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Global attractive set of neural networks with neutral item*

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Abstract. This paper investigates the global attractive set of neural networks with neutral item. To better deal with the neutral terms, different types of activation functions are considered. Based on matrix measures, inequality techniques, and Lyapunov theory, three new types of Lyapunov functions are designed to find the global attractive set of the system. We give out a simulation example to verify the validity of theory results. The result is very inclusive, whether the system has equilibrium or not. As long as the system is stable, we can find its global attractive set.

Keywords: global attractive set, neural networks, neutral, linear matrix inequality.

1 Introduction

The artificial neural network is a highly simplified approximation of human brain neural network through mathematical and physical methods from the perspective of information processing. It has the advantages of fault tolerance, versatility, and adaptability. It has been widely used in speech recognition, visual technology, artificial intelligence, image recognition, associative memory, speech translation, and other fields. The application of the neural networks is related to their dynamic behavior. According to different application requirements, its dynamic characteristics have different requirements. Many scholars have some interesting conclusions in this area [9, 17, 18]. Especially, the global

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exponential stability of several types of neural networks is discussed in [2,4,5]. By using delayed impulsive control, quantized intermittent control and synchronization control are investigated in [7, 16, 24–26].

The focus of neural network research is the dynamic characteristics of orbits near the equilibrium point of the system. Nonetheless, in many practical problems, the equilibrium of the system is unstable or even nonexistent, so the stability of the system needs to be considered. If the global attractive set of the system is identified, we just need to focus on the dynamics within the set because the equilibrium point of the system only exists in the global attractive set [13, 15].

In real life, the development of many things is affected by some past states, that is, the phenomenon of time delay. Due to the limitation of signal propagation speed, time delay is widespread [14]. For example, when sending emails, pictures, or videos via WeChat, there will be a time delay from the beginning to the success of sending. When the time delay phenomenon occurs, the system will be oscillated or unstable, resulting in network paralysis and other phenomena. There are many results in stability analysis of neural networks with time delays in [6, 8, 19]. In the study of these delays, one kind of delay is worth paying attention to, that is, neutral delay. It not only considers the influence of past state on present state of nonlinear time-delay systems, but also considers the influence of past state changes on present state. The neural networks with neutral items are more complex than the general neural networks of the same dimension. It is a new hot research direction [3,10,27]. Liao et al. considered the exponential stability for neutral-type neural networks [12]. Ali et al. discussed finite-time stability of systems with random delays [1].

Based on the above analysis, this paper studies the global attractive set of neural networks with neutral item. The main contributions are presented as follows:

(I) The neural networks with neutral items are more complex than the general neural networks. We will make some efforts on finding various Lyapunov functions and global attractive sets for system with neutral items.

(II) External inputs affect the stability of the system. This paper attempts to study the effect of external input with bounded function on the global attractive set and stability.

By matrix measures, inequality techniques, and Lyapunov theory, we can find the global attractive set of the system. We give out simulation examples to verify the validity of theory results.

Notations. In this paper, $\mathbb{R}^n$ represent the n-dimensional real space. $\mathbb{R}^{m \times n}$ represent the set of $m \times n$ real matrix. T and $*$ represent the transposition of matrix and conjugation transposition.

2 Preliminaries

In this paper, we consider the following neural networks model:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -Cx(t) + Af\big(x(t)\big) + Bf\big(x\big(t-h(t)\big)\big) + Mg\big(\dot{x}\big(t-\sigma(t)\big)\big) + U(t), \quad (1)$$

where $x(t) = \{x_1(t), x_2(t), \ldots, x_n(t)\}^{\mathrm{T}} \in \mathbb{R}^n$. $C = \mathrm{diag}\{c_1, c_2, \ldots, c_n\}$, $c_i > 0$, $A = (a_{ij})_{n \times n} \in \mathbb{R}^{n \times n}$, $B = (b_{ij})_{n \times n} \in \mathbb{R}^{n \times n}$, and $M = (m_{kj})_{n \times n} \in \mathbb{R}^{n \times n}$ are connection weight matrices. $f(x(t)) \in \mathbb{R}^n$, $f(x(t - h(t))) \in \mathbb{R}^n$, and $g(\dot{x}(t - \sigma(t))) \in \mathbb{R}^n$ are activation functions. $h(t)$ is the delay, $0 \leqslant h(t) \leqslant \bar{h}$. $\sigma(t)$ is the neutral delay with $0 \leqslant \sigma(t) \leqslant \sigma$, and let $\tau = \max\{\bar{h}, \sigma\}$. $U(t) \in \mathbb{R}^n$ is the external input, $U(t) \leqslant U$, $U = (u_1, u_2, \ldots, u_n)^{\mathrm{T}} \in \mathbb{R}^n$.

The initial condition of neural networks (1) is given as

$$x(s) = \varphi(s), \quad \dot{x}(s) = \phi(s), \quad s \in [-\tau, 0].$$

We consider two classes of continuous activation functions:

$$\mathfrak{B} = \big\{ f(\cdot) \mid \exists\, k_i > 0,\ \big|f_i(x_i)\big| \leqslant k_i\ \forall x_i \in \mathbb{R},\ i \in \Gamma = \{1, 2, \ldots, n\} \big\},$$
$$K = \mathrm{diag}\{k_1, k_2, \ldots, k_n\}$$
$$\mathfrak{S} = \left\{ f(\cdot) \,\middle|\, \exists\, l_i, h_i, l_i \leqslant \frac{f_i(x) - f_i(y)}{x - y} \leqslant h_i,\ x \neq y,\ i \in \Gamma \right\},$$
$$L = \mathrm{diag}\{l_1, l_2, \ldots, l_n\}, \quad H = \mathrm{diag}\{h_1, h_2, \ldots, h_n\}.$$

Remark 1. If $f(\cdot) \in \mathfrak{S}$, we can know that $f(\cdot)$ is neither bounded nor monotonously, which can make the results with better applicability. The constants l_i and h_i can be any real value.

