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On the non-closure under convolution for strong subexponential distributions*

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Abstract. In this paper, we consider the convolution closure problem for the class of strong subexponential distributions, denoted as $\mathscr{S}^*$. First, we show that, if $F, G \in \mathscr{L}$, then inclusions of $F * G$, FG, and $pF + (1-p)G$ for all (some) $p \in (0,1)$ into the class $\mathscr{S}^*$ are equivalent. Then, using examples constructed by Klüppelberg and Villasenor [The full solution of the convolution closure problem for convolution-equivalent distributions, *J. Math. Anal. Appl.*, 41:79–92, 1991], we show that $\mathscr{S}^*$ is not closed under convolution.

Keywords: class of strong subexponential distributions, class of subexponential distributions, convolution closure.

1 Introduction and the main result

Throughout the paper, we will say that a distribution F is on $\mathbb{R} := (-\infty, \infty)$ if $\overline{F}(x) := 1 - F(x) > 0$ for all x; we will say that a distribution F is on $\mathbb{R}_+$ if its support is contained in $\mathbb{R}_+ := [0, \infty)$ and $\overline{F}(x) > 0$ for all x. For two positive functions $a(x)$ and $b(x)$, we write $a(x) \sim b(x)$ if $\lim_{x\to\infty} a(x)/b(x) = 1$, we write $a(x) \asymp b(x)$ if $0 < \liminf_{x\to\infty} a(x)/b(x) \leqslant \limsup_{x\to\infty} a(x)/b(x) < \infty$. For any two distribution F and G, by $F * G$ we denote their convolution:

$$F * G(x) = \int_{-\infty}^{\infty} F(x-y)\,\mathrm{d}G(y), \quad x \in \mathbb{R}.$$

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We say that distribution F on $\mathbb{R}$ belongs to the class of *long-tailed* distributions, denoted $\mathscr{L}$, if its right tail $\overline{F} = 1 - F$ satisfies

$$\lim_{x\to\infty} \frac{\overline{F}(x-y)}{\overline{F}(x)} = 1$$

for any $y > 0$. We say that distribution F on $\mathbb{R}$ belongs to the *subexponential* class of distributions, denoted $\mathscr{S}$, if $F \in \mathscr{L}$ and

$$\lim_{x\to\infty} \frac{\overline{F * F}(x)}{\overline{F}(x)} = 2. \tag{1}$$

The class of distributions, characterized by (1), was introduced by Chistyakov [2] and later, in more general setup, by Athreya and Ney [1] and Chover et al. [3, 4].

A distribution F on $\mathbb{R}$ is said to belong to the *strong subexponential* class of distributions, introduced by Klüppelberg [9] and denoted $\mathscr{S}^*$, if $\mu_F = \int_0^\infty \overline{F}(y)\,\mathrm{d}y \in (0,\infty)$ and

$$\lim_{x\to\infty} \frac{1}{\overline{F}(x)} \int_0^x \overline{F}(x-y)\overline{F}(y)\,\mathrm{d}y = 2\mu_F. \tag{2}$$

The properties of class $\mathscr{S}^*$ and related classes were studied in [6, Sect. 3.4], [8–14], [18–20], and other papers. In particular, it is well known that, under $\mu_F < \infty$, it holds that $\mathscr{S}^* \subset \mathscr{S} \subset \mathscr{L}$.

In the following theorem, we present equivalent conditions for the convolution $F * G$ to be in the class $\mathscr{S}^*$ under the initial assumption $F, G \in \mathscr{L}$.

Theorem 1. *Suppose that F and G are two distributions on $\mathbb{R}$. Let $F, G \in \mathscr{L}$. Then the following statements are equivalent*:

(i) $F * G \in \mathscr{S}^*$,
(ii) $FG \in \mathscr{S}^*$,
(iii) $pF + (1-p)G \in \mathscr{S}^*$ *for some* $0 < p < 1$,
(iv) $pF + (1-p)G \in \mathscr{S}^*$ *for all* $0 < p < 1$.

Moreover, any of these equivalent statements implies the relations

$$\overline{F * G}(x) \sim \overline{F}(x) + \overline{G}(x), \tag{3}$$

$$\int_0^x \overline{F}(x-y)\overline{G}(y)\,\mathrm{d}y \sim \mu_G\overline{F}(x) + \mu_F\overline{G}(x), \tag{4}$$

where $\mu_F := \int_0^\infty \overline{F}(y)\,\mathrm{d}y$, $\mu_G := \int_0^\infty \overline{G}(y)\,\mathrm{d}y$.

In the corollary below the assumption $F, G \in \mathscr{L}$ of Theorem 1 is replaced by a stricter assumption $F, G \in \mathscr{S}^*$. In this case, the asymptotic relation (4) is equivalent to any of statements (i)–(iv) of Theorem 1 (see also [7, Thm. 3], which refers to [10]).

Corollary 1. *Suppose that F and G are two distributions on $\mathbb{R}$. Let $F, G \in \mathscr{S}^*$. Then any of statements* (i)–(iv) *of Theorem* 1 *is equivalent to* (4).

Proof. We need only to prove that (4) implies (iii) of Theorem 1. Obviously,

$$\begin{aligned} I &:= \int_0^x \frac{1}{2}(\overline{F}+\overline{G})(x-y)\frac{1}{2}(\overline{F}+\overline{G})(y)\,\mathrm{d}y \\ &= \frac{1}{4}\int_0^x \overline{F}(x-y)\overline{F}(y)\,\mathrm{d}y + \frac{1}{2}\int_0^x \overline{F}(x-y)\overline{G}(y)\,\mathrm{d}y + \frac{1}{4}\int_0^x \overline{G}(x-y)\overline{G}(y)\,\mathrm{d}y \\ &=: I_1 + I_2 + I_3. \end{aligned}$$

Since $F, G \in \mathscr{S}^*$, by relation (4), we get

$$\begin{aligned} &\limsup_{x\to\infty} \frac{I}{(\mu_F+\mu_G)(\overline{F}(x)+\overline{G}(x))} \\ &\quad \leqslant \limsup_{x\to\infty} \max\left\{ \frac{I_1}{\mu_F\overline{F}(x)}, \frac{I_2}{\mu_G\overline{F}(x)+\mu_F\overline{G}(x)}, \frac{I_3}{\mu_G\overline{G}(x)} \right\} \leqslant \frac{1}{2}. \end{aligned}$$

Similarly, we obtain that

$$\liminf_{x\to\infty} \frac{I}{(\mu_F+\mu_G)(\overline{F}(x)+\overline{G}(x))} \geqslant \frac{1}{2}.$$

The derived estimates imply that

$$\begin{aligned} &\int_0^x \frac{1}{2}(\overline{F}+\overline{G})(x-y)\frac{1}{2}(\overline{F}+\overline{G})(y)\,\mathrm{d}y \\ &\quad \sim (\mu_F+\mu_G)\frac{1}{2}(\overline{F}+\overline{G})(x) = 2\mu_{(F+G)/2}\overline{(F+G)/2}(x), \end{aligned}$$

and, consequently, $(F+G)/2 \in \mathscr{S}^*$ by definition. $\square$

We use Theorem 1 for proving main Theorem 2 on the convolution non-closure of class $\mathscr{S}^*$. Indeed, by Theorem 1, for distributions $F, G \in \mathscr{S}^*$, their convolution $F * G$ is not in $\mathscr{S}^*$ if and only if $pF + (1-p)G$ is not in $\mathscr{S}^*$ for some $p \in (0,1)$. Hence, in order to prove Theorem 2, we construct two distributions F and G such that $F/2 + G/2 \notin \mathscr{S}^*$ or, equivalently, $F * G \notin \mathscr{S}^*$.

