Nonlinear Analysis: Modelling and Control, Vol. 28, No. 2, 288–307
https://doi.org/10.15388/namc.2023.28.31450

A new conversation on the existence of Hilfer fractional stochastic Volterra–Fredholm integro-differential inclusions via almost sectorial operators

Sivajiganesan Sivasankar[a], **Ramalingam Udhayakumar**[a,1], **Venkatesan Muthukumaran**[b]

[a]Department of Mathematics, School of Advanced Sciences,
Vellore Institute of Technology,
Vellore - 632 014, Tamil Nadu, India
sivajisivasankar@gmail.com; udhayaram.v@gmail.com

[b]Department of Mathematics,
College of Engineering and Technology,
SRM Institute of Science and Technology,
Kattankulathur - 603203, Tamilnadu, India
muthu.v2404@gmail.com

Received: April 17, 2022 / **Revised:** January 20, 2023 / **Published online:** February 22, 2023

Abstract. The existence of Hilfer fractional stochastic Volterra–Fredholm integro-differential inclusions via almost sectorial operators is the topic of our paper. The researchers used fractional calculus, stochastic analysis theory, and Bohnenblust–Karlin's fixed point theorem for multivalued maps to support their findings. To begin with, we must establish the existence of a mild solution. In addition, to show the principle, an application is presented.

Keywords: Hilfer fractional system, stochastic system, integro-differential system, almost sectorial operators, multivalued maps.

1 Introduction

In 1695, fractional calculus was presented as a major field of mathematics. It happened approximately simultaneously with the development of classical calculus. Researchers have discovered that fractional calculus may accurately portray a range of nonlocal phenomena in the fields of natural science and technology, and the notion of fractional calculus has recently been successfully applied to a variety of sectors. The most common fields of fractional calculus are rheology, dynamical cycles in identity and heterogeneous structures, diffusive transport equivalent to dispersion, liquid stream, optics, viscoelasticity, and others. Because diagnostic arrangements can be tough to come by in many fields, the successful use of fractional systems has prompted many investigators to reconsider

[1]Corresponding author.

their mathematical estimation methods. In [8, 23], readers can find some interesting conclusions related to fractional dynamical systems and research articles related to fractional differential systems theory. Recently, Guo et al. [12] investigated the existence and Hyers–Ulam stability of solution for an impulsive Riemann–Liouville fractional neutral functional stochastic differential equation with infinite delay of order $1 < \beta < 2$ by using the Mönch fixed point theorem.

Throughout the past decade, fractional calculus has been one of the most important frameworks for analysing brief operations. Such models pique the interests of architects, scientists, and pure mathematicians alike. The most essential of these models are fractional equations with fractional-order derivatives. Furthermore, academics are focusing on qualitative behaviours such as fractional dynamical systems, stability, existence, and controllability in [21]. In practical use, since stochastic fluctuation is unavoidable, we must investigate deterministic problems for stochastic differential equations [20, 24]. Because of their applicability in several disciplines of science and engineering, stochastic differential equations have piqued people's curiosity. Furthermore, it should be noted that in nature, even in artificial systems, noise or stochastic discomfort cannot be prohibited. Stochastic differential systems have sparked interest due to their widespread application in presenting a wide range of sophisticated dynamical systems in scientific, physical, and pharmaceutical domains [3, 6, 11]. Differential inclusion tools make it simpler to investigate dynamical systems with kinematics that are not solely determined by the state of the system.

Other fractional-order derivatives, such as the RL derivatives and Caputo fractional derivatives, were started by Hilfer [1, 2, 14, 29]. Furthermore, theoretical simulations of thermoelasticity in crystal compounds, chemical processing, rheological constitutive modelling, engineering, and other domains have uncovered the usefulness and applicability of the Hilfer fractional derivative. Gu and Trujillo [10] recently employed a noncompact measure approach and a fixed point technique to show that there is an integral solution to the Hilfer fractional derivative evolution problem. To designate the derivative's order, he developed the latest variable $\mu \in [0, 1]$ as well as a fractional variable λ, so that $\mu = 0$ provides the RL derivative and $\lambda = 1$ yields the Caputo derivative. Hilfer fractional calculus [6, 8–11, 13] has been the subject of several articles. Researchers revealed the existence of the mild solution for Hilfer fractional differential systems via almost sectorial operators using a fixed point approach in [4, 15, 16]. The authors investigated the solvability and controllability of differential systems using a fixed point technique in [17, 27].