Definition 1. (See [22].) The set $\Omega \subset R^n$ is called to be a global attractive set (GAS) of (1) if Ω posses an open neighborhood D for all $\varphi \in D$ and $x(t, t_0, \varphi)$ converges to Ω as $t \to \infty$.

Definition 2. (See [21].) For a real matrix $C = (c_{ij})_{n \times n}$, the matrix measure C is defined as $\mu_p(C) = \lim_{\Delta t \to 0^+} (\|E_n + \Delta t C\|_p - 1)/\Delta t$, where $p = 1, 2, \infty$. The matrix measures are obtained as $\mu_1(C) = \max_j\{c_{jj} + \sum_{i=1, i \neq j}^n |c_{ij}|\}$, $\mu_2(C) = \lambda_{\max}(C^{\mathrm{T}} + C)/2$, $\mu_\infty(C) = \max_i\{c_{ii} + \sum_{j=1, j \neq i}^n |c_{ij}|\}$.

$\|C\|_p$ is the matrix norm of C, where $\|C\|_1 = \max_j \sum_{i=1}^n |c_{ij}|$, $\|C\|_2 = \sqrt{\lambda_{\max}(C^{\mathrm{T}}C)}$, $\|C\|_\infty = \max_i \sum_{j=1}^n |c_{ij}|$. For $x \in \mathbb{R}^n$, $\|x\|_1 = \sum_{i=1}^n |x_i|$, $\|x\|_2 = \sqrt{x^{\mathrm{T}}x}$, $\|x\|_\infty = \max_{1 \leqslant i \leqslant n} |x_i|$.

Remark 2. Take $C = \begin{pmatrix} 2 & 2 \\ 1 & -9 \end{pmatrix}$ for example, therefore $\|-C\|_1 = \|C\|_1 = 11$, $\|-C\|_2 = \|C\|_2 = 9.2364$, $\|C\|_\infty = \|-C\|_\infty = 10$, $\mu_1(C) = 3$, $\mu_2(C) = 2.2009$, $\mu_\infty(C) = 4$, $\mu_1(-C) = 11$, $\mu_2(-C) = 9.2009$, $\mu_\infty(-C) = 10$. Hence, we can get that $\|-C\|_p = \|C\|_p$, $\mu_p(-C) \neq \mu_p(C)$.

It shows that matrix measure is more sensitive to sign, and the results base on matrix measure are more accurate and applicable.

Lemma 1. (See [20].) Let $r, \xi, \eta, \tau > 0$. When the function $V(t) \geqslant 0$ and $D^+V(t) \leqslant r - \xi V(t) + \eta \bar{V}(t)$, $t \geqslant t_0$, there exists $\sigma > 0$ such that $-\xi + \eta \leqslant -\sigma$, and one has

$$V(t) \leqslant \begin{cases} \dfrac{r}{\sigma} + (\bar{V}(t_0) - \dfrac{r}{\sigma})\mathrm{e}^{-\mu^*(t-t_0)} & \text{if } \bar{V}(t_0) > \dfrac{r}{\sigma}, \\[2mm] \dfrac{r}{\sigma} & \text{if } \bar{V}(t_0) \leqslant \dfrac{r}{\sigma}. \end{cases}$$

Here μ^* is the unique positive root of $\mu - \xi + \eta e^{\mu\tau} = 0$, $\bar{V}(t) = \sup_{t-\tau \leqslant s \leqslant t} V(s)$, $D^+ v(t) = \overline{\lim}_{\Delta t \to 0^+}(v(t + \Delta t) - v(t))/\Delta t$.

Lemma 2. (*See* [20].) *For all $A \in [\underline{A}, \bar{A}]$, we have*

$$\|A\|_p \leqslant \|A^*\|_p + \|A_*\|_p,$$

where $A_* = (\bar{A} - \underline{A})/2$, $A^* = (\bar{A} + \underline{A})/2$, $p = 1, 2, \infty$.

Lemma 3. (*See* [11].) *Let $S = \begin{pmatrix} S_{11} & S_{12} \\ S_{12}^\mathrm{T} & S_{22} \end{pmatrix} < 0$ with $S_{11} = S_{11}^\mathrm{T}$, $S_{22} = S_{22}^\mathrm{T}$. Then the following obtained*:

$$S_{22} < 0, \qquad S_{11} - S_{12} S_{22}^{-1} S_{12}^\mathrm{T} < 0,$$
$$S_{11} < 0, \qquad S_{22} - S_{12}^\mathrm{T} S_{11}^{-1} S_{12} < 0.$$

Lemma 4. *For matrices $A_1, A_2, B_1, B_2, C_1, C_2 \in \mathbb{R}^{n \times n}$ and reversible matrices X, Y, Z, denote*

$$\Phi_1 \triangleq \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} X^{-1} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}^\mathrm{T} + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} Y^{-1} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}^\mathrm{T} + \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} Z^{-1} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}^\mathrm{T},$$

$$\Phi_2 \triangleq \begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{pmatrix} \begin{pmatrix} X^{-1} & 0 & 0 \\ 0 & Y^{-1} & 0 \\ 0 & 0 & Z^{-1} \end{pmatrix} \begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{pmatrix}^\mathrm{T},$$

Then $\Phi_1 = \Phi_2$.