Theorem 2. *Distribution class $\mathscr{S}^*$ is not closed under convolution, i.e. there exist distributions $F, G \in \mathscr{S}^*$ such that $F * G \notin \mathscr{S}^*$.*

Remark 1. The first counterexample for the closure of the subexponential class with respect to convolution was provided by Leslie [16]. Another counterexample for the closure of the convolution equivalent class of distributions with respect to convolution was given a few years later by Klüppelberg and Villasenor [10].

2 Proof of Theorem 1

2.1 Auxiliary lemmas

Before proving our main result, we state two auxiliary lemmas.

Lemma 1. *Suppose that F and G are two distributions on $\mathbb{R}$.*

(i) *If $F \in \mathscr{S}^*$, $G \in \mathscr{L}$, and $\overline{G}(x) \asymp \overline{F}(x)$, then $G \in \mathscr{S}^*$.*
(ii) *If $F \in \mathscr{L}$, $G \in \mathscr{L}$, then $F * G \in \mathscr{L}$.*
(iii) *If $F \in \mathscr{L}$, $G \in \mathscr{L}$, and $F * G \in \mathscr{S}$, then $\overline{F * G}(x) \sim \overline{F}(x) + \overline{G}(x)$.*

Proof. For the proof of part (i), see [9, Thm. 2.1(b)] (see also [6, Thm. 3.25]). For part (ii), see [5, Thm. 3(b)] (see also [6, Cor. 2.42] or [17, Lemma 4.2]). Part (iii) can be found in Theorem 1.1 of Leipus and Šiaulys [15]. □

Lemma 2. *Suppose F is distribution on $\mathbb{R}$ with finite μ_F. The following statements are equivalent*:

(i) *$F \in \mathscr{S}^*$.*
(ii) *$F \in \mathscr{L}$ and*

$$\lim_{v\to\infty} \limsup_{x\to\infty} \int\limits_{v}^{x-v} \frac{\overline{F}(x-y)\overline{F}(y)}{\overline{F}(x)} \, \mathrm{d}y = 0.$$

Proof. An equivalent assertion by choosing a special form function instead of the additional variable v is given in [6, Them. 3.24]. For the sake of completeness, we briefly present the main steps of the lemma proof. According to considerations in [6] (see the proof of Theorem 3.24), [7] (see the proof of Lemma 4), and [9] (see the proof of Theorem 3.2(b)), the assertion of the lemma follows from the estimate

$$\int\limits_{0}^{x/2} \frac{\overline{F}(x-z)\overline{F}(z)}{\overline{F}(x)} \, \mathrm{d}z \geqslant \int\limits_{0}^{y} \overline{F}(z) \, \mathrm{d}z + \frac{\overline{F}(x-y)}{\overline{F}(x)} \int\limits_{y}^{x/2} \overline{F}(z) \, \mathrm{d}z, \quad x > 2y > 0,$$

implying that

$$1 \leqslant \frac{\overline{F}(x-y)}{\overline{F}(x)} \leqslant \frac{(\overline{F}(x))^{-1} \int_0^{x/2} \overline{F}(x-z)\overline{F}(z) \, \mathrm{d}z - \int_0^y \overline{F}(z) \, \mathrm{d}z}{\int_0^{x/2} \overline{F}(z) \, \mathrm{d}z - \int_0^y \overline{F}(z) \, \mathrm{d}z},$$

and from equality

$$\int\limits_{0}^{x} \overline{F}(x-z)\overline{F}(z) \, \mathrm{d}z = 2 \int\limits_{0}^{v} \overline{F}(x-z)\overline{F}(z) \, \mathrm{d}z + \int\limits_{v}^{x-v} \overline{F}(x-z)\overline{F}(z) \, \mathrm{d}z,$$

where $x > 2v > 0$. □

2.2 Proof of the theorem

(ii) $\Rightarrow$ (i) Assume that $FG \in \mathscr{S}^*$. Lemma 1(ii) implies $F * G \in \mathscr{L}$, thus the proof will follow from

$$\overline{F * G}(x) \asymp \overline{FG}(x) \tag{5}$$

and Lemma 1(i). To prove (5), assume that X and Y are independent random variables with distributions F and G, correspondingly, and write

$$\begin{aligned}\overline{F * G}(x) &= \mathbf{P}(X + Y > x) \geqslant \mathbf{P}\bigl(\{X > x,\, Y > 0\} \cup \{X > 0,\, Y > x\}\bigr)\\ &= \overline{G}(0)\overline{F}(x) + \overline{F}(0)\overline{G}(x) - \overline{F}(x)\overline{G}(x),\\ \overline{FG}(x) &= \overline{F}(x) + \overline{G}(x) - \overline{F}(x)\overline{G}(x).\end{aligned}$$

Thus,

$$\begin{aligned}\liminf_{x\to\infty} \frac{\overline{F * G}(x)}{\overline{FG}(x)} &\geqslant \liminf_{x\to\infty} \frac{\overline{G}(0)\overline{F}(x) + \overline{F}(0)\overline{G}(x) - \overline{F}(x)\overline{G}(x)}{\overline{F}(x) + \overline{G}(x) - \overline{F}(x)\overline{G}(x)}\\ &\geqslant \min\bigl\{\overline{F}(0), \overline{G}(0)\bigr\} > 0.\end{aligned} \tag{6}$$

On the other hand,

$$\begin{aligned}\limsup_{x\to\infty} \frac{\overline{F * G}(x)}{\overline{FG}(x)} &= \limsup_{x\to\infty} \frac{\mathbf{P}(X_1 + Y_2 > x)}{\mathbf{P}(X \vee Y > x)}\\ &\leqslant \limsup_{x\to\infty} \frac{\mathbf{P}(X_1 \vee Y_1 + X_2 \vee Y_2 > x)}{\mathbf{P}(X \vee Y > x)},\end{aligned} \tag{7}$$

where (X_1, Y_1) and (X_2, Y_2) are independent copies of (X, Y).

