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Finite-time lag projective synchronization of delayed fractional-order quaternion-valued neural networks with parameter uncertainties*

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Abstract. This paper discusses a class issue of finite-time lag projective synchronization (FTLPS) of delayed fractional-order quaternion-valued neural networks (FOQVNNs) with parameter uncertainties, which is solved by a non-decomposition method. Firstly, a new delayed FOQVNNs model with uncertain parameters is designed. Secondly, two types of feedback controller and adaptive controller without sign functions are designed in the quaternion domain. Based on the Lyapunov analysis method, the non-decomposition method is applied to replace the decomposition method that requires complex calculations, combined with some quaternion inequality techniques, to accurately estimate the settling time of FTLPS. Finally, the correctness of the obtained theoretical results is testified by a numerical simulation example.

Keywords: quaternion-valued neural networks, finite-time lag projective synchronization, parameter uncertainties.

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1 Introduction

Quaternion-valued neural networks(QVNNs) are one of the hot topics during the last few years, which have been studied by many scholars [9, 15, 20, 28]. A quaternion consists of one real number plus three imaginary units because of the characteristics of its expression form, QVNNs perform better than real-valued neural networks(RVNNs) and complex-valued neural networks (CVNNs) in processing multi-dimensional data. For this reason, QVNNs are more and more widely used in reality, for example, in pattern recognition [8], associative memory [10], and so on.

Synchronization is one of the important tools to study dynamical behavior. Recently, scholars have explored various types of synchronization and made some achievements, such as lag synchronization [18, 29], finite-time synchronization [17], Mittag-Leffler synchronization [21, 23], projective synchronization [4, 14], and so on. Among them, lag projective synchronization has important research significance in practical applications. For instance, secure communication can achieve faster communication speed by adjusting parameters, that is, converging the synchronization error to a small area under appropriate control conditions. Based on the above conclusions, if it wants to achieve synchronization as soon as possible in practical applications, it is very important to study the FTLPS of QVNNs.

In fact, due to the limited speed of information transmission between neurons, time delay inevitably exists in NNs models. It is well known that the existence of time delay can generate oscillation, which brings about the instability of NNs. Furthermore, due to the model errors and environmental factors that can interfere with the system, there is always uncertainty in system parameters, and parameter uncertainties are unavoidable. Uncertainty will affect the synchronization, stability, and other dynamical behaviors of the system. Therefore, it is very necessary to consider both time delay and parameter uncertainties.

By comparison, fractional calculus has better genetic memory characteristics than integer calculus, which can accurately describe the system and improve its performance. Thereupon, it has been widely used in various fields, such as image encryption [2,31], system recognition [5, 12], NNs [11, 24]. Nowadays, some researchers combined fractional calculus with quaternion to form FOQVNNs, which can describe the rich dynamic performance and give better play to its excellent characteristics [1,7,25]. [7] addressed a type of FOQVNNs with uncertain parameters by employing the non-separation method. Together with some inequality techniques and the properties of fractional calculus, some algebraic criteria established for robust finite-time synchronization. In [1], based on fractional-calculus theory, some new criteria are proposed for the global dissipativity and exponential stability of delayed FOQVNNs by constructing a novel Lyapunov function. The global Mittag-Leffler stability of FOQVNNs with leakage and time-varying delays was studied in [25] by employing the non-decomposition method, and some sufficient criteria to guarantee the stability of the system were given. However, up to now, results seldom consider the FTLPS of FOQVNNs.

Inspired by the aforementioned analysis, this paper considers a class of FTLPS of delayed FOQVNNs with parameter uncertainties. The innovation and main achievements of this paper are summarized as follows:

1. Inspired by [7], a more practical FOQVNNs model is proposed, that is, the model considers both the time delay and the uncertainty of system parameters.
2. Different from [32], the controllers designed in this paper do not contain sign functions, and the feedback controller and adaptive controller with quaternion-valued are designed respectively.
3. In view of the complexity and the huge account operations of the decomposition method, based on the stability theory and the construction of the Lyapunov function, this paper will use the non-decomposition method to study the FTLPS of FOQVNNs.

Notations. Let $\mathbb{R}$ and $\mathbb{Q}$ denote the real number and quaternion number, respectively. For a quaternion, its expression is $x = x^R + ix^I + jx^J + kx^K \in \mathbb{Q}$, where i, j, k are imaginary units, $x^R, x^I, x^J, x^K \in \mathbb{R}$. $\overline{x} = x^R - ix^I - jx^J - kx^K$ stands for the conjugate of x. For any $x \in \mathbb{Q}$, $|x|_1 = |x^R| + |x^I| + |x^J| + |x^K|$, $|x|_2 = \sqrt{x\overline{x}}$. For any $x \in \mathbb{Q}^n$, $\|x\|_1 = \sum_{r=1}^{M} |x_r|_1$, $\|x_r\|_2 = (\sum_{r=1}^{M} |x_r|_2^2)^{1/2}$.

2 Preliminaries

In this section, some definitions and lemmas are recalled.

Definition 1. (See [6, 13].) The Riemann–Liouville fractional-order integral of order $\mu > 0$ for function $o(t)$ is defined as

$$ {}_{t_0}^{RL}D_t^{-\mu} o(t) = \frac{1}{\Gamma(\mu)} \int_{t_0}^{t} (t-\tau)^{\mu-1} o(\tau)\,\mathrm{d}\tau, \quad t > t_0. $$

Definition 2. (See [6, 13].) The Caputo derivative of function $o(t)$ with fractional order $0 < \mu < 1$ is defined by

$$ {}_{t_0}^{c}D_t^{\mu} o(t) = \frac{1}{\Gamma(1-\mu)} \int_{t_0}^{t} (t-\tau)^{-\mu} o'(\tau)\,\mathrm{d}\tau, \quad t > t_0. $$

Lemma 1. (*See* [3].) *Suppose* $\alpha, \beta \in \mathbb{Q}$*, then*

$$ \alpha\overline{\beta} + \overline{\alpha}\beta \leqslant \zeta\alpha\overline{\alpha} + \frac{1}{\zeta}\beta\overline{\beta}. $$

Lemma 2. (*See* [32].) *For any* $x \in \mathbb{Q}$*,*

$$ x + \overline{x} = 2\,\mathrm{Re}(x) \leqslant 2|x|. $$

Lemma 3. (*See* [27].) *When* $h(t) \in \mathbb{Q}$ *is a continuous differentiable function, one has*

$$ {}_{t_0}^{c}D_t^{\mu}\overline{h(t)}\,h(t) \leqslant \overline{h(t)}\,{}_{t_0}^{c}D_t^{\mu}h(t) + \big({}_{t_0}^{c}D_t^{\mu}\overline{h(t)}\,\big)h(t), \quad 0 < \mu < 1. $$

Lemma 4. (*See* [32].) *Suppose $V(t)$ is a continuously nonnegative definite function and satisfies*

$$ {}_{t_0}^{c}D_t^{\mu} V(t) \leqslant -\lambda V^{\kappa}(t) $$

in which $\lambda > 0$, $t \geqslant t_0$, and $0 \leqslant \kappa < \mu < 1$. Then the following two cases hold:

(i) *For all $t \geqslant t_1$, when $\kappa = 0$, $V(t) = 0$,*

$$ t_1 = t_0 + \left(\frac{V(t_0)\Gamma(1+\mu)}{\lambda} \right)^{1/\mu}; $$

(ii) *For all $t \geqslant t_2$, when $0 \leqslant \kappa < \mu < 1$, $V(t) = 0$,*

$$ t_2 = t_0 + \left(\frac{\mu}{\lambda} V^{\mu-\kappa}(t_0) B(\mu,\, 1-\kappa) \right)^{1/\mu}, $$

and B is the beta function defined for any real $p, q > 0$ as follows:

$$ B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,\mathrm{d}x. $$

3 Model description

In this part, a class of delayed FOQVNNs with parameter uncertainties is described by

$$ \begin{aligned} {}_{t_0}^{c}D_t^{\mu} \phi_h(t) = {} & -k_h \phi_h(t) + \sum_{r=1}^{m} \big(a_{hr} + \Delta a_{hr}(t)\big) f_r\big(\phi_r(t)\big) \\ & + \sum_{r=1}^{m} \big(b_{hr} + \Delta b_{hr}(t)\big) g_r\big(\phi_r(t-\tau)\big) + W_h(t), \end{aligned} \tag{1} $$

where $t \geqslant t_0$, $\mu \in (0,1)$, $k_h > 0$ is the self-feedback inhibition, $\phi_h \in \mathbb{Q}$ is the quaternion-valued state variable, $a_{hr}, b_{hr} \in \mathbb{Q}$ are the connection weights, $\Delta a_{hr}(t), \Delta b_{hr}(t) \in \mathbb{Q}$ are the uncertain parameters, $f_r(\phi_r(t)), g_r(\phi_r(t-\tau)) \in \mathbb{Q}$ are the quaternion-valued activation functions, $W_h(t)$ is the external input.