Proof. According to the matrix algorithm, we have

$$\Phi_1 = \begin{pmatrix} A_1 X^{-1} A_1 & A_1 X^{-1} A_2 \\ A_2 X^{-1} A_1 & A_2 X^{-1} A_2 \end{pmatrix} + \begin{pmatrix} B_1 Y^{-1} B_1 & B_1 Y^{-1} B_2 \\ B_2 Y^{-1} B_1 & B_2 Y^{-1} B_2 \end{pmatrix}$$
$$+ \begin{pmatrix} C_1 Z^{-1} C_1 & C_1 Z^{-1} C_2 \\ C_2 Z^{-1} C_1 & C_2 Z^{-1} C_2 \end{pmatrix},$$

$$\Phi_2 = \begin{pmatrix} A_1 X^{-1} & B_1 Y^{-1} & C_1 Z^{-1} \\ A_2 X^{-1} & B_2 Y^{-1} & C_2 Z^{-1} \end{pmatrix} \begin{pmatrix} A_1 & A_2 \\ B_1 & B_2 \\ C_1 & C_2 \end{pmatrix} = \begin{pmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{pmatrix},$$

where

$$\Delta_{11} = A_1 X^{-1} A_1 + B_1 Y^{-1} B_1 + C_1 Z^{-1} C_1,$$
$$\Delta_{12} = A_1 X^{-1} A_2 + B_1 Y^{-1} B_2 + C_1 Z^{-1} C_2,$$
$$\Delta_{21} = A_2 X^{-1} A_1 + B_2 Y^{-1} B_1 + C_2 Z^{-1} C_1,$$
$$\Delta_{22} = A_2 X^{-1} A_2 + B_2 Y^{-1} B_2 + C_2 Z^{-1} C_2.$$

Then, by the properties of matrix operations, $\Phi_1 = \Phi_2$. □

Lemma 5. (*See* [23].) *If P is a positive definite matrix, $a, b \in \mathbb{R}^n$, we have*

$$2a^\mathrm{T} b \leqslant a^\mathrm{T} P^{-1} a + b^\mathrm{T} P b.$$

3 Main results

Theorem 1. *If* $f(\cdot) \in \mathfrak{S}$, $g(\cdot) \in \mathfrak{B}$, *there exists* $\delta_1 > 0$ *such that* $\Phi_1 + \Phi_2 \leqslant -\delta_1 < 0$, *then the set*

$$\Omega_1 = \left\{ x \in \mathbb{R}^n \;\middle|\; \|x\|_p \leqslant \frac{\|\gamma_1\|_p}{\delta_1} \right\}$$

is a GAS of system (1). *Here* $\gamma_1 = \|M\|_p K + \|U\|_p$, $\Phi_1 = \mu_p(-C) + \bar{H}\|A\|_p$, $\Phi_2 = \bar{H}\|B\|_p$, $\bar{H} = \max_{1 \leqslant i \leqslant n}\{|l_i|, |h_i|\}$ *with* $p = 1, 2, \infty$.

Proof. Let $V(t) = \|x(t)\|_p$. Since $f(\cdot) \in \mathfrak{S}$, we can get that $\|f(x(t))\|_p \leqslant \bar{H}\|x(t)\|_p$, $\|f(x(t - h(t)))\|_p \leqslant \bar{H}\|x(t - h(t))\|_p$. In the view of Definition 2 and Lemma 2, the following obtained:

$$
\begin{aligned}
\mathrm{D}^+ V(t) &\leqslant \varlimsup_{\Delta t \to 0^+} \frac{\|x(t + \Delta t)\|_p - \|x(t)\|_p}{\Delta t} \\
&= \varlimsup_{\Delta t \to 0^+} \frac{\|x(t) + \Delta t \dot{x}(t) + o(\Delta t)\|_p - \|x(t)\|_p}{\Delta t} \\
&\leqslant \varlimsup_{\Delta t \to 0^+} \frac{\|x(t) + \Delta t(-C)x(t) - \|x(t)\|_p}{\Delta t} + \varlimsup_{\Delta t \to 0^+} \frac{\|Af(x(t))\|_p \Delta t}{\Delta t} \\
&\quad + \varlimsup_{\Delta t \to 0^+} \frac{\|Bf(x(t - h(t)))\|_p \Delta t}{\Delta t} + \|M\|_p K + \|U\|_p \\
&\leqslant \varlimsup_{\Delta t \to 0^+} \frac{\|E_n + \Delta t(-C)\|_p - 1}{\Delta t}\|x(t)\|_p + \|A\|_p \bar{H}\|x(t)\|_p \\
&\quad + \|B\|_p \bar{H}\|x(t - h(t))\|_p + \|M\|_p K + \|U\|_p \\
&\leqslant \Phi_1 \|V(t)\|_p + \Phi_2 \overline{\|V(t)\|}_p + \|M\|_p K + \|U\|_p.
\end{aligned}
$$

According to Lemma 1, for $\|\varphi(s)\|_p > \|\gamma_1\|_p/\delta_1$, $s \in [-\tau, 0]$, $V(t) \leqslant \|\gamma_1\|_p/\delta_1 + M_1 \mathrm{e}^{-\mu_1 t}$, where $M_1 = \sup(V(s) - \|\gamma_1\|_p/\delta_1)$, $-\tau \leqslant s \leqslant 0$, μ_1 is the root of $\mu + \Phi_1 + \Phi_2 \mathrm{e}^{\mu\tau} = 0$. Then $\Omega_1 = \{x \in \mathbb{R}^n \mid \|x\|_p \leqslant \|\gamma_1\|_p/\delta_1\}$ is a GAS of system (1).

For $\|\varphi(s)\|_p \leqslant \|\gamma_1\|_p/\delta_1$, $s \in [-\tau, 0]$, following Lemma 1, $\|x\|_p \leqslant \|\gamma_1\|_p/\delta_1$, $t \geqslant 0$. The set Ω_1 is also a GAS of system (1). $\qquad\square$

Theorem 2. *If* $f(\cdot) \in \mathfrak{S}$, $g(\cdot) \in \mathfrak{B}$, $c_i > h_i \sum_{j=1}^n |a_{ji}| + (h_i/(1 - \mu) \sum_{j=1}^n |b_{ji}|$, $0 \leqslant \dot{h}(t) \leqslant \mu \leqslant 1$, *then*

$$\Omega_2 = \left\{ x \in \mathbb{R}^n \;\middle|\; \sum_{i=1}^n |x_i(t)| \leqslant \frac{\sum_{j=1}^n |m_{ij}|k_j + \sum_{i=1}^n |u_i|}{\min_{1 \leqslant i \leqslant n} \Xi} \right\}$$

is a GAS of system (1), *where* $\Xi = c_i - h_i \sum_{j=1}^n |a_{ji}| - (h_i/(1 - \mu)) \sum_{j=1}^n |b_{ji}|$.

