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Maximal elements and equilibria for generalized majorized maps

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Abstract. This papers presents new maximal element type results for general majorized type maps, and as an application, we consider equilibria for some one-person games.

Keywords: fixed point theory, coincidence theory, maximal elements, majorized maps.

1 Introduction

In this paper, we use fixed and coincidence point results of the author to present new maximal element type results for general majorized type maps (these general majorized type maps generalize majorized maps in the literature [2,3,7–9,13,14]). As an application, we show how our new maximal element type results will guarantee equilibria for some one-person games.

In this paper, we consider $\Phi^\star$ maps from the literature [1] and also admissible maps in the sense of Gorniewicz [6]. First, we describe the maps. Let H be the Čech homology functor with compact carriers and coefficients in the field of rational numbers K from the category of Hausdorff topological spaces and continuous maps to the category of graded vector spaces and linear maps of degree zero. Thus $H(X) = \{H_q(X)\}$ (here X is a Hausdorff topological space) is a graded vector space, $H_q(X)$ being the q-dimensional Čech homology group with compact carriers of X. For a continuous map $f : X \to X$, $H(f)$ is the induced linear map $f_\star = \{f_{\star\, q}\}$, where $f_{\star\, q} : H_q(X) \to H_q(X)$. A space X is acyclic if X is nonempty, $H_q(X) = 0$ for every $q \geqslant 1$, and $H_0(X) \approx K$.

Let X, Y and Γ be Hausdorff topological spaces. A continuous single-valued map $p : \Gamma \to X$ is called a Vietoris map (written $p : \Gamma \Rightarrow X$) if the following two conditions are satisfied:

(i) for each $x \in X$, the set $p^{-1}(x)$ is acyclic;
(ii) p is a perfect map, i.e., p is closed and for every $x \in X$ the set $p^{-1}(x)$ is nonempty and compact.

Let $\phi : X \to Y$ be a multivalued map (for each $x \in X$, we assume that $\phi(x)$ is a nonempty subset of Y). A pair (p, q) of single-valued continuous maps of the form $X \overset{p}{\leftarrow} \Gamma \overset{q}{\rightarrow} Y$ is called a selected pair of ϕ (written $(p, q) \subset \phi$) if the following two conditions hold:

(i) p is a Vietoris map, and
(ii) $q(p^{-1}(x)) \subset \phi(x)$ for any $x \in X$.

Now we define the admissible maps of Gorniewicz [6]. An upper semicontinuous map $\phi : X \to Y$ with compact values is said to be admissible (and we write $\phi \in Ad(X, Y)$), provided there exists a selected pair (p, q) of ϕ. An example of an admissible map is a Kakutani map. A upper semicontinuous map $\phi : X \to K(Y)$ is said to be Kakutani (and we write $\phi \in Kak(X, Y)$); here Y is a Hausdorff topological vector space, and $K(Y)$ denotes the family of nonempty, convex, compact subsets of Y.

The following class of maps will play a major role in this paper. Let Z and W be subsets of Hausdorff topological vector spaces Y_1 and Y_2, and G is a multifunction. We say $G \in \Phi^\star(Z, W)$ [2] if W is convex and there exists a map $S : Z \to W$ with $S(x) \subseteq G(x)$ for $x \in Z$, $S(x) \neq \emptyset$ and has convex values for each $x \in Z$, and the fibre $S^{-1}(w) = \{z \in Z\colon\ w \in S(z)\}$ is open (in Z) for each $w \in W$.

2 Fixed, maximal and coincidence theory

In this section, we present old and new fixed, coincidence and maximal type point results. In particular, we will focus on majorized type maps, and in addition, our maps considered will be either coercive, compact or condensing type. Also, we will apply our results to some one-person games.

We begin with some ideas on a generalization of majorized type maps [2,3,7,9,13,14]. Let Z and W be sets in a Hausdorff topological vector space with W convex and Z compact. Suppose $H : Z \to W$, and for each $y \in Z$, assume that there exist a map $A_y : Z \to W$ and an open set U_y containing y with $H(z) \subseteq A_y(z)$ for every $z \in U_y$, A_y is convex-valued, and $(A_y)^{-1}(x)$ is open (in Z) for each $x \in W$. We now claim that there exists a map $T : Z \to W$ with $H(z) \subseteq T(z)$ for $z \in Z$, T is convex-valued and $T^{-1}(x)$ is open (in Z) for each $x \in W$. To see this, note $\{U_y\}_{y\in Z}$ is an open covering of Z, and since Z is compact, there exist a finite set $\{y_1, \dots, y_n\}$ (with $y_i \in Z$ for $i \in \{1, \dots, n\}$) and an open covering $\{V_{y_i}\}_{i=1}^n$ of Z with $y_i \in V_{y_i}$ and $\Omega_{y_i} = \overline{V_{y_i}} \subseteq U_{y_i}$ for $i \in \{1, \dots, n\}$ [4,5]. Fix $i \in \{1, \dots, n\}$ and let

$$Q_{y_i}(z) = \begin{cases} A_{y_i}(z), & z \in \Omega_{y_i}, \\ W, & z \in Z \setminus \Omega_{y_i}. \end{cases}$$

Now Q_{y_i} is convex-valued, and $H(z) \subseteq Q_{y_i}(z)$ for every $z \in Z$ (note if $z \in \Omega_{y_i}$, then since $\Omega_{y_i} \subseteq U_{y_i}$ and since $H(w) \subseteq A_{y_i}(w)$ for $w \in U_{y_i}$, we have $H(z) \subseteq Q_{y_i}(z)$, whereas if $z \in Z \setminus \Omega_{y_i}$, then it is immediate since $Q_{y_i}(z) = W$). Note that for any $x \in W$,

$$\begin{aligned}(Q_{y_i})^{-1}(x) &= \{z \in Z\colon\ x \in Q_{y_i}(z)\} \\ &= \{z \in Z \setminus \Omega_{y_i}\colon\ x \in Q_{y_i}(z) = W\} \cup \{z \in \Omega_{y_i}\colon\ x \in Q_{y_i}(z) = A_{y_i}(z)\} \\ &= (Z \setminus \Omega_{y_i}) \cup \{z \in \Omega_{y_i}\colon\ x \in A_{y_i}(z)\} \\ &= (Z \setminus \Omega_{y_i}) \cup [\Omega_{y_i} \cap \{z \in Z\colon\ x \in A_{y_i}(z)\}] \\ &= (Z \setminus \Omega_{y_i}) \cup [\Omega_{y_i} \cap A_{y_i}^{-1}(x)] = Z \cap [(Z \setminus \Omega_{y_i}) \cup A_{y_i}^{-1}(x)] \\ &= (Z \setminus \Omega_{y_i}) \cup A_{y_i}^{-1}(x),\end{aligned}$$

which is open in Z (note $A_{y_i}^{-1}(x)$ is open in Z, and Ω_{y_i} is closed in Z). Let $T : Z \to W$ be given by

$$T(z) = \bigcap_{i=1}^{n} Q_{y_i}(z) \quad \text{for } z \in Z.$$

Now T is convex-valued, $H(z) \subseteq T(z)$ for every $z \in Z$, and for $x \in W$, we have

$$\begin{aligned}T^{-1}(x) &= \{z \in Z\colon\ x \in T(z)\} = \left\{z \in Z\colon\ x \in \bigcap_{i=1}^{n} Q_{y_i}(z)\right\} \\ &= \bigcap_{i=1}^{n} \{z \in Z\colon\ x \in Q_{y_i}(z)\} = \bigcap_{i=1}^{n} (Q_{y_i})^{-1}(x),\end{aligned}$$

which is open in Z.

We begin by considering the case when the sets are compact. After we discuss the compact case, we will then consider the cases when the maps are either coercive or condensing. In [9], we established the following fixed point result.

Theorem 1. *Let $\{X_i\}_{i=1}^N$ be a family of convex compact sets each in a Hausdorff topological vector space E_i. For each $i \in \{1, \dots, N\}$, suppose $F_i : X \equiv \prod_{i=1}^N X_i \to X_i$, and in addition, there exists a map $S_i : X \to X_i$ with $S_i(x) \subseteq F_i(x)$ for $x \in X$, $S_i(x)$ has convex values for $x \in X$, and $S_i^{-1}(w)$ is open (in X) for each $w \in X_i$. Finally, suppose for each $x \in X$, there exists $i \in \{1, \dots, N\}$ with $S_i(x) \neq \emptyset$. Then there exist $x \in X$ and $i \in \{1, \dots, N\}$ with $x_i \in F_i(x)$ (here x_i is the projection of x on X_i).*

Now Theorem 1 immediately yields a maximal element type result.

