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Unique positive solutions for boundary value problem of p-Laplacian fractional differential equation with a sign-changed nonlinearity*

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Abstract. This paper investigates the existence of a unique positive solution for a class of boundary value problems of p-Laplacian fractional differential equations, where its nonlinearity is sign-changed and involves a fractional derivative term, and its boundary involves a nonlinear fractional integral term. By constructing an appropriate auxiliary boundary value problem and applying a generalized fixed point theorem of sum operator and properties of Mittag-Leffler function, some sufficient conditions for the existence of a unique positive solution are presented, and a monotone iterative sequence uniformly converging to the unique solution is constructed. In addition, an example is given to illustrate the main result.

Keywords: fractional boundary value problem, p-Laplacian operator, positive solution, fixed point theorem of sum operator, Mittag-Leffler function.

1 Introduction and preliminaries

Because of the extensive application in many fields such as physics, biology and engineering, etc., fractional differential equation has attracted considerable attention and has become an important area of investigation in differential equation theories. For a small sample of such work, we refer the reader to [1–3, 11, 13, 16, 20] and the references therein. At the same time, the differential equations with p-Laplacian operator are recognized as important mathematical models in various fields of non-Newtonian mechanics, population biology, elasticity theory, and so forth. More and more emphases have been put on the research of positive solutions for fractional boundary value problems with p-Laplacian operator, and excellent results from research into it emerge continuously. For some recent works on the subject, readers can see [4, 7, 8, 10, 12, 14, 15, 17, 18, 24] and the references therein. In these literature, there are a few papers on the existence of a unique positive

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solution [7, 17, 24]. Xu and Dong [24] investigated the existence and uniqueness for the following Riemann–Liouville fractional boundary value problem with p-Laplacian operator

$$\begin{aligned}
&D_{0+}^{\alpha}\left(\phi_p\left(D_{0+}^{\beta}x(t)\right)\right) = f\left(t, x(t)\right), \quad t \in (0,1),\\
&x(0) = x(1) = x'(0) = x'(1) = 0, \qquad D_{0+}^{\beta}x(0) = 0,\\
&D_{0+}^{\beta}x(1) = bD_{0+}^{\beta}x(\eta),
\end{aligned}$$

where $\alpha \in (1,2]$, $\beta \in (3,4]$, $\eta \in (0,1)$, $b \in (0, \eta^{(1-\alpha)/(p-1)})$, $f \in C([0,1]\times\mathbb{R}^+, \mathbb{R}^+)$. Their analysis based on the Schauder fixed point theorem, the upper and lower solutions method and the idea of concave and increasing operator theory. But to our knowledge, there are few papers reported on the existence of a unique positive solution for p-Laplacian fractional boundary value problems involving a fractional derivative term in the nonlinearity and a nonlinear integral term in the boundary conditions.

As is well known, the existence of a unique positive solution for nonlinear boundary value problems plays a very important role in theory and application, and the fixed point theory of operators with monotonicity and concavity (convexity) is an effective tool to deal with such problems. Many researchers have studied the existence and uniqueness of positive solutions by using different fixed point theorems of operators with monotonicity and concavity (convexity), for example, fixed point theorems of concave (such as φ-concave, δ-concave, u_0-concave, $\psi-(h,r)$-concave) and increasing operators, see [5, 7, 17, 25]; fixed point theorems of generalized δ-concave and increasing (generalized $-\delta$-convex and decreasing) operators, see [22]; the fixed point theorem of sum operators (i.e., Lemma 7), see [26] and [27]; fixed point theorems of sum operators with concavity–convexity and mixed monotonicity, see [9, 28, 29]. As usual, while using this tool to study unique positive solutions of a boundary value problem, it is essential to require its nonlinearity to be nonnegative and satisfy monotonicity conditions and concavity (convexity) conditions. But, when the nonlinearity of the boundary value problem is a sign-changed function without monotonicity and concavity (convexity), we want to know whether the boundary value problem has a unique positive solution. More specifically, under what conditions and how to use this tool to prove the existence of the unique positive solution? To the best of authors' knowledge, there are no answers to these questions.

Motivated by the above literature, this paper will investigate the following p-Laplacian fractional boundary value problem (BVP) involving a fractional derivative term in the nonlinearity and a nonlinear integral term in the boundary conditions

$$\begin{aligned}
&D_{0+}^{\alpha}\left(\phi_p\left(-D_{0+}^{\beta}x(t)\right)\right) = f\left(t, x(t), -D_{0+}^{\beta}x(t)\right), \quad t \in (0,1),\\
&x(0) = 0, \qquad \phi_p\left(D_{0+}^{\beta}x(0)\right) = 0,\\
&D_{0+}^{\beta-1}x(1) = I_{0+}^{\omega}g\left(\xi, x(\xi)\right) + k,
\end{aligned} \tag{1}$$

where D_{0+}^{α} is the Riemann–Liouville fractional derivative of order α, I_{0+}^{ω} is the Riemann–Liouville fractional integral of order ω; $0 < \alpha \leqslant 1 < \beta \leqslant 2$, $0 < \xi \leqslant 1$, $\omega, k > 0$; $f \in C([0,1]\times\mathbb{R}^+\times\mathbb{R}^+, \mathbb{R})$, $g \in C([0,1]\times\mathbb{R}^+, \mathbb{R}^+)$, $\mathbb{R}^+ = [0,+\infty)$; $\phi_p(s) = |s|^{p-2}s$,

$p > 1$. Obviously, $(\phi_p)^{-1} = \phi_q$, $1/p + 1/q = 1$. Basic notations on Riemann–Liouville fractional integral and fractional derivative can be found in [11].

The purpose of this paper is to establish some sufficient conditions for the existence of a unique positive solution of BVP (1) where the nonlinearity $f(t,x,y)$ may be sign-changed and has neither monotonicity nor concavity (convexity), and construct a monotone iterative sequence uniformly converging to the unique positive solution. Our analysis relies on the cone theory, properties of Mittag-Leffler function, and a generalized fixed point theorem of a sum operator defined on an equivalence class in cone.

For convenience, we first list hypotheses used in this article as follows:

(H1) $f(t,0,0) > 0, t \in [0,1]$; there exists $L \geqslant 0$ such that

$$f(t,x_1,y_1) - f(t,x_2,y_2) \leqslant -L\bigl(\phi_p(y_1) - \phi_p(y_2)\bigr) \tag{2}$$

for $t \in [0,1], 0 \leqslant x_1 \leqslant x_2, 0 \leqslant y_1 \leqslant y_2$; there exists $\delta \in (0,1)$ such that

$$f(t,rx,ry) + L\phi_p(ry) \geqslant \phi_p\bigl(r^\delta\bigr)\bigl(f(t,x,y) + L\phi_p(y)\bigr) \tag{3}$$

for $r \in (0,1), t \in [0,1], x,y \in \mathbb{R}^+$.

(H2) there exists $\mu \geqslant 0$ satisfying $\Gamma(\beta+\omega) > \mu\xi^{\beta+\omega-1}$ such that

$$g(t,x_2) - g(t,x_1) \geqslant \mu(x_2 - x_1), \quad t \in [0,1],\ 0 \leqslant x_1 \leqslant x_2; \tag{4}$$

there exists a function $\varphi \geqslant 0$, $\varphi \in L[0,1]$, satisfying $\int_0^\xi (\xi - s)^{\omega-1}\varphi(s)\,\mathrm{d}s > 0$ such that

$$g(t,x) - \mu x \leqslant \varphi(t), \quad t \in [0,1],\ x \in \mathbb{R}^+. \tag{5}$$

(H3) $g(t,\tau x) \geqslant \tau g(t,x)$ for $\tau \in (0,1)$, $t \in [0,1]$, $x \in \mathbb{R}^+$.