Since, by (ii), $F_{X\vee Y} \in \mathscr{S}^* \subset \mathscr{S}$, we have

$$\mathbf{P}(X_1 \vee Y_1 + X_2 \vee Y_2 > x) \sim 2\mathbf{P}(X \vee Y > x). \tag{8}$$

Hence, by (6)–(8),

$$\min\bigl\{\overline{F}(0), \overline{G}(0)\bigr\} \leqslant \liminf_{x\to\infty} \frac{\overline{F * G}(x)}{\overline{FG}(x)} \leqslant \limsup_{x\to\infty} \frac{\overline{F * G}(x)}{\overline{FG}(x)} \leqslant 2$$

and (5) follows.

(i) $\Rightarrow$ (ii) Let $F * G \in \mathscr{S}^*$. Since $\mathscr{S}^* \subset \mathscr{S}$, by Lemma 1(iii),

$$\overline{F * G}(x) \sim \overline{F}(x) + \overline{G}(x) \sim \overline{FG}(x),$$

which further implies $FG \in \mathscr{L}$ by the above second equivalence. Therefore, $FG \in \mathscr{S}^*$ follows from Lemma 1(i) immediately.

(ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) follows because of Lemma 1(i).

Finally, relation (3) holds by Lemma 1(iii). It remains to prove relation (4). First, observe that any of the equivalent statements in (i)–(iv) from Theorem 1 implies the existence of finite μ_F and μ_G. Further, for $M > 0$ and $x > 2M$, we have

$$\begin{aligned}\int_0^x \overline{F}(x-y)\overline{G}(y)\,\mathrm{d}y &= \int_0^M \overline{F}(x-y)\overline{G}(y)\,\mathrm{d}y + \int_M^{x-M} \overline{F}(x-y)\overline{G}(y)\,\mathrm{d}y \\ &\quad + \int_0^M \overline{F}(y)\overline{G}(x-y)\,\mathrm{d}y \\ &=: J_1 + J_2 + J_3.\end{aligned}$$

Therefore,

$$\begin{aligned}\liminf_{x\to\infty} \frac{\int_0^x \overline{F}(x-y)\overline{G}(y)\,\mathrm{d}y}{\mu_G\overline{F}(x)+\mu_F\overline{G}(x)} &\geqslant \liminf_{x\to\infty} \frac{J_1+J_3}{\mu_G\overline{F}(x)+\mu_F\overline{G}(x)} \\ &\geqslant \liminf_{x\to\infty} \min\left\{\frac{J_1}{\mu_G\overline{F}(x)}, \frac{J_3}{\mu_F\overline{G}(x)}\right\} \\ &\geqslant \min\left\{\frac{\int_0^M \overline{G}(y)\,\mathrm{d}y}{\mu_G}, \frac{\int_0^M \overline{F}(y)\,\mathrm{d}y}{\mu_F}\right\}. \end{aligned} \tag{9}$$

Letting $M \to \infty$, we get from (9) that

$$\liminf_{x\to\infty} \frac{\int_0^x \overline{F}(x-y)\overline{G}(y)\,\mathrm{d}y}{\mu_G\overline{F}(x)+\mu_F\overline{G}(x)} \geqslant 1. \tag{10}$$

For the corresponding upper bound, we obtain

$$\begin{aligned}\limsup_{x\to\infty} \frac{\int_0^x \overline{F}(x-y)\overline{G}(y)\,\mathrm{d}y}{\mu_G\overline{F}(x)+\mu_F\overline{G}(x)} &\leqslant \limsup_{x\to\infty} \max\left\{\frac{J_1}{\mu_G\overline{F}(x)}, \frac{J_3}{\mu_F\overline{G}(x)}\right\} \\ &\quad + \limsup_{x\to\infty} \frac{J_2}{\mu_G\overline{F}(x)+\mu_F\overline{G}(x)}. \end{aligned} \tag{11}$$

By condition $F \in \mathscr{L}$, we get

$$\limsup_{x\to\infty} \frac{J_1}{\mu_G\overline{F}(x)} \leqslant \limsup_{x\to\infty} \frac{\overline{F}(x-M)}{\overline{F}(x)} \frac{1}{\mu_G} \int_0^M \overline{G}(y)\,\mathrm{d}y = \frac{1}{\mu_G} \int_0^M \overline{G}(y)\,\mathrm{d}y.$$

Now, letting $M \to \infty$, we obtain

$$\lim_{M\to\infty} \limsup_{x\to\infty} \frac{J_1}{\mu_G\overline{F}(x)} \leqslant 1. \tag{12}$$

Similarly, condition $G \in \mathscr{L}$ implies

$$\lim_{M\to\infty} \limsup_{x\to\infty} \frac{J_3}{\mu_F\overline{G}(x)} \leqslant 1. \tag{13}$$

Further, according to Theorem 1(iii), we have that $(F+G)/2 \in \mathscr{S}^*$. Hence, due to Lemma 2,

$$\begin{aligned}
&\lim_{M\to\infty}\limsup_{x\to\infty}\frac{J_2}{\mu_G\overline{F}(x)+\mu_F\overline{G}(x)}\\
&\quad\leqslant \frac{1}{\min\{\mu_F,\mu_G\}}\lim_{M\to\infty}\limsup_{x\to\infty}\frac{\int_M^{x-M}\overline{F}(x-y)\overline{G}(y)\,\mathrm{d}y}{\overline{F}(x)+\overline{G}(x)}\\
&\quad\leqslant \frac{2}{\min\{\mu_F,\mu_G\}}\lim_{M\to\infty}\limsup_{x\to\infty}\int\limits_M^{x-M}\frac{\frac{1}{2}(\overline{F}+\overline{G})(x-y)\frac{1}{2}(\overline{F}+\overline{G})(y)}{\frac{1}{2}(\overline{F}+\overline{G})(x)}\,\mathrm{d}y\\
&\quad= 0. \qquad (14)
\end{aligned}$$

Estimates (11)–(14) imply

$$\limsup_{x\to\infty}\frac{\int_0^x\overline{F}(x-y)\overline{G}(y)\,\mathrm{d}y}{\mu_G\overline{F}(x)+\mu_F\overline{G}(x)}\leqslant 1. \qquad (15)$$

Hence, the desired relation (4) of the theorem follows immediately from (10) and (15). Theorem 1 is proved.

3 Proof of Theorem 2

3.1 Auxiliary lemmas

In this subsection, we present two additional lemmas, which play a crucial role in the proof of Theorem 2. The statement of the first lemma is similar to that in Lemma 2. Note that equivalent condition for $F \in \mathscr{S}^*$ does not require additional condition $F \in \mathscr{L}$, comparing to Lemma 2.

Lemma 3. *Suppose F is distribution of $\mathbb{R}$ such that $\mu_F<\infty$. Then $F\in\mathscr{S}^*$ if and only if*

$$\lim_{x\to\infty}\int\limits_0^{x/2}\frac{\overline{F}(x-y)-\overline{F}(x)}{\overline{F}(x)}\overline{F}(y)\,\mathrm{d}y=0.$$

Proof. The proof is similar to the proof of Lemma 3 from [10]. Obviously, equality (2) is equivalent to

$$\frac{\int_0^{x/2}\overline{F}(x-y)\overline{F}(y)\,\mathrm{d}y}{\overline{F}(x)}=\mu_F.$$

Thus,

$$\begin{aligned}
F\in\mathscr{S}^* \quad&\Longleftrightarrow\quad \lim_{x\to\infty}\int\limits_0^{x/2}\frac{\overline{F}(x-y)}{\overline{F}(x)}\overline{F}(y)\,\mathrm{d}y=\lim_{x\to\infty}\int\limits_0^{x/2}\overline{F}(y)\,\mathrm{d}y\\
&\Longleftrightarrow\quad \lim_{x\to\infty}\int\limits_0^{x/2}\biggl(\frac{\overline{F}(x-y)}{\overline{F}(x)}-1\biggr)\overline{F}(y)\,\mathrm{d}y=0. \qquad \square
\end{aligned}$$

The second lemma is a technical result about behaviour of the special sequences.

Lemma 4. *Let* $\{a_n,\ n \geqslant 1\}$ *be an unboundedly increasing sequence of positive numbers, and let*

$$h_n := \max\bigl\{k\colon\ (k+1)! \leqslant a_n(\log a_n)^\beta\bigr\}$$

with some positive $\beta > 0$*. Then, for all sufficiently large* n*,*

$$\frac{\log a_n}{\log\log a_n} \leqslant h_n \leqslant \frac{2\log a_n}{\log\log a_n}. \tag{16}$$

Proof. The proof is constructed along to similar lines as in Lemma 5 from [10]. Namely, the Stirling's formula implies that

$$\log(k+1)! = (k+1)\log(k+1) - (k+1) - O(\log k)$$

for $k \to \infty$. Define

$$\widehat{h}_n = \frac{2\log a_n}{\log\log a_n}.$$

For some positive constant c_1 and for sufficiently large n, we have

$$\begin{aligned}
\log(\widehat{h}_n+1)! &\geqslant (\widehat{h}_n+1)\log(\widehat{h}_n+1) - (\widehat{h}_n+1) - c_1\log\widehat{h}_n \geqslant \frac{9}{10}\,\widehat{h}_n\log\widehat{h}_n \\
&= \frac{9}{5}\,\log\bigl(a_n(\log a_n)^\beta\bigr)\frac{\log a_n}{\log\log a_n}\,\frac{\log 2 + \log\log a_n - \log\log\log a_n}{\log a_n + \beta\log\log a_n} \\
&\geqslant \log\bigl(a_n(\log a_n)^\beta\bigr),
\end{aligned}$$

which implies the upper bound in (16).

Similarly, using Stirling's formula again, for

$$\widetilde{h}_n = \frac{\log a_n}{\log\log a_n},$$

we obtain

$$\begin{aligned}
\log(\widetilde{h}_n+1)! &\leqslant \widetilde{h}_n\log\widetilde{h}_n + c_2\log\widetilde{h}_n \\
&= \log a_n\left(1 - \frac{\log\log\log a_n}{\log\log a_n}\right)\left(1 + \frac{c_2}{\widetilde{h}_n}\right)
\end{aligned}$$

with some positive c_2 and sufficiently large n. Therefore, for large n,

$$\log(\widetilde{h}_n+1)! \leqslant \log\bigl(a_n(\log a_n)^\beta\bigr),$$

which implies the lower bound in (16). Lemma is proved. □

3.2 Proof of the theorem

Define two distributions $\mathcal{F}$ and $\mathcal{G}$ with tails:

$$\overline{\mathcal{F}}(x) := \mathbf{1}_{(-\infty,\,6!)}(x) + (6!)^2\Biggl\{\sum_{n=6}^{\infty}\frac{1}{(n!)^2}\mathbf{1}_{[n!,\,(n+1)!-b_nd_n)}(x) + \sum_{n=6}^{\infty}\frac{1}{((n+1)!)^2}\biggl(1+\frac{(n+1)!-x}{d_n}\biggr)\mathbf{1}_{[(n+1)!-b_nd_n,\,(n+1)!)}(x)\Biggr\},$$

$$\overline{\mathcal{G}}(x) := \mathbf{1}_{(-\infty,8!)}(x) + (8!)^2\Biggl\{\sum_{n=3}^{\infty}\frac{1}{((2^n)!)^2}\mathbf{1}_{[(2^n)!,\,(2^n+1)!-\widehat{b}_n\widehat{d}_n)}(x) + \sum_{n=3}^{\infty}\frac{1}{((2^n+1)!)^2}\biggl(1+\frac{(2^n+1)!-x}{\widehat{d}_n}\biggr)\mathbf{1}_{[(2^n+1)!-\widehat{b}_n\widehat{d}_n,\,(2^n+1)!)}(x) + \sum_{n=3}^{\infty}\frac{1}{x^2}\mathbf{1}_{[(2^n+1)!,\,(2^{n+1})!)}(x)\Biggr\},$$

where $b_n := n^2+2n$, $d_n := (\log b_n)^3$, $\widehat{b}_n = b_{2^n} = 2^n(2^n+2)$, and $\widehat{d}_n = (\log\widehat{b}_n)^2$. The functions above are constructed according to the scheme presented in [10] and [16].

Because of Theorem 1, it suffices to prove that $\mathcal{F},\mathcal{G} \in \mathscr{S}^*$ and $(\mathcal{F}+\mathcal{G})/2 \notin \mathscr{S}^*$. According to Lemma 3, we have to prove the following relations:

$$\mu_{\mathcal{F}} < \infty, \qquad \mu_{\mathcal{G}} < \infty, \tag{17}$$

$$\limsup_{x\to\infty}\int_0^{x/2}\frac{\overline{\mathcal{F}}(x-y)-\overline{\mathcal{F}}(x)}{\overline{\mathcal{F}}(x)}\,\overline{\mathcal{F}}(y)\,\mathrm{d}y = 0, \tag{18}$$

$$\limsup_{x\to\infty}\int_0^{x/2}\frac{\overline{\mathcal{G}}(x-y)-\overline{\mathcal{G}}(x)}{\overline{\mathcal{G}}(x)}\,\overline{\mathcal{G}}(y)\,\mathrm{d}y = 0, \tag{19}$$

$$\limsup_{x\to\infty}\int_0^{x/2}\frac{(\overline{\mathcal{F}}+\overline{\mathcal{G}})(x-y)-(\overline{\mathcal{F}}+\overline{\mathcal{G}})(x)}{(\overline{\mathcal{F}}+\overline{\mathcal{G}})(x)}\,(\overline{\mathcal{F}}+\overline{\mathcal{G}})(y)\,\mathrm{d}y \;>\; 0. \tag{20}$$

Denote

$$\Delta_{\mathcal{F}}(x,y) := \frac{\overline{\mathcal{F}}(x-y)-\overline{\mathcal{F}}(x)}{\overline{\mathcal{F}}(x)}, \qquad \Delta_{\mathcal{G}}(x,y) := \frac{\overline{\mathcal{G}}(x-y)-\overline{\mathcal{G}}(x)}{\overline{\mathcal{G}}(x)}.$$

Proof of (17). According to definitions of $\overline{\mathcal{F}}(x)$ and $\overline{\mathcal{G}}(x)$,

$$\mu_{\mathcal{F}} = \int_0^{\infty}\overline{\mathcal{F}}(y)\,\mathrm{d}y \leqslant 6! + 6!\sum_{n=3}^{\infty}\frac{1}{(n!)^2}\bigl((n+1)!-n!\bigr) < 1238,$$

$$\mu_{\mathcal{G}} = \int_0^\infty \overline{\mathcal{G}}(y)\,\mathrm{d}y \leqslant 8! + 8!\sum_{n=3}^\infty \frac{1}{((2^n)!)^2}\big((2^n+1)! - (2^n)!\big) + 8!\sum_{n=3}^\infty \left(\frac{1}{(2^n+1)!} - \frac{1}{(2^{n+1})!}\right) < 98243,$$

implying (17). □

Proof of (18). Suppose that n is sufficiently large and let

$$(n+1)! - b_n d_n \leqslant x < (n+1)!. \tag{21}$$

For such x, we have

$$\Delta_{\mathcal{F}}(x,y) = \begin{cases} \frac{y}{d_n} & \text{if } 0 \leqslant y \leqslant x - ((n+1)! - b_n d_n), \\ \frac{x-((n+1)!-b_n d_n)}{d_n+(n+1)!-x} & \text{if } x - ((n+1)! - b_n d_n) < y \leqslant \frac{x}{2}. \end{cases}$$

Therefore, for x in (21), we have

$$\begin{aligned} J_{\mathcal{F}}(x) &:= \int_0^{x/2} \Delta_{\mathcal{F}}(x,y)\overline{\mathcal{F}}(y)\,\mathrm{d}y \\ &\leqslant \frac{1}{d_n}\int_0^{b_n d_n} y\overline{\mathcal{F}}(y)\,\mathrm{d}y + b_n \int_{b_n d_n}^\infty \overline{\mathcal{F}}(y)\,\mathrm{d}y =: K_1 + K_2. \end{aligned} \tag{22}$$

Define $k_n := \max\{k\colon\ (k+1)! \leqslant b_n d_n\}$ and write

$$\begin{aligned} K_1 \leqslant{}& \frac{(6!)^2}{2d_n}\Biggl\{1 + \sum_{k=6}^{k_n+1} \frac{1}{(k!)^2}\big(\big((k+1)! - b_k d_k\big)^2 - (k!)^2\big) \\ &+ \sum_{k=6}^{k_n+1} \frac{1}{((k+1)!)^2}\left(1 + \frac{(k+1)!}{d_k}\right)\big(\big((k+1)!\big)^2 - \big((k+1)! - b_k d_k\big)^2\big)\Biggr\} \\ \leqslant{}& \frac{(6!)^2}{2d_n}\left(1 + 3\sum_{k=6}^{k_n+1} b_k\right) = \frac{(6!)^2}{2d_n}\left(1 + 3\sum_{k=6}^{k_n+1} (k^2+2k)\right) \end{aligned} \tag{23}$$

because

$$\frac{1}{(k!)^2}\big(((k+1)! - b_k d_k)^2 - (k!)^2\big) \leqslant (k+1)^2 - 1 = b_k$$

and

$$\left(1 + \frac{(k+1)!}{d_k}\right)\big(\big((k+1)!\big)^2 - \big((k+1)! - b_k d_k\big)^2\big) \leqslant 2\big((k+1)!\big)^2 b_k.$$

Thus, by (23) and Lemma 4,

$$K_1 \leqslant (6!)^2 \frac{k_n^3}{d_n} \leqslant (6!)^2 \left(\frac{2\log b_n}{\log\log b_n}\right)^3 \frac{1}{(\log b_n)^3} = \frac{8(6!)^2}{(\log\log b_n)^3}. \tag{24}$$

For the second integral in (22), we have

$$\begin{aligned} K_2 &\leqslant b_n(6! - b_n d_n)^+ + b_n \sum_{k=k_n+1}^{\infty} \frac{6!}{(k!)^2}\big((k+1)! - k!\big) \\ &= b_n(6! - b_n d_n)^+ + b_n 6! \sum_{k=k_n}^{\infty} \frac{1}{k!} \\ &\leqslant b_n(6! - b_n d_n)^+ + \frac{24(6!)}{(\log\log b_n)^2 \log b_n} \end{aligned} \tag{25}$$

because of the following estimate:

$$\begin{aligned} b_n \sum_{k=k_n}^{\infty} \frac{1}{k!} \leqslant \frac{\mathrm{e}}{k_n!} \frac{b_n d_n}{d_n} &\leqslant \frac{\mathrm{e}}{k_n!} \frac{(k_n+2)!}{d_n} \\ &\leqslant \frac{2\mathrm{e} k_n^2}{d_n} \leqslant 6\left(\frac{2\log b_n}{\log\log b_n}\right)^2 \frac{1}{(\log b_n)^3} \\ &= \frac{24}{(\log\log b_n)^2 \log b_n}. \end{aligned}$$

Here we have used that, by definition of k_n, $b_n d_n \leqslant (k_n+2)! \leqslant 2k_n!$ and then applied Lemma 4. Substituting estimates (24)–(25) into (22), we get that for x from (21), it holds

$$J_{\mathcal{F}}(x) \leqslant \frac{c_1}{(\log\log b_n)^3} \tag{26}$$

for some positive constant c_1.

Now, consider x satisfying

$$(n+1)! \leqslant x < (n+2)! - b_{n+1} d_{n+1}. \tag{27}$$

We split this interval into three subintervals

$$(n+1)! \leqslant x < 2\big((n+1)! - b_n d_n\big), \tag{28}$$

$$2\big((n+1)! - b_n d_n\big) \leqslant x < 2(n+1)!, \tag{29}$$

$$2(n+1)! \leqslant x < (n+2)! - b_{n+1} d_{n+1} \tag{30}$$

and estimate $J_{\mathcal{F}}(x)$ in each case separately.

In case (28), we have

$$\varDelta_{\mathcal{F}}(x,y) = \begin{cases} \frac{y}{d_n} & \text{if}\ , 0 \leqslant y \leqslant x - (n+1)!, \\ \frac{(n+1)! - x + y}{d_n} & \text{if}\ x - (n+1)! < y \leqslant x - ((n+1)! - b_n d_n), \\ b_n & \text{if}\ x - ((n+1)! - b_n d_n) < y \leqslant \frac{x}{2}. \end{cases}$$

Since

$$\frac{(n+1)!-x+y}{d_n} \leqslant \min\left\{\frac{y}{d_n}, b_n\right\},$$

for the x from (28) and for $x-(n+1)! < y \leqslant x-\big((n+1)!-b_n d_n\big)$, we get that $J_{\mathcal{F}}(x) \leqslant K_1+K_2$ and estimate (26) holds again.