Theorem 2. *Let $\{X_i\}_{i=1}^N$ be a family of convex compact sets each in a Hausdorff topological vector space. For each $i \in \{1, \dots, N\}$, suppose $F_i : X \equiv \prod_{i=1}^N X_i \to X_i$, and in addition, there exists a map $S_i : X \to X_i$ with $S_i(x) \subseteq F_i(x)$ for $x \in X$, $S_i(x)$ has convex values for $x \in X$, and $S_i^{-1}(w)$ is open (in X) for each $w \in X_i$. Now suppose for all $i \in \{1, \dots, N\}$, $x_i \notin F_i(x)$ for each $x \in X$. Then there exists $x \in X$ with $S_i(x) = \emptyset$ for all $i \in \{1, \dots, N\}$.*

Proof. Suppose the conclusion is false. Then for each $x \in X$, there exists $i \in \{1, \dots, N\}$ with $S_i(x) \neq \emptyset$. Now Theorem 1 guarantees that $x \in X$ and $i \in \{1, \dots, N\}$ with $x_i \in F_i(x)$, a contradiction. □

Remark 1. In Theorem 1 (and Theorem 2), one could replace $\{X_i\}_{i=1}^N$ with $\{X_i\}_{i\in I}$, where I is an index set if we rephrase Theorem 1 (and Theorem 2) appropriately (see [9, 11]). This is also true for other theorems in this paper, but we will not refer to it again.

Now we use Theorem 2 to obtain a result for very general majorized type maps.

Theorem 3. *Let $\{X_i\}_{i=1}^N$ be a family of convex compact sets each in a Hausdorff topological vector space. For each $i \in \{1,\dots,N\}$, suppose $H_i : X \equiv \prod_{i=1}^N X_i \to X_i$, and in addition, assume that there exists a map $T_i : X \to X_i$ with $H_i(w) \subseteq T_i(w)$ for $w \in X$, $T_i(x)$ has convex values for $x \in X$, $T_i^{-1}(z)$ is open (in X) for each $z \in X_i$, and $w_i \notin T_i(w)$ for each $w \in X$. Then there exists $x \in X$ with $H_i(x) = \emptyset$ for all $i \in \{1,\dots,N\}$.*

Proof. Apply Theorem 2 with $F_i = S_i = T_i$, so there exists $x \in X$ with $T_j(x) = \emptyset$ for all $j \in \{1,\dots,N\}$. Now since $H_j(w) \subseteq T_j(w)$ for $w \in X$, then $H_j(x) = \emptyset$ for all $j \in \{1,\dots,N\}$. □

Remark 2. Suppose that for each $i \in \{1,\dots,N\}$ and for each $x \in X$, there exist a map $A_{i,x} : X \to X_i$ and an open set $U_{i,x}$ containing x with $H_i(z) \subseteq A_{i,x}(z)$ for every $z \in U_{i,x}$, $A_{i,x}$ is convex-valued, $(A_{i,x})^{-1}(z)$ is open (in X) for each $z \in X_i$, and $w_i \notin A_{i,x}(w)$ for each $w \in U_{i,x}$.

From the discussion before Theorem 1 (with $Z = X$, $W = X_i$, $H = H_i$) there exists a map $T_i : X \to X_i$ with $H_i(w) \subseteq T_i(w)$ for $w \in X$, T_i is convex-valued, $(T_i)^{-1}(z)$ is open for each $z \in X_i$; here for $i \in \{1,\dots,N\}$, we have that $\{U_{i,x}\}_{x\in X}$ is an open covering of X, so there exist a finite set $\{y_{i,1},\dots,y_{i,n_i}\}$ (with $y_{i,j} \in X$ for $j \in \{1,\dots,n_i\}$), an open covering $\{V_{i,y_{i,j}}\}_{i=1}^{n_i}$ of X and $\Omega_{i,y_{i,j}} = \overline{V_{i,y_{i,j}}} \subseteq U_{i,y_{i,j}}$ for $j \in \{1,\dots,n_i\}$, and for fixed $j \in \{1,\dots,n_i\}$,

$$Q_{i,y_{i,j}}(z) = \begin{cases} A_{i,y_{i,j}}(z), & z \in \Omega_{i,y_{i,j}}, \\ X_i, & z \in X \setminus \Omega_{i,y_{i,j}}, \end{cases}$$

and

$$T_i(z) = \bigcap_{j=1}^{n_i} Q_{i,y_{i,j}}(z) \quad \text{for } z \in X.$$

We claim that $w_i \notin T_i(w)$ for each $w \in X$ and $i \in \{1,\dots,N\}$. To see this, let $i \in \{1,\dots,N\}$ and $w \in X$. Note there exists $k \in \{1,\dots,n\}$ with $y_{i,k} \in X$ and $w \in \Omega_{i,y_{i,k}}$ so

$$T_i(w) = \bigcap_{j=1}^{n_i} Q_{i,y_{i,j}}(w) \subseteq Q_{i,y_{i,k}}(w) = A_{i,y_{i,k}}(w),$$

and since $z_i \notin A_{i,x}(z)$ for each $z \in U_{i,x}$, we have $w_i \notin T_i(w)$.

Corollary 1. *Let X be a convex compact set in a Hausdorff topological vector space. Suppose that $H : X \to X$, and in addition, assume that there exists a map $T : X \to X$ with $H(w) \subseteq T(w)$ for $w \in X$, $T(x)$ has convex values for $x \in X$, $T^{-1}(z)$ is open (in X) for each $z \in X$, and $w \notin T(w)$ for each $w \in X$. Then there exists $x \in X$ with $H(x) = \emptyset$.*

Proof. This follows from Theorem 3 with $N = 1$. □

Remark 3. For each $x \in X$, suppose there exist a map $A_x : X \to X$ and an open set U_x containing x with $H(z) \subseteq A_x(z)$ for every $z \in U_x$, A_x is convex-valued, $A_x^{-1}(z)$ is open (in X) for each $z \in X$, and $w \notin A_x(w)$ for each $w \in U_x$. Then, as in Remark 2, there exists a map $T : X \to X$ with $H(w) \subseteq T(w)$ for $w \in X$, T is convex-valued, $T^{-1}(z)$ is open for each $z \in X$, and $w \notin T(w)$ for $w \in X$.

In [9], we established the following coincidence result.

Theorem 4. *Let $\{X_i\}_{i=1}^{N}$, $\{Y_i\}_{i=1}^{N_0}$ be families of convex sets each in a Hausdorff topological vector space E_i with $\prod_{i=1}^{N} X_i$ paracompact, and in addition, $\{Y_i\}_{i=1}^{N_0}$ is also a family of compact sets. For each $i \in \{1, \dots, N_0\}$, suppose $F_i : X \equiv \prod_{i=1}^{N} X_i \to Y_i$, and there exists a map $T_i : X \to Y_i$ with $T_i(x) \subseteq F_i(x)$ for $x \in X$, $T_i(x)$ has convex values for each $x \in X$, and $T_i^{-1}(w)$ is open (in X) for each $w \in Y_i$. For each $j \in \{1, \dots, N\}$, suppose $G_j : Y \equiv \prod_{i=1}^{N_0} Y_i \to X_j$, and there exists a map $S_j : Y \to X_j$ with $S_j(y) \subseteq G_j(y)$ for $y \in Y$, $S_j(y)$ has convex values for each $y \in Y$, and $S_j^{-1}(w)$ is open (in Y) for each $w \in X_j$. Finally, suppose that for each $x \in X$, there exists $i \in \{1, \dots, N_0\}$ with $T_i(x) \neq \emptyset$, and suppose for each $y \in Y$, there exists $j \in \{1, \dots, N\}$ with $S_j(y) \neq \emptyset$. Then there exist $x \in X$, $y \in Y$, $j_0 \in \{1, \dots, N_0\}$ and $i_0 \in \{1, \dots, N\}$ with $y_{j_0} \in F_{j_0}(x)$ and $x_{i_0} \in G_{i_0}(y)$.*

We can rewrite Theorem 4 as follows.