Remark 1. Clearly, it is a special case of (2) in (H1) with $L = 0$ that $f(t,x,y)$ is increasing with respect to x and y. In addition, (H1) implies that $f(t,x,y) + L\phi_p(y) \geqslant f(t,0,0) > 0$ for $t \in [0,1], x,y \in \mathbb{R}^+$, and (H2) implies that $g(t,x) - \mu x \geqslant g(t,0) \geqslant 0$ for $t \in [0,1], x \in \mathbb{R}^+$.

Remark 2. In [21], authors studied the existence of a unique positive solution for the following problem:

$$\begin{gathered} {}^C D_{0+}^\alpha x(t) + \lambda f\bigl(t,x(t)\bigr) = 0, \quad 0 < t < 1, \\ ax(0) - bx'(0) = 0, \qquad x(1) = \int_0^1 k(s)g\bigl(x(s)\bigr)\,\mathrm{d}s + \mu \end{gathered}$$

when the nonlinear function $g(x)$ was bounded and increasing. Different from [21], the nonlinear function $g(t,x)$ satisfying (H2) in BVP (1) may be unbounded.

Remark 3. When (H1), (H2), and (H3) are satisfied, it is difficult to prove the existence of a unique positive solution for BVP (1). Firstly, due to (2) and (4), the common method of constructing equivalent operator equations used in [5,7,17,21,22,24–29] fails to BVP (1). Secondly, since f involves $D^{\beta}_{0+}x$, the partially order used in this paper is related to $D^{\beta}_{0+}x$. In addition, note that BVP (1) involves ϕ_p and $I^{\omega}_{0+}g(\xi, x(\xi))$, so it is difficult to construct a valid equivalence class in cone, but it is essential for our work. Finally, to our knowledge, fixed point theorems in the existing literature can not be directly applied to our analysis.

The paper is organized as follows. In Section 2, we recall some useful preliminaries and lemmas. In particular, we generalize a fixed point theorem of sum operators on cone. In Section 3, based on the generalized fixed point theorem, some results on the existence of a unique positive solution for BVP (1) are presented and proved. In Section 4, an example is given to illustrate our main result.

2 Preliminaries and fixed point theorems

Lemma 1. (*See* [11].) *Let $n-1<\alpha\leqslant n$, $L\in\mathbb{R}$, $h:\mathbb{R}^+\to\mathbb{R}$, then the fractional equation*

$$D^{\alpha}_{0+}v(t)-Lv(t)=h(t),\quad t>0,$$

is solvable, and its general solution is given by

$$v(t)=\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\big[L(t-s)^{\alpha}\big]h(s)\,\mathrm{d}s+\sum_{j=1}^{n}c_j t^{\alpha-j}E_{\alpha,\alpha+1-j}\big(Lt^{\alpha}\big),$$

where $c_1,c_2,\dots,c_n\in\mathbb{R}$, provided the above integral exists.

Here $E_{\alpha_1,\alpha_2}(u)=\sum_{i=0}^{\infty}u^i/\Gamma(i\alpha_1+\alpha_2)$, $\alpha_1,\alpha_2>0$, is the Mittag-Leffler function.

Lemma 2. (*See* [23].) *Let $0<\alpha\leqslant 1$, then*

$$E_{\alpha,\alpha}(u)>0,\qquad \frac{\mathrm{d}E_{\alpha,\alpha}(u)}{\mathrm{d}u}=\sum_{i=0}^{\infty}\frac{iu^{i-1}}{\Gamma((i+1)\alpha)}>0,\quad u\in\mathbb{R}.$$

The following result can be easily derived by Lemma 1.

Lemma 3. *Let $0<\alpha\leqslant 1, h\in C[0,1], L\in\mathbb{R}$, then the unique solution of the initial value problem*

$$\begin{aligned}&D^{\alpha}_{0+}v(t)+Lv(t)=h(t),\quad t\in(0,1],\\ &v(0)=0\end{aligned}$$

is given by

$$v(t)=\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\big[-L(t-s)^{\alpha}\big]h(s)\,\mathrm{d}s,\quad t\in[0,1].$$

Arguing similarly to the proof of Lemma 1 in [19], we can show the following result.

Lemma 4. *Let $z \in C[0,1]$, $\mu, \lambda \in \mathbb{R}$, $1 < \beta \leqslant 2$, $\omega > 0$, $0 < \xi \leqslant 1$. If $\Gamma(\beta+\omega) \neq \mu\xi^{\beta+\omega-1}$, then the following problem*

$$-D_{0+}^{\beta} x(t) = z(t), \quad t \in (0,1),$$
$$x(0) = 0, \qquad D_{0+}^{\beta-1} x(1) = \mu I_{0+}^{\omega} x(\xi) + \lambda$$

has a unique solution

$$x(t) = \int_0^1 H(t,s) z(s)\,\mathrm{d}s + \frac{\Gamma(\beta+\omega)\lambda t^{\beta-1}}{\rho},$$

where

$$\rho = \Gamma(\beta)\big(\Gamma(\beta+\omega) - \mu\xi^{\beta+\omega-1}\big),$$

$$H(t,s) = \frac{1}{\rho}\begin{cases} [\Gamma(\beta+\omega) - \mu(\xi-s)^{\beta+\omega-1}]t^{\beta-1} \\ \quad -[\Gamma(\beta+\omega) - \mu\xi^{\beta+\omega-1}](t-s)^{\beta-1}, & s \leqslant t, s \leqslant \xi, \\ [\Gamma(\beta+\omega) - \mu(\xi-s)^{\beta+\omega-1}]t^{\beta-1}, & t \leqslant s \leqslant \xi, \\ \Gamma(\beta+\omega)[t^{\beta-1} - (t-s)^{\beta-1}] \\ \quad +\mu\xi^{\beta+\omega-1}(t-s)^{\beta-1}, & \xi \leqslant s \leqslant t, \\ \Gamma(\beta+\omega)t^{\beta-1}, & s \geqslant t, s \geqslant \xi. \end{cases} \tag{6}$$

Remark 4. Lemma 4 is Lemma 2.2 in [8] when $\beta \neq 2$.