Proof. Functional

$$V(t) = \sum_{i=1}^n \left(|x_i(t)| + \frac{1}{1 - \mu} \sum_{j=1}^n |b_{ij}| \int_{t-h(t)}^t |f_j(x_j(s))| \,\mathrm{d}s \right)$$

is adopted, then

$$
\begin{aligned}
D^+V(t) \\
&\leqslant \sum_{i=1}^{n}\left(-c_i\big|x_i(t)\big| + \sum_{j=1}^{n}|a_{ij}|\big|f_j\big(x_j(t)\big)\big| + \sum_{j=1}^{n}|b_{ij}|\big|f_j\big(x_j\big(t-h(t)\big)\big)\big| \right. \\
&\quad + \sum_{j=1}^{n}|m_{ij}|\big|g_j\big(\dot{x}_j\big(t-\sigma(t)\big)\big)\big| + |u_i|\bigg) + \frac{1}{1-\mu}\sum_{j=1}^{n}|b_{ij}|\big|f_j\big(x_j(t)\big)\big| \\
&\quad - \frac{1}{1-\mu}\sum_{j=1}^{n}|b_{ij}|\big(1-\dot{h}(t)\big)\big|f_j\big(x_j\big(t-h(t)\big)\big)\big| \\
&\leqslant \sum_{i=1}^{n}\left(-c_i\big|x_i(t)\big| + \sum_{j=1}^{n}|a_{ij}|h_j\big|x_j(t)\big| + \frac{1}{1-\mu}\sum_{j=1}^{n}h_j|b_{ij}|\big|x_j(t)\big| \right. \\
&\quad + \sum_{j=1}^{n}|m_{ij}|k_j + |u_i|\bigg) + \sum_{j=1}^{n}|b_{ij}|\left(1-\frac{1}{1-\mu}\big(1-\dot{h}(t)\big)\right)\big|f_j\big(x_j\big(t-h(t)\big)\big)\big| \\
&\leqslant -\sum_{i=1}^{n}\left(c_i - h_i\sum_{j=1}^{n}|a_{ij}| - \frac{h_i}{1-\mu}\sum_{j=1}^{n}|b_{ij}|\right)\big|x_j(t)\big| + \sum_{j=1}^{n}|m_{ij}|k_j + \sum_{i=1}^{n}|u_i| \\
&\leqslant -\min_{1\leqslant i\leqslant n}\left\{c_i - h_i\sum_{j=1}^{n}|a_{ji}| - \frac{h_i}{1-\mu}\sum_{j=1}^{n}|b_{ji}|\right\}\sum_{i=1}^{n}\big|x_i(t)\big| \\
&\quad + \sum_{j=1}^{n}|m_{ij}|k_j + \sum_{i=1}^{n}|u_i|.
\end{aligned}
$$

If $x \in \mathbb{R}^n \setminus \Omega_2$, then $D^+V(t)|_{(1)} < 0$, that is, for any $\varphi(s) \in \Omega_2$, $x(t,t_0,\varphi) \subseteq \Omega_2$, $t \geqslant t_0$. For $\varphi(s) \notin \Omega_2$, there exists $T > 0$, $x(t,t_0,\varphi) \subseteq \Omega_2$, where $t \geqslant t_0 + T$. In view of Definition 1, we can know that Ω_2 is a GAS of system (1). $\qquad\square$

Theorem 3. *Let $f(\cdot) \in \mathfrak{S}, g(\cdot) \in \mathfrak{B}$, R is a positive definite matrix, D, F, Q, P, S are positive diagonal matrices, and the following LMI holds:*

$$
\begin{pmatrix}
\widetilde{\Sigma} & (R-LD)A - CD & (R-LD)B & (R-LD)M & R-LD \\
* & DA + A^{\mathrm{T}}D - P & DB & DM & D \\
* & * & -Q & 0 & 0 \\
* & * & 0 & -S & 0 \\
* & * & 0 & 0 & -F
\end{pmatrix} < 0, \quad (2)
$$

$$
WQW \leqslant R. \tag{3}
$$

Then

$$
\Omega_3 = \left\{ x(t) \in \mathbb{R}^n \;\middle|\; x^{\mathrm{T}}(t)Rx(t) \leqslant \frac{\gamma_2}{\delta_2} \right\}
$$

is a GAS of system (1), where $\delta_2 \leqslant \varepsilon$, $\gamma_2 = K^\mathrm{T}SK + U^\mathrm{T}FU$, $\widetilde{\Sigma} = (1+\varepsilon)(R + D(H-L) - RC - CR + 2LDC + WPW)$, $w_i = \max|l_i|, |h_i|$, $W = \mathrm{diag}\{w_1, w_2, \ldots, w_n\}$, $i \in \Gamma$.

Proof. Let for $f(\cdot) \in \mathfrak{S}$,

$$V(t) = x^\mathrm{T}(t)Rx(t) + 2\sum_{i=1}^{n} d_i \int_0^{x_i(t)} \big(f_i(s) - L_i s\big)\,\mathrm{d}s,$$

P is a positive diagonal matrix, and let $\zeta = (x(t), f(x(t))^\mathrm{T}$, $\check{R} = R - LD$. Then

$$\begin{aligned}
2\big(x^\mathrm{T}(t)R &+ f^\mathrm{T}\big(x(t)D - x^\mathrm{T}(t)LD\big)\big)\big(-Cx(t) + Af\big(x(t)\big)\big)\\
&= 2x^\mathrm{T}(t)(-RC + LDC)x(t) + 2f^\mathrm{T}\big(x(t)\big)(DA)f\big(x(t)\big)\\
&\quad + 2x^\mathrm{T}(t)(RA - LDA - CD)f\big(x(t)\big)\\
&\leqslant 2x^\mathrm{T}(t)(-RC + LDC)x(t) + 2f^\mathrm{T}\big(x(t)\big)(DA)f\big(x(t)\big)\\
&\quad + 2x^\mathrm{T}(t)(RA - LDA - CD)f\big(x(t)\big)\\
&\quad + x^\mathrm{T}(t)WPWx(t) - f^\mathrm{T}\big(x(t)\big)Pf\big(x(t)\big)\\
&= \zeta^\mathrm{T}\Sigma_1\zeta,
\end{aligned} \tag{4}$$

where

$$\Sigma_1 = \begin{pmatrix} \Lambda & \check{R}A - CD \\ * & DA + A^\mathrm{T}D - P \end{pmatrix}, \qquad \Lambda = -RC - CR + 2LDC + WPW.$$

Using Lemma 5, for a given positive diagonal matrix Q, S, F, the following inequality holds:

$$\begin{aligned}
2\big(x^\mathrm{T}(t)RB &+ f^\mathrm{T}\big(x(t)DB - x^\mathrm{T}(t)LDB\big)\big)f\big(x\big(t - h(t)\big)\big)\\
&\leqslant \zeta^\mathrm{T}\begin{pmatrix} \check{R}B \\ DB \end{pmatrix}Q^{-1}\begin{pmatrix} \check{R}B \\ DB \end{pmatrix}^\mathrm{T}\zeta x^\mathrm{T}\big(t - h(t)\big)WQWx\big(t - h(t)\big),
\end{aligned}$$

$$\begin{aligned}
2\big(x^\mathrm{T}(t)RM &+ f^\mathrm{T}\big(x(t)DM - x^\mathrm{T}(t)LDM\big)\big)g\big(\dot{x}\big(t - \sigma(t)\big)\big)\\
&\leqslant \big(x^\mathrm{T}(t)RM + f^\mathrm{T}\big(x(t)DM - x^\mathrm{T}(t)LDM\big)\big)S^{-1}\big(x^\mathrm{T}(t)RM\\
&\quad + f^\mathrm{T}\big(x(t)DM - x^\mathrm{T}(t)LDM\big)\big)^\mathrm{T} + g\big(\dot{x}\big(t - \sigma(t)\big)\big)^\mathrm{T}Sg\big(\dot{x}\big(t - \sigma(t)\big)\big)\\
&\leqslant \zeta^\mathrm{T}\begin{pmatrix} \check{R}M \\ DM \end{pmatrix}S^{-1}\begin{pmatrix} \check{R}M \\ DM \end{pmatrix}^\mathrm{T}\zeta + K^\mathrm{T}SK
\end{aligned}$$

and

$$2\big(x^\mathrm{T}(t)R + f^\mathrm{T}\big(x(t)D - x^\mathrm{T}(t)LD\big)\big)U \leqslant \zeta^\mathrm{T}\begin{pmatrix} \check{R} \\ D \end{pmatrix}F^{-1}\begin{pmatrix} \check{R} \\ D \end{pmatrix}^\mathrm{T}\zeta + U^\mathrm{T}FU. \tag{5}$$

Based on Lemma 4 and (4) $\sim$ (5), one has

$$\begin{aligned}
\frac{\mathrm{d}V(t)}{\mathrm{d}t} &\leqslant x^\mathrm{T}\big(t - h(t)\big)WQWx\big(t - h(t)\big)\\
&\quad + \zeta^\mathrm{T}\Sigma_1\zeta + \zeta^\mathrm{T}\Sigma_2\zeta + K^\mathrm{T}SK + U^\mathrm{T}FU,
\end{aligned} \tag{6}$$

where

$$\Sigma_2 = \Sigma_3 \begin{pmatrix} Q^{-1} & 0 & 0 \\ 0 & S^{-1} & 0 \\ 0 & 0 & F^{-1} \end{pmatrix} \Sigma_3^{\mathrm{T}} \quad \text{and} \quad \Sigma_3 = \begin{pmatrix} \check{R}B & \check{R}M & \check{R} \\ DB & DM & D \end{pmatrix}.$$

Using Lemma 3 to (2), let $\widetilde{R} = R + D(H - L)$, then we get

$$\begin{pmatrix} -(1+\varepsilon)(\widetilde{R} + \varLambda) & \check{R}A - CD \\ * & DA + A^{\mathrm{T}}D - P \end{pmatrix} + \Sigma_2 < 0,$$

and it is equal to

$$\Sigma_1 + \Sigma_2 < \begin{pmatrix} -(1+\varepsilon)\widetilde{R} & 0 \\ 0 & 0 \end{pmatrix}. \tag{7}$$

Based on (3), (6), and (7), we have

$$\begin{aligned} \frac{\mathrm{d}V(t)}{\mathrm{d}t} &\leqslant -(1+\varepsilon)x^{\mathrm{T}}(t)\widetilde{R}x(t) + x^{\mathrm{T}}\big(t - h(t)\big)Rx\big(t - h(t)\big) \\ &\quad + K^{\mathrm{T}}SK + U^{\mathrm{T}}FU. \end{aligned}$$

Since $V(t) \leqslant x^{\mathrm{T}}(t)\widetilde{R}x(t)$,

$$\frac{\mathrm{d}V(t)}{\mathrm{d}t} \leqslant -(1+\varepsilon)V(t) + \bar{V}(t) + K^{\mathrm{T}}SK + U^{\mathrm{T}}FU.$$

According to Lemma 1, if $x^{\mathrm{T}}(t)Rx(t) > \gamma_2/\delta_2$, then $V(t) \leqslant \gamma_2/\delta_2 + M_2\mathrm{e}^{-\mu_2 t}$, and for $x^{\mathrm{T}}(t)Rx(t) \leqslant \gamma_2/\delta_2$, we have $V(t) \leqslant \gamma_2/\delta_2$, where $M_2 = \sup(V(s) - \gamma_2/\delta_2)$, $-\tau \leqslant s \leqslant 0$, μ_2 is the root of $\mu - (1+\varepsilon) + \mathrm{e}^{\mu\tau} = 0$. According to Definition 1, $\varOmega_3 = \{x(t) \in \mathbb{R}^n \mid x^{\mathrm{T}}(t)Px(t) \leqslant \gamma_2/\delta_2\}$ is a GAS of system (1). □

4 Illustrative examples

We will give two examples and simulations to demonstrate the validity of our theory results.