Theorem 5. *Let $\{X_i\}_{i=1}^{N}$, $\{Y_i\}_{i=1}^{N_0}$ be families of convex sets each in a Hausdorff topological vector space with $\prod_{i=1}^{N} X_i$ paracompact, and in addition, $\{Y_i\}_{i=1}^{N_0}$ is also a family of compact sets. For each $i \in \{1, \dots, N_0\}$, suppose that $F_i : X \equiv \prod_{i=1}^{N} X_i \to Y_i$, and there exists a map $T_i : X \to Y_i$ with $T_i(x) \subseteq F_i(x)$ for $x \in X$, $T_i(x)$ has convex values for each $x \in X$, and $T_i^{-1}(w)$ is open (in X) for each $w \in Y_i$. For each $j \in \{1, \dots, N\}$, suppose $G_j : Y \equiv \prod_{i=1}^{N_0} Y_i \to X_j$, and there exists a map $S_j : Y \to X_j$ with $S_j(y) \subseteq G_j(y)$ for $y \in Y$, $S_j(y)$ has convex values for each $y \in Y$, and $S_j^{-1}(w)$ is open (in Y) for each $w \in X_j$. Now suppose either for all $j \in \{1, \dots, N_0\}$, we have $y_j \notin F_j(x)$ for each $(x, y) \in X \times Y$ with $x_{i_0} \in G_{i_0}(y)$ for some $i_0 \in \{1, \dots, N\}$ or for all $i \in \{1, \dots, N\}$, we have $x_i \notin G_i(y)$ for each $(x, y) \in X \times Y$ with $y_{j_0} \in F_{j_0}(x)$ for some $j_0 \in \{1, \dots, N_0\}$. Then either there exists $x \in X$ with $T_i(x) = \emptyset$ for all $i \in \{1, \dots, N_0\}$ or there exists $y \in Y$ with $S_j(y) = \emptyset$ for all $j \in \{1, \dots, N\}$.*

Proof. Suppose the conclusion is false. Then for each $x \in X$, there exists $i \in \{1, \dots, N_0\}$ with $T_i(x) \neq \emptyset$, and for each $y \in Y$, there exists $j \in \{1, \dots, N\}$ with $S_j(y) \neq \emptyset$. Now Theorem 4 guarantees that $x \in X$, $y \in Y$, $j_0 \in \{1, \dots, N_0\}$, $i_0 \in \{1, \dots, N\}$ with $y_{j_0} \in F_{j_0}(x)$ and $x_{i_0} \in G_{i_0}(y)$, a contradiction. $\square$

Remark 4.

(i) To get a contradiction in the proof of Theorem 5, one only needs the statement "there exist $x \in X$, $y \in Y$, $j_0 \in \{1, \dots, N_0\}$, and $i_0 \in \{1, \dots, N\}$ with $y_{j_0} \in F_{j_0}(x)$, and $x_{i_0} \in G_{i_0}(y)$" to be false, so one could list other conditions to guarantee the contradiction.

(ii) Note (see Theorem 5) part of the assumption in [9, Thm. 2.6] was inadvertently omitted (but in fact, it is a condition mentioned in part (i)).

Theorem 6. *Let $\{X_i\}_{i=1}^N$, $\{Y_i\}_{i=1}^{N_0}$ be families of convex sets each in a Hausdorff topological vector space with $\prod_{i=1}^N X_i$ paracompact, and in addition, $\{Y_i\}_{i=1}^{N_0}$ is also a family of compact sets. For each $i \in \{1,\dots,N_0\}$ and for each $j \in \{1,\dots,N\}$, suppose $H_i : X \equiv \prod_{i=1}^N X_i \to Y_i$ and $\Psi_j : Y \equiv \prod_{i=1}^{N_0} Y_i \to X_j$, and in addition, there exists a map $T_i : X \to Y_i$ with $H_i(z) \subseteq T_i(z)$ for $z \in X$, $T_i(x)$ has convex values for each $x \in X$, $T_i^{-1}(w)$ is open (in X) for each $w \in Y_i$, and there exists a map $S_j : Y \to X_j$ with $\Psi_j(y) \subseteq S_j(y)$ for $y \in Y$, $S_j(y)$ has convex values for each $y \in Y$, and $S_j^{-1}(w)$ is open (in Y) for each $w \in X_j$. Now suppose either for all $j \in \{1,\dots,N_0\}$, we have $y_j \notin T_j(x)$ for each $(x,y) \in X \times Y$ with $x_{i_0} \in S_{i_0}(y)$ for some $i_0 \in \{1,\dots,N\}$ or for all $i \in \{1,\dots,N\}$, we have $x_i \notin S_i(y)$ for each $(x,y) \in X \times Y$ with $y_{j_0} \in T_{j_0}(x)$ for some $j_0 \in \{1,\dots,N_0\}$. Then either there exists $x \in X$ with $H_i(x) = \emptyset$ for all $i \in \{1,\dots,N_0\}$ or there exists $y \in Y$ with $\Psi_j(y) = \emptyset$ for all $j \in \{1,\dots,N\}$.*

Proof. Apply Theorem 5 (with $F_i = T_i$ and $G_j = S_j$), so either there exists $x \in X$ with $T_i(x) = \emptyset$ for all $i \in \{1,\dots,N_0\}$ or there exists $y \in Y$ with $S_j(y) = \emptyset$ for all $j \in \{1,\dots,N\}$. Now since $H_i(z) \subseteq T_i(z)$ for $z \in X$ and $\Psi_j(w) \subseteq S_j(w)$ for $w \in Y$ the conclusion follows. □

Remark 5.

(i) Note Theorem 6 improves Theorem 2.7 in [9].

(ii) Note we could consider maps of the type before the statement of Theorem 1 to create the maps T_i and S_j in Theorem 6 (see [9, Thm. 2.7]), where part of the assumption was inadvertently omitted (but in fact, it is a condition mentioned in Remark 4(i)).

Now one could also consider coincidence results between other classes. In [9], we established the following.

Theorem 7. *Let $\{X_i\}_{i=1}^N$, $\{Y_i\}_{i=1}^{N_0}$ be families of convex sets each in a Hausdorff topological vector space E_i, and in addition, $\{Y_i\}_{i=1}^{N_0}$ is also a family of compact sets. For each $i \in \{1,\dots,N_0\}$, suppose $F_i : X \equiv \prod_{i=1}^N X_i \to Y_i$ and $F_i \in Ad(X, Y_i)$. For each $j \in \{1,\dots,N\}$, suppose $G_j : Y \equiv \prod_{i=1}^{N_0} Y_i \to X_j$, and there exists a map $S_j : Y \to X_j$ with $S_j(y) \subseteq G_j(y)$ for $y \in Y$, $S_j(y)$ has convex values for each $y \in Y$, and $S_j^{-1}(w)$ is open (in Y) for each $w \in X_j$. Finally, suppose that for each $y \in Y$, there exists $j \in \{1,\dots,N\}$ with $S_j(y) \neq \emptyset$. Then there exist $x \in X$, $y \in Y$, $i_0 \in \{1,\dots,N\}$ with $y_j \in F_j(x)$ for all $j \in \{1,\dots,N_0\}$ and $x_{i_0} \in G_{i_0}(y)$.*

Now Theorem 7 immediately yields the following result.

Theorem 8. *Let $\{X_i\}_{i=1}^N$, $\{Y_i\}_{i=1}^{N_0}$ be families of convex sets each in a Hausdorff topological vector space, and in addition, $\{Y_i\}_{i=1}^{N_0}$ is also a family of compact sets. For each $i \in \{1,\dots,N_0\}$, suppose $F_i : X \equiv \prod_{i=1}^N X_i \to Y_i$ and $F_i \in Ad(X, Y_i)$. For each $j \in \{1,\dots,N\}$, suppose $G_j : Y \equiv \prod_{i=1}^{N_0} Y_i \to X_j$, and there exists a map $S_j : Y \to X_j$*

with $S_j(y) \subseteq G_j(y)$ for $y \in Y$, $S_j(y)$ has convex values for each $y \in Y$, and $S_j^{-1}(w)$ is open (in Y) for each $w \in X_j$. Now suppose either for all $i \in \{1,\dots,N\}$, we have $x_i \notin G_i(y)$ for each $(x,y) \in X \times Y$ with $y_j \in F_j(x)$ for all $j \in \{1,\dots,N_0\}$ or for each $(x,y) \in X \times Y$ with $x_{i_0} \in G_{i_0}(y)$ for some $i_0 \in \{1,\dots,N\}$, there exists $j \in \{1,\dots,N_0\}$ with $y_j \notin F_j(x)$. Then there exists $y \in Y$ with $S_i(y) = \emptyset$ for all $i \in \{1,\dots,N\}$.