Lemma 5. *Let $h \in C[0,1]$, $0 < \alpha \leqslant 1 < \beta \leqslant 2$, $\omega > 0$, $L, \mu, \lambda \in \mathbb{R}$. If $\Gamma(\beta+\omega) \neq \mu\xi^{\beta+\omega-1}$, then the fractional boundary value problem*

$$\begin{aligned} &D_{0+}^{\alpha}\big(\phi_p\big(-D_{0+}^{\beta} x(t)\big)\big) + L\phi_p\big(-D_{0+}^{\beta} x(t)\big) = h(t), \quad t \in (0,1), \\ &x(0) = 0, \quad D_{0+}^{\beta} x(0) = 0, \qquad D_{0+}^{\beta-1} x(1) = \mu I_{0+}^{\omega} x(\xi) + \lambda \end{aligned} \tag{7}$$

has a unique solution

$$x(t) = \int_0^1 H(t,s)\phi_q\left(\int_0^s (s-\tau)^{\alpha-1} E_{\alpha,\alpha}\big[-L(s-\tau)^{\alpha}\big] h(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s + \frac{\Gamma(\beta+\omega)\lambda t^{\beta-1}}{\rho}, \quad t \in [0,1].$$

Proof. Set $\phi_p(-D_{0+}^{\beta} x(t)) = v(t)$. Note that $(\phi_p)^{-1} = \phi_q$, then BVP (7) is equivalent to the following problem:

$$\begin{aligned} &-D_{0+}^{\beta} x(t) = \phi_q\big(v(t)\big), \quad t \in (0,1), \\ &D_{0+}^{\alpha} v(t) + Lv(t) = h(t), \quad t \in (0,1), \\ &x(0) = 0, \quad v(0) = 0, \qquad D_{0+}^{\beta-1} x(1) = \mu I_{0+}^{\omega} x(\xi) + \lambda. \end{aligned} \tag{8}$$

By Lemma 3 the unique solution of the initial value problem

$$\begin{aligned}
&D_{0+}^{\alpha}v(t) + Lv(t) = h(t), \quad t \in (0,1],\\
&v(0) = 0
\end{aligned}$$

can be written as

$$v(t) = \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\bigl[-L(t-s)^{\alpha}\bigr] h(s)\,\mathrm{d}s.$$

By Lemma 4 the unique solution of the boundary value problem

$$\begin{aligned}
&-D_{0+}^{\beta}x(t) = \phi_q\bigl(v(t)\bigr), \quad t \in (0,1),\\
&x(0) = 0, \qquad D_{0+}^{\beta-1}x(1) = \mu I_{0+}^{\omega}x(\xi) + \lambda
\end{aligned}$$

is given by

$$x(t) = \int_0^1 H(t,s)\phi_q\bigl(v(s)\bigr)\,\mathrm{d}s + \frac{\Gamma(\beta+\omega)\lambda t^{\beta-1}}{\rho}.$$

Consequently, problem (8) has a unique solution

$$\begin{aligned}
x(t) = {}&\int_0^1 H(t,s)\phi_q\Biggl(\int_0^s (s-\tau)^{\alpha-1} E_{\alpha,\alpha}\bigl[-L(s-\tau)^{\alpha}\bigr] h(\tau)\,\mathrm{d}\tau\Biggr)\,\mathrm{d}s\\
&+ \frac{\Gamma(\beta+\omega)\lambda t^{\beta-1}}{\rho}, \quad t \in [0,1],
\end{aligned}$$

which is the unique solution of BVP (7). The proof is complete. □

Remark 5. According to the proof of Lemma 5, if x is a solution of BVP (7), then

$$D_{0+}^{\beta}x(t) = -\phi_q\Biggl(\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\bigl[-L(t-s)^{\alpha}\bigr] h(s)\,\mathrm{d}s\Biggr), \quad t \in [0,1].$$

Lemma 6. *Let* $1 < \beta \leqslant 2$, $\omega > 0$, $\mu \geqslant 0$. *If* $\Gamma(\beta+\omega) > \mu\xi^{\beta+\omega-1}$, *then* $H(t,s)$ *given by* (6) *is continuous and*

$$0 \leqslant H(t,s) \leqslant \frac{\Gamma(\beta+\omega)t^{\beta-1}}{\rho}, \quad t,s \in [0,1].$$

It is obvious that Lemma 6 follows from (6).

In the sequel, we present some concepts in ordered Banach spaces, which can be found in [6] and [26].

Let $(E, \|\cdot\|)$ be a real Banach space which is partially ordered by a cone $P \subset E$, that is, $x \preccurlyeq y$ iff $y - x \in P$. If $x \preccurlyeq y$ and $x \neq y$, then we denote $x \prec y$ or $y \succ x$. By θ we denote the zero element of E. A cone P is said to be normal if there exists a constant $N > 0$ such that $\theta \preccurlyeq x \preccurlyeq y$ implies $\|x\| \leqslant N\|y\|$. In this case, the smallest constant satisfying this inequality is called the normality constant of P. For all $x, y \in E$, the notation $x \sim y$ means that there exist $l_1 > 0, l_2 > 0$ such that $l_1 x \preccurlyeq y \preccurlyeq l_2 x$. Clearly, $\sim$ is an equivalence relation. Given $e \succ \theta$ (i.e., $e \in P$ and $e \neq \theta$), and the equivalence class of the element e is denoted by the set P_e, that is,

$$P_e = \{x \in E \mid \exists l_1(x) > 0,\ l_2(x) > 0 \text{ such that } l_1(x)e \preccurlyeq x \preccurlyeq l_2(x)e\}. \tag{9}$$

Let $D \subset E$. An operator $T : D \to E$ is said to be increasing if $x, y \in D, x \preccurlyeq y \Rightarrow Tx \preccurlyeq Ty$. An element $x^* \in D$ is called a fixed point of T if $Tx^* = x^*$.

In [26], Zhai and Anderson obtained the following result.

Lemma 7. (*See* [26].) *Let P be a normal cone in E, $A : P \to P$ and $B : P \to P$ be increasing operators. Assume that*

(G1) *there is $e \succ \theta$ such that $Ae \in P_e$ and $Be \in P_e$;*
(G2) *there exists a constant $\delta \in [0, 1)$ such that $A(\tau x) \succcurlyeq \tau^\delta Ax$ and $B(\tau x) \succcurlyeq \tau Bx$ for $x \in P$ and $\tau \in (0, 1)$;*
(G3) *there exists a constant $\sigma_0 > 0$ such that $Ax \succcurlyeq \sigma_0 Bx$ for $x \in P$.*

Then the operator equation $Ax + Bx = x$ has a unique solution x^* in P_e. Moreover, for any initial value $x_0 \in P_e$, constructing successively the sequence $x_n = Ax_{n-1} + Bx_{n-1}$ $(n = 1, 2, \dots)$, we have $\lim_{n\to+\infty} \|x_n - x^*\| = 0$.

However, in this paper, the operator B defined by (18) does not satisfy condition (G1) since $Be \notin P_e$ for any $e \succ \theta$. Therefore, we need to simply generalize Lemma 7.

Set

$$\overline{P}_e = \{x \in E \mid \exists l(x) > 0 \text{ such that } \theta \preccurlyeq x \preccurlyeq l(x)e\}. \tag{10}$$

Clearly, $P_e \subset \overline{P}_e \subset P$. So, the following condition (G1′) is more extensive than (G1).

(G1′) there is $e \succ \theta$ such that $Ae \in P_e$, $Be \in \overline{P}_e$.

In order to complete our analysis, we present the following result.