A growing number of researchers are advancing fractional existence for fractional calculus using almost sectorial operators. For the system under examination, the investigators established a new technique for identifying mild solution. Furthermore, the investigators developed a theory to derive various properties of related semigroups created by almost sectorial operators using fractional calculus, semigroups, multivalued analysis, a measure of noncompactness, the Laplace transform, a Wright-type function, and a fixed point theorem. As examples, we can look at [7, 19, 22, 26, 28, 30, 31]. Furthermore, in [5, 6], researchers studied fractional differential inclusion papers using Bohnenblust–Karlin's fixed point theorem for multivalued maps. Sivasankar and Udhayakumar [25] recently used the fixed point approach to investigate the existence of Hilfer fractional stochastic differential systems via almost sectorial operators. However, to the best of our knowledge, so far, no work has been reported in the literature about the existence of Hilfer fractional stochastic Volterra–Fredholm integro-differential inclusions via almost sectorial operators.

Inspired by the above-mentioned work, this paper aims to fill this gap. The purpose of this paper is to show the existence of Hilfer fractional stochastic Volterra–Fredholm integro-differential inclusions via almost sectorial operators of the form

$$
\begin{aligned}
{}^{H}D_{0+}^{\lambda,\mu} y(\rho) \in Ay(\rho) + \mathcal{G}\left(\rho, y(\rho), \int_0^{\rho} f\big(\rho,\nu,y(\nu)\big)\,\mathrm{d}\nu, \int_0^{c} h\big(\rho,\nu,y(\nu)\big)\,\mathrm{d}\nu\right) \\
\times \frac{\mathrm{d}W(\rho)}{\mathrm{d}\rho}, \quad \rho \in \mathcal{V}' = (0,c], \qquad (1)
\end{aligned}
$$

$$
I_{0+}^{(1-\lambda)(1-\mu)} y(0) = y_0, \qquad (2)
$$

where A is an almost sectorial operator of the analytic semigroup $\{T(\rho),\ \rho \geqslant 0\}$ on Y. ${}^{H}D_{0+}^{\lambda,\mu}$ denotes the Hilfer fractional derivative (HFD) of order $\lambda \in (0,1)$ and type $\mu \in [0,1]$, the state $y(\cdot)$ takes the value in a Hilbert space Y with $\|\cdot\|$. Let $\mathcal{V} = [0,c]$ be the interval, $\mathcal{G} : \mathcal{V} \times Y \times Y \times Y \to 2^{Y} \setminus \{\emptyset\}$ is a bounded, nonempty, convex closed multivalued map, and $f, h : \mathcal{V} \times \mathcal{V} \times Y \to Y$ are the appropriate functions. For clarity, we take

$$
(Fy)(\rho) = \int_0^{\rho} f\big(\rho,\nu,y(\nu)\big)\,\mathrm{d}\nu,
$$

$$
(Hy)(\rho) = \int_0^{c} h\big(\rho,\nu,y(\nu)\big)\,\mathrm{d}\nu.
$$

The structure of the article is divided into the following: In Section 2, we cover the principles of fractional calculus, sectorial operators, and multivalued maps. In Section 3, we present the existence of the mild solution. We provide an application in Section 4 to highlight our main concepts. Lastly, there are some conclusions to be drawn.

2 Preliminaries

We offer the required theorems and results in this section, which will be used throughout the essay to obtain the new results.

Two real separable Hilbert spaces are represented by $(Y, \|\cdot\|)$ and $(U, \|\cdot\|)$. Assume that $(\Omega, \mathscr{E}, \mathbf{P})$ is a complete probability space connected with the proper family of right-continuous increasing sub σ-algebras $\{\mathscr{E}_\rho,\ \rho \in \mathcal{V}\}$ satisfying $\mathscr{E}_\rho \subset \mathscr{E}$. Let $W = (W_\rho)_{\rho \geqslant 0}$ be a Q-Wiener process defined on $(\Omega, \mathscr{E}, \mathbf{P})$ with the covariance operator Q such that $\mathrm{Tr}(Q) < \infty$. We suppose that there exists a proper orthonormal system e_n, $n \geqslant 1$, in U, a bounded sequence of nonnegative real numbers δ_n such that $Qe_n = \delta_n e_n$, $n = 1, 2, \dots,$ and $\{\widehat{\beta}_n\}$ of independent Brownian motions such that

$$\big(W(\rho), e\big)_U = \sum_{n=1}^{\infty} \sqrt{\delta_n}(e_n, e)\widehat{\beta}_n(\rho), \quad e \in U,\ \rho \geqslant 0.$$

Assume that $L_2^0 = L_2(Q^{1/2}U, Y)$, which stands for the space of all Q-Hilbert–Schmidt operators $\phi : Q^{1/2}U \to Y$ with the inner product $\|\phi\|_Q^2 = \langle \phi, \phi \rangle = \mathrm{Tr}(\phi Q \phi)$, is a Hilbert space. Consider $0 \in \varrho(A)$, the resolvent set of A, where $\mathcal{S}(\cdot)$ is uniformly bounded, i.e., $\|\mathcal{S}(\rho)\| \leqslant M$, $M \geqslant 1$, and $\rho \geqslant 0$. The fractional-power operator A^η on its domain $D(A^\eta)$ may then be determined for $\eta \in (0, 1]$. In addition, $D(A^\eta)$ is dense in Y.