Example 1. Consider the following neural networks model:

$$\begin{aligned} \frac{\mathrm{d}x(t)}{\mathrm{d}t} &= -Cx(t) + Af\big(x(t)\big) + Bf\big(x\big(t - h(t)\big)\big) \\ &\quad + Mg\big(\dot{x}\big(t - \sigma(t)\big)\big) + U(t), \end{aligned} \tag{8}$$

where $f(x) = (0.05(|x-1| - |x+1|),\ 0.25(|x-1| - |x+1|))^{\mathrm{T}}$, $g(x) = 0.4\tanh x$. $h(t) = (0.25\sin(t) + 0.8,\ 0.25\sin(t) + 0.65)^{\mathrm{T}}$, $\sigma(t) = (0.4\cos(t) + 0.6,\ 0.2\cos(t) + 0.8)^{\mathrm{T}}$, $U(t) = (\sin(3t),\ -\cos(3t))^{\mathrm{T}}$,

$$C = \begin{pmatrix} 5.98 & 0 \\ 0 & 8.85 \end{pmatrix}, \qquad A = \begin{pmatrix} 2.78 & -0.52 \\ 2.28 & 1.59 \end{pmatrix},$$

$$B = \begin{pmatrix} 1.55 & -0.84 \\ 1.06 & 0.77 \end{pmatrix}, \qquad M = \begin{pmatrix} 1.43 & 1.31 \\ -1.36 & 0.87 \end{pmatrix}.$$

Obviously, $U = (1,1)^{\mathrm{T}}$, $\mu = 0.25$, $\sigma = 1$. The activation function $f(x) \in \mathfrak{S}$ with $-l_i = h_i = \mathrm{diag}\{0.1, 0.5\}$, and $g(x) \in \mathfrak{B}$ with $k_i = 0.4$. According to Theorem 2, the GAS of system (8) is $\Omega = \{x(t) \in \mathbb{R}^2 | |x(1)| + |x(2)| \leqslant 2.1886\}$. Figure 1 shows the simulation results.

Example 2. Consider the following neural networks model:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -Cx(t) + Af\big(x(t)\big) + Bf\big(x\big(t - h(t)\big)\big)$$
$$+ Mg\big(\dot{x}\big(t - \sigma(t)\big)\big) + U(t), \tag{9}$$

where

$$C = \begin{pmatrix} 6.68 & 0 & 0 \\ 0 & 4.76 & 0 \\ 0 & 0 & 5.74 \end{pmatrix}, \qquad A = \begin{pmatrix} 3.05 & 3.22 & 1.65 \\ 2.23 & 1.57 & 2.32 \\ 3.56 & 1.23 & 0.36 \end{pmatrix},$$

$$B = \begin{pmatrix} 3.25 & -0.63 & 1.37 \\ 1.26 & 0.47 & 0.68 \\ 0.26 & 1.26 & 0.77 \end{pmatrix}, \qquad M = \begin{pmatrix} 1.43 & 0.75 & 1.61 \\ 0.42 & 0.63 & 0.38 \\ 0.34 & -1.16 & 0.47 \end{pmatrix},$$

$U(t) = (-2 - \sin(3t)'\, 2 - \cos(3t),\, 2 + \cos(3t))^{\mathrm{T}}$, $h(t) = 0.65 + 0.35\sin^2(t)$, $\sigma(t) = 0.6 + 0.4\sin(t)$, $f(x) = 0.1(-2 + |x - 1| + |x + 1|)$, $g(x) = 0.2\tanh(x)$. Obviously, $U = (-1, 1, 1)^{\mathrm{T}}$, $\mu = 0.7$, $\sigma = 1$. The activation function $f(x) \in \mathfrak{S}$ with $-l_i = h_i = \mathrm{diag}\{0.2, 0.2, 0.2\}$, and $g(x) \in \mathfrak{B}$ with $k_i = 0.2$.

We have the following results via the LMI control toolbox:

$$P = \begin{pmatrix} 0.3470 & 0.0197 & 0.0662 \\ 0.0197 & 0.4544 & 0.0435 \\ 0.0662 & 0.0435 & 0.2776 \end{pmatrix}, \qquad D = \begin{pmatrix} 0.0095 & 0 & 0 \\ 0 & -0.2049 & 0 \\ 0 & 0 & -0.2473 \end{pmatrix},$$

$$Q = \begin{pmatrix} 1.4901 & 0 & 0 \\ 0 & 1.5397 & 0 \\ 0 & 0 & 0.8676 \end{pmatrix}, \qquad R = \begin{pmatrix} 9.6567 & 0 & 0 \\ 0 & 7.8347 & 0 \\ 0 & 0 & 7.2108 \end{pmatrix},$$

$$S = \begin{pmatrix} 4.3926 & 0 & 0 \\ 0 & 4.9744 & 0 \\ 0 & 0 & 4.4830 \end{pmatrix}, \qquad H = \begin{pmatrix} 4.0371 & 0 & 0 \\ 0 & 4.0986 & 0 \\ 0 & 0 & 4.1039 \end{pmatrix}.$$