Theorem 1. *Let P be a normal cone in E, $A : P \to P$ and $B : P \to P$ be increasing operators. Assume that* (G1′), (G2), *and* (G3) *hold. Then the operator equation $Ax + Bx = x$ has a unique solution x^* in P_e. Moreover, for any initial value $x_0 \in P_e$, constructing successively the sequence $x_n = Ax_{n-1} + Bx_{n-1}$ $(n = 1, 2, \dots)$, we have $\lim_{n\to+\infty} \|x_n - x^*\| = 0$.*

Proof. Since $Ae \in P_e$ and $Be \in \overline{P}_e$, it is follows from (9) and (10) that there exist constants $l_1 > 0$, $l_2 > 0$ and $l_3 > 0$ such that $l_1 e \preccurlyeq Ae \preccurlyeq l_2 e$ and $0 \preccurlyeq Be \preccurlyeq l_3 e$, which implies that

$$l_1 e \preccurlyeq Ae + Be \preccurlyeq (l_2 + l_3)e.$$

So, $Ae + Be \in P_e$. Define an operator $T = A + B$ by $Tx = Ax + Bx$, then $T : P \to P$ and $Te \in P_e$. Next, to show that $T(P_e) \subset P_e$. It is easy to see from (G2) that

$$A(\tau^{-1}x) \preccurlyeq \tau^{-\delta}Ax \quad \text{and} \quad B(\tau^{-1}x) \preccurlyeq \tau^{-1}Bx \quad \text{for } \tau \in (0,1),\ x \in P.$$

For any $x \in P_e$, we can choose a sufficiently small number $\tau_0 \in (0,1)$ such that

$$\tau_0 e \preccurlyeq x \preccurlyeq \tau_0^{-1}e.$$

Noticing that $T : P \to P$ is increasing, we have

$$\begin{aligned} Tx &\preccurlyeq A(\tau_0^{-1}e) + B(\tau_0^{-1}e) \preccurlyeq \tau_0^{-\delta}Ae + \tau_0^{-1}Be \preccurlyeq (l_2\tau_0^{-\delta} + l_3\tau_0^{-1})e,\\ Tx &\succcurlyeq A(\tau_0 e) + B(\tau_0 e) \succcurlyeq \tau_0^{\delta}Ae + \tau_0 Be \succcurlyeq l_1\tau_0^{\delta}e. \end{aligned}$$

Since $l_2\tau_0^{-\delta} + l_3\tau_0^{-1} > 0$, $l_1\tau_0^{\delta} > 0$, we get $Tx \in P_e$, that is, $T(P_e) \subset P_e$. The rest of the proof is almost the same as that of Theorem 2.1 in [26]. The proof is complete. □

3 Main results

In this section, by constructing an auxiliary boundary value problem and applying Theorem 1 we obtain some new results on unique positive solution for BVP (1).

Set $X = \{x \mid x \in C[0,1],\ D_{0+}^{\beta}x(t) \in C[0,1]\}$, then it is a Banach space with the norm

$$\|x\| = \max_{t\in[0,1]}|x(t)| + \max_{t\in[0,1]}|D_{0+}^{\beta}x(t)|.$$

Let

$$P = \{x \in X \mid x(t) \geqslant 0,\ D_{0+}^{\beta}x(t) \leqslant 0,\ t \in [0,1]\}.$$

Clearly, P is a cone, and X is endowed with a partial order given by the cone P, that is,

$$x, y \in X, \quad x \preccurlyeq y \quad \Longleftrightarrow \quad x(t) \leqslant y(t), \quad -D_{0+}^{\beta}x(t) \leqslant -D_{0+}^{\beta}y(t), \quad t \in [0,1].$$

Moreover, P is a normal cone and the normality constant is 1.

Definition 1. Let x be a solution of BVP (1). x is called a positive solution of BVP (1) if $x(t) > 0$ for $t \in (0,1)$.

Theorem 2. *Assume that* (H1), (H2), *and* (H3) *hold. Then BVP* (1) *has a unique positive solution x^*, and there exist two constants $\gamma^* > 0$ and $\eta^* > 0$ such that for $t \in [0,1]$,*

$$\gamma^*\left(2t^{\beta-1} - t^{\alpha/(p-1)+\beta}\right) \leqslant x^*(t) \leqslant \eta^*\left(2t^{\beta-1} - t^{\alpha/(p-1)+\beta}\right), \tag{11}$$

$$\frac{\gamma^*\Gamma(\frac{\alpha}{p-1}+\beta+1)}{\Gamma(\frac{\alpha}{p-1}+1)}t^{\alpha/(p-1)} \leqslant -D_{0+}^{\beta}x^*(t) \leqslant \frac{\eta^*\Gamma(\frac{\alpha}{p-1}+\beta+1)}{\Gamma(\frac{\alpha}{p-1}+1)}t^{\alpha/(p-1)}. \tag{12}$$

Moreover, for any $x_0 \in P$, constructing successively the sequence

$$x_{n+1}(t) = \int_0^1 H(t,s)\phi_q\Biggl(\int_0^s (s-\tau)^{\alpha-1}E_{\alpha,\alpha}\bigl(-L(s-\tau)^\alpha\bigr) \times \bigl[f\bigl(\tau, x_n(\tau), -D_{0+}^\beta x_n(\tau)\bigr) + L\phi_p\bigl(-D_{0+}^\beta x_n(\tau)\bigr)\bigr]\,\mathrm{d}\tau\Biggr)\,\mathrm{d}s + \frac{\Gamma(\beta+\omega)}{\rho}t^{\beta-1}\bigl[I_{0+}^\omega\bigl(g\bigl(\xi, x_n(\xi)\bigr) - \mu x_n(\xi)\bigr) + k\bigr], \quad n = 0,1,2,\dots, \tag{13}$$

we have

$$\lim_{n\to+\infty}\max_{t\in[0,1]}\bigl|x_{n+1}(t) - x^*(t)\bigr| = 0, \qquad \lim_{n\to+\infty}\max_{t\in[0,1]}\bigl|D_{0+}^\beta x_{n+1}(t) - D_{0+}^\beta x^*(t)\bigr| = 0. \tag{14}$$

Proof. For any given $x \in P$, consider the auxiliary boundary value problem

$$\begin{aligned} &D_{0+}^\alpha\bigl(\phi_p\bigl(-D_{0+}^\beta y(t)\bigr)\bigr) \\ &\quad = f\bigl(t, x(t), -D_{0+}^\beta x(t)\bigr) + L\bigl[\phi_p\bigl(-D_{0+}^\beta x(t)\bigr) - \phi_p\bigl(-D_{0+}^\beta y(t)\bigr)\bigr], \quad t \in (0,1), \\ &y(0) = 0, \quad D_{0+}^\beta y(0) = 0, \\ &D_{0+}^{\beta-1} y(1) = I_{0+}^\omega\bigl(g\bigl(\xi, x(\xi)\bigr) + \mu\bigl[y(\xi) - x(\xi)\bigr]\bigr) + k, \end{aligned} \tag{15}$$

where L and μ are given in (H1) and (H2), respectively. By Lemma 5, BVP (15) has a unique solution given by

$$y(t) = \int_0^1 H(t,s)\phi_q\Biggl(\int_0^s (s-\tau)^{\alpha-1}E_{\alpha,\alpha}\bigl(-L(s-\tau)^\alpha\bigr) \times \bigl[f\bigl(\tau, x(\tau), -D_{0+}^\beta x(\tau)\bigr) + L\phi_p\bigl(-D_{0+}^\beta x(\tau)\bigr)\bigr]\,\mathrm{d}\tau\Biggr)\,\mathrm{d}s + \frac{\Gamma(\beta+\omega)t^{\beta-1}I_{0+}^\omega(g(\xi, x(\xi)) - \mu x(\xi))}{\rho} + \frac{\Gamma(\beta+\omega)kt^{\beta-1}}{\rho}, \quad t \in [0,1]. \tag{16}$$