The following theorem lists fundamental properties of A^η.

Theorem 1.

(i) *Suppose that* $0 < \eta \leqslant 1$*, corresponding* $Y_\eta := D(A^\eta)$ *is a Banach space with* $\|y\|_\eta = \|A^\eta y\|$, $y \in Y_\eta$.

(ii) *Suppose* $0 < \gamma < \eta \leqslant 1$*, corresponding* $D(A^\eta) \to D(A^\gamma)$*, and the embedding is compact every time that* A *is compact.*

(iii) *For all* $0 < \eta \leqslant 1$*, there exists* $C_\eta > 0$ *such that* $\|A^\eta \mathcal{S}(\rho)\| \leqslant C_\eta/\rho^\eta$, $0 < \rho \leqslant c$.

The set of all strongly-measurable, square-integrable, Y-valued random variables, denoted by $L_2(\Omega, Y)$, is a Banach space connected with $\|y(\cdot)\|_{L_2(\Omega,Y)} = (\mathbf{E}\|y(\cdot, W)\|^2)^{1/2}$, where $\mathbf{E}$ is defined as $\mathbf{E}(y) = \int_\Omega y(W)\,\mathrm{d}\mathbf{P}$. An essential subspace of $L_2(\Omega, Y)$ is provided by

$$L_2^0(\Omega, Y) = \big\{y \in L_2(\Omega, Y),\ y \text{ is } \mathscr{E}_0\text{-measurable}\big\}.$$

For $c > 0$, let $\mathcal{V} = [0, c]$ and $\mathcal{V}' = (0, c]$. Denote $C(\mathcal{V}, Y) = \mathcal{C}$ as the Banach space of all continuous functions from $\mathcal{V}$ to Y that satisfies the condition $\sup_{\rho\in\mathcal{V}} \mathbf{E}\|y(\rho)\|^2 < \infty$. Let

$$\Delta = \Big\{y \in C(\mathcal{V}', Y)\colon\ \lim_{\rho\to 0} \rho^{1-\mu+\lambda\mu-\lambda\vartheta} y(\rho) \text{ exists and finite}\Big\}$$

is a Banach space with $\|\cdot\|_\Delta$ and $\|y\|_\Delta = (\sup_{\rho\in\mathcal{V}'} \mathbf{E}\|\rho^{1-\mu+\lambda\mu-\lambda\vartheta} y(\rho)\|^2)^{1/2}$. Set

$$B_r(\mathcal{V}) = \big\{u \in \mathcal{C}\colon\ \|u\| \leqslant r\big\} \quad \text{and} \quad B_r^\Delta(\mathcal{V}) = \big\{y \in \Delta\colon\ \|y\|_\Omega \leqslant r\big\}.$$

Definition 1. (See [22].) For $0 < \vartheta < 1$, $0 < \omega < \pi/2$, we define the group of closed linear operators $\Theta_\omega^{-\vartheta}$, the sector $S_\omega = \{v \in \mathbb{C} \setminus \{0\}\colon\ |\arg v| \leqslant \omega\}$, and the operator $A : \mathcal{D}(\mathcal{A}) \subset Y \to Y$ in such a way that the following holds:

(i) $\sigma(A) \subseteq S_\omega$;

(ii) There exists a constant M_δ such that $\|(vI - A)^{-1}\| \leqslant M_\delta |v|^{-\vartheta}$ for all $\omega < \delta < \pi$.

Then $A \in \Theta_\omega^{-\vartheta}$ is called an almost sectorial operator on Y.

Proposition 1. (*See* [22].) *Let* $T(\rho)$ *be the compact semigroup,* $A \in \Theta_\omega^{-\vartheta}$ *for* $0 < \vartheta < 1$ *and* $0 < \omega < \pi/2$*. Then the following is satisfied*:

(i) $T(\rho+\nu)=T(\rho)T(\nu)$ *for all* $\nu,\rho\in S_{\pi/2-\omega}$;
(ii) $\|T(\rho)\|_{L(Y)}\leqslant\kappa_0\rho^{\vartheta-1}$, $\rho>0$, *where* $\kappa_0>0$ *be the constant*;
(iii) *The range* $R(T(\rho))$ *of* $T(\rho)$, $\rho\in S_{\pi/2-\omega}$, *is contained in* $D(A^\infty)$. *Particularly,* $R(T(\rho))\subset D(A^\theta)$ *for all* $\theta\in\mathbb{C}$ *with* $\mