^{(h+1)/2}\\
&\leqslant -k_3\bigl(2V_2(t)\bigr)^{1/2} - \eta(Nn)^{(1-h)/2}\bigl(2V_2(t)\bigr)^{(h+1)/2}.
\end{aligned}$$

Using Lemma 2, we obtain that the sliding mode surface $\bar{\sigma}(t) = 0$ can be reached in fixed time. Furthermore, the settling time can be estimated as

$$T_3 \leqslant \frac{1}{k_3\sqrt{2}}\left[\frac{k_3}{\eta(Nn)^{(1-h)/2}2^{h/2}}\right]^{1/h}\left(2 + \frac{2}{h-1}\right).\tag{22}$$

Consequently, for $t \geqslant T_3$, the sliding mode surface $\bar{\sigma}(t) = 0$ is reached. Then we have $\dot{e}_x(t) = -\mu\operatorname{sig}^{\alpha}(e_x(t)) - \mu\operatorname{sig}^{\beta}(e_x(t))$. Similar to analysis of Theorem 2, we also can prove that $e_x(t) \to 0$ and $e_v(t) \to 0$ in fixed time. The estimation of settling time is

$$T \leqslant T_3 + \frac{1}{\mu 2^{(\alpha+1)/2}}\left[\frac{1}{(Nn)^{(1-\beta)/2}2^{(\beta-\alpha)/2}}\right]^{(1-\alpha)/(\beta-\alpha)}\left(\frac{2}{1-\alpha} + \frac{2}{\beta-1}\right). \quad \square$$

Remark 4. In Theorems 1 and 2, some finite-time and fixed-time consensus conditions are obtained for SONMASs with undirected networks, respectively. Since network topology is undirected, the convergence analysis mainly utilizes the symmetry of the Lapalcian matrix. However, the Lapalcian matrix is asymmetry in directed networks. Therefore, the design of control protocol (18) and the selection of Lyapunov (21) are different form previous ones.

Remark 5. Since the sign function is employed in protocol (10), (14), and (18), there exists chattering in the control process. How to reduce or eliminate chattering is an interesting issue. At present, some methods including the fuzzy control [4] and the neural network control [23] are used to solve the chattering problem. In our future work, we will further improve the control protocol and use some special functions such as saturation function to reduce the chattering phenomenon.

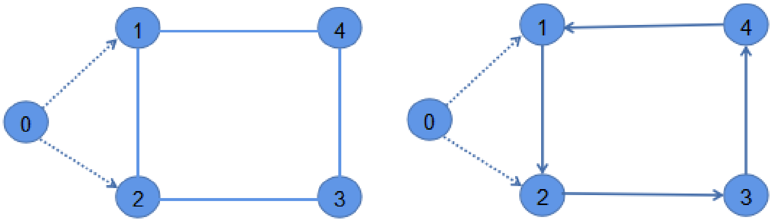
Remark 6. In protocols (10), (14), and (18), we may find that when the states of the followers move to $e_x(t) = 0$ and $e_v(t) \neq 0$, the denominator control law equation will be zero. That means $e_x(t) \to 0$ will lead to a singular problem, thus rendering a big peak torque value. Whereas, in practice it is difficult to implement a large value of torque. How to design a nonsingular terminal sliding mode control protocol will be further considered in our future work.

4 Numerical example

In this section, we give three numerical examples to demonstrate the effectiveness of the theoretical results.

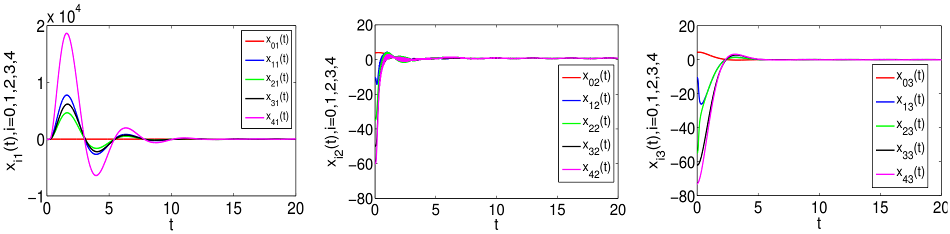
Example 1. Consider the SONMAS (2)–(3) with one leader and four followers. The interaction topology among the leader and followers is shown in Fig. 1(a). The nonlinear function are defined as follows:

$$f\big(x_i(t), v_i(t)\big) = \begin{pmatrix} \cos(x_{i1}(t)) - v_{i1}(t) - 2x_{i1}(t) \\ \cos(x_{i2}(t)) - v_{i2}(t) - x_{i2}(t) \\ -2v_{i3}(t) - 2x_{i3}(t) \end{pmatrix}, \quad i = 0, 1, \dots, 4.$$