Proof. Suppose the conclusion is false. Then for each $y \in Y$, there exists $j \in \{1,\dots,N\}$ with $S_j(y) \neq \emptyset$. Now Theorem 7 guarantees $x \in X$, $y \in Y$, $i_0 \in \{1,\dots,N\}$ with $y_j \in F_j(x)$ for all $j \in \{1,\dots,N_0\}$ and $x_{i_0} \in G_{i_0}(y)$, a contradiction. □

Remark 6.

(i) To get a contradiction in the proof of Theorem 8, one only needs the statement "there exist $x \in X$, $y \in Y$, $i_0 \in \{1,\dots,N\}$ with $y_j \in F_j(x)$ for all $j \in \{1,\dots,N_0\}$ and $x_{i_0} \in G_{i_0}(y)$" to be false, so one could list other conditions to guarantee the contradiction.

(ii) Note (see Theorem 8) part of the assumption in [9, Thm. 2.10] was inadvertently omitted (but in fact, it is a condition mentioned in part (i)).

Theorem 9. *Let $\{X_i\}_{i=1}^N$, $\{Y_i\}_{i=1}^{N_0}$ be families of convex sets each in a Hausdorff topological vector space, and in addition, $\{Y_i\}_{i=1}^{N_0}$ is also a family of compact sets. For each $i \in \{1,\dots,N_0\}$, suppose $F_i : X \equiv \prod_{i=1}^N X_i \to Y_i$ and $F_i \in Ad(X, Y_i)$. For each $j \in \{1,\dots,N\}$, suppose $\Psi_j : Y \equiv \prod_{i=1}^{N_0} Y_i \to X_j$, and there exists a map $S_j : Y \to X_j$ with $\Psi_j(z) \subseteq S_j(z)$ for $z \in Y$, $S_j(y)$ has convex values for each $y \in Y$, and $S_j^{-1}(w)$ is open (in Y) for each $w \in X_j$. Now suppose either for all $i \in \{1,\dots,N\}$, we have $x_i \notin S_i(y)$ for each $(x,y) \in X \times Y$ with $y_j \in F_j(x)$ for all $j \in \{1,\dots,N_0\}$ or for each $(x,y) \in X \times Y$ with $x_{i_0} \in S_{i_0}(y)$ for some $i_0 \in \{1,\dots,N\}$, there exists $j \in \{1,\dots,N_0\}$ with $y_j \notin F_j(x)$. Then there exists $y \in Y$ with $\Psi_i(y) = \emptyset$ for all $i \in \{1,\dots,N\}$.*

Proof. Apply Theorem 8 with $G_j = S_j$, so there exists $y \in Y$ with $S_i(y) = \emptyset$ for all $i \in \{1,\dots,N\}$. The result now follows since $\Psi_j(z) \subseteq S_j(z)$ for $z \in Y$. □

Remark 7.

(i) Note Theorem 9 improves [9, Thm. 2.11].

(ii) Note we could consider maps of the type before the statement of Theorem 1 to create the maps S_j in Theorem 16 (see [9, Thm. 2.11]), where part of the assumption was inadvertently omitted (but in fact, it is a condition mentioned in Remark 6(i)).

In our next results, we will consider the case when our maps are coercive [2,8,10,11]. We first redo the analysis before Theorem 1. Let Z and W be convex sets in a Hausdorff topological vector space with Z paracompact. Suppose $H : Z \to W$, and for each $y \in Z$, assume that there exist a map $A_y : Z \to W$ and an open set U_y containing y with $H(z) \subseteq A_y(z)$ for every $z \in U_y$, A_y is convex-valued, and $(A_y)^{-1}(x)$ is open (in Z) for each $x \in W$. We now claim that there exists a map $T : Z \to W$ with $H(z) \subseteq T(z)$

for $z \in Z$, T is convex-valued, and $T^{-1}(x)$ is open (in Z) for each $x \in W$. To see this, note $\{U_y\}_{y \in Z}$ is an open covering of Z, and since Z is paracompact, there exists a locally finite open covering $\{V_y\}_{y \in Z}$ of Z with $y \in V_y$ and $\Omega_y = \overline{V_y} \subseteq U_y$ for each $y \in Z$ [4,5]. Now for each $y \in Z$, let

$$Q_y(z) = \begin{cases} A_y(z), & z \in \Omega_y, \\ W, & z \in Z \setminus \Omega_y. \end{cases}$$

Note, as in the argument before Theorem 1, for any $x \in W$, we have

$$(Q_y)^{-1}(x) = (Z \setminus \Omega_y) \cup (A_y)^{-1}(x),$$

which is open in Z, Q_y is convex-valued, and $H(z) \subseteq Q_y(z)$ for every $z \in Z$ (to see this, note if $z \in \Omega_y$, then it is immediate since $\Omega_y \subseteq U_y$, whereas if $z \in Z \setminus \Omega_y$, then it is immediate since $Q_y(z) = W$). Let $T : Z \to W$ be given by

$$T(z) = \bigcap_{y \in Z} Q_y(z) \quad \text{for } z \in Z.$$

Now T is convex-valued, and $H(z) \subseteq T(z)$ for every $z \in Z$. It remains to show that $T^{-1}(x)$ is open for each $x \in W$. Fix $x \in W$ and let $u \in T^{-1}(x)$. We now claim that there exists an open set W_u containing u with $u \in W_u \subseteq T^{-1}(x)$, so then as a result, $T^{-1}(x)$ is open. To prove our claim, note since $\{V_y\}_{y \in Z}$ is locally finite, there exists an open neighborhood N_u of u (in Z) such that $\{y \in Z\colon N_u \cap V_y \neq \emptyset\} = \{y_1, \dots, y_m\}$ (a finite set). Now if $y \notin \{y_1, \dots, y_m\}$, then $\emptyset = V_y \cap N_u = \overline{V_y} \cap N_u = \Omega_y \cap N_u$, so $Q_y(z) = W$ for all $z \in N_u$, and as a result,

$$T(z) = \bigcap_{y \in Z} Q_y(z) = \bigcap_{i=1}^{m} Q_{y_i}(z) \quad \text{for all } z \in N_u.$$

Now $T^{-1}(x) = \{z \in Z\colon x \in T(z)\}$, whereas

$$\bigl\{z \in N_u\colon x \in T(z)\bigr\} = \left\{z \in N_u\colon x \in \bigcap_{i=1}^{m} Q_{y_i}(z)\right\} = N_u \cap \left[\bigcap_{i=1}^{m} (Q_{y_i})^{-1}(x)\right],$$

so

$$u \in N_u \cap \left[\bigcap_{i=1}^{m} (Q_{y_i})^{-1}(x)\right] \subseteq T^{-1}(x),$$

and our claim is true (note $N_u \cap [\cap_{i=1}^{m} (Q_{y_i})^{-1}(x)]$ is an open neighborhood of u).

In [11, Thm. 2.12], we established the following fixed point result for coercive maps. We note that coercive maps could be replaced by compact type maps if we use [11, Thm. 2.7] (we leave these obvious analogue statements of Theorems 10, 11 and 12 below to the reader).

Theorem 10. *Let $\{X_i\}_{i=1}^N$ be a family of convex sets each in a Hausdorff topological vector space E_i with $X = \prod_{i=1}^N X_i$ paracompact. For each $i \in \{1, \dots, N\}$, suppose $F_i : X \to X_i$, and in addition, there exists a map $S_i : X \to X_i$ with $S_i(x) \subseteq F_i(x)$ for $x \in X$, $S_i(x)$ has convex values for $x \in X$, and $S_i^{-1}(w)$ is open (in X) for each $w \in X_i$. Also, assume that there are a compact subset K of X and, for each $i \in \{1, \dots, N\}$, a convex compact subset Y_i of X_i such that for each $x \in X \setminus K$, there exists $j \in \{1, \dots, N\}$ with $S_j(x) \cap Y_j \neq \emptyset$. Suppose for each $x \in X$, there exists $i \in \{1, \dots, N\}$ with $S_i(x) \neq \emptyset$. Then there exist $x \in X$ and $i \in \{1, \dots, N\}$ with $x_i \in F_i(x)$.*

Now Theorem 10 immediately yields a maximal element type result for coercive maps (see [8] also).