Assumption 1. For any $v, \tilde{v} \in \mathbb{Q}$, $r = 1, 2, \dots, m$, the activation functions $f_r(\cdot)$ and $g_r(\cdot)$ satisfy

$$ \begin{aligned} \big| f_r(v) - f_r(\tilde{v}) \big| &\leqslant \omega_r |v - \tilde{v}|, \\ \big| g_r(v) - g_r(\tilde{v}) \big| &\leqslant \theta_r |v - \tilde{v}| \end{aligned} $$

in which $\omega_r, \theta_r > 0$ are Lipschitz constants.

Assumption 2. If there exist the constants $\widetilde{a_{hr}}$, $\widetilde{b_{hr}}$ such that $\Delta a_{hr}(t) = \widetilde{a_{hr}} \alpha_{hr}(t)$, $\Delta b_{hr}(t) = \widetilde{b_{hr}} \beta_{hr}(t)$, where $\alpha_{hr}(t)$, $\beta_{hr}(t)$ are uncertain quaternion-valued functions and satisfy $\overline{\alpha_{hr}(t)}\alpha_{hr}(t) \leqslant 1$, $\overline{\beta_{hr}(t)}\beta_{hr}(t) \leqslant 1$, $h, r = 1, 2, \dots, m$.

The associated response system is depicted by

$$ {}_{t_0}^{c}D_t^{\mu}\varphi_h(t) = -k_h\varphi_h(t) + \sum_{r=1}^{m}\big(a_{hr} + \Delta a_{hr}(t)\big)f_r\big(\varphi_r(t)\big) + \sum_{r=1}^{m}\big(b_{hr} + \Delta b_{hr}(t)\big)g_r\big(\varphi_r(t-\tau)\big) + W_h(t) + I_h(t), \tag{2} $$

where $I_h(t)$ is an external controller.

Definition 3. (See [16].) For any solutions $\phi_h(t)$ and $\varphi_h(t)$ of systems (1) and (2), the delayed FOQVNNs with uncertain parameters (1) and (2) can achieve FTLPS if there exist constants $\sigma \neq 0$, $\gamma \geqslant 0$ and a real number $T \in (0, +\infty)$ such that

$$ \lim_{t\to T}\big\|\varphi_h(t) - \sigma\phi_h(t-\gamma)\big\| = 0, $$
$$ \big\|\varphi_h(t) - \sigma\phi_h(t-\gamma)\big\| = 0, \quad t \geqslant T, $$

where σ is the projection coefficient, γ is the lag term, and T is the synchronization time.

4 Main results

From the definition of FTLPS the synchronization error is expressed as

$$ e_h(t) = \varphi_h(t) - \sigma\phi_h(t-\gamma), $$

then

$$ {}_{t_0}^{c}D_t^{\mu}e_h(t) = -k_h e_h(t) + \sum_{r=1}^{m}\big(a_{hr} + \Delta a_{hr}(t)\big)\big[f_r\big(\varphi_r(t)\big) - f_r\big(\sigma\phi_r(t-\gamma)\big)\big] + \sum_{r=1}^{m}\big(b_{hr} + \Delta b_{hr}(t)\big)\big[g_r\big(\varphi_r(t-\tau)\big) - g_r\big(\sigma\phi_r(t-\tau-\gamma)\big)\big] + I_h(t), $$

where $0 < \mu < 1$.

To achieve FTLPS between systems (1) and (2), the following quaternion-valued feedback controller is designed:

$$ I_h(t) = \begin{cases} -\varepsilon_h e_h(t) + \delta_h e_h(t-\tau) - \frac{\vartheta e_h(t)}{(\overline{e_h(t)}e_h(t))^{\lambda}}, & e_h(t) \neq 0, \\ 0, & e_h(t) = 0, \end{cases} \tag{3} $$

where $\varepsilon_h, \delta_h, \vartheta > 0$, $1 - \mu < \lambda < 1$, $h = 1, 2, \dots, m$.

Theorem 1. *Suppose Assumption* 1 *is held and under controller* (3). *Then systems* (1) *and* (2) *are FTLPS, and the following conditions are satisfied*:

$$ \rho_h < 2k_h + 2\varepsilon_h - \delta_h - 2m\omega_h^2 - \sum_{r=1}^{m}\big(|a_{hr}|^2 + |b_{hr}|^2 + |\widetilde{a_{hr}}|^2 + |\widetilde{b_{hr}}|^2\big), $$
$$ \varrho_h > \delta_h + 2m\theta_h^2 $$

in which the time T is reckoned by

$$T_1 = t_0 + \left(\frac{\mu}{2\vartheta} V^{\mu-1+\lambda}(t_0) B(\mu, \lambda) \right)^{1/\mu}.$$

Proof. Let us construct the following Lyapunov function:

$$V(t) = \sum_{h=1}^{m} \bigl(\overline{e_h(t)} e_h(t) \bigr),$$

$$\begin{aligned}
&{}_{t_0}^{c}D_t^{\mu} V(t) \\
&\quad \leqslant \sum_{h=1}^{m} \bigl[\overline{e_h(t)} {}_{t_0}^{c}D_t^{\mu} e_h(t) + \bigl({}_{t_0}^{c}D_t^{\mu} \overline{e_h(t)} \bigr) e_h(t) \bigr] \\
&\quad = \sum_{h=1}^{m} \Biggl[\overline{e_h(t)} \Biggl(-k_h e_h(t) + \sum_{r=1}^{m} \bigl(a_{hr} + \Delta a_{hr}(t) \bigr) \bigl[f_r\bigl(\varphi_r(t)\bigr) - f_r\bigl(\sigma \phi_r(t-\gamma)\bigr) \bigr] \\
&\qquad + \sum_{r=1}^{m} \bigl(b_{hr} + \Delta b_{hr}(t) \bigr) \bigl[g_r\bigl(\varphi_r(t-\tau)\bigr) - g_r\bigl(\sigma \phi_r(t-\tau-\gamma)\bigr) \bigr] \\
&\qquad - \varepsilon_h e_h(t) + \delta_h e_h(t-\tau) - \frac{\vartheta e_h(t)}{(\overline{e_h(t)} e_h(t))^{\lambda}} \Biggr) \\
&\qquad + \Biggl(-k_h \overline{e_h(t)} + \sum_{r=1}^{m} \overline{\bigl[f_r\bigl(\varphi_r(t)\bigr) - f_r\bigl(\sigma \phi_r(t-\gamma)\bigr) \bigr] \bigl(a_{hr} + \Delta a_{hr}(t) \bigr)} \\
&\qquad + \sum_{r=1}^{m} \overline{\bigl[g_r\bigl(\varphi_r(t-\tau)\bigr) - g_r\bigl(\sigma \phi_r(t-\tau-\gamma)\bigr) \bigr] \bigl(b_{hr} + \Delta b_{hr}(t) \bigr)} \\
&\qquad - \varepsilon_h \overline{e_h(t)} + \delta_h \overline{e_h(t-\tau)} - \frac{\vartheta e_h(t)}{(\overline{e_h(t)} e_h(t))^{\lambda}} \Biggr) e_h(t) \Biggr].
\end{aligned}$$

From Assumption 1 one has

$$\begin{aligned}
&{}_{t_0}^{c}D_t^{\mu} V(t) \\
&\quad \leqslant -\sum_{h=1}^{m} (2k_h + 2\varepsilon_h) \overline{e_h(t)} e_h(t) - 2\vartheta \sum_{h=1}^{m} \bigl(\overline{e_h(t)} e_h(t) \bigr)^{1-\lambda} \\
&\qquad + \sum_{h=1}^{m} \overline{e_h(t)} \delta_h e_h(t-\tau) + \sum_{h=1}^{m} \delta_h \overline{e_h(t-\tau)} e_q(t) \\
&\qquad + \sum_{h=1}^{m} \sum_{r=1}^{m} \overline{e_h(t)} a_{hr} \omega_h e_r(t) + \sum_{h=1}^{m} \sum_{r=1}^{m} \overline{e_r(t)}\, \overline{\omega_r}\, \overline{a_{hr}} e_h(t) \\
&\qquad + \sum_{h=1}^{m} \sum_{r=1}^{m} \overline{e_h(t)} \Delta a_{hr}(t) \omega_h e_r(t) + \sum_{h=1}^{m} \sum_{r=1}^{m} \overline{e_r(t)}\, \overline{\omega_r}\, \overline{\Delta a_{hr}(t)} e_h(t)
\end{aligned}$$

$$+\sum_{h=1}^{m}\sum_{r=1}^{m}\overline{e_h(t)}b_{hr}\theta_r e_r(t-\tau)+\sum_{h=1}^{m}\sum_{r=1}^{m}\overline{e_r(t-\tau)}\,\overline{\theta_r b_{hr}}e_h(t)$$
$$+\sum_{h=1}^{m}\sum_{r=1}^{m}\overline{e_h(t)}\Delta b_{hr}(t)\theta_r e_r(t-\tau)+\sum_{h=1}^{m}\sum_{r=1}^{m}\overline{e_r(t-\tau)}\,\overline{\theta_r}\,\overline{\Delta b_{hr}(t)}e_h(t). \quad (4)$$