Define two operators A and B by

$$(Ax)(t) = \int_0^1 H(t,s)\phi_q\Biggl(\int_0^s (s-\tau)^{\alpha-1}E_{\alpha,\alpha}\bigl(-L(s-\tau)^\alpha\bigr) \times \bigl[f\bigl(\tau, x(\tau), -D_{0+}^\beta x(\tau)\bigr) + L\phi_p\bigl(-D_{0+}^\beta x(\tau)\bigr)\bigr]\,\mathrm{d}\tau\Biggr)\,\mathrm{d}s + \frac{\Gamma(\beta+\omega)kt^{\beta-1}}{\rho}, \quad t \in [0,1],\ x \in P, \tag{17}$$

$$(Bx)(t) = \frac{\Gamma(\beta+\omega) I_{0^+}^{\omega}(g(\xi, x(\xi)) - \mu x(\xi))}{\rho} t^{\beta-1}, \quad t \in [0,1],\ x \in P. \tag{18}$$

In view of Remark 5, we have

$$\begin{aligned}
&-D^{\beta}(Ax)(t) \\
&\quad = \phi_q\Biggl(\int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\bigl(-L(t-\tau)^{\alpha}\bigr)\bigl[f\bigl(\tau, x(\tau), -D_{0^+}^{\beta} x(\tau)\bigr) \\
&\qquad + L\phi_p\bigl(-D_{0^+}^{\beta} x(\tau)\bigr)\bigr]\,\mathrm{d}\tau\Biggr), \quad t \in [0,1],\ x \in P, \\
&-D_{0^+}^{\beta}(Bx)(t) = 0, \quad t \in [0,1],\ x \in P.
\end{aligned}$$

Moreover, according to Lemmas 2, 6 and Remark 1, it is easy to show that $A : P \to P$ and $B : P \to P$. In addition, from (15)–(18) we can assert that $x^* \in P$ is a fixed point of $A + B$ if and only if x^* is a solution of BVP (1) in P.

In order to get the conclusions, we verify that operators A and B satisfy all assumptions of Theorem 1 in the sequel.

Firstly, we show that operators A and B are increasing. For all $x_1, x_2 \in P$ with $x_1 \preccurlyeq x_2$, we have

$$0 \leqslant x_1(t) \leqslant x_2(t), \quad 0 \leqslant -D_{0^+}^{\beta} x_1(t) \leqslant -D_{0^+}^{\beta} x_2(t), \quad t \in [0,1].$$

Furthermore, by (2) and (4) we obtain

$$\begin{aligned}
(Ax_1)(t) &= \int_0^1 H(t,s)\phi_q\Biggl(\int_0^s (s-\tau)^{\alpha-1} E_{\alpha,\alpha}\bigl(-L(s-\tau)^{\alpha}\bigr) \\
&\qquad \times \bigl[f\bigl(\tau, x_1(\tau), -D_{0^+}^{\beta} x_1(\tau)\bigr) + L\phi_p\bigl(-D_{0^+}^{\beta} x_1(\tau)\bigr)\bigr]\,\mathrm{d}\tau\Biggr)\,\mathrm{d}s \\
&\qquad + \frac{\Gamma(\beta+\omega) k t^{\beta-1}}{\rho} \\
&\leqslant \int_0^1 H(t,s)\phi_q\Biggl(\int_0^s (s-\tau)^{\alpha-1} E_{\alpha,\alpha}\bigl(-L(s-\tau)^{\alpha}\bigr) \\
&\qquad \times \bigl[f\bigl(\tau, x_2(\tau), -D_{0^+}^{\beta} x_2(\tau)\bigr) + L\phi_p\bigl(-D_{0^+}^{\beta} x_2(\tau)\bigr)\bigr]\,\mathrm{d}\tau\Biggr)\,\mathrm{d}s \\
&\qquad + \frac{\Gamma(\beta+\omega) k t^{\beta-1}}{\rho} \\
&= (Ax_2)(t), \quad t \in [0,1],
\end{aligned}$$

$$
\begin{aligned}
-D_{0+}^{\beta}(Ax_1)(t) &= \phi_q\Biggl(\int\limits_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}\bigl(-L(t-\tau)^{\alpha}\bigr) \\
&\qquad \times \bigl[f\bigl(\tau, x_1(\tau), -D_{0+}^{\beta}x_1(\tau)\bigr) + L\phi_p\bigl(-D_{0+}^{\beta}x_1(\tau)\bigr)\bigr]\,\mathrm{d}\tau\Biggr) \\
&\leqslant \phi_q\Biggl(\int\limits_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}\bigl(-L(t-\tau)^{\alpha}\bigr) \\
&\qquad \times \bigl[f\bigl(\tau, x_2(\tau), -D_{0+}^{\beta}x_2(\tau)\bigr) + L\phi_p\bigl(-D_{0+}^{\beta}x_2(\tau)\bigr)\bigr]\,\mathrm{d}\tau\Biggr) \\
&= -D_{0+}^{\beta}(Ax_2)(t), \quad t \in [0,1],
\end{aligned}
$$

$$
\begin{aligned}
(Bx_1)(t) &= \frac{\Gamma(\beta+\omega)t^{\beta-1}I_{0+}^{\omega}\bigl(g(\xi, x_1(\xi)) - \mu x_1(\xi)\bigr)}{\rho} \\
&\leqslant \frac{\Gamma(\beta+\omega)t^{\beta-1}I_{0+}^{\omega}\bigl(g(\xi, x_2(\xi)) - \mu x_2(\xi)\bigr)}{\rho} = (Bx_2)(t), \quad t \in [0,1],
\end{aligned}
$$

$$
-D_{0+}^{\beta}(Bx_1)(t) = 0 = -D_{0+}^{\beta}(Bx_2)(t), \quad t \in [0,1],
$$

that is, $Ax_1 \preccurlyeq Ax_2$ and $Bx_1 \preccurlyeq Bx_2$.

Secondly, we prove that there exists $e \succ \theta$ such that $Ae \in P_e$ and $Be \in \overline{P}_e$, that is, assumption (G1′) holds. Set

$$
e(t) = 2t^{\beta-1} - t^{\alpha/(p-1)+\beta}, \quad t \in [0,1],
$$

then

$$
-D_{0+}^{\beta}e(t) = mt^{\alpha/(p-1)}, \quad t \in [0,1],
$$

where

$$
m = \biggl(\Gamma\biggl(\frac{\alpha}{p-1}+1\biggr)\biggr)^{-1}\Gamma\biggl(\frac{\alpha}{p-1}+\beta+1\biggr).
$$

Clearly,

$$
\begin{aligned}
&0 \leqslant t^{\beta-1} \leqslant e(t) \leqslant 2t^{\beta-1} \leqslant 2, \quad t \in [0,1], \\
&0 \leqslant -D_{0+}^{\beta}e(t) = mt^{\alpha/(p-1)} \leqslant m, \quad t \in [0,1],
\end{aligned} \tag{19}
$$

which means that $e \succ \theta$.