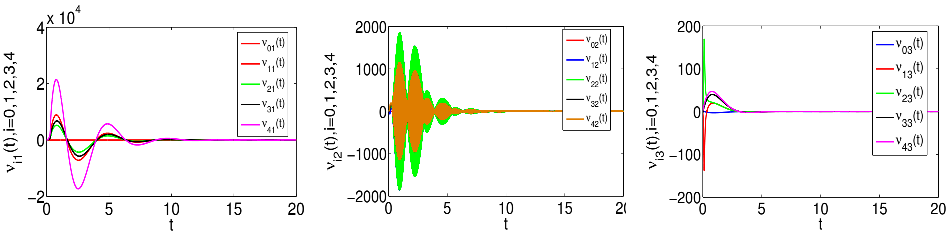
(a) (b)

Figure 1. The network topology.



(a) (b) (c)

Figure 2. The states of $x_i(t)$ with control protocol (10).



(a) (b) (c)

Figure 3. The states of $v_i(t)$ with the control protocol (10).

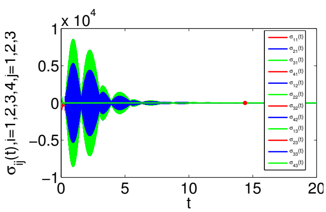


Figure 4. The states of sliding mode variables.

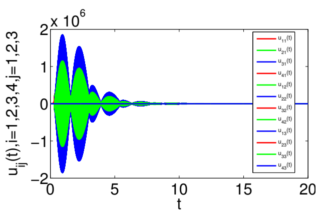
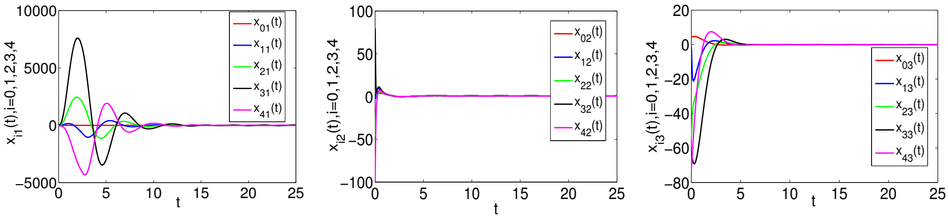


Figure 5. The control input $u_i(t)$.

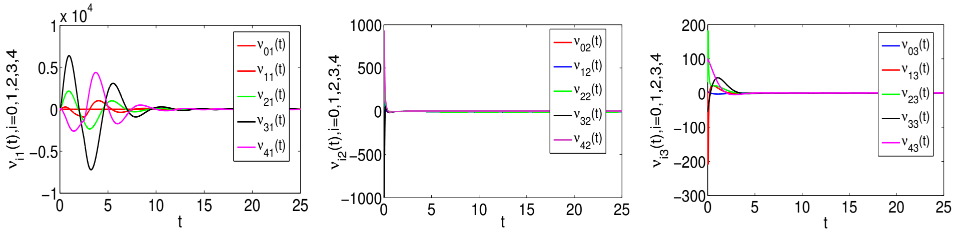
The external disturbances are chosen to satisfy $\|w_i(t)\|_\infty \leqslant 0.5$, $i = 1, 2, 3, 4$, and $\|w_0(t)\|_\infty \leqslant 2.5$. In control protocol (10), we choose $k_1 = 5.4$, $k_2 = 4.8$, $k_3 = 4$, $k_4 = 1.1$, $k_5 = 1.8$, $\alpha = 0.9$, and $\beta = 1.5$. It can be verified that all conditions of Theorem 1 are satisfied. The simulation results are presented in Figs. 2–5. Specifically, Fig. 2 describes the position states of four followers and one leader. Figure 3 describes the velocity states of four followers and one leader. Figure 4 describes the evolution of sliding mode variables, and the estimation of the setting time is $T \leqslant 14.85$. Figure 5 describes the evolution of the control protocol (10).

Example 2. Consider the SONMAS (2)–(3) with one leader and four followers. The nonlinear function, external disturbances, and interaction topology are same as the ones in Example 1. In control protocol (14), we choose $k_1 = 3.7$, $k_2 = 4.8$, $k_3 = 5$, $k_4 = 1.1$, $k_5 = 1.8$, $\alpha = 0.9$, $\beta = 1.5$, $\theta = 1$, and $h = 1.3$. Then all conditions of Theorem 2 are satisfied. The simulation results are presented in Figs. 6–9. Specifically,



(a) (b) (c)

Figure 6. The states of $x_i(t)$ with control protocol (14).