According to Assumption 2 and Lemma 2,

$$\sum_{h=1}^{m}\overline{e_h(t)}\delta_h e_h(t-\tau)+\sum_{h=1}^{m}\delta_h\overline{e_h(t-\tau)}e_h(t)$$
$$\leqslant\sum_{h=1}^{m}\delta_h\overline{e_h(t)}e_h(t)+\sum_{h=1}^{m}\overline{e_h(t-\tau)}e_h(t-\tau), \quad (5)$$

$$\sum_{h=1}^{m}\sum_{r=1}^{m}\overline{e_h(t)}a_{hr}\omega_r e_r(t)+\sum_{h=1}^{m}\sum_{r=1}^{m}\overline{e_r(t)}\,\overline{\omega_r}\,\overline{a_{hr}}e_h(t)$$
$$\leqslant m\sum_{r=1}^{m}\overline{e_r(t)}\,\overline{\omega_r}\,\omega_r e_r(t)+\sum_{h=1}^{m}\sum_{r=1}^{m}\overline{e_h(t)}\,a_{hr}\overline{a_{hr}}\,e_h(t)$$
$$\leqslant m\sum_{h=1}^{m}\omega_h^2\overline{e_h(t)}e_h(t)+\sum_{h=1}^{m}\sum_{r=1}^{m}|a_{hr}|^2\,\overline{e_h(t)}\,e_h(t), \quad (6)$$

$$\sum_{h=1}^{m}\sum_{r=1}^{m}\overline{e_h(t)}\,\Delta a_{hr}(t)\omega_r\,e_r(t)+\sum_{h=1}^{m}\sum_{r=1}^{m}\overline{e_r(t)}\,\overline{\omega_r}\,\overline{\Delta a_{hr}(t)}e_h(t)$$
$$\leqslant m\sum_{r=1}^{m}\overline{e_r(t)}\,\overline{\omega_r}\,\omega_r e_r(t)+\sum_{h=1}^{m}\sum_{r=1}^{m}\overline{e_h(t)}\Delta a_{hr}(t)\overline{\Delta a_{hr}(t)}\,e_h(t)$$
$$\leqslant m\sum_{h=1}^{m}\omega_h^2\overline{e_h(t)}e_h(t)+\sum_{h=1}^{m}\sum_{r=1}^{m}|\widetilde{a_{hr}}|^2\,\overline{e_h(t)}\,e_h(t). \quad (7)$$

Similarly,

$$\sum_{h=1}^{m}\sum_{r=1}^{m}\overline{e_h(t)}b_{hr}\theta_r e_r(t-\tau)+\sum_{h=1}^{m}\sum_{r=1}^{m}\overline{e_r(t-\tau)}\,\overline{\theta_r}\,\overline{b_{hr}}e_h(t)$$
$$\leqslant m\sum_{h=1}^{m}\theta_h^2\overline{e_h(t-\tau)}e_h(t-\tau)+\sum_{h=1}^{m}\sum_{r=1}^{m}|b_{hr}|^2\overline{e_h(t)}e_h(t), \quad (8)$$

$$\sum_{h=1}^{m}\sum_{r=1}^{m}\overline{e_h(t)}\Delta b_{hr}(t)\theta_r e_r(t-\tau)+\sum_{h=1}^{m}\sum_{r=1}^{m}\overline{e_r(t-\tau)}\,\overline{\theta_r}\,\overline{\Delta b_{hr}(t)}e_h(t)$$
$$\leqslant m\sum_{h=1}^{m}\theta_h^2\overline{e_h(t-\tau)}e_h(t-\tau)+\sum_{h=1}^{m}\sum_{r=1}^{m}|\widetilde{b_{hr}}|^2\overline{e_h(t)}e_h(t). \quad (9)$$

Submitting (5)–(9) into (4), one has

$$\begin{aligned}
{}_{t_0}^{c}D_t^{\mu} V(t) \leqslant &-\sum_{h=1}^{m}\Bigg[2k_h + 2\varepsilon_h - \delta_h - 2m\omega_h^2 \\
&- \sum_{r=1}^{m}\big(|a_{hr}|^2 + |b_{hr}|^2 + |\widetilde{a_{hr}}|^2 + |\widetilde{b_{hr}}|^2\big)\Bigg]\overline{e_h(t)}e_h(t) \\
&+ \sum_{h=1}^{m}(\delta_h + 2m\theta_h^2)\overline{e_h(t-\tau)}e_h(t-\tau) - 2\vartheta\sum_{h=1}^{m}\big(\overline{e_h(t)}e_h(t)\big)^{1-\lambda} \\
\leqslant &-\rho_h V(t) + \varrho_h V(t-\tau) - 2\vartheta V^{1-\lambda}(t),
\end{aligned}$$

and there exists $\eta > 1$ such that $\rho_h - \varrho_h\eta > 0$. Then using fractional-order Razumikhin theorem, for all $t > t_0$, one has

$${}_{t_0}^{c}D_t^{\mu} V(t) \leqslant -(\rho_h - \varrho_h\eta)V(t) - 2\vartheta V^{1-\lambda}(t) \leqslant -2\vartheta V^{1-\lambda}(t).$$

Therefore, according to Lemma 4, systems (8) and (9) can achieve the FTLPS, the setting time T is estimated by

$$T_1 = t_0 + \left(\frac{\mu}{2\vartheta}V^{\mu-1+\lambda}(t_0)B(\mu,\lambda)\right)^{1/\mu}. \qquad \square$$

If there are no indeterminate parameters, i.e., $\Delta a_{hr} = 0$, $\Delta b_{hr} = 0$, then system (1) can be converted into the following form:

$$\begin{aligned}
{}_{t_0}^{c}D_t^{\mu}\phi_h(t) = &-k_h\phi_h(t) + \sum_{r=1}^{m} a_{hr}f_r\big(\phi_r(t)\big) \\
&+ \sum_{r=1}^{m} b_{hr}g_r\big(\phi_r(t-\tau)\big) + W_h(t).
\end{aligned} \tag{10}$$

Similarly, system (2) can be written as

$$\begin{aligned}
{}_{t_0}^{c}D_t^{\mu}\varphi_h(t) = &-k_h\varphi_h(t) + \sum_{r=1}^{m} a_{hr}f_r\big(\varphi_r(t)\big) \\
&+ \sum_{r=1}^{m} b_{hr}g_r\big(\varphi_r(t-\tau)\big) + W_h(t) + I_h(t).
\end{aligned} \tag{11}$$

Next, we can derive the following corollary.

Corollary 1. *Suppose Assumption* 1 *is held and under controller* (3). *Systems* (10) *and* (11) *are FTLPS and meet the following conditions*:

$$\widetilde{\rho_h} < 2k_h + 2\varepsilon_h - \delta_h - m\omega_h^2 - \sum_{r=1}^{m}\big(|a_{hr}|^2 + |b_{hr}|^2\big),$$
$$\widetilde{\varrho_h} > \delta_h + m\theta_h^2$$

in which the time T_2 is reckoned by

$$T_2 = t_0 + \left(\frac{\mu}{2\vartheta} V^{\mu-1+\lambda}(t_0) B(\mu, \lambda)\right)^{1/\mu}.$$

Remark 1. Among the research results known so far, there are many conclusions about finite-time synchronization with uncertain parameters of FONNs, but there are still few papers that can take into account the systems with uncertain parameters and time delay. To apply our model more accurately and more effectively in solving practical problems, it is worth our further study to consider these two factors at the same time.