Define P_e and $\overline{P}_e$ as (9) and (10), respectively. In view of Lemma 2, it is obvious that

$$
E_{\alpha,\alpha}\bigl(-L\bigr) \leqslant E_{\alpha,\alpha}\bigl(-Lt^{\alpha}\bigr) \leqslant \frac{1}{\Gamma(\alpha)}, \quad t \in [0,1],
$$

which, together with Lemma 6, (H1) and (19), yields

$$\begin{aligned}(Ae)(t) &\leqslant \frac{\Gamma(\beta+\omega)}{\rho}\left\{\phi_q\left(\frac{1}{\Gamma(\alpha+1)}\left[\max_{s\in[0,1]} f(s,2,m)+L\phi_p(m)\right]\right)+k\right\}t^{\beta-1} \\ &\leqslant \frac{\Gamma(\beta+\omega)}{\rho}\left\{\phi_q\left(\frac{\max_{s\in[0,1]} f(s,2,m)+L\phi_p(m)}{\Gamma(\alpha+1)}\right)+k\right\}e(t), \quad t\in[0,1], \\ (Ae)(t) &\geqslant \frac{\Gamma(\beta+\omega)kt^{\beta-1}}{\rho} \geqslant \frac{\Gamma(\beta+\omega)k}{2\rho}e(t), \quad t\in[0,1],\end{aligned}$$

$$\begin{aligned}-D^{\beta}_{0^+}(Ae)(t) &\leqslant \phi_q\left(\frac{\max_{s\in[0,1]} f(s,2,m)+L\phi_p(m)}{\Gamma(\alpha)}\int\limits_0^t (t-s)^{\alpha-1}\,\mathrm{d}s\right) \\ &= \phi_q\left(\frac{\max_{s\in[0,1]} f(s,2,m)+L\phi_p(m)}{\Gamma(\alpha+1)}\right)t^{\alpha/(p-1)} \\ &= \frac{1}{m}\phi_q\left(\frac{\max_{s\in[0,1]} f(s,2,m)+L\phi_p(m)}{\Gamma(\alpha+1)}\right)\left(-D^{\beta}_{0^+}e(t)\right), \quad t\in[0,1],\end{aligned}$$

$$\begin{aligned}-D^{\beta}_{0^+}(Ae)(t) &\geqslant \phi_q\left(E_{\alpha,\alpha}(-L)\int\limits_0^t (t-s)^{\alpha-1}f(s,0,0)\,\mathrm{d}s\right) \\ &\geqslant \phi_q\left(\frac{E_{\alpha,\alpha}(-L)\min_{s\in[0,1]} f(s,0,0)}{\alpha}\right)t^{\alpha/(p-1)} \\ &= \frac{1}{m}\phi_q\left(\frac{E_{\alpha,\alpha}(-L)\min_{s\in[0,1]} f(s,0,0)}{\alpha}\right)\left(-D^{\beta}_{0^+}e(t)\right), \quad t\in[0,1].\end{aligned}$$

Consequently,

$$\gamma_0 e(t) \leqslant (Ae)(t) \leqslant \eta_0 e(t), \quad \gamma_0\left(-D^{\beta}_{0^+}e(t)\right) \leqslant -D^{\beta}_{0^+}(Ae)(t) \leqslant \eta_0\left(-D^{\beta}_{0^+}e(t)\right)$$

for $t\in[0,1]$, where

$$\gamma_0 = \min\left\{\frac{\Gamma(\beta+\omega)k}{2\rho},\ \frac{1}{m}\phi_q\left(\frac{E_{\alpha,\alpha}(-L)\min_{s\in[0,1]} f(s,0,0)}{\alpha}\right)\right\} > 0,$$

$$\eta_0 = \max\left\{\frac{\Phi_0}{m},\ \frac{\Gamma(\beta+\omega)}{\rho}[\Phi_0+k]\right\}, \quad \Phi_0 = \phi_q\left(\frac{\max_{s\in[0,1]} f(s,2,m)+L\phi_p(m)}{\Gamma(\alpha+1)}\right),$$

which means that $Ae\in P_e$.

On the other hand, from (5) and (19) we have

$$0 \leqslant (Be)(t) \leqslant \frac{\Gamma(\beta+\omega)I^{\omega}_{0^+}\varphi(\xi)}{\rho}t^{\beta-1} \leqslant \frac{\Gamma(\beta+\omega)I^{\omega}_{0^+}\varphi(\xi)}{\rho}e(t), \quad t\in[0,1],$$

$$0 = -D^{\beta}_{0^+}(Be)(t) \leqslant \frac{\Gamma(\beta+\omega)I^{\omega}_{0^+}\varphi(\xi)}{\rho}\left(-D^{\beta}_{0^+}e(t)\right), \quad t\in[0,1].$$

So, $\theta \preccurlyeq Be \preccurlyeq (\Gamma(\beta+\omega)I^{\omega}_{0^+}\varphi(\xi)/\rho)e$, that is, $Be\in \overline{P}_e$.

Next, we demonstrate that assumption (G2) of Theorem 1 is satisfied. For any $r \in (0,1)$, $x \in P$, and $t \in [0,1]$, from (3) we obtain that

$$\begin{aligned} A(rx)(t) &\geqslant r^\delta \int_0^1 H(t,s)\phi_q\Biggl(\int_0^s (s-\tau)^{\alpha-1}E_{\alpha,\alpha}\bigl(-L(s-\tau)^\alpha\bigr) \\ &\quad \times \bigl[f\bigl(\tau, x(\tau), -D_{0+}^\beta(x)(\tau)\bigr) + L\phi_p\bigl(-D_{0+}^\beta x(\tau)\bigr)\bigr]\,\mathrm{d}\tau\Biggr)\,\mathrm{d}s \\ &\quad + \frac{\Gamma(\beta+\omega)kt^{\beta-1}}{\rho} \geqslant r^\delta (Ax)(t), \\ -D_{0+}^\beta\bigl(A(rx)\bigr)(t) &\geqslant r^\delta \phi_q\Biggl(\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}\bigl(-L(t-\tau)^\alpha\bigr) \\ &\quad \times \bigl[f\bigl(\tau, x(\tau), -D_{0+}^\beta x(\tau)\bigr) + L\phi_p\bigl(-D_{0+}^\beta x(\tau)\bigr)\bigr]\,\mathrm{d}\tau\Biggr) \\ &= r^\delta\bigl(-D_{0+}^\beta (Ax)(t)\bigr). \end{aligned}$$

That is, $A(rx) \succcurlyeq r^\delta Ax$ for $r \in (0,1)$ and $x \in P$. Also, for any $r \in (0,1)$, $x \in P$, and $t \in [0,1]$, by (H3) we have

$$\begin{aligned} B(rx)(t) &\geqslant \frac{r\Gamma(\beta+\omega)t^{\beta-1} I_{0+}^\omega (g(\xi, x(\xi)) - \mu x(\xi))}{\rho} = r(Bx)(t), \\ -D_{0+}^\beta\bigl(B(rx)\bigr)(t) &= 0 = r\bigl(-D_{0+}^\beta (Bx)(t)\bigr). \end{aligned}$$

That is, $B(rx) \succcurlyeq rBx$ for $r \in (0,1)$ and $x \in P$.