Consider now case (29). We have

$$\Delta_{\mathcal{F}}(x,y) = \begin{cases} 0 & \text{if } 0 \leqslant y \leqslant x-(n+1)!, \\ \frac{(n+1)!-x+y}{d_n} & \text{if } x-(n+1)! < y \leqslant \frac{x}{2}. \end{cases}$$

Thus,

$$\begin{aligned} J_{\mathcal{F}}(x) &= \int\limits_{x-(n+1)!}^{x/2} \frac{(n+1)!-x+y}{d_n}\,\overline{\mathcal{F}}(y)\,\mathrm{d}y \\ &\leqslant b_n \int\limits_{b_n d_n}^{\infty} \overline{\mathcal{F}}(y)\,\mathrm{d}y = K_2 \leqslant \frac{c_2}{\log b_n(\log\log b_n)^2}, \end{aligned}$$

according to estimate (25), where c_2 is some positive constant.

Finally, in case (30), $\Delta_{\mathcal{F}}(x,y)=0$ for all $0 \leqslant y \leqslant x/2$, implying $J_{\mathcal{F}}(x)=0$.

Summarizing, estimate (26) holds for all x in (27) and for all sufficiently large n. This implies relation (18). □

Proof of (19). Suppose that n is sufficiently large and split the interval

$$\big(2^n\big)! \leqslant x < \big(2^{n+1}\big)!$$

into following subintervals:

$$\big(2^n\big)! \leqslant x < 2\big(2^n\big)!, \tag{31}$$

$$2\big(2^n\big)! \leqslant x < \big(2^n+1\big)! - \widehat{b}_n\widehat{d}_n, \tag{32}$$

$$\big(2^n+1\big)! - \widehat{b}_n\widehat{d}_n \leqslant x < \big(2^n+1\big)!, \tag{33}$$

$$\big(2^n+1\big)! \leqslant x < 2\big(\big(2^n+1\big)! - \widehat{b}_n\widehat{d}_n\big), \tag{34}$$

$$2\big(\big(2^n+1\big)! - \widehat{b}_n\widehat{d}_n\big) \leqslant x < 2\big(2^n+1\big)!, \tag{35}$$

$$2\big(2^n+1\big)! \leqslant x < \big(2^{n+1}\big)!. \tag{36}$$

As in the case of $\mathcal{F}$, for each subset above, we will obtain the exact expressions for $\Delta_{\mathcal{G}}(x,y)$ and then, the upper bounds for $J_{\mathcal{G}}(x)$.

In case (31),

$$\Delta_{\mathcal{G}}(x,y) = \begin{cases} 0 & \text{if } 0 \leqslant y \leqslant x-(2^n)!, \\ \frac{((2^n)!)^2-(x-y)^2}{(x-y)^2} & \text{if } x-(2^n)! < y \leqslant \frac{x}{2}, \end{cases}$$

and, consequently,

$$J_{\mathcal{G}}(x) \leqslant \int_0^{x/2} \left(\left(\frac{x}{x-y} \right)^2 - 1 \right) \overline{\mathcal{G}}(y)\,\mathrm{d}y.$$

Since $(x/(x-y))^2 \leqslant 4$, by the dominated convergence theorem, we have that

$$\sup_{(2^n)! \leqslant x < 2(2^n)!} J_{\mathcal{G}}(x) = \epsilon_1(n) \to 0, \quad n \to \infty. \tag{37}$$

In case (32), $\Delta_{\mathcal{G}}(x,y) = 0$ for $0 \leqslant y \leqslant x/2$, implying

$$\sup_{2(2^n)! \leqslant x < (2^n+1)! - \widehat{b}_n \widehat{d}_n} J_{\mathcal{G}}(x) = 0. \tag{38}$$

In case (33), we have

$$\Delta_{\mathcal{G}}(x,y) = \begin{cases} \frac{y}{(2^n+1)! + \widehat{d}_n - x} & \text{if } 0 \leqslant y < x - \big((2^n+1)! - \widehat{b}_n \widehat{d}_n\big), \\ \frac{x - ((2^n+1)! - \widehat{b}_n \widehat{d}_n)}{(2^n+1)! + \widehat{d}_n - x} & \text{if } x - ((2^n+1)! - \widehat{b}_n \widehat{d}_n) \leqslant y < \frac{x}{2}, \end{cases}$$

implying that

$$\begin{aligned} J_{\mathcal{G}}(x) &\leqslant \int_0^{x/2} \min\left\{ \frac{y}{\widehat{d}_n}, \widehat{b}_n \right\} \overline{\mathcal{G}}(y)\,\mathrm{d}y \leqslant \frac{1}{\widehat{d}_n} \int_0^{\widehat{d}_n \widehat{b}_n} y \overline{\mathcal{G}(y)}\,\mathrm{d}y + \widehat{b}_n \int_{\widehat{d}_n \widehat{b}_n}^{\infty} \overline{\mathcal{G}}(y)\,\mathrm{d}y \\ &=: L_1 + L_2. \end{aligned}$$

Analogously to k_n, define $\widehat{k}_n := \max\{k\colon (2^k+1)! \leqslant \widehat{b}_n \widehat{d}_n\}$. We get

$$\begin{aligned} L_1 &\leqslant \frac{1}{\widehat{d}_n} \int_0^{\widehat{b}_n \widehat{d}_n} y \Bigg\{ \mathbf{1}_{(-\infty,8!)}(y) + (8!)^2 \sum_{k=3}^{\infty} \frac{1}{((2^k)!)^2} \mathbf{1}_{[(2^k)!,(2^k+1)! - \widehat{b}_k \widehat{d}_k)}(y) \\ &\quad + (8!)^2 \sum_{k=3}^{\infty} \frac{1+\widehat{b}_k}{((2^k+1)!)^2} \mathbf{1}_{[(2^k+1)! - \widehat{b}_k \widehat{d}_k, (2^k+1)!)}(y) \\ &\quad + (8!)^2 \sum_{k=3}^{\infty} \frac{1}{y^2} \mathbf{1}_{[(2^k+1)!,(2^{k+1})!)}(y) \Bigg\}\,\mathrm{d}y \\ &\leqslant \frac{(8!)^2}{2\widehat{d}_n} + \frac{(8!)^2}{\widehat{d}_n} \sum_{k=3}^{\widehat{k}_n+1} \frac{1}{((2^k)!)^2} \int_{(2^k)!}^{(2^k+1)!} y\,\mathrm{d}y + \frac{(8!)^2}{\widehat{d}_n} \int_{9!}^{\widehat{b}_n \widehat{d}_n} \frac{1}{y}\,\mathrm{d}y \\ &\leqslant \frac{(8!)^2}{2\widehat{d}_n} + \frac{(8!)^2}{\widehat{d}_n} \log(\widehat{b}_n \widehat{d}_n) + \frac{(8!)^2}{2\widehat{d}_n} \sum_{k=3}^{\widehat{k}_n+1} \widehat{b}_k = \epsilon_2(n) \to 0, \quad n \to \infty, \end{aligned} \tag{39}$$

because for sufficiently large n,

$$\sum_{k=3}^{\widehat{k}_n+1} \widehat{b}_k \leqslant \frac{16}{3}\, 2^{2\widehat{k}_n} + 8 \cdot 2^{\widehat{k}_n} \leqslant 6 \cdot 2^{2\widehat{k}_n} \leqslant \frac{24(\log \widehat{b}_n)^2}{(\log\log \widehat{b}_n)^2}$$

due to Lemma 4.