To achieve FTLPS between systems (1) and (2), the following quaternion-valued adaptive controller is designed:

$$I_h(t) = \begin{cases} -\varepsilon_h(t) e_h(t) - \frac{\vartheta e_h(t)}{(\overline{e_h(t)} e_h(t))^{\lambda}}, & e_h(t) \neq 0, \\ 0, & e_h(t) = 0, \end{cases} \tag{12}$$

where ${}_{t_0}^{c}D_t^{\mu} \varepsilon_h(t) = \varepsilon'_h \overline{e_h(t)} e_h(t)$, and $\vartheta, \varepsilon'_h > 0$, $h = 1, 2, \ldots, m$.

Theorem 2. *Suppose Assumption* 1 *is held and under controller* (12). *Systems* (1) *and* (2) *are FTLPS, and the following conditions are satisfied*:

$$\zeta_h = 2k_h, \qquad \xi_h > \sum_{r=1}^{m} \theta_r \big(|b_{hr}| + |\widetilde{b_{hr}}|\big)$$

in which the time T is reckoned by

$$T_3 = t_0 + \left(\frac{\mu}{2\vartheta} V^{\mu-1+\lambda}(t_0) B(\mu, \lambda)\right)^{1/\mu}.$$

Proof. Let us construct the following Lyapunov function:

$$V(t) = \sum_{h=1}^{m} \big(\overline{e_h(t)} e_h(t)\big) + \sum_{h=1}^{m} \frac{1}{\varepsilon'_h} \big(\varepsilon_h(t) - \varepsilon_h^*\big)^2,$$

where

$$2\varepsilon_h^* = \sum_{r=1}^{m} \omega_r \big(|a_{hr}| + |\widetilde{a_{hr}}|\big) - \sum_{r=1}^{m} \omega_h \big(|a_{rh}| + |\widetilde{a_{rh}}|\big) - \sum_{r=1}^{m} \theta_r \big(|b_{hr}| + |\widetilde{b_{hr}}|\big),$$

$$\begin{aligned} &{}_{t_0}^{c}D_t^{\mu} V(t) \\ &\quad \leqslant \sum_{h=1}^{m} \big[\overline{e_h(t)}\, {}_{t_0}^{c}D_t^{\mu} e_h(t) + \big({}_{t_0}^{c}D_t^{\mu} \overline{e_h(t)}\,\big) e_h(t)\big] + \sum_{h=1}^{m} \frac{2}{\varepsilon'_h} \big(\varepsilon_h(t) - \varepsilon'_h\big)\, {}_{t_0}^{c}D_t^{\mu} \varepsilon_h(t) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{h=1}^{m} \Biggl[\overline{e_h(t)} \Biggl(-k_h e_h(t) + \sum_{r=1}^{m} \bigl(a_{hr} + \Delta a_{hr}(t)\bigr) \bigl[f_r\bigl(\varphi_r(t)\bigr) - f_r\bigl(\sigma\phi_r(t-\gamma)\bigr) \bigr] \\
&\quad + \sum_{r=1}^{m} \bigl(b_{hr} + \Delta b_{hr}(t)\bigr) \bigl[g_r\bigl(\varphi_r(t-\tau)\bigr) - g_r\bigl(\sigma\phi_r(t-\tau-\gamma)\bigr) \bigr] \\
&\quad - \varepsilon_h(t) e_h(t) - \frac{\vartheta e_h(t)}{(\overline{e_h(t)} e_h(t))^{\lambda}} \Biggr) \\
&\quad + \Biggl(-k_h \overline{e_h(t)} + \sum_{r=1}^{m} \overline{\bigl[f_r\bigl(\varphi_r(t)\bigr) - f_r(\sigma\phi_r(t-\gamma)) \bigr] \bigl(a_{hr} + \Delta a_{hr}(t)\bigr)} \\
&\quad + \sum_{r=1}^{m} \overline{\bigl[g_r\bigl(\varphi_r(t-\tau)\bigr) - g_r(\sigma\phi_r(t-\tau-\gamma)) \bigr] \bigl(b_{hr} + \Delta b_{hr}(t)\bigr)} \\
&\quad - \varepsilon_h(t) \overline{e_h(t)} - \frac{\vartheta e_h(t)}{(\overline{e_h(t)} e_h(t))^{\lambda}} \Biggr) e_h(t) \Biggr] + 2 \sum_{h=1}^{m} \bigl(\varepsilon_h(t) - \varepsilon_h^*\bigr) \overline{e_h(t)} e_h(t).
\end{aligned}
$$

From Assumption 1 one has

$$
\begin{aligned}
&{}_{t_0}^{c}D_t^{\mu} V(t) \\
&\quad \leqslant -\sum_{h=1}^{m} (2k_h + 2\varepsilon_h^*) \overline{e_h(t)} e_h(t) - 2\vartheta \sum_{h=1}^{m} (\overline{e_h(t)} e_h(t))^{1-\lambda} \\
&\quad\quad + \sum_{h=1}^{m} \sum_{r=1}^{m} \overline{e_h(t)} a_{hr} \omega_r e_r(t) + \sum_{h=1}^{m} \sum_{r=1}^{m} \overline{e_r(t)} \overline{\omega_r a_{hr}} e_h(t) \\
&\quad\quad + \sum_{h=1}^{m} \sum_{r=1}^{m} \overline{e_h(t)} \Delta a_{hr}(t) \omega_r e_r(t) + \sum_{h=1}^{m} \sum_{r=1}^{m} \overline{e_r(t)}\, \overline{\omega_r}\, \overline{\Delta a_{hr}(t)} e_h(t) \\
&\quad\quad + \sum_{h=1}^{m} \sum_{r=1}^{m} \overline{e_h(t)} b_r \theta_r e_r(t-\tau) + \sum_{h=1}^{m} \sum_{r=1}^{m} \overline{e_r(t-\tau)}\, \overline{\theta_r}\, \overline{b_{hr}}\, e_h(t) \\
&\quad\quad + \sum_{h=1}^{m} \sum_{r=1}^{m} \overline{e_h(t)} \Delta b_{hr}(t) \theta_r e_r(t-\tau) + \sum_{h=1}^{m} \sum_{r=1}^{m} \overline{e_r(t-\tau)} \overline{\theta_r}\, \overline{\Delta b_{hr}(t)}\, e_h(t).
\end{aligned} \tag{13}
$$

According to Assumption 1 and Lemma 2,

$$
\begin{aligned}
&\sum_{h=1}^{m} \sum_{r=1}^{m} \overline{e_h(t)} a_{hr} \omega_r e_r(t) + \sum_{h=1}^{m} \sum_{r=1}^{m} \overline{e_r(t)}\, \overline{\omega_r}\, \overline{a_{hr}} e_h(t) \\
&\quad \leqslant 2 \sum_{h=1}^{m} \sum_{r=1}^{m} \sqrt{\overline{e_r(t)} \overline{\omega_r a_{hr}} e_h(t) \overline{e_h(t)} a_{hr} \omega_r e_r(t)} \\
&\quad \leqslant 2 \sum_{h=1}^{m} \sum_{r=1}^{m} \omega_r |a_{hr}| |e_h(t)| |e_r(t)|
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant \sum_{h=1}^{m}\sum_{r=1}^{m}\omega_r|a_{hr}|\big(\overline{e_h(t)}e_h(t)+\overline{e_r(t)}e_r(t)\big)\\
&=\sum_{h=1}^{m}\sum_{r=1}^{m}\omega_r|a_{hr}|\overline{e_h(t)}e_h(t)+\sum_{h=1}^{m}\sum_{r=1}^{m}\omega_h|a_{rh}|\overline{e_h(t)}e_h(t)
\end{aligned}\tag{14}
$$

and

$$
\begin{aligned}
&\sum_{h=1}^{m}\sum_{r=1}^{m}\overline{e_h(t)}\Delta a_{hr}(t)\omega_r e_r(t)+\sum_{h=1}^{m}\sum_{r=1}^{m}\overline{e_r(t)}\overline{\omega_r}\overline{\Delta a_{hr}(t)}e_h(t)\\
&\quad\leqslant 2\sum_{h=1}^{m}\sum_{r=1}^{m}\sqrt{\overline{e_r(t)}\overline{\omega_r}\overline{\Delta a_{hr}(t)}e_h(t)\overline{e_h(t)}\Delta a_{hr}(t)\omega_r e_r(t)}\\
&\quad\leqslant 2\sum_{h=1}^{m}\sum_{r=1}^{m}\omega_r\big|\widetilde{a_{hr}}\big|\big|e_h(t)\big|\big|e_r(t)\big|\\
&\quad\leqslant \sum_{h=1}^{m}\sum_{r=1}^{m}\omega_r|\widetilde{a_{hr}}|\big(\overline{e_h(t)}e_h(t)+\overline{e_r(t)}e_r(t)\big)\\
&\quad=\sum_{h=1}^{m}\sum_{r=1}^{m}\omega_r|\widetilde{a_{hr}}|\overline{e_h(t)}e_h(t)+\sum_{h=1}^{m}\sum_{r=1}^{m}\omega_h|\widetilde{a_{rh}}|\overline{e_h(t)}e_h(t).
\end{aligned}\tag{15}
$$

Similarly,

$$
\begin{aligned}
&\sum_{h=1}^{m}\sum_{r=1}^{m}\overline{e_h(t)}b_{hr}\theta_r e_r(t-\tau)+\sum_{h=1}^{m}\sum_{r=1}^{m}\overline{e_r(t-\tau)}\,\overline{\theta_r}\,\overline{b_{hr}}e_h(t)\\
&\quad\leqslant 2\sum_{h=1}^{m}\sum_{r=1}^{m}\sqrt{\overline{e_r(t-\tau)}\overline{\theta_r}\overline{b_{hr}}e_h(t)\overline{e_h(t)}a_{hr}\theta_r e_r(t-\tau)}\\
&\quad\leqslant 2\sum_{h=1}^{m}\sum_{r=1}^{m}\theta_r|b_{hr}|\big|e_h(t)\big|\big|e_r(t-\tau)\big|\\
&\quad\leqslant \sum_{h=1}^{m}\sum_{r=1}^{m}\theta_r|b_{hr}|\big(\overline{e_h(t)}e_h(t)+\overline{e_r(t-\tau)}e_r(t-\tau)\big)\\
&\quad=\sum_{h=1}^{m}\sum_{r=1}^{m}\theta_r|b_{hr}|\overline{e_h(t)}e_h(t)+\sum_{h=1}^{m}\sum_{r=1}^{m}\theta_h|b_{rh}|\overline{e_h(t-\tau)}e_h(t-\tau)
\end{aligned}\tag{16}
$$

and

$$
\begin{aligned}
&\sum_{h=1}^{m}\sum_{r=1}^{m}\overline{e_h(t)}\Delta b_{hr}(t)\theta_r e_r(t-\tau)+\sum_{h=1}^{m}\sum_{r=1}^{m}\overline{e_r(t-\tau)}\,\overline{\theta_r}\,\overline{\Delta b_{hr}(t)}e_h(t)\\
&\quad\leqslant 2\sum_{h=1}^{m}\sum_{r=1}^{m}\sqrt{\overline{e_r(t)}\,\overline{\theta_r}\,\overline{\Delta b_{hr}(t)}e_h(t)\overline{e_h(t)}\Delta b_{hr}(t)\theta_r e_r(t-\tau)}
\end{aligned}
$$

$$\begin{aligned}
&\leqslant 2\sum_{h=1}^{m}\sum_{r=1}^{m}\theta_r|\widetilde{b_{hr}}|\big|e_h(t)\big|\big|e_r(t-\tau)\big| \\
&\leqslant \sum_{h=1}^{m}\sum_{r=1}^{m}\theta_r|\widetilde{b_{hr}}|\big(\overline{e_h(t)}e_h(t)+\overline{e_r(t-\tau)}e_r(t-\tau)\big) \\
&= \sum_{h=1}^{m}\sum_{r=1}^{m}\theta_r|\widetilde{b_{hr}}|\,\overline{e_h(t)}e_h(t)+\sum_{h=1}^{m}\sum_{r=1}^{m}\theta_h|\widetilde{b_{rh}}|\,\overline{e_h(t-\tau)}e_h(t-\tau). \qquad (17)
\end{aligned}$$

Submitting (14)–(17) into (13), one has

$$\begin{aligned}
{}_{t_0}^{c}D_t^{\mu}V(t) \leqslant{}& -\sum_{h=1}^{m}\Bigg[2k_h+2\varepsilon_h^*-\sum_{r=1}^{m}\omega_r\big(|a_{hr}|+|\widetilde{a_{hr}}|\big) \\
&-\sum_{r=1}^{m}\omega_h\big(|a_{rh}|+|\widetilde{a_{rh}}|\big)-\sum_{r=1}^{m}\theta_r\big(|b_{hr}|+|\widetilde{b_{hr}}|\big)\Bigg]\overline{e_h(t)}e_h(t) \\
&+\sum_{r=1}^{m}\theta_r\big(|b_{hr}|+|\widetilde{b_{hr}}|\big)\overline{e_h(t-\tau)}e_h(t-\tau)-2\vartheta\sum_{h=1}^{m}\big(\overline{e_h(t)}e_h(t)\big)^{1-\lambda} \\
\leqslant{}& -\zeta_h V(t)+\xi_h V(t-\tau)-2\vartheta V^{1-\lambda}(t),
\end{aligned}$$

and there exists $\eta > 1$ such that $\zeta_h - \xi_h\eta > 0$. Then using fractional-order Razumikhin theorem, for all $t > t_0$, we have

$$ {}_{t_0}^{c}D_t^{\mu}V(t) \leqslant -(\zeta_h-\xi_h\eta)V(t)-2\vartheta V^{1-\lambda}(t) \leqslant -2\vartheta V^{1-\lambda}(t).$$

Therefore, according to Lemma 4, systems (8) and (9) are FTLPS, the setting time T is reckoned by

$$T_3 = t_0 + \left(\frac{\mu}{2\vartheta}V^{\mu-1+\lambda}(t_0)B(\mu,\lambda)\right)^{1/\mu}. \qquad \square$$

Corollary 2. *Suppose Assumption* 1 *is held and under controller* (12). *Systems* (10) *and* (11) *are finite-time lag projective synchronization and meet the following conditions*:

$$\widetilde{\zeta_h} = 2k_h, \qquad \widetilde{\xi_h} > \sum_{r=1}^{m}\theta_r|b_{qh}|$$

in which

$$2\varepsilon_h^* = \sum_{r=1}^{m}\omega_r|a_{hr}| - \sum_{r=1}^{m}\omega_r|a_{rh}| - \sum_{r=1}^{m}\theta_r|b_{hr}|,$$

and the time T_4 is reckoned by

$$T_4 = t_0 + \left(\frac{\mu}{2\vartheta}V^{\mu-1+\lambda}(t_0)B(\mu,\lambda)\right)^{1/\mu}.$$

Remark 2. Judging from the existing research results, there are few conclusions about the FTLPS analysis of FOQVNNs with uncertain parameters. [19] studied the synchronization and quasi-synchronization problems of FOQVNNs with uncertain parameters and proposed lexicographical ordering approach, which can be used to fix the range of two different quaternion-values. By constructing a Lyapunov function and comparison theory, the global asymptotic synchronization and stability of delayed FOMQVNNs with uncertain parameters are studied, and some appropriate criteria are established in [22].

Remark 3. Different from [22], this paper uses the non-decomposition method instead of the decomposition method, that is, directly regards the system as a whole and no longer divides the system into other subsystems for analysis, which greatly reduces the computational complexity to a certain extent.

Remark 4. In [30], the projection synchronization of the FOQVNNs model when $\gamma = 0$ was studied, obviously the lag projection synchronization considered in this paper can be regarded as a generalization of the above reference, and the theoretical conclusions obtained were more general.

Remark 5. The finite-time synchronization problem of delayed FOQVNNs was analyzed in [26]. Compared with the FOQVNNs model considered in this paper, this paper not only added the time delay term to the model, but also considered the uncertainty term that will affect the stability of the neural networks. Therefore, the model designed in this paper was more realistic and easier to be verified in practical applications.

5 Numerical simulations

In this section, the example of the FTLPS is given to show the validity of the results obtained in Theorems 1 and 2.