According to Theorem 3, when $0 < \delta_2 \leqslant 0.1$, the GAS of the system (9) is $\Omega = \{x(t) \in \mathbb{R}^3 \mid x(t)^{\mathrm{T}} P x(t) \leqslant 12.7936/\delta_2\}$. Figure 2 shows the results of system (9) with $s \in [-2, 0]$. In particular, if the weight matrix are delayed form and the initial value is left unchanged, the value of the external input is set as $U = (0, 0, 0)^{\mathrm{T}}$. Figure 3 shows the simulation results.
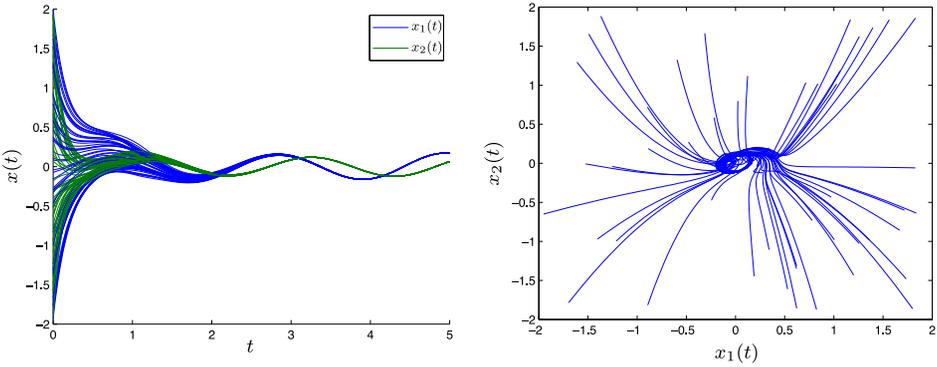
Figure 1. The transient behavior trajectories of the system (8).
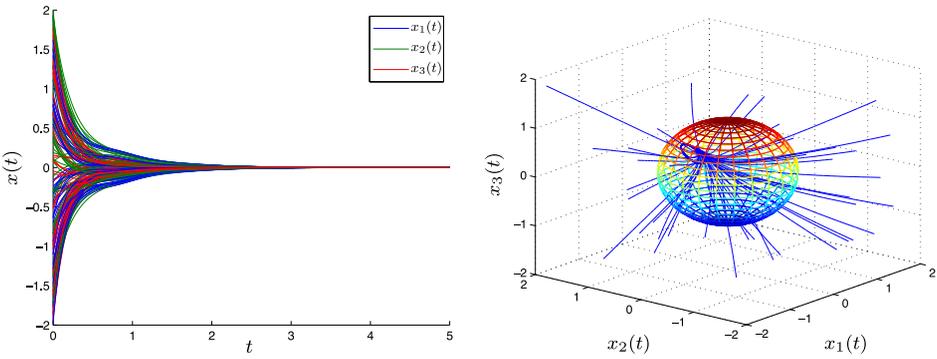

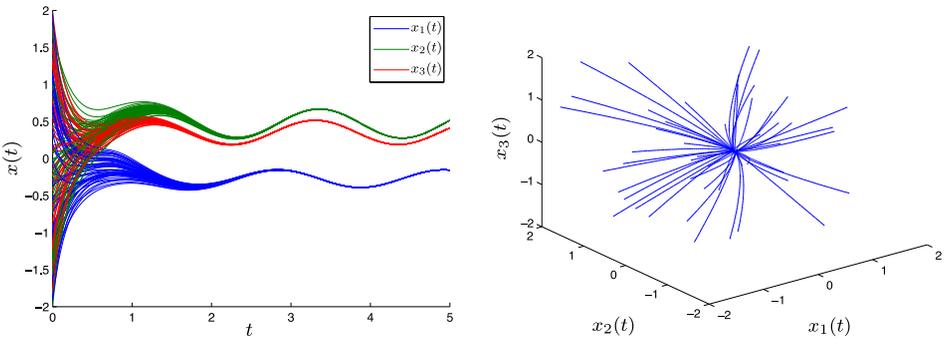**Figure 2.** The transient behavior trajectories of the system (9).

Figure 3. The transient behavior trajectories of the system (9).

Remark 3. From Fig. 2 we can know that system (9) is stable, and we can easily find its global attractive set. The solution converges to the equilibrium point fast when the value of external input is more concise as shown in Fig. 3.

5 Conclusions

This paper discussed the global attractive set of the neural networks with neutral items. We found three types of Lyapunov functions via matrix measures, inequality techniques, and Lyapunov theory. The specific estimations for the GAS of neural networks with neutral item are presented.

In addition, much more work would be considered, such as the dynamical behaviors of system with neutral items in the field of complex-valued or quaternion-valued and how to find the global attractive set of the neural networks when the neutral types activation function with general case.

References

1. M.S. Ali, S. Saravanan, Q. Zhu, Finite-time stability of neutral-type neural networks with random time-varying delays, *Int. J. Syst. Sci.*, **48**(1):1–17, 2017, `https://doi.org/10.1080/00207721.2017.1367434`.

2. S. Arik, An analysis of exponential stability of delayed neural networks with time varying delays, *Neural Netw.*, **17**(7):1027–1031, 2004, `https://doi.org/10.1016/j.neunet.2004.02.001`.

3. S. Arik, An analysis of stability of neutral-type neural systems with constant time delays, *J. Franklin Inst.*, **351**(11):4949–4959, 2014, `https://doi.org/10.1016/j.jfranklin.2014.08.013`.

4. J. Cao, Q. Song, Stability in Cohen–Grossberg-type bidirectional associative memory neural networks with time-varying delays, *Nonlinearity*, **19**(7):1601, 2006, `https://doi.org/10.1088/0951-7715/19/7/008`.

5. J. Cao, Y. Wan, Matrix measure strategies for stability and synchronization of inertial BAM neural network with time delays, *Neural Netw.*, **53**:165–172, 2014, `https://doi.org/10.1016/j.neunet.2014.02.003`.