Theorem 11. *Let $\{X_i\}_{i=1}^N$ be a family of convex sets each in a Hausdorff topological vector space with $X = \prod_{i=1}^N X_i$ paracompact. For each $i \in \{1, \dots, N\}$, suppose $F_i : X \to X_i$, and in addition, there exists a map $S_i : X \to X_i$ with $S_i(x) \subseteq F_i(x)$ for $x \in X$, $S_i(x)$ has convex values for $x \in X$, and $S_i^{-1}(w)$ is open (in X) for each $w \in X_i$. Also, assume that there are a compact subset K of X and, for each $i \in \{1, \dots, N\}$, a convex compact subset Y_i of X_i such that for each $x \in X \setminus K$, there exists $j \in \{1, \dots, N\}$ with $S_j(x) \cap Y_j \neq \emptyset$. Now suppose for all $i \in \{1, \dots, N\}$, $x_i \notin F_i(x)$ for each $x \in X$. Then there exists $x \in X$ with $S_i(x) = \emptyset$ for all $i \in \{1, \dots, N\}$.*

Now we use Theorem 11 to obtain a result for majorized type maps in the coercive situation.

Theorem 12. *Let $\{X_i\}_{i=1}^N$ be a family of convex sets each in a Hausdorff topological vector space with $X = \prod_{i=1}^N X_i$ paracompact. For each $i \in \{1, \dots, N\}$, suppose $H_i : X \to X_i$, and in addition, there exists a map $T_i : X \to X_i$ with $H_i(w) \subseteq T_i(w)$ for $w \in X$, $T_i(x)$ has convex values for $x \in X$, $T_i^{-1}(w)$ is open (in X) for each $w \in X_i$, and $w_i \notin T_i(w)$ for each $w \in X$. Also, assume that there are a compact subset K of X and, for each $i \in \{1, \dots, N\}$, a convex compact subset Y_i of X_i such that for each $x \in X \setminus K$, there exists $j \in \{1, \dots, N\}$ with $H_j(x) \cap Y_j \neq \emptyset$. Then there exists $x \in X$ with $H_i(x) = \emptyset$ for all $i \in \{1, \dots, N\}$.*

Proof. Apply Theorem 11 with $F_i = S_i = T_i$ (note for $x \in X \setminus K$, there exists $j \in \{1, \dots, N\}$ with $H_j(x) \cap Y_j \neq \emptyset$ with K and Y_j as in the statement of Theorem 12, so then $T_j(x) \cap Y_j \neq \emptyset$ since $H_j(w) \subseteq T_j(w)$ for $w \in X$), so there exists $x \in X$ with $T_i(x) = \emptyset$ for all $i \in \{1, \dots, N\}$. Now since $H_j(w) \subseteq T_j(w)$ for $w \in X$, then $H_i(x) = \emptyset$ for all $i \in \{1, \dots, N\}$. □

Remark 8. Note in the statement of Theorem 12 (see the proof above), we could replace "Also, assume that there are a compact subset K of X and, for each $i \in \{1, \dots, N\}$, a convex compact subset Y_i of X_i such that for each $x \in X \setminus K$, there exists $j \in \{1, \dots, N\}$ with $H_j(x) \cap Y_j \neq \emptyset$" with "Also, assume that there are a compact subset K of X and, for each $i \in \{1, \dots, N\}$, a convex compact subset Y_i of X_i such that for each $x \in X \setminus K$, there exists $j \in \{1, \dots, N\}$ with $T_j(x) \cap Y_j \neq \emptyset$".

Remark 9. Suppose for each $i \in \{1, \dots, N\}$ and for each $x \in X$, there exist a map $A_{i,x} : X \to X_i$ and an open set $U_{i,x}$ containing x with $H_i(z) \subseteq A_{i,x}(z)$ for every $z \in U_{i,x}$, $A_{i,x}$ is convex-valued, $(A_{i,x})^{-1}(z)$ is open (in X) for each $z \in X_i$, and $w_i \notin A_{i,x}(w)$ for each $w \in U_{i,x}$. From the discussion before Theorem 10 (with $Z = X$, $W = X_i$, $H = H_i$) there exists a map $T_i : X \to X_i$ with $H_i(w) \subseteq T_i(w)$ for $w \in X$, T_i is convex-valued, and $(T_i)^{-1}(z)$ is open for each $z \in X_i$. Here

$$Q_{i,x}(z) = \begin{cases} A_{i,x}(z), & z \in \Omega_{i,x}, \\ X_i, & z \in X \setminus \Omega_{i,x}, \end{cases} \quad \text{and} \quad T_i(z) = \bigcap_{x \in X} Q_{i,x}(z) \quad \text{for } z \in X,$$

where $\{V_{i,x}\}_{x \in X}$ is a locally finite open covering of X with $x \in V_{i,x}$, and $\Omega_{i,x} = \overline{V_{i,x}} \subseteq U_{i,x}$ for each $x \in X$. Now let $i \in \{1, \dots, N\}$ and $w \in X$. Note there exists $y \in X$ with $w \in \Omega_{i,y}$ (recall $\{V_{i,x}\}_{x \in X}$ is a locally finite open covering of X), so $T_i(w) = \cap_{x \in X}, Q_{i,x}(w) \subseteq Q_{i,y}(w) = A_{i,y}(w)$, and since $w_i \notin A_{i,y}(w)$ for each $w \in U_{i,y}$, we have $w_i \notin T_i(w)$ for $w \in X$.

Corollary 2. *Let X be a convex paracompact set in a Hausdorff topological vector space with $H : X \to X$, and in addition, there exists a map $T : X \to X$ with $H(w) \subseteq T(w)$ for $w \in X$, $T(x)$ has convex values for $x \in X$, $T^{-1}(w)$ is open (in X) for each $w \in X$, and $w \notin T(w)$ for each $w \in X$. Also, assume that there are a compact subset K of X and a convex compact set Y of X such that for each $x \in X \setminus K$, we have $H(x) \cap Y \neq \emptyset$ (or, alternatively, $T(x) \cap Y \neq \emptyset$). Then there exists $x \in X$ with $H(x) = \emptyset$.*

Proof. This follows from Theorem 12 with $N = 1$. □

Remark 10. For each $x \in X$, suppose there exist a map $A_x : X \to X$ and an open set U_x containing x with $H(z) \subseteq A_x(z)$ for every $z \in U_x$, A_x is convex-valued, $A_x^{-1}(z)$ is open (in X) for each $z \in X$, and $w \notin A_x(w)$ for each $w \in U_x$. Then as in Remark 9, there exists a map $T : X \to X$ with $H(w) \subseteq T(w)$ for $w \in X$, T is convex-valued, $T^{-1}(z)$ is open for each $z \in X$, and $w \notin T(w)$ for $w \in X$.

In [10, Thm. 2.15], we established the following coincidence type result in the coercive case. We note that the coercive type map could be replaced by a compactness type map if we use [10, Thm. 2.9] (we leave the obvious statements of Theorems 13, 14 and 15 below in the compactness map type setting to the reader).

Theorem 13. *Let $\{X_i\}_{i=1}^N$, $\{Y_i\}_{i=1}^{N_0}$ be families of convex sets each in a Hausdorff topological vector space E_i with $\prod_{i=1}^N X_i$ and $\prod_{i=1}^{N_0} Y_i$ paracompact. For each $i \in \{1, \dots, N_0\}$, suppose $F_i : X \equiv \prod_{i=1}^N X_i \to Y_i$, and there exists a map $T_i : X \to Y_i$ with $T_i(x) \subseteq F_i(x)$ for $x \in X$, $T_i(x)$ has convex values for each $x \in X$, and $T_i^{-1}(w)$ is open (in X) for each $w \in Y_i$. For each $j \in \{1, \dots, N\}$, suppose $G_j : Y \equiv \prod_{i=1}^{N_0} Y_i \to X_j$, and there exists a map $S_j : Y \to X_j$ with $S_j(y) \subseteq G_j(y)$ for $y \in Y$, $S_j(y)$ has convex values for each $y \in Y$, and $S_j^{-1}(w)$ is open (in Y) for each $w \in X_j$. In addition, assume that there are a compact subset K of Y and, for each $i \in \{1, \dots, N\}$, a convex compact subset Z_i of X_i such that for each $y \in Y \setminus K$, there exists $i \in \{1, \dots, N\}$ with*

$S_i(y) \cap Z_i \neq \emptyset$. Finally, suppose for each $x \in X$, there exists $i \in \{1, \dots, N_0\}$ with $T_i(x) \neq \emptyset$, and suppose for each $y \in Y$, there exists $j \in \{1, \dots, N\}$ with $S_j(y) \neq \emptyset$. Then there exist $x \in X$, $y \in Y$, $j_0 \in \{1, \dots, N_0\}$ and $i_0 \in \{1, \dots, N\}$ with $y_{j_0} \in F_{j_0}(x)$ and $x_{i_0} \in G_{i_0}(y)$.