Example. Consider the following delayed FOQVNNs with uncertain parameters as drive and response systems, respectively:

$$\begin{aligned} {}_{t_0}^{c}D_t^{\mu}\phi_h(t) &= -k_h\phi_h(t) + \sum_{r=1}^{m}(a_{hr} + \varDelta a_{hr}(t))f_r\big(\phi_r(t)\big) \\ &\quad + \sum_{r=1}^{m}\big(b_{hr} + \varDelta b_{hr}(t)\big)g_r\big(\phi_r(t-\tau)\big) + W_h(t), \end{aligned} \tag{18}$$

$$\begin{aligned} {}_{t_0}^{c}D_t^{\mu}\varphi_h(t) &= -k_h\varphi_h(t) + \sum_{r=1}^{m}(a_{hr} + \varDelta a_{hr}(t))f_r\big(\varphi_r(t)\big) \\ &\quad + \sum_{r=1}^{m}\big(b_{hr} + \varDelta b_{hr}(t)\big)g_r\big(\varphi_r(t-\tau)\big) + W_h(t) + I_h(t), \end{aligned} \tag{19}$$

where $\phi_q(t) = \phi_q^R(t) + i\phi_q^I(t) + j\phi_q^J(t) + k\phi_q^K(t)$ with $\phi_q^R(t)$, $\phi_q^I(t)$; $\phi_q^J(t)$, $\phi_q^K(t) \in \mathbb{R}$, $\varphi_q(t) = \varphi_q^R(t) + i\varphi_q^I(t) + j\varphi_q^J(t) + k\varphi_q^K(t)$ with $\varphi_q^R(t)$, $\varphi_q^I(t)$, $\varphi_q^J(t)$, $\varphi_q^K(t) \in \mathbb{R}$;

the activation functions $f(\cdot) = g(\cdot) = \tanh(\phi_h^R(t)) + i\tanh(\phi_h^I(t)) + j\tanh(\phi_h^J(t)) + k\tanh(\phi_h^K(t))$; $h = 1, 2$, $W_1(t) = W_2(t) = 0$, $\mu = 0.92$, $\tau = 1$,

$$K = \operatorname{diag}(k_1, k_2) = \begin{pmatrix} 15 & 0 \\ 0 & 10 \end{pmatrix},$$

$$A = (a_{hr})_{2\times 2} = \begin{pmatrix} 2.4 - 0.1i + 2.2j + 3k & -2 + 1.5i + 0.3j + 1.5k \\ -1.1 + i + j + 2k & -2.3 + 2i + 2.1j - 2k \end{pmatrix},$$

$$B = \big(\varDelta a_{hr}(t)\big)_{2\times 2},$$

where $\varDelta a_{11} = 0.5\sin(t) + 0.5\cos(t)i + 0.8\cos(t)j + 0.5\sin(t)k$, $\varDelta a_{12} = 0.5\sin(t) - 0.5\sin(t)i + \cos(t)j - 0.5\sin(t)k$, $\varDelta a_{21} = 0.5\cos(t) + 0.5\cos(t)i + 0.5\cos(t)j + 0.5\cos(t)k$, $\varDelta a_{22} = 0.5\sin(t) - 0.5\sin(t)i - 0.5\sin(t)j - 0.5\sin(t)k$,

$$C = (b_{hr})_{2\times 2} = \begin{pmatrix} 2 - 0.6i + 2j + 2k & -5 + 6i + 2j \\ -2 - 3i + j - 1.1k & -1 + 1.4i + 1.9j - 0.1k \end{pmatrix},$$

$$D = \big(\varDelta b_{hr}(t)\big)_{2\times 2},$$

where $\varDelta b_{11} = 0.5\sin(t) + 0.5\sin(t)i + 0.8\sin(t)j + 0.5\sin(t)k$, $\varDelta b_{12} = -0.5\sin(t) - 0.5\sin(t)i - 0.5\sin(t)j - 0.5\sin(t)k$, $\varDelta b_{21} = 0.5\cos(t) + 0.5\cos(t)i + 0.5\cos(t)j - 0.5\sin(t)k$, $\varDelta b_{22} = 0.5\sin(t) - 0.5\sin(t)i - 0.5\sin(t)j - 0.5\sin(t)k$.

Under the quaternion-valued feedback controller (3), let the initial values of system (18) and (19) are $\phi(0) = (4.2 + 2i + 2j - 2.3k,\ 2.2 - 2i + 2j + 3.1k)$, $\varphi(0) = (2.8 + 4.1i + 5j - 4.5k,\ 4 - 4.5i + 4.5j + 4.3k)$. The state tracks of synchronization error without controller (3) are displayed in Figs. 1 and 2, which derived that systems (19) and (18) cannot achieve FTLPS. Next, we set values to the parameters in controller (3), that is, $\varepsilon_1 = 46$, $\varepsilon_2 = 35$, $\delta_1 = 0.5$, $\delta_2 = 2.5$, $\vartheta = 12$, $\lambda = 0.9$. Then through the calculation, one has $V(0) = 36.65$, and two parameters that satisfy the conditions in Theorem 1 are obtained, namely, $\rho_1 = 6.5$, $\rho_2 = 32$, $\varrho_1 = 7$, $\varrho_2 = 7$. In addition, there is $\eta = 1.1$ such that $\rho_h - \varrho_h\eta > 0$ making fractional-order Razumikhin theorem established. Therefore, systems (18) and (19) can realize FTLPS and the settling time $T_1 \approx 0.8730$, which is shown in Fig. 3, and the state variables curves are depicted in Figs. 4 and 5.

In the quaternion-valued adaptive controller (12), we set $\varepsilon_1(0) = 43$, $\varepsilon_2(0) = 38$, $\vartheta = 6$, $\lambda = 0.4$, and the initial values of system (18) and (19) are $\phi(0) = (1 - 1.8i + 3j - 2.3k,\ 2 - 2.2i + 2j + 3.1k)$, $\varphi(0) = (3 + 5.4i + 5.3j - 2.5k,\ 2.4 - 4.7i + 5.5j + 4.3k)$. Similarly, through simple calculations, we can get $V(0) = 115.6342$, $\varepsilon_1^* = -4.515$, $\varepsilon_2^* = -3.555$ and the constants $\zeta_1 = 30$, $\zeta_2 = 14$, $\xi_1 = 20$, $\xi_2 = 9$ satisfying the conditions of Theorem 2. Also, there exists $\eta = 1.2$ such that $\zeta_h - \xi_h\eta > 0$ making fractional-order Razumikhin theorem established. It can be seen in Figs. 6 and 7 that the trajectories of various state variables without the controller are disordered and cannot achieve synchronization. From Fig. 8 the system error gradually tends to zero, that is, the synchronization between systems (18) and (19) has been achieved in a finite time, and the settling-time can be calculated by the formula as $T_3 \approx 0.8814$. Meanwhile, Figs. 9 and 10 depicted that the trajectory of each state variable can indeed achieve synchronization at this time.

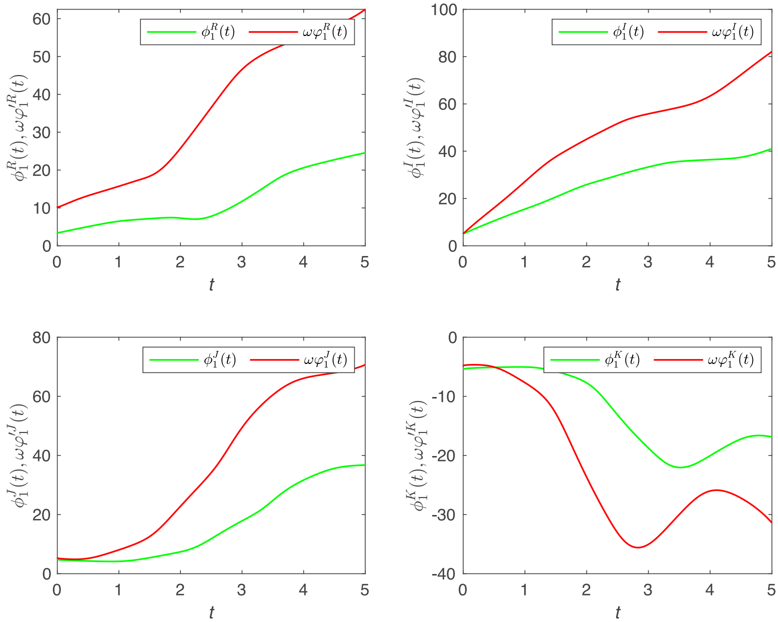

Figure 1. Under $\sigma = 2$, $\gamma = 1$, state trajectories of ϕ_1^R, $\sigma\varphi_1^R$, ϕ_1^I, $\sigma\varphi_1^I$, ϕ_1^J, $\sigma\varphi_1^J$, ϕ_1^K, $\sigma\varphi_1^K$ without controller (3).