For the integral L_2, we obtain

$$\begin{aligned} L_2 &\leqslant \widehat{b}_n \Biggl((8! - \widehat{b}_n \widehat{d}_n)^+ + (8!)^2 \sum_{k=\widehat{k}_n+1}^{\infty} \frac{(2^k+1)! - (2^k)!}{((2^k)!)^2} + (8!)^2 \int_{\widehat{b}_n \widehat{d}_n}^{\infty} \frac{\mathrm{d}y}{y^2} \Biggr) \\ &\leqslant \widehat{b}_n \Biggl((8! - \widehat{b}_n \widehat{d}_n)^+ + (8!)^2 \sum_{k=\widehat{k}_n+1}^{\infty} \frac{1}{(2^k-1)!} + \frac{(8!)^2}{\widehat{b}_n \widehat{d}_n} \Biggr) \\ &\leqslant \widehat{b}_n (8! - \widehat{b}_n \widehat{d}_n)^+ + \frac{(8!)^2 \mathrm{e}\, \widehat{b}_n}{(2^{\widehat{k}_n+1}-1)!} + \frac{(8!)^2}{\widehat{d}_n} = \epsilon_3(n) \to 0, \quad n \to \infty, \end{aligned} \tag{40}$$

because $(2^{\widehat{k}_n+1} - 1)! > \widehat{b}_n \widehat{d}_n$ for large n, according to definition of the sequence $\widehat{k}_n$. Relations (39)–(40) imply that

$$\sup_{2(2^n)! \leqslant x < (2^n+1)! - b_n d_n} J_{\mathcal{G}}(x) \leqslant \epsilon_2(n) + \epsilon_3(n) \to 0, \quad n \to \infty. \tag{41}$$

In case (34), we obtain

$$\varDelta_{\mathcal{G}}(x,y) = \begin{cases} \frac{y(2x-y)}{(x-y)^2} & \text{if } 0 \leqslant y \leqslant x-(2^n+1)!, \\ \frac{x^2(\widehat{d}_n+(2^n+1)!-x+y)}{\widehat{d}_n((2^n+1)!)^2} & \text{if } x-(2^n+1)! < y \leqslant x-((2^n+1)!-\widehat{b}_n\widehat{d}_n), \\ (\frac{x}{(2^n)!})^2 - 1 & \text{if } x-((2^n+1)!-\widehat{b}_n\widehat{d}_n) < y \leqslant \frac{x}{2}. \end{cases} \tag{42}$$

Hence,

$$\begin{aligned} J_{\mathcal{G}}(x) = & \int_0^{x-(2^n+1)!} \frac{y(2x-y)}{(x-y)^2}\, \overline{\mathcal{G}}(y)\,\mathrm{d}y \\ & + \int_{x-(2^n+1)!}^{x-((2^n+1)!-\widehat{b}_n\widehat{d}_n)} \biggl(\frac{x^2(\widehat{d}_n + (2^n+1)! - x + y)}{\widehat{d}_n((2^n+1)!)^2} - 1 \biggr) \overline{\mathcal{G}}(y)\,\mathrm{d}y \\ & + \int_{x-((2^n+1)!-\widehat{b}_n\widehat{d}_n)}^{x/2} \biggl(\biggl(\frac{x}{(2^n)!} \biggr)^2 - 1 \biggr) \overline{\mathcal{G}}(y)\,\mathrm{d}y. \end{aligned} \tag{43}$$

In particular case of (34), when $(2^n+1)! \leqslant x < (2^n+1)! + \widehat{b}_n \widehat{d}_n$, by estimating the above integrals separately, we get

$$J_{\mathcal{G}}(x) \leqslant \int_0^\infty \left(\frac{x}{x-y}\right)^2 \mathbf{1}_{[0,x/2]}(y)\overline{\mathcal{G}}(y)\,\mathrm{d}y - \int_0^\infty \overline{\mathcal{G}}(y)\,\mathrm{d}y + \epsilon_4(n) + \frac{2}{\widehat{d}_n} \int_0^{\widehat{b}_n \widehat{d}_n} y\overline{\mathcal{G}}(y)\,\mathrm{d}y + 2\widehat{b}_n \int_{\widehat{b}_n \widehat{d}_n}^{2\widehat{b}_n \widehat{d}_n} \overline{\mathcal{G}}(y)\,\mathrm{d}y + \widehat{b}_n \int_{\widehat{b}_n \widehat{d}_n}^{\infty} \overline{\mathcal{G}}(y)\,\mathrm{d}y$$

for some vanishing function $\epsilon_4(n)$. Thus, for large n and for all $x \in [(2^n+1)!, (2^n+1)! + \widehat{b}_n \widehat{d}_n)$, we have that

$$\begin{aligned} J_{\mathcal{G}}(x) &\leqslant \epsilon_1(n) + \epsilon_4(n) + 3(L_1 + L_2) \\ &\leqslant \epsilon_1(n) + \epsilon_4(n) + 3\bigl(\epsilon_2(n) + \epsilon_3(n)\bigr) \to 0. \end{aligned} \tag{44}$$

For the remaining subinterval of (34), where $(2^n+1)! + \widehat{b}_n \widehat{d}_n \leqslant x < 2((2^n+1)! - \widehat{b}_n \widehat{d}_n)$, using expressions (42) and (43), we obtain

$$\begin{aligned} J_{\mathcal{G}}(x) &\leqslant \int_0^\infty \left(\frac{x}{x-y}\right)^2 \mathbf{1}_{[0,x/2]}(y)\overline{\mathcal{G}}(y)\,\mathrm{d}y - \int_0^\infty \overline{\mathcal{G}}(y)\,\mathrm{d}y + 3\widehat{b}_n \int_{\widehat{b}_n \widehat{d}_n}^{\infty} \overline{\mathcal{G}}(y)\,\mathrm{d}y \\ &\leqslant \epsilon_1(n) + 3\,\epsilon_3(n) \to 0. \end{aligned} \tag{45}$$

Relations (44) and (45) imply that

$$\sup_{2(2^n+1)! \leqslant x < 2((2^n+1)! - \widehat{b}_n \widehat{d}_n)} J_{\mathcal{G}}(x) \leqslant \epsilon_5(n) \tag{46}$$

with some vanishing function ϵ_5.