Further, we prove that assumption (G3) of Theorem 1 is also satisfied. It follows from (H2) that

$$\sigma := \frac{I_{0+}^\omega \varphi(\xi)}{k} = \frac{\int_0^\xi (\xi-s)^{\omega-1}\varphi(s)\,\mathrm{d}s}{k\Gamma(\omega)} > 0.$$

Moreover, set $\sigma_0 = 1/\sigma$, then for any $x \in P$, (5), together with (17) and (18), yields that

$$\begin{aligned} (Bx)(t) &= \frac{\Gamma(\beta+\omega)t^{\beta-1}}{\rho\Gamma(\omega)} \int_0^\xi (\xi-s)^{\omega-1}\bigl(g\bigl(s,x(s)\bigr) - \mu x(s)\bigr)\,\mathrm{d}s \\ &\leqslant \frac{\Gamma(\beta+\omega)t^{\beta-1}}{\rho\Gamma(\omega)} \int_0^\xi (\xi-s)^{\omega-1}\varphi(s)\,\mathrm{d}s = \frac{\sigma\Gamma(\beta+\omega)kt^{\beta-1}}{\rho} \\ &\leqslant \sigma(Ax)(t), \quad t \in [0,1], \\ -D_{0+}^\beta (Bx)(t) &= 0 \leqslant \sigma\bigl(-D_{0+}^\beta (Ax)(t)\bigr), \quad t \in [0,1]. \end{aligned}$$

Hence, $Ax \succcurlyeq (1/\sigma)Bx = \sigma_0 Bx$ for $x \in P$.

Finally, applying Theorem 1, we obtain that $A + B$ has a unique fixed point x^* in P_e, and for any $x_0 \in P_e$, setting $x_{n+1} = Ax_n + Bx_n$, $n = 0, 1, 2, \dots$, we have $\lim_{n\to\infty} \|x_{n+1} - x^*\| = 0$. In addition, by using similar arguments as the proof of $\gamma_0 e \preccurlyeq Ae$, we can get $\gamma_0 e \preccurlyeq Ax$ for $x \in P$. Moreover, $(A + B)(P) \subset P_e$. Therefore, BVP (1) has a unique positive solution x^* satisfying (11) and (12), and for any $x_0 \in P$, we construct successively the sequence $\{x_{n+1}\}$ as (13), then (14) is tenable. This ends the proof. □

The following results can be derived from Theorem 2.

Corollary 1. *Assume* (H2), (H3) *and suppose that*

(H4) *for every* $t \in [0, 1]$, $f(t, 0, 0) > 0$ *and* $f(t, x_1, y_1) \leqslant f(t, x_2, y_2)$ *for* $0 \leqslant x_1 \leqslant x_2$, $0 \leqslant y_1 \leqslant y_2$; *there exists* $\delta \in (0, 1)$ *such that* $f(t, rx, ry) \geqslant \phi_p(r^\delta) f(t, x, y)$ *for* $r \in (0, 1)$, $x, y \in \mathbb{R}^+$, $t \in [0, 1]$.

Then BVP (1) *has a unique positive solution* x^* *satisfying* (11) *and* (12). *Moreover, for any initial value* $x_0 \in P$, *set*

$$x_{n+1}(t) = \int_0^1 H(t,s)\phi_q\left(\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1} f\bigl(\tau, x_n(\tau), -D_{0+}^\beta x_n(\tau)\bigr)\,\mathrm{d}\tau\right)\mathrm{d}s + \frac{\Gamma(\beta+\omega)t^{\beta-1}}{\rho}\bigl[I^\omega\bigl(g\bigl(\xi, x_n(\xi)\bigr) - \mu x_n(\xi)\bigr) + k\bigr], \quad n = 0, 1, 2, \dots .$$

Then (14) *holds.*

Corollary 2. *Assume* (H1), (H3) *and suppose that*

(H5) $g(t, x)$ *is increasing with respect to* x, *and there exists a nonnegative function* $\varphi \in L[0, 1]$ *satisfying* $\int_0^\xi (\xi - s)^{\omega-1}\varphi(s)\,\mathrm{d}s > 0$ *such that* $g(t, x) \leqslant \varphi(t)$ *for* $t \in [0, 1]$ *and* $x \in \mathbb{R}^+$.

Then BVP (1) *has a unique positive solution* x^* *satisfying* (11) *and* (12). *Moreover, for any initial value* $x_0 \in P$, *set*

$$\begin{aligned} x_{n+1}(t) = \int_0^1 H_0(t,s)\phi_q\biggl(\int_0^s (s-\tau)^{\alpha-1} E_{\alpha,\alpha}\bigl(-L(s-\tau)^\alpha\bigr) \\ \times \bigl[f\bigl(\tau, x_n(\tau), -D_{0+}^\beta x_n(\tau)\bigr) + L\phi_p\bigl(-D_{0+}^\beta x_n(\tau)\bigr)\bigr]\,\mathrm{d}\tau\biggr)\mathrm{d}s \\ + \frac{t^{\beta-1}}{\Gamma(\beta)}\bigl[I^\omega g\bigl(\xi, x_n(\xi)\bigr) + k\bigr], \quad n = 0, 1, 2, \dots, \end{aligned}$$

where

$$H_0(t,s) = \frac{1}{\Gamma(\beta)}\begin{cases} t^{\beta-1} - (t-s)^{\beta-1}, & 0 \leqslant s \leqslant t \leqslant 1, \\ t^{\beta-1}, & 0 \leqslant t \leqslant s \leqslant 1. \end{cases}$$

Then (14) *is also valid.*

Corollary 3. *Assume that* (H3)*,* (H4)*, and* (H5) *hold. Then BVP* (1) *has a unique positive solution* x^* *satisfying* (11) *and* (12)*. Moreover, for any initial value* $x_0 \in P$*, set*

$$x_{n+1}(t) = \int_0^1 H_0(t,s)\phi_q\left(\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1} f\big(\tau, x_n(\tau), -D_{0+}^{\beta} x_n(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s + \frac{t^{\beta-1}}{\Gamma(\beta)}\big[I^{\omega} g\big(\xi, x_n(\xi)\big) + k\big], \quad n = 0,1,2,\dots.$$

Then (14) *holds.*

Remark 6. Corollary 1 is the special case of Theorem 2 where $L = 0$ in (H1), Corollary 2 is the special case of Theorem 2 where $\mu = 0$ in (H2), and Corollary 3 is the special case of Theorem 2 where $L = 0$ in (H1), and $\mu = 0$ in (H2). Although the above three corollaries are the special cases of Theorem 2, they are still new results.