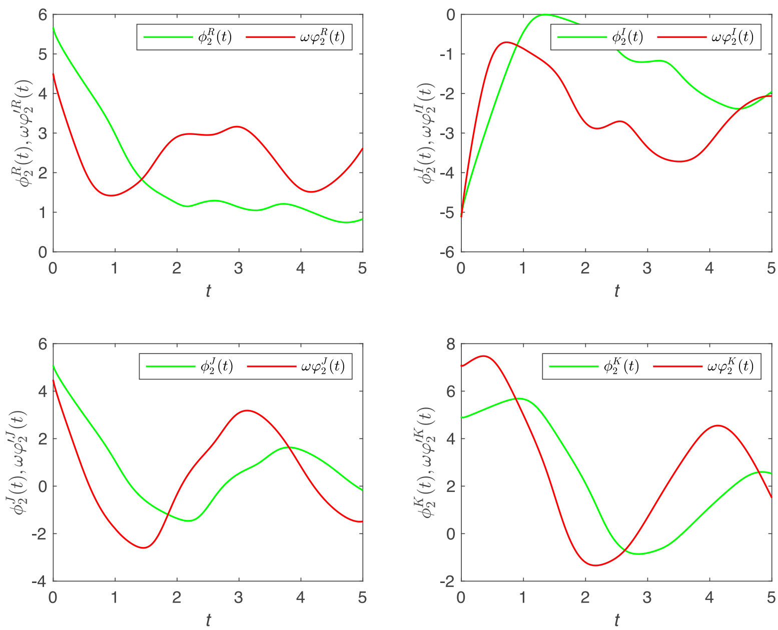

Figure 2. Under $\sigma = 2$, $\gamma = 1$, state trajectories of ϕ_2^R, $\sigma\varphi_2^R$, ϕ_2^I, $\sigma\varphi_2^I$, ϕ_2^J, $\sigma\varphi_2^J$, ϕ_2^K, $\sigma\varphi_2^K$ without controller (3).

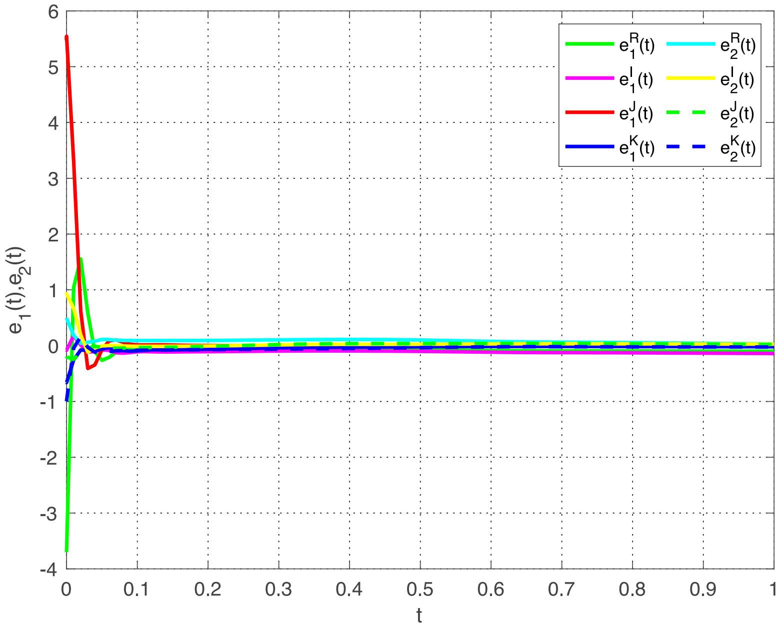

Figure 3. Finite-time synchronization error e_1^R, e_2^R, e_1^I, e_2^I, e_1^J, e_2^J, e_1^K, e_2^K under controller (3).

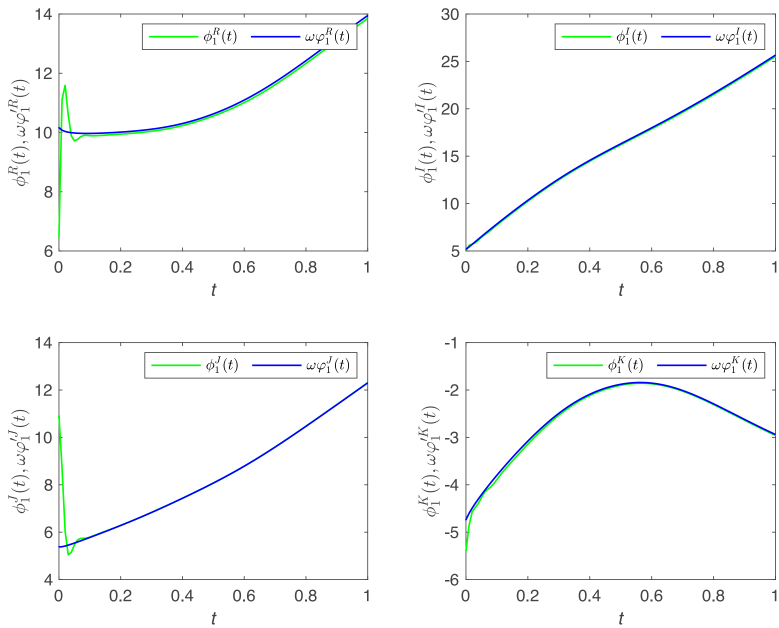

Figure 4. Under $\sigma = 2$, $\gamma = 1$, state trajectories of ϕ_1^R, $\sigma\varphi_1^R$, ϕ_1^I, $\sigma\varphi_1^I$, ϕ_1^J, $\sigma\varphi_1^J$, ϕ_1^K, $\sigma\varphi_1^K$.

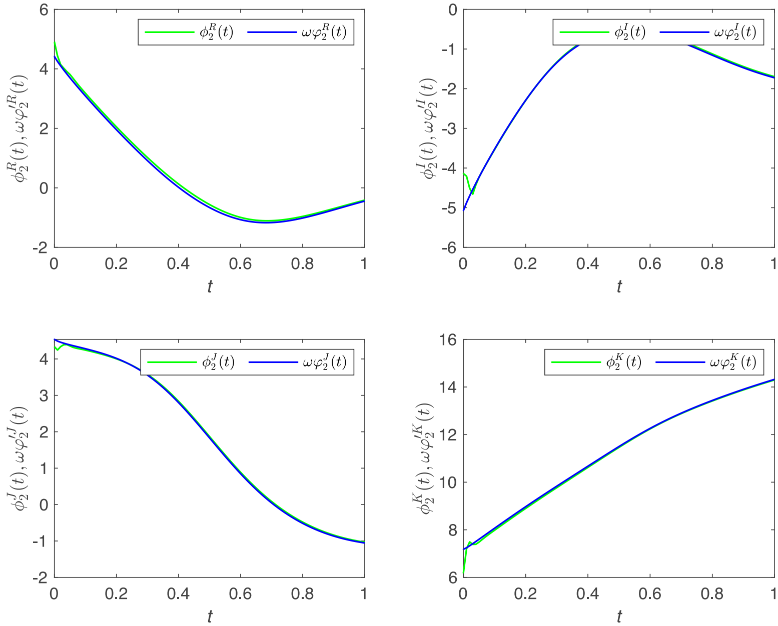

Figure 5. Under $\sigma = 2$, $\gamma = 1$, state trajectories of ϕ_2^R, $\sigma\varphi_2^R$, ϕ_2^I, $\sigma\varphi_2^I$, ϕ_2^J, $\sigma\varphi_2^J$, ϕ_2^K, $\sigma\varphi_2^K$.

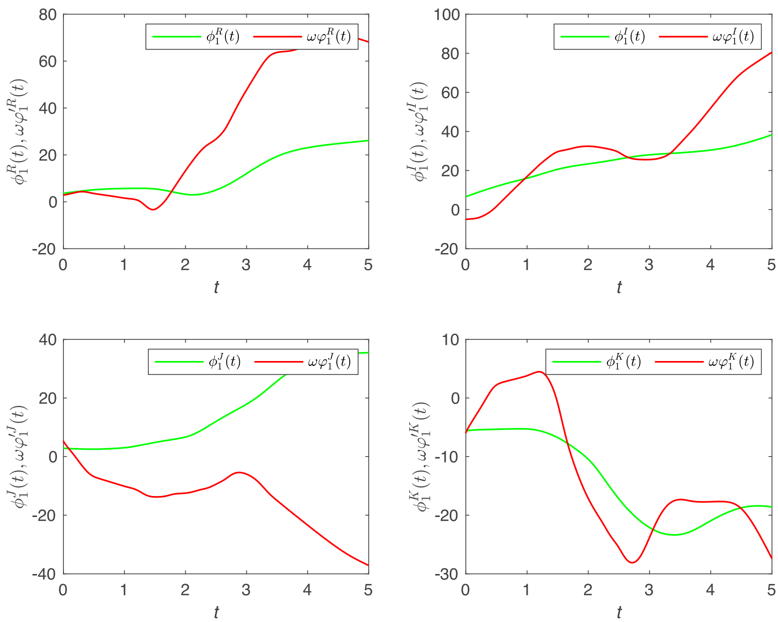

Figure 6. Under $\sigma = 2.3$, $\gamma = 1$, state trajectories of ϕ_1^R, $\sigma\varphi_1^R$, ϕ_1^I, $\sigma\varphi_1^I$, ϕ_1^J, $\sigma\varphi_1^J$, ϕ_1^K, $\sigma\varphi_1^K$ without controller (12).