We can rewrite Theorem 13 as follows.

Theorem 14. *Let $\{X_i\}_{i=1}^{N}$, $\{Y_i\}_{i=1}^{N_0}$ be families of convex sets each in a Hausdorff topological vector space with $\prod_{i=1}^{N} X_i$ and $\prod_{i=1}^{N_0} Y_i$ paracompact. For each $i \in \{1, \dots, N_0\}$, suppose $F_i : X \equiv \prod_{i=1}^{N} X_i \to Y_i$, and there exists a map $T_i : X \to Y_i$ with $T_i(x) \subseteq F_i(x)$ for $x \in X$, $T_i(x)$ has convex values for each $x \in X$, and $T_i^{-1}(w)$ is open (in X) for each $w \in Y_i$. For each $j \in \{1, \dots, N\}$, suppose $G_j : Y \equiv \prod_{i=1}^{N_0} Y_i \to X_j$, and there exists a map $S_j : Y \to X_j$ with $S_j(y) \subseteq G_j(y)$ for $y \in Y$, $S_j(y)$ has convex values for each $y \in Y$, and $S_j^{-1}(w)$ is open (in Y) for each $w \in X_j$. In addition, assume that there are a compact subset K of Y and, for each $i \in \{1, \dots, N\}$, a convex compact subset Z_i of X_i such that for each $y \in Y \setminus K$, there exists $i \in \{1, \dots, N\}$ with $S_i(y) \cap Z_i \neq \emptyset$. Now suppose either for all $j \in \{1, \dots, N_0\}$, we have $y_j \notin F_j(x)$ for each $(x, y) \in X \times Y$ with $x_{i_0} \in G_{i_0}(y)$ for some $i_0 \in \{1, \dots, N\}$ or for all $i \in \{1, \dots, N\}$, we have $x_i \notin G_i(y)$ for each $(x, y) \in X \times Y$ with $y_{j_0} \in F_{j_0}(x)$ for some $j_0 \in \{1, \dots, N_0\}$. Then either there exists $x \in X$ with $T_i(x) = \emptyset$ for all $i \in \{1, \dots, N_0\}$ or there exists $y \in Y$ with $S_j(y) = \emptyset$ for all $j \in \{1, \dots, N\}$.*

Proof. Suppose the conclusion is false. Then for each $x \in X$, there exists $i \in \{1, \dots, N_0\}$ with $T_i(x) \neq \emptyset$, and for each $y \in Y$, there exists $j \in \{1, \dots, N\}$ with $S_j(y) \neq \emptyset$. Now Theorem 13 guarantees $x \in X$, $y \in Y$, $j_0 \in \{1, \dots, N_0\}$ and $i_0 \in \{1, \dots, N\}$ with $y_{j_0} \in F_{j_0}(x)$ and $x_{i_0} \in G_{i_0}(y)$, a contradiction. □

Remark 11.

(i) To get a contradiction in the proof of Theorem 14, one only needs the statement "there exist $x \in X$, $y \in Y$, $i_0 \in \{1, \dots, N\}$, $j_0 \in \{1, \dots, N_0\}$ with $y_{j_0} \in F_{j_0}(x)$ and $x_{i_0} \in G_{i_0}(y)$" to be false, so one could list other conditions to guarantee the contradiction.

(ii) Note (see Theorem 14) part of the assumption in [8, Thm. 3.4] was inadvertently omitted (but in fact, it is a condition mentioned in part (i)).

Theorem 15. *Let $\{X_i\}_{i=1}^{N}$, $\{Y_i\}_{i=1}^{N_0}$ be families of convex sets each in a Hausdorff topological vector space with $\prod_{i=1}^{N} X_i$ and $\prod_{i=1}^{N_0} Y_i$ paracompact. For each $i \in \{1, \dots, N_0\}$ and for each $j \in \{1, \dots, N\}$, suppose $H_i : X \equiv \prod_{i=1}^{N} X_i \to Y_i$ and $\Psi_j : Y \equiv \prod_{i=1}^{N_0} Y_i \to X_j$, and in addition, assume that there exists a map $T_i : X \to Y_i$ with $H_i(z) \subseteq T_i(z)$ for $z \in X$, $T_i(x)$ has convex values for each $x \in X$, and $T_i^{-1}(w)$ is open (in X) for each $w \in Y_i$, and there exists a map $S_j : Y \to X_j$ with $\Psi_j(w) \subseteq S_j(w)$ for $w \in Y$, $S_j(y)$ has convex values for each $y \in Y$, and $S_j^{-1}(w)$ is open (in Y) for each $w \in X_j$. In addition, assume that there are a compact subset K of Y and, for each $i \in \{1, \dots, N\}$, a convex compact subset Z_i of X_i such that for each $y \in Y \setminus K$, there exists $i \in \{1, \dots, N\}$ with $\Psi_i(y) \cap Z_i \neq \emptyset$ (or, alternatively, $S_i(y) \cap Z_i \neq \emptyset$). Now* *suppose either for all $j \in \{1, \dots, N_0\}$, we have $y_j \notin T_j(x)$ for each $(x, y) \in X \times Y$ with $x_{i_0} \in S_{i_0}(y)$ for some $i_0 \in \{1, \dots, N\}$ or for all $i \in \{1, \dots, N\}$, we have $x_i \notin S_i(y)$ for each $(x, y) \in X \times Y$ with $y_{j_0} \in T_{j_0}(x)$ for some $j_0 \in \{1, \dots, N_0\}$. Then either there exists $x \in X$ with $H_i(x) = \emptyset$ for all $i \in \{1, \dots, N_0\}$ or there exists $y \in Y$ with $\Psi_j(y) = \emptyset$ for all $j \in \{1, \dots, N\}$.*

Proof. Apply Theorem 14 (with $F_i = T_i$ and $G_j = S_j$), so either there exists $x \in X$ with $T_i(x) = \emptyset$ for all $i \in \{1, \dots, N_0\}$ or there exists $y \in Y$ with $S_j(y) = \emptyset$ for all $j \in \{1, \dots, N\}$. Now since $H_i(z) \subseteq T_i(z)$ for $z \in X$ and $\Psi_j(w) \subseteq S_j(w)$ for $w \in Y$, the conclusion follows. □

Remark 12.

(i) Note Theorem 15 improves [8, Thm. 3.4].

(ii) We could consider maps of the type before Theorem 10 to create the maps T_i and S_i in Theorem 15 (see [8, Thm. 3.4]).

We now present another coincidence result established in [8].

Theorem 16. *Let $\{X_i\}_{i=1}^N$, $\{Y_i\}_{i=1}^{N_0}$ be families of convex sets each in a Hausdorff topological vector space with $\prod_{i=1}^{N_0} Y_i$ paracompact. For each $i \in \{1, \dots, N_0\}$, suppose $F_i : X \equiv \prod_{i=1}^N X_i \to Y_i$ and $F_i \in Ad(X, Y_i)$. For each $j \in \{1, \dots, N\}$, suppose $G_j : Y \equiv \prod_{i=1}^{N_0} Y_i \to X_j$, and there exists a map $S_j : Y \to X_j$ with $S_j(y) \subseteq G_j(y)$ for $y \in Y$, $S_j(y)$ has convex values for each $y \in Y$, and $S_j^{-1}(w)$ is open (in Y) for each $w \in X_j$. Also, assume that there are a compact subset K of Y and, for each $i \in \{1, \dots, N\}$, a convex compact subset Z_i of X_i such that for each $y \in Y \setminus K$, there exists $i \in \{1, \dots, N\}$ with $S_i(y) \cap Z_i \neq \emptyset$. Finally, suppose for each $y \in Y$, there exists $j \in \{1, \dots, N\}$ with $S_j(y) \neq \emptyset$. Then there exist $x \in X$, $y \in Y$, $i_0 \in \{1, \dots, N\}$ with $y_j \in F_j(x)$ for all $j \in \{1, \dots, N_0\}$ and $x_{i_0} \in G_{i_0}(y)$.*

Theorem 16 can be rephrased as follows.