6. X. Chen, Z. Li, Q. Song, J. Hu, Y. Tan, Robust stability analysis of quaternion-valued neural networks with time delays and parameter uncertainties, *Neural Netw.*, **91**:55–65, 2017, `https://doi.org/10.1016/j.neunet.2017.04.006`.

7. Y. Feng, X. Yang, Q. Song, J. Cao, Synchronization of memristive neural networks with mixed delays via quantized intermittent control, *Appl. Math. Comput.*, **339**:874–887, 2018, `https://doi.org/10.1016/j.amc.2018.08.009`.

8. H. He, F. Gang, C. Jinde, Robust state estimation for uncertain neural networks with time-varying delay, *IEEE Trans. Neural Networks*, **19**(8):1329–1339, 2019, `https://doi.org/10.1109/TNN.2008.2000206`.

9. J.J. Hopfield, Neurons with graded response have collective computational properties like those of two-state neurons, *Proc. Natl. Acad. Sci. USA*, **81**(10):3088–3092, 1984, `https://doi.org/10.1073/pnas.81.10.3088`.

10. S. Lakshmanan, T. Senthilkumar, P. Balasubramaniam, Improved results on robust stability of neutral systems with mixed time-varying delays and nonlinear perturbations, *Appl. Math. Modelling*, **35**(11):5355–5368, 2011, `https://doi.org/10.1016/j.apm.2011.04.043`.

11. E. Last, Linear matrix inequalities in system and control theory, *Proc. IEEE*, **86**(12):2473–2474, 1994, `https://doi.org/10.1109/JPROC.1998.735454`.

12. X. Liao, Y. Liu, H. Wang, T. Huang, Exponential estimates and exponential stability for neutral-type neural networks with multiple delays, *Neurocomputing*, **149**:868–883, 2015, `https://doi.org/10.1016/j.neucom.2014.07.048`.

13. X. Liao, Q. Luo, Z. Zeng, Y. Guo, Global exponential stability in Lagrange sense for recurrent neural networks with time delays, *Nonlinear Anal., Real World Appl.*, **9**(4):1535–1557, 2008, `https://doi.org/10.1016/j.nonrwa.2007.03.018`.

14. J. Lu, X. Guo, T. Huang, Z. Wang, Consensus of signed networked multi-agent systems with nonlinear coupling and communication delays, *Appl. Math. Comput.*, **350**:153–162, 2019, `https://doi.org/10.1016/j.amc.2019.01.006`.

15. Q. Luo, Z. Zeng, X. Liao, Global exponential stability in Lagrange sense for neutral type recurrent neural networks, *Neurocomputing*, **74**(4):638–645, 2011, `https://doi.org/10.1016/j.neucom.2010.10.001`.

16. T. Peng, J. Qiu, J. Lu, Z. Tu, J. Cao, Finite-time and fixed-time synchronization of quaternion-valued neural networks with/without mixed delays: An improved one-norm method, *IEEE Trans. Neural Netw. Learn. Syst.*, **33**(12):7475–7487, 2021, `https://doi.org/10.1109/TNNLS.2021.3085253`.

17. V. Ruiz, E. Aranda-Escolástico, J. Salt, S. Dormido, Stability and synchronization of switched multi-rate recurrent neural networks, *IEEE Access*, **9**:45614–45621, 2021, `https://doi.org/10.1109/ACCESS.2021.3067452`.

18. D.E. Rumelhart, G.E. Hinton, R.J. Williams, Learning representations by back-propagating errors, *Nature*, **323**(6088):533–536, 1986, `https://doi.org/10.1038/323533a0`.

19. Q. Song, Z. Wang, Stability analysis of impulsive stochastic Cohen–Grossberg neural networks with mixed time delays, *Physica A*, **387**(13):3314–3326, 2008, `https://doi.org/10.1016/j.physa.2008.01.079`.

20. Z. Tu, J. Cao, T. Hayat, Matrix measure based dissipativity analysis for inertial delayed uncertain neural networks, *Neural Netw.*, **75**:47–55, 2016, `https://doi.org/10.1016/j.neunet.2015.12.001`.

21. M. Vidyasagar, C.A. Desoer, Nonlinear systems analysis, *IEEE Trans. Syst. Man Cybern.*, **42**(8):537–538, 1978, `https://doi.org/10.1109/TSMC.1980.4308549`.

22. D. Xu, H. Zhao, Invariant set and attractivity of nonlinear differential equations with delays, *Appl. Math. Lett.*, **15**(3):321–325, 2002, `https://doi.org/10.1016/S0893-9659(01)00138-0`.

23. S. Xu, J. Lam, A new approach to exponential stability analysis of neural networks with time-varying delays, *Neural Netw.*, **19**(1):76–83, 2006, `https://doi.org/10.1016/j.neunet.2005.05.005`.

24. X. Yang, J. Cao, J. Qiu, pth moment exponential stochastic synchronization of coupled memristor-based neural networks with mixed delays via delayed impulsive control, *Neural Netw.*, **65**:80–91, 2015, `https://doi.org/10.1016/j.neunet.2015.01.008`.

25. X. Yang, J. Cao, C. Xu, J. Feng, Finite-time stabilization of switched dynamical networks with quantized couplings via quantized controller, *Sci. China, Technol. Sci.*, **61**(2):299–308, 2018, `https://doi.org/10.1007/s11431-016-9054-y`.

26. X. Yang, Z. Yang, Synchronization of TS fuzzy complex dynamical networks with time-varying impulsive delays and stochastic effects, *Fuzzy Sets Syst.*, **235**:25–43, 2014, `https://doi.org/10.1016/j.fss.2013.06.008`.

27. H. Zhou, Z. Zhou, W. Jiang, Almost periodic solutions for neutral type BAM neural networks with distributed leakage delays on time scales, *Neurocomputing*, **157**:223–230, 2015, `https://doi.org/10.1016/j.neucom.2015.01.013`.