(a) (b) (c)

Figure 7. The states of $v_i(t)$ with control protocol (14).

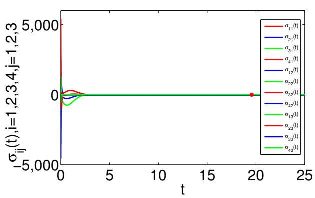


Figure 8. The states of sliding mode variables.

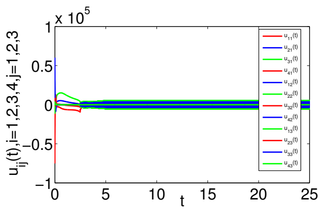
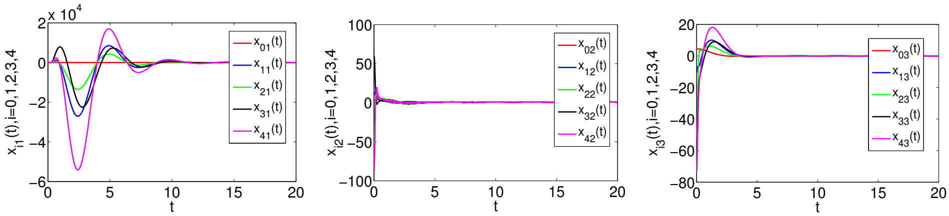


Figure 9. The control input $u_i(t)$.

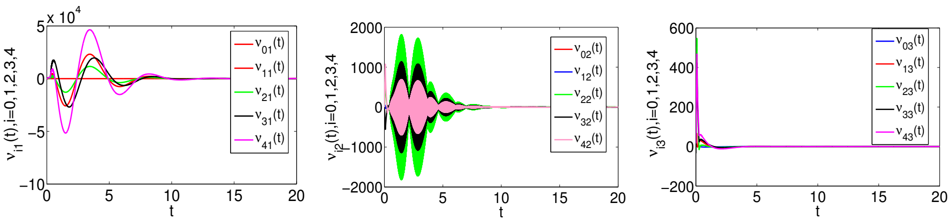
Fig. 6 describes the position states of four followers and one leader. Figure 7 describes the velocity states of four followers and one leader. Figure 8 describes the evolution of sliding mode variables, and the estimation of the setting time is $T \leqslant 19.56$. Figure 9 describes the evolution of the control protocol (14).

Example 3. Consider the SONMAS (2)–(3) with one leader and four followers. The directed interaction topology among the leader and followers is shown in Fig. 1(b). The nonlinear function and external disturbances are same as the ones in Example 1. In control protocol (18), we choose $k_1 = 5.4$, $k_2 = 4.8$, $k_3 = 8$, $k_4 = 3$, $k_5 = 2.5$, $\alpha = 0.9$, $\beta = 1.5$, $\mu = 2$, $h = 1.2$, and $\eta = 1$. Then all conditions of Theorem 3 are satisfied. The simulation results are presented in Figs. 10–13. Figure 10 describes the position states of four followers and the leader. Figure 11 describes the velocity states of four followers and the leader. Figure 12 describes the evolution of sliding mode variables, and the estimation of the setting time is $T \leqslant 17.10$. Figure 13 describes the evolution of the control protocol (18).



(a) (b) (c)

Figure 10. The states of $x_i(t)$ with control protocol (18).



(a) (b) (c)

Figure 11. The states of $v_i(t)$ with control protocol (18).

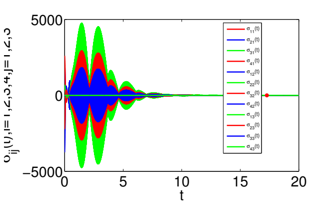


Figure 12. The states of sliding mode variables.

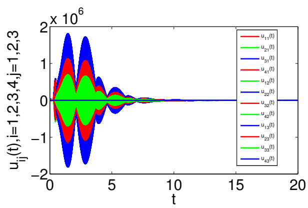


Figure 13. The control input $u_i(t)$.