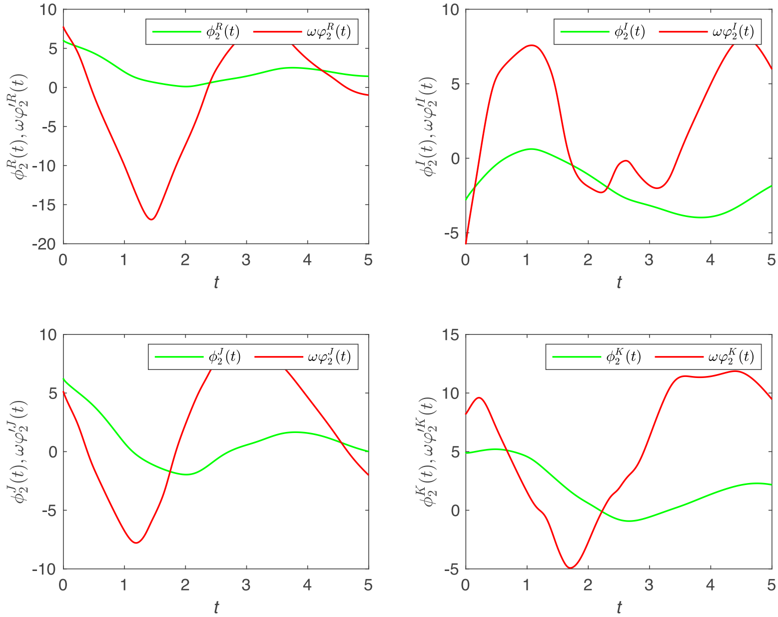

Figure 7. Under $\sigma = 2.3$, $\gamma = 1$, state trajectories of ϕ_2^R, $\sigma\varphi_2^R$, ϕ_2^I, $\sigma\varphi_2^I$, ϕ_2^J, $\sigma\varphi_2^J$, ϕ_2^K, $\sigma\varphi_2^K$ without controller (12).

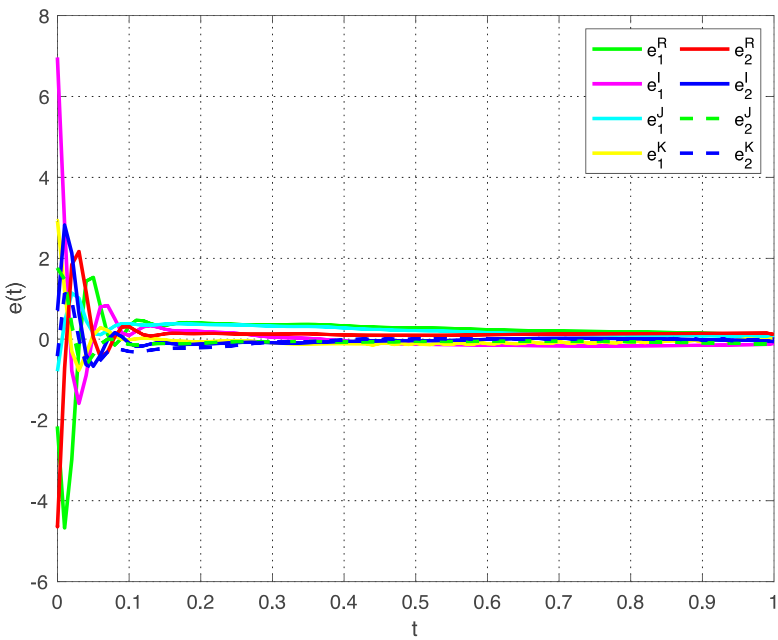

Figure 8. Finite-time synchronization error e_1^R, e_2^R, e_1^I, e_2^I, e_1^J, e_2^J, e_1^K, e_2^K under controller (12).

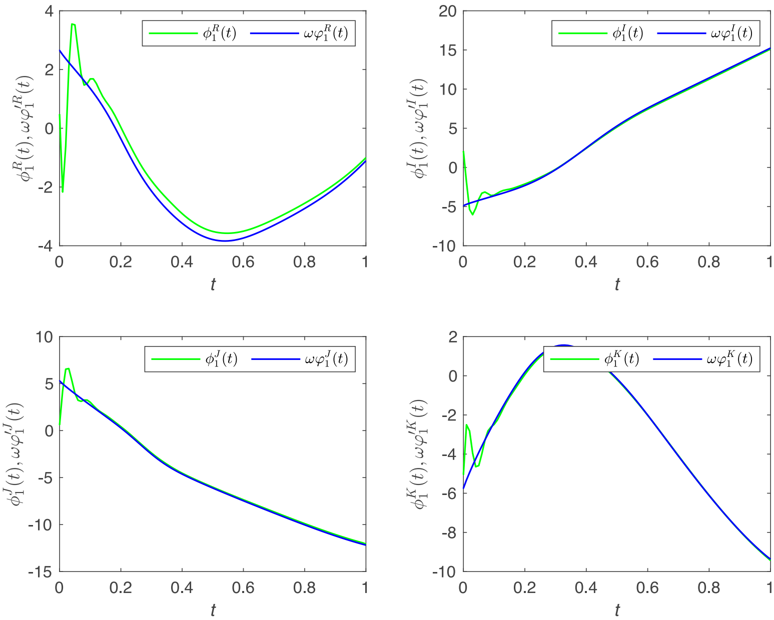

Figure 9. Under $\sigma = 2.3$, $\gamma = 1$, state trajectories of ϕ_1^R, $\sigma\varphi_1^R$, ϕ_1^I, $\sigma\varphi_1^I$, ϕ_1^J, $\sigma\varphi_1^J$, ϕ_1^K, $\sigma\varphi_1^K$.

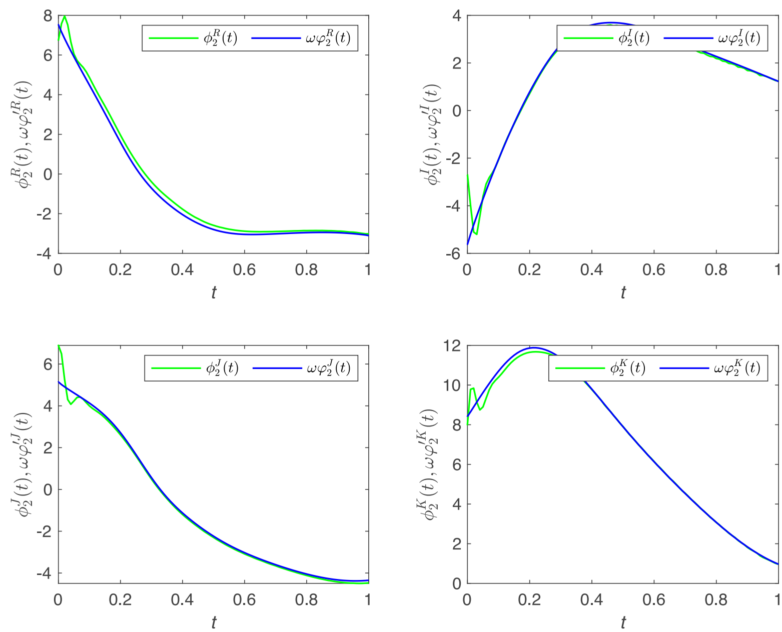

Figure 10. Under $\sigma = 2.3$, $\gamma = 1$, state trajectories of ϕ_2^R, $\sigma\varphi_2^R$, ϕ_2^I, $\sigma\varphi_2^I$, ϕ_2^J, $\sigma\varphi_2^J$, ϕ_2^K, $\sigma\varphi_2^K$.

6 Conclusion

This paper discussed the FTLPS problem of time-delayed FOQVNNs with uncertain parameters. First, in order to ensure the stability of the NNs, both time-delay terms and uncertainties are considered in the model. Secondly, using a non-decomposition method under two different types of controllers gives two criteria that can ensure that FOQVNNs lag projection synchronization in finite time. In addition, the obtained theoretical results are effectively proved by a numerical simulation. In our future work, the proposed method can be used to study other dynamical behaviors of delayed FOQVNNs with uncertain parameters, such as stability, stabilization, and so on.

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