Consider now case (35). For such x,

$$\Delta_{\mathcal{G}}(x,y) = \begin{cases} \frac{y(2x-y)}{(x-y)^2} & \text{if } 0 \leqslant y \leqslant x - (2^n+1)!, \\ \frac{x^2(\widehat{d}_n + (2^n+1)! - x + y)}{\widehat{d}_n((2^n+1)!)^2} - 1 & \text{if } x - (2^n+1)! < y \leqslant \frac{x}{2}, \end{cases}$$

implying that

$$J_{\mathcal{G}}(x) = \int_0^{x-(2^n+1)!} \frac{y(2x-y)}{(x-y)^2}\,\overline{\mathcal{G}}(y)\,\mathrm{d}y + \int_{x-(2^n+1)!}^{x/2} \left(\frac{x^2(\widehat{d}_n + (2^n+1)! - x + y)}{\widehat{d}_n((2^n+1)!)^2} - 1\right)\overline{\mathcal{G}}(y)\,\mathrm{d}y.$$

In the case under consideration, we have that $x - (2^n+1)! \geqslant \widehat{b}_n \widehat{d}_n$ and

$$\frac{x^2(\widehat{d}_n + (2^n+1)! - x + y)}{\widehat{d}_n((2^n+1)!)^2} - 1 \leqslant 4\,\frac{\widehat{d}_n + (2^n+1)! - x/2}{\widehat{d}_n} \leqslant \frac{\widehat{d}_n + \widehat{b}_n \widehat{d}_n}{\widehat{d}_n} \leqslant 5\widehat{b}_n.$$

The derived estimates yield

$$J_{\mathcal{G}}(x) \leqslant \int_0^\infty \left(\frac{x}{x-y}\right)^2 \mathbf{1}_{[0,x/2]}(y)\overline{\mathcal{G}}(y)\,\mathrm{d}y - \int_0^\infty \overline{\mathcal{G}}(y)\,\mathrm{d}y + 5L_2,$$

which implies that

$$\sup_{2((2^n+1)! - \widehat{b}_n \widehat{d}_n) \leqslant x < 2(2^n+1)!} J_{\mathcal{G}}(x) \leqslant \epsilon_1(n) + 5\epsilon_3(n). \tag{47}$$

Finally, consider case (36). For these x and for all $0 \leqslant y \leqslant x/2$,

$$\Delta_{\mathcal{G}}(x,y) = \frac{y(2x-y)}{(x-y)^2}.$$

Thus,

$$\sup_{2(2^n+1)! \leqslant x < (2^{n+1})!} J_{\mathcal{G}}(x) \leqslant \epsilon_1(n). \tag{48}$$

The derived estimates (37), (38), (41), (46), (47), and (48) imply that

$$\lim_{n\to\infty} \sup_{(2^n)! \leqslant x < (2^{n+1})!} J_{\mathcal{G}}(x) \to 0,$$

showing the validity of (19).

It remains to prove inequality (20). Integral from this inequality is bounded from below by

$$J_{\mathcal{F},\mathcal{G}}(x) := \int_0^{x/2} \frac{\overline{\mathcal{G}}(x-y) - \overline{\mathcal{G}}(x)}{\overline{\mathcal{F}}(x) + \overline{\mathcal{G}}(x)}\,\overline{\mathcal{F}}(y)\,\mathrm{d}y.$$

Take $x_n := (2^n+1)!$. Then $\overline{\mathcal{F}}(x_n) = \overline{\mathcal{G}}(x_n) = 1/x_n^2$, implying that

$$J_{\mathcal{F},\mathcal{G}}(x_n) = \frac{1}{2}\int_0^{x_n/2} \frac{\overline{\mathcal{G}}(x_n-y) - \overline{\mathcal{G}}(x_n)}{\overline{\mathcal{G}}(x_n)}\,\overline{\mathcal{F}}(y)\,\mathrm{d}y \geqslant \frac{1}{2}\int_0^{\widehat{b}_n \widehat{d}_n} \Delta_{\mathcal{G}}(x_n,y)\overline{\mathcal{F}}(y)\,\mathrm{d}y$$

for large n. According to (42),

$$\Delta_{\mathcal{G}}(x_n, y) = \frac{\widehat{d}_n + y}{\widehat{d}_n} \geqslant \frac{y}{\widehat{d}_n}.$$

Consequently, denoting $\widetilde{k}_n = \max\{k\colon (k+1)! \leqslant \widehat{b}_n \widehat{d}_n\}$, we get

$$\begin{aligned}
J_{\mathcal{F},\mathcal{G}}(x_n) &\geqslant \frac{1}{2\widehat{d}_n} \int\limits_0^{\widehat{b}_n \widehat{d}_n} y \overline{\mathcal{F}}(y)\,\mathrm{d}y \\
&= \frac{1}{2\widehat{d}_n} \int\limits_0^{\widehat{b}_n \widehat{d}_n} y \Biggl\{ \mathbf{1}_{(-\infty,6!)}(y) + (6!)^2 \sum_{k=6}^{\infty} \frac{1}{(k!)^2} \mathbf{1}_{[k!,(k+1)!-b_k d_k)}(y) \\
&\quad + (6!)^2 \sum_{k=6}^{\infty} \frac{1}{((k+1)!)^2} \biggl(1 + \frac{(k+1)! - y}{d_k}\biggr) \mathbf{1}_{[(k+1)!-b_k d_k,(k+1)!)}(y) \Biggr\}\,\mathrm{d}y \\
&\geqslant \frac{(6!)^2}{4\widehat{d}_n} \sum_{k=6}^{\widetilde{k}_n} \frac{((k+1)! - b_k d_k)^2 - (k!)^2}{(k!)^2} \\
&= \frac{(6!)^2}{4\widehat{d}_n} \sum_{k=6}^{\widetilde{k}_n} \biggl(\biggl(k + 1 - \frac{b_k d_k}{k!}\biggr)^2 - 1 \biggr) \\
&\geqslant \frac{(6!)^2}{4\widehat{d}_n} \Biggl(\sum_{k=6}^{\widetilde{k}_n} k^2 - 2 \sum_{k=6}^{\widetilde{k}_n} \frac{k+1}{k!}\, b_k d_k \Biggr).
\end{aligned}$$

Since the series

$$\sum_{k=6}^{\infty} \frac{k+1}{k!}\, b_k d_k$$

converges, we have that

$$J_{\mathcal{F},\mathcal{G}}(x_n) \geqslant c_2 \biggl(\frac{\widetilde{k}_n^3}{\widehat{d}_n} - c_3 \biggr)$$

for large n with some positive constants c_2 and c_3. As $\widehat{b}_n \widehat{d}_n = \widehat{b}_n (\log \widehat{b}_n)^2$, applying Lemma 4 with $a_n = \widehat{b}_n$ and $\beta = 2$ to the sequence $\widetilde{k}_n$, we get

$$\frac{\widetilde{k}_n^3}{\widehat{d}_n} \geqslant \frac{\log \widehat{b}_n}{(\log\log \widehat{b}_n)^3} \to \infty, \quad n \to \infty.$$

Therefore,

$$\lim_{n\to\infty} J_{\mathcal{F},\mathcal{G}}(x_n) = \infty,$$

and the desired inequality (20) follows. Theorem 2 is proved. $\square$

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