5 Conclusion

In this paper, the leader-following consensus of SONMASs with external disturbances was studied. Three control protocols were developed to guarantee the consensus performance of SONMASs over undirected and directed networks, respectively. Firstly, a fixed-time sliding mode manifold was designed, and some control protocols over undirected networks were proposed, which can make all agents achieve consensus in finite time and fixed time, respectively. Moreover, we improved the proposed protocol and studied the fixed-time consensus of SONMASs with directed networks. In addition, a fixed-time control protocol was designed for directed networks. The settling time was estimated by using an improved lemma. Finally, some numerical simulations were given to show the effectiveness of the proposed control protocols. In the future work, the consensus of higher-order MASs with dynamic event-triggered communication mechanism will be considered.

References

1. J. Bai, G. Wen, A. Rahmani, Leaderless consensus for the fractional-order nonlinear multi-agent systems under directed interaction topology, *Int. J. Syst. Sci.*, **49**(5):954–963, 2018, https://doi.org/10.1080/00207721.2018.1435837.
2. S. Bhat, D. Bernstein, Finite-time stability of continuous autonomous systems, *SIAM J. Control Optim.*, **38**(3):751–766, 2000, https://doi.org/10.1137/S0363012997321358.
3. T. Chen, X. Liu, W. Lu, Pinning complex networks by a single controller, *IEEE Trans. Circuits Syst. I Regul. Pap.*, **54**(6):1317–1326, 2007, https://doi.org/10.1109/TCSI.2007.895383.
4. W. Cheng, H. Xue, H. Liang, W. Wang, Prescribed performance adaptive fuzzy control of stochastic nonlinear multi-agent systems with input hysteresis and saturation, *Int. J. Fuzzy Syst.*, **24**:91–104, 2022, https://doi.org/10.1007/s40815-021-01112-y.
5. Z. Ding, Output regulation of uncertain nonlinear systems with nonlinear exosystems, *IEEE Trans. Autom. Control*, **51**(3):498–503, 2006, https://doi.org/10.1109/TAC.2005.864199.
6. Z. Ding, Consensus disturbance rejection with disturbance observers, *IEEE Trans. Ind. Electron.*, **62**(9):5829–5837, 2015, https://doi.org/10.1109/TIE.2015.2442218.
7. Y. Dong, J. Chen, J. Cao, Fixed-time consensus of nonlinear multi-agent systems with stochastically switching topologies, *Int. J. Control*, pp. 1–21, 2021, https://doi.org/10.1080/00207179.2021.1939165.
8. V. Dygalo, M. Lyashenko, V. Shekhovtsov, Formation of basic performance properties of wheeled vehicles in braking mode, *Transp. Res. Procedia*, **50**:130–135, 2020, https://doi.org/10.1016/j.trpro.2020.10.016.
9. X. Ge, Q.-L. Han, Distributed formation control of networked multi-agent systems using a dynamic event-triggered communication mechanism, *IEEE Trans. Ind. Electron.*, **64**(10):8118–8127, 2017, https://doi.org/10.1109/TIE.2017.2701778.
10. M. Grzes, M. Slowik, Z. Gosiewsk, Multirotor UAV sensor fusion for precision landing, *Aircr. Eng. Aerosp. Technol.*, **91**(2):241–248, 2018, https://doi.org/10.1108/AEAT-01-2018-0070.
11. W. He, G. Chen, Q.-L. Han, F. Qian, Network-based leader-following consensus of nonlinear multi-agent systems via distributed impulsive control, *Inf. Sci.*, **380**:145–158, 2017, https://doi.org/10.1016/j.ins.2015.06.005.
12. W. He, C. Xu, Q.-L. Han, F. Qian, Z. Lang, Finite-time $\mathcal{L}_2$ leader-follower consensus of networked Euler–Lagrange systems with external disturbances, *IEEE Trans. Syst. Man Cybern.*, **48**(11):1920–1928, 2018, https://doi.org/10.1109/TSMC.2017.2774251.
13. Y. Hong, X. Wang, Z. Jiang, Distributed output regulation of leader-follower multi-agent systems, *Int. J. Robust Nonlinear Control*, **23**:48–66, 2012, https://doi.org/10.1002/rnc.1814.
14. T. Li, W. Wang, Y. Zhang, X. Tan, LMI-based sliding mode robust control for a class of multi-agent linear systems, *Open Access Library Journal*, **9**(1):1–7, 2022, https://doi.org/10.4236/oalib.1108342.