Theorem 17. *Let $\{X_i\}_{i=1}^N$, $\{Y_i\}_{i=1}^{N_0}$ be families of convex sets each in a Hausdorff topological vector space with $\prod_{i=1}^{N_0} Y_i$ paracompact. For each $i \in \{1, \dots, N_0\}$, suppose $F_i : X \equiv \prod_{i=1}^N X_i \to Y_i$ and $F_i \in Ad(X, Y_i)$. For each $j \in \{1, \dots, N\}$, suppose $G_j : Y \equiv \prod_{i=1}^{N_0} Y_i \to X_j$, and there exists a map $S_j : Y \to X_j$ with $S_j(y) \subseteq G_j(y)$ for $y \in Y$, $S_j(y)$ has convex values for each $y \in Y$, and $S_j^{-1}(w)$ is open (in Y) for each $w \in X_j$. Also, assume that there is a compact subset K of Y and, for each $i \in \{1, \dots, N\}$, a convex compact subset Z_i of X_i such that for each $y \in Y \setminus K$, there exists $i \in \{1, \dots, N\}$ with $S_i(y) \cap Z_i \neq \emptyset$. Now suppose either for all $i \in \{1, \dots, N\}$, we have $x_i \notin G_i(y)$ for each $(x, y) \in X \times Y$ with $y_j \in F_j(x)$ for all $j \in \{1, \dots, N\}$ or there exists $j \in \{1, \dots, N_0\}$ with $y_j \notin F_j(x)$ for each $(x, y) \in X \times Y$ with $x_{i_0} \in G_{i_0}(y)$ for some $i_0 \in \{1, \dots, N\}$. Then there exists $y \in Y$ with $S_i(y) = \emptyset$ for all $i \in \{1, \dots, N\}$.*

Proof. Suppose the conclusion is false. Then for each $y \in Y$, there exists $j \in \{1, \dots, N\}$ with $S_j(y) \neq \emptyset$. Now Theorem 16 guarantees $x \in X$, $y \in Y$, $i_0 \in \{1, \dots, N\}$ with $y_j \in F_j(x)$ for all $j \in \{1, \dots, N_0\}$ and $x_{i_0} \in G_{i_0}(y)$, a contradiction. □

Remark 13.

(i) To get a contradiction in the proof of Theorem 17, one only needs the statement "there exist $x \in X$, $y \in Y$, $i_0 \in \{1, \dots, N\}$ with $y_j \in F_j(x)$ for all $j \in \{1, \dots, N_0\}$ and $x_{i_0} \in G_{i_0}(y)$" to be false, so one could list other conditions to guarantee the contradiction.

(ii) Note (see Theorem 17) part of the assumption in [8, Thm. 3.5] was inadvertently omitted (but in fact, it is a condition mentioned in part (i)).

Theorem 18. *Let $\{X_i\}_{i=1}^N$, $\{Y_i\}_{i=1}^{N_0}$ be families of convex sets each in a Hausdorff topological vector space with $\prod_{i=1}^{N_0} Y_i$ paracompact. For each $i \in \{1, \dots, N_0\}$, suppose $F_i : X \equiv \prod_{i=1}^N X_i \to Y_i$ and $F_i \in Ad(X, Y_i)$. For each $j \in \{1, \dots, N\}$, suppose $\Psi_j : Y \equiv \prod_{i=1}^{N_0} Y_i \to X_j$, and there exists a map $S_j : Y \to X_j$ with $\Psi_j(z) \subseteq S_j(z)$ for $z \in Y$, $S_j(y)$ has convex values for each $y \in Y$, and $S_j^{-1}(w)$ is open (in Y) for each $w \in X_j$. Also, assume that there are a compact subset K of Y and, for each $i \in \{1, \dots, N\}$, a convex compact subset Z_i of X_i such that for each $y \in Y \setminus K$, there exists $i \in \{1, \dots, N\}$ with $\Psi_i(y) \cap Z_i \neq \emptyset$ (or, alternatively, $S_i(y) \cap Z_i \neq \emptyset$). Now suppose either for all $i \in \{1, \dots, N\}$, we have $x_i \notin S_i(y)$ for each $(x, y) \in X \times Y$ with $y_j \in F_j(x)$ for all $j \in \{1, \dots, N\}$ or there exists $j \in \{1, \dots, N_0\}$ with $y_j \notin F_j(x)$ for each $(x, y) \in X \times Y$ with $x_{i_0} \in S_{i_0}(y)$ for some $i_0 \in \{1, \dots, N\}$. Then there exists $y \in Y$ with $\Psi_i(y) = \emptyset$ for all $i \in \{1, \dots, N\}$.*

Proof. Apply Theorem 17 with $G_j = S_j$, so there exists $y \in Y$ with $S_i(y) = \emptyset$ for all $i \in \{1, \dots, N\}$. The result follows since $\Psi_j(z) \subseteq S_j(z)$ for $z \in Y$. $\square$

Remark 14.

(i) Note Theorem 18 improves [8, Thm. 3.6].

(ii) We could consider maps of the type before Theorem 10 to create the maps S_i in Theorem 18 (see [8, Thm. 3.6]).

Finally, before we consider an application in games, we recall some fixed point results for condensing type maps in [12] (one could also consider the coincidence type results in [12] and obtain results similar to those obtained for coercive type maps; we leave this to the reader).

Theorem 19. *Let $\{X_i\}_{i=1}^N$ be a family of convex sets each in a Hausdorff topological vector space E_i. For each $i \in \{1, \dots, N\}$, suppose $F_i : X \equiv \prod_{i=1}^N X_i \to X_i$ and $F_i \in \Phi^\star(X, X_i)$. In addition, assume that there is a compact convex subset K of X with $F(K) \subseteq K$, where $F(x) = \prod_{i=1}^N F_i(x)$ for $x \in X$. Then there exists $x \in X$ with $x_i \in F_i(x)$ for $i \in \{1, \dots, N\}$.*

Next, we state the following special version when $N = 1$.

Corollary 3. *Let X be a convex set in a Hausdorff topological vector space and suppose $F : X \to X$ with $F \in \Phi^\star(X, X)$. Also, assume that there is a compact convex subset K of X with $F(K) \subseteq K$. Then there exists $x \in X$ with $x \in F(x)$.*

We can rewrite Corollary 3 as a maximal element type result.

Theorem 20. *Let X be a convex set in a Hausdorff topological vector space and suppose $F : X \to X$. In addition, assume that there exists a map $S : X \to X$ with $S(z) \subseteq F(z)$ for $z \in X$, $S(x)$ has convex values for each $x \in X$, and $S^{-1}(w)$ is open (in X) for each $w \in X$. Also, assume that there is a compact convex subset K of X with $F(K) \subseteq K$. Finally, assume $x \notin F(x)$ for $x \in X$. Then there exists $y \in X$ with $S(y) = \emptyset$.*

Proof. Assume that the conclusion is false. Then $S(y) \neq \emptyset$ for each $y \in X$, so $F \in \Phi^\star(X, X)$. Now Corollary 3 guarantees $x \in X$ with $x \in F(x)$, a contradiction. □

Similar, we have immediately a maximal element type result from Theorem 19.

Theorem 21. *Let $\{X_i\}_{i=1}^N$ be a family of convex sets each in a Hausdorff topological vector space E_i. For each $i \in \{1, \ldots, N\}$, suppose $F_i : X \equiv \prod_{i=1}^N X_i \to X_i$, and there exists a map $S_i : X \to X_i$ with $S_i(z) \subseteq F_i(z)$ for $z \in X$, $S_i(x)$ has convex values for each $x \in X$, and $S_i^{-1}(w)$ is open (in X) for each $w \in X_i$. Also, assume that there is a compact convex subset K of X with $F(K) \subseteq K$, where $F(x) = \prod_{i=1}^N F_i(x)$ for $x \in X$. Finally, suppose for each $x \in X$, there exists $i \in \{1, \ldots, N\}$ with $x_i \notin F_i(x)$. Then there exists $y \in X$ with $S_{i_0}(y) = \emptyset$ for some $i_0 \in \{1, \ldots, N\}$.*

Proof. Assume that the conclusion is false. Then for each $i \in \{1, \ldots, N\}$, we have $S_i(y) \neq \emptyset$ for each $y \in X$ and so $F_i \in \Phi^\star(X, X_i)$. Now Theorem 19 guarantees $x \in X$ with $x_i \in F_i(x)$ for $i \in \{1, \ldots, N\}$, a contradiction. □

Now we use Theorem 20 to obtain a result for majorized maps in the condensing setting.