4 Example

Consider the following fractional p-Laplacian boundary value problem:

$$\begin{aligned}
&D_{0+}^{1/2}\big(\phi_{10/3}\big(-D_{0+}^{3/2}x(t)\big)\big) = \sqrt{t}\big[x^2(t) + x^{5/6}(t)\big] - \frac{t\sin t}{5}\big(D_{0+}^{3/2}x(t)\big)^{1/3} \\
&\quad + 3\big(D_{0+}^{3/2}x(t)\big)^{7/3} + \frac{t+1}{99}, \quad t \in (0,1), \\
&x(0) = 0, \qquad D_{0+}^{3/2}x(0) = 0, \\
&D_{0+}^{1/2}x(1) = \frac{1}{\Gamma(\frac{7}{2})}\int_0^{3/4}\left(\frac{3}{4} - s\right)^{5/2}\frac{x(s)(5s\mathrm{e}^s + 63(1 + x(s)))}{3(1 + x(s))}\,\mathrm{d}s + \frac{3}{1000},
\end{aligned} \tag{20}$$

that is, in BVP (1), $\alpha = 1/2$, $\beta = 3/2$, $\omega = 7/2$, $\xi = 3/4$, $k = 3/1000$, $p = 10/3$;

$$\begin{aligned}
f(t,x,y) &= \sqrt{t}\big(x^2 + x^{5/6}\big) + \frac{t\sin t}{5}y^{1/3} - 3y^{7/3} + \frac{t+1}{99}, \quad t \in [0,1],\ x, y \in \mathbb{R}^+, \\
g(t,x) &= \frac{5t\mathrm{e}^t x + 63(x^2 + x)}{3(1 + x)}, \quad t \in [0,1],\ x \in \mathbb{R}^+.
\end{aligned}$$

Obviously, $f \in C([0,1] \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R})$, $g \in C([0,1] \times \mathbb{R}^+, \mathbb{R}^+)$. Put

$$e(t) = 2t^{\beta-1} - t^{\alpha/(p-1)+\beta} = 2\sqrt{t} - t^{12/7}, \quad t \in [0,1],$$

then

$$-D_{0+}^{\beta}e(t) = \left(\Gamma\left(\frac{17}{14}\right)\right)^{-1}\Gamma\left(\frac{19}{7}\right)t^{3/14}, \quad t \in [0,1].$$

Take $L = 3$, then for any $0 \leqslant x_1 \leqslant x_2$, $0 \leqslant y_1 \leqslant y_2$, $t \in [0,1]$, we have

$$\begin{aligned} f(t,x_1,y_1) - f(t,x_2,y_2) &\leqslant -3\bigl(y_1^{7/3} - y_2^{7/3}\bigr) \\ &= -L\bigl(\phi_p(y_1) - \phi_p(y_2)\bigr). \end{aligned}$$

Moreover, set $\delta = 6/7$, then for any $r \in (0,1)$, $x \geqslant 0$, $y \geqslant 0$, $t \in [0,1]$, we get

$$\begin{aligned} f(t,rx,ry) + L\phi_p(ry) &\geqslant \phi_{10/3}\bigl(r^{6/7}\bigr)\bigl(f(t,x,y) + L\phi_p(y)\bigr) \\ &= \phi_p\bigl(r^{\delta}\bigr)\bigl(f(t,x,y) + L\phi_p(y)\bigr). \end{aligned}$$

In addition, it is clear that $f(t,0,0) > 0$ for $t \in [0,1]$. Hence, condition (H1) is satisfied.

Take $\mu = 21$, then $\Gamma(\beta+\omega) = 24 > 21 > \mu\xi^{\beta+\omega-1}$. Further, for any $0 \leqslant x_1 \leqslant x_2$, $t \in [0,1]$, it is obvious that

$$\begin{aligned} g(t,x_2) - g(t,x_1) &\geqslant 21(x_2 - x_1) + \frac{5t\mathrm{e}^t}{3}\biggl(\frac{x_2}{1+x_2} - \frac{x_1}{1+x_1}\biggr) \\ &\geqslant \mu(x_2 - x_1). \end{aligned}$$

Now, we take $\varphi(t) = (5/3)t\mathrm{e}^t$, $t \in [0,1]$. It is easy to check that $\int_0^{\xi}(\xi-s)^{\omega-1}\varphi(s)\,\mathrm{d}s > 0$, and

$$g(t,x) - \mu x = \frac{5t\mathrm{e}^t x}{3(1+x)} \leqslant \varphi(t), \quad t \in [0,1],\ x \geqslant 0.$$

Noticing that

$$\begin{aligned} g(t,rx) - \mu rx = \frac{5rt\mathrm{e}^t x}{3(1+rx)} &\geqslant \frac{5rt\mathrm{e}^t x}{3(1+x)} \\ &= r\bigl(g(t,x) - \mu x\bigr), \quad t \in [0,1], x \geqslant 0, \end{aligned}$$

we can get

$$g(t,rx) \geqslant rg(t,x), \quad t \in [0,1],\ x \geqslant 0.$$

Consequently, conditions (H2) and (H3) are satisfied.

Since all the conditions of Theorem 2 are satisfied, we obtain that BVP (20) has a unique positive solution x^*, and there exist $\gamma^* > 0$ and $\eta^* > 0$ such that

$$\gamma^*\bigl(2\sqrt{t} - t^{12/7}\bigr) \leqslant x^*(t) \leqslant \eta^*\bigl(2\sqrt{t} - t^{12/7}\bigr), \quad t \in [0,1],$$

$$\begin{aligned} &\gamma^*\biggl(\Gamma\biggl(\frac{17}{14}\biggr)\biggr)^{-1}\Gamma\biggl(\frac{19}{7}\biggr)t^{3/14} \\ &\quad \leqslant -D_{0^+}^{3/2}x^*(t) \leqslant \eta^*\biggl(\Gamma\biggl(\frac{17}{14}\biggr)\biggr)^{-1}\Gamma\biggl(\frac{19}{7}\biggr)t^{3/14}, \quad t \in [0,1]. \end{aligned}$$

Moreover, for any $x_0 \in P$, constructing successively the sequence

$$x_{n+1}(t) = \int_0^1 H(t,s)\phi_{10/7}\Biggl(\int_0^s (s-\tau)^{-1/2}E_{1/2,1/2}(-3\sqrt{s-\tau}\,) \times \biggl[\sqrt{\tau}\bigl(x_n^2(\tau)+x_n^{5/6}(\tau)\bigr) - \frac{\tau\sin\tau}{5}\bigl(D_{0^+}^{3/2}x_n(\tau)\bigr)^{1/3} + \frac{\tau+1}{99}\biggr]\,\mathrm{d}\tau\Biggr)\,\mathrm{d}s + \frac{24\sqrt{t}}{\rho}\Biggl(\int_0^{3/4} \frac{8se^s(\frac{3}{4}-s)^{5/2}x_n(s)}{9\sqrt{\pi}(1+x_n(s))}\,\mathrm{d}s + \frac{3}{1000}\Biggr), \quad n = 0,1,2,\dots,$$

here

$$H(t,s) = \frac{1}{\rho}\begin{cases} [24-21(\xi-s)^4]\sqrt{t} - \frac{4443}{256}\sqrt{t-s}, & s \leqslant t,\ s \leqslant \frac{3}{4}, \\ [24-21(\frac{3}{4}-s)^4]\sqrt{t}, & t \leqslant s \leqslant \frac{3}{4}, \\ 24\bigl[\sqrt{t}-\sqrt{t-s}\bigr] + \frac{1701}{256}\sqrt{t-s}, & \frac{3}{4} \leqslant s \leqslant t, \\ 24\sqrt{t}, & s \geqslant t,\ s \geqslant \frac{3}{4}, \end{cases}$$

$$\rho = \Gamma(\beta)\bigl(\Gamma(\beta+\omega) - \mu\xi^{\beta+\omega-1}\bigr) = \frac{4443\sqrt{\pi}}{512},$$

we have

$$\lim_{n\to+\infty}\max_{t\in[0,1]}\bigl|x_n(t) - x^*(t)\bigr| = 0$$

and

$$\lim_{n\to+\infty}\max_{t\in[0,1]}\bigl|D_{0^+}^{3/2}x_n(t) - D_{0^+}^{3/2}x^*(t)\bigr| = 0.$$

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