15. H. Lü, W. He, Q.-L. Han, X. Ge, C. Peng, Finite-time containment control for nonlinear multi-agent systems with external disturbances, *Inf. Sci.*, **512**:338–351, 2020, `https://doi.org/10.1016/j.ins.2019.05.049`.
16. H. Lü, W. He, Q.-L. Han, C. Peng, Fixed-time synchronization for coupled delayed neural networks with discontinuous or continuous activations, *Neurocomputing*, **314**:143–153, 2018, `https://doi.org/10.1016/j.neucom.2018.06.037`.
17. R. Mishra, A. Sinha, Event-triggered sliding mode based consensus tracking in second order heterogeneous nonlinear multi-agent systems, *Eur. J. Control*, **45**:30–44, 2019, `https://doi.org/10.1016/j.ejcon.2018.10.003`.
18. D. Mu, G. Wang, Y. Fan, Formation control strategy for underactuated unmanned surface vehicles subject to unknown dynamics and external disturbances with input saturation, *Int. J. Control Autom. Syst.*, **18**:2742–2752, 2020, `https://doi.org/10.1007/s12555-019-0611-6`.
19. A. Polyakov, Nonlinear feedback design for fixed-time stabilization of linear control systems, *IEEE Trans. Autom. Control*, **57**(8):2106–2110, 2012, `https://doi.org/10.1109/TAC.2011.2179869`.
20. H. Rastgoftar, E. Atkins, Safe multi-cluster UAV continuum deformation coordination, *Aerosp. Sci. Technol.*, **91**:640–655, 2019, `https://doi.org/10.1016/j.ast.2019.05.002`.
21. C. Ren, C. Chen, Sliding mode leader-following consensus controllers for second-order nonlinear multi-agent systems, *IET Control Theory Appl.*, **9**(10):1544–1552, 2015, `https://doi.org/10.1049/iet-cta.2014.0523`.
22. Y. Shang, Fixed-time group consensus for multi-agent systems with non-linear dynamics and uncertainties, *IET Control Theory Appl.*, **12**(3):305–309, 2018, `https://doi.org/10.1049/iet-cta.2017.1021`.
23. Z. Shi, G. Zhou, J. Guo, Neural network observer based consensus control of unknown nonlinear multi-agent systems with prescribed performance and input quantization, *Int. J. Control Autom. Syst.*, **19**:1944–1952, 2021, `https://doi.org/10.1007/s12555-020-0326-8`.
24. Y. Su, J. Huang, Cooperative semi-global robust output regulation for a class of nonlinear uncertain multi-agent systems, *Automatica*, **50**:1053–1065, 2014, `https://doi.org/10.1016/j.automatica.2014.02.010`.
25. F. Sun, P. Liu, H. Li, W. Zhu, Fixed-time consensus of heterogeneous multi-agent systems based on distributed observer, *Int. J. Syst. Sci.*, **52**(9):1780–1789, 2021, `https://doi.org/10.1080/00207721.2020.1871105`.
26. A. Xiang, Y. Chen, Y. Zhang, Spherical formation tracking control of non-holonomic aircraft-like vehicles in a spatiotemporal flowfield, *J. Franklin Inst.*, **357**(7):3924–3952, 2020, `https://doi.org/10.1016/j.jfranklin.2020.01.002`.
27. N. Yang, J. Li, New distributed adaptive protocols for uncertain nonlinear leader-follower multi-agent systems via a repetitive learning control approach, *J. Franklin Inst.*, **356**(12):6571–6590, 2019, `https://doi.org/10.1016/j.jfranklin.2019.01.052`.
28. W. Yu, W. Ren, W. Zheng, G. Chen, J. Lü, Distributed control gains design for consensus in multi-agent systems with second-order nonlinear dynamics, *Automatica*, **49**(7):2107–2115, 2013, `https://doi.org/10.1016/j.automatica.2013.03.005`.

29. Z. Yu, S. Yu, H. Jiang, X. Mei, Distributed fixed-time optimization for multiagent systems over a directed network, *Nonlinear Dyn.*, **103**:775–789, 2021, `https://doi.org/10.1007/s11071-020-06116-1`.
30. L. Zhao, Y. Jia, J. Yu, J. Du, H_∞ sliding mode based scaled consensus control for linear multi-agent systems with disturbances, *Appl. Math. Comput.*, **292**:375–389, 2017, `https://doi.org/10.1016/j.amc.2016.08.002`.
31. A. Zou, K. Kumar, Z. Hou, Distributed consensus control for multi-agent systems using terminal sliding mode and chebyshev neural networks, *Int. J. Robust Nonlinear Control*, **23**(3):334–357, 2013, `https://doi.org/10.1002/rnc.1829`.
32. Z. Zou, Nonsingular fixed-time consensus tracking for second-order multi-agent networks, *Automatica*, **54**:395–405, 2015, `https://doi.org/10.1016/j.automatica.2015.01.021`.