Theorem 22. *Let X be a convex set in a Hausdorff topological vector space and suppose $H : X \to X$. In addition, assume that there exists a map $T : X \to X$ with $H(z) \subseteq T(z)$ for $z \in X$, $T(x)$ has convex values for each $x \in X$, $T^{-1}(w)$ is open (in X) for each $w \in X$, and $w \notin T(w)$ for $w \in X$. Also, assume that there is a compact convex subset K of X with $T(K) \subseteq K$. Then there exists $y \in X$ with $H(y) = \emptyset$.*

Proof. Apply Theorem 20 with $F = S = T$, so there exists $x \in X$ with $T(x) = \emptyset$. Now since $H(z) \subseteq T(z)$ for $z \in X$, the conclusion follows. □

Similarly, from Theorem 21 we have the following result.

Theorem 23. *Let $\{X_i\}_{i=1}^N$ be a family of convex sets each in a Hausdorff topological vector space E_i. For each $i \in \{1, \ldots, N\}$, suppose $H_i : X \equiv \prod_{i=1}^N X_i \to X_i$, and there exists a map $T_i : X \to X_i$ with $H_i(z) \subseteq T_i(z)$ for $z \in X$, $T_i(x)$ has convex values for each $x \in X$, and $T_i^{-1}(w)$ is open (in X) for each $w \in X_i$. Also, assume that there is a compact convex subset K of X with $T(K) \subseteq K$, where $T(x) = \prod_{i=1}^N T_i(x)$ for $x \in X$. Finally, suppose for each $x \in X$, there exists $i \in \{1, \ldots, N\}$ with $x_i \notin T_i(x)$. Then there exists $y \in X$ with $H_{i_0}(y) = \emptyset$ for some $i_0 \in \{1, \ldots, N\}$.*

Proof. Apply Theorem 21 with $F_i = S_i = T_i$, so there exists $y \in X$ with $T_{i_0}(y) = \emptyset$ for some $i_0 \in \{1, \dots, N\}$. The result follows since $H_i(z) \subseteq T_i(z)$ for $z \in X$. □

We will now use Corollary 2 and Theorem 22 to obtain equilibrium theorems for a one-person game. A one-person game is given by $\Gamma = (X, A, B, P)$, where we have one player (agent). The agent has a nonempty choice set or strategy set X, which is a nonempty subset of a Hausdorff topological vector space E. Now, $A, B : X \to E$ are constraint correspondences (multivalued maps), and $P : X \to E$ is a preference correspondence (multivalued map). An equilibrium of Γ is a point $x \in X$ such that $x \in \overline{B}(x)$ and $A(x) \cap P(x) = \emptyset$.

Theorem 24. *Let X be a convex paracompact set in a Hausdorff topological vector space E. Let $A, B, P : X \to E$ with* $\operatorname{cl} B$ $(\equiv \overline{B}) : X \to CK(X)$ *upper semicontinuous*; *here $CK(X)$ denotes the family of nonempty convex compact subsets of X. Also, assume that the following conditions are satisfied*:

(i) *$A : X \to X$ has nonempty convex values, and $A^{-1}(x)$ is open* (*in X*) *for each $x \in X$;*
(ii) *$A(x) \subseteq \overline{B}(x)$ for $x \in X$;*
(iii) *There exists a map $S : X \to X$ with $(A \cap P)(z) \subseteq S(z)$ for $z \in X$, $S(x)$ is convex-valued for each $x \in X$, $S^{-1}(z)$ is open* (*in X*) *for each $z \in X$, and $x \notin S(x)$ for $x \in X$.*

Assume that there are a compact subset K of X and a convex compact set Y of X such that for each $x \in X \setminus K$, we have $(A \cap P)(x) \cap Y \neq \emptyset$. Then there exists an equilibrium point x, i.e., $x \in X$ with $x \in \overline{B}(x)$ and $A(x) \cap P(x) = \emptyset$.

Proof. Let $M = \{x \in X\colon\ x \notin \overline{B}(x)\}$ and note M is open in X since $\overline{B} : X \to CK(X)$ is upper semicontinuous. Let $H : X \to X$ and $T : X \to X$ be given by

$$H(x) = \begin{cases} A(x) \cap P(x), & x \notin M, \\ A(x), & x \in M, \end{cases} \quad \text{and} \quad T(x) = \begin{cases} A(x) \cap S(x), & x \notin M, \\ A(x), & x \in M. \end{cases}$$

First, note $H(w) \subseteq T(w)$ for $w \in X$, and $T(x)$ has convex values for each $x \in X$. Next, we show that $T^{-1}(y)$ is open (in X) for each $y \in X$. To see this, let $y \in X$ and note

$$\begin{aligned} T^{-1}(y) &= \bigl\{z \in X\colon\ y \in T(z)\bigr\} \\ &= \bigl\{z \in M\colon\ y \in A(z)\bigr\} \cup \bigl\{z \in X \setminus M\colon\ y \in A(z) \cap S(z)\bigr\} \\ &= \bigl[M \cap \bigl\{z \in X\colon\ y \in A(z)\bigr\}\bigr] \cup \bigl[(X \setminus M) \cap \bigl\{z \in X\colon\ y \in A(z) \cap S(z)\bigr\}\bigr] \\ &= \bigl[M \cap A^{-1}(y)\bigr] \cup \bigl[(X \setminus M) \cap \bigl[A^{-1}(y) \cap S^{-1}(y)\bigr]\bigr] \\ &= \bigl[M \cup S^{-1}(y)\bigr] \cap A^{-1}(y) \end{aligned}$$

(note $A^{-1}(y) \cap S^{-1}(y) \subseteq A^{-1}(y)$), which is open in X. Now we claim $w \notin T(w)$ for $w \in X$. First, consider $w \in M$. Then $w \notin \overline{B}(w)$, so $w \notin A(w)$ from (ii), i.e., $w \notin T(w)$

if $w \in M$. Next, consider $w \notin M$. Then $w \notin (A \cap S)(w)$ since $(A \cap S)(w) \subseteq S(w)$, and $x \notin S(x)$ for $x \in X$, i.e., $w \notin T(w)$ if $w \notin M$. Consequently, $w \notin T(w)$ for $w \in X$.

Now let K and Y be as in the statement of Theorem 24. We note $H(x) \cap Y \neq \emptyset$ for $x \in X \setminus K$ since this is immediate if $x \notin M$, whereas if $x \in M$, it is also true since $\emptyset \neq (A \cap P)(x) \cap Y \subseteq A(x) \cap Y$. Now Corollary 2 guarantees $x \in X$ with $H(x) = \emptyset$. Now, see condition (i), A has nonempty values, so in fact, $x \notin M$. Thus $x \notin M$ with $H(x) = \emptyset$, i.e., $x \in \overline{B}(x)$ and $A(x) \cap P(x) = \emptyset$. □

Theorem 25. *Let X be a convex set in a Hausdorff topological vector space E. Let $A, B, P : X \to E$ with* $\operatorname{cl} B\,(\equiv \overline{B}) : X \to CK(X)$ *upper semicontinuous. Suppose conditions* (i)–(iii) *hold, and in addition, assume that there is a compact convex subset K of X with $A(K) \subseteq K$. Then there exists $x \in X$ with $x \in \overline{B}(x)$ and $A(x) \cap P(x) = \emptyset$.*

Proof. Let M, H and T be as in Theorem 24 and note $H(w) \subseteq T(w)$ for $w \in X$, $T(x)$ has convex values for each $x \in X$, $T^{-1}(y)$ is open (in X) for each $y \in X$, and $w \notin T(w)$ for $w \in X$. Now let K be as in the statement of Theorem 25 and note $T(K) \subseteq K$ since $T(K) \subseteq A(K) \subseteq K$. Thus all the conditions of Theorem 22 are satisfied, so there exists $x \in X$ with $H(x) = \emptyset$. Thus $x \notin M$ with $H(x) = \emptyset$. □

Remark 15. Notice that in Theorems 24 and 25, $\overline{B}$ is upper semicontinuous, and A is a Ky Fan map, which are well-known maps in the literature (see, e.g., [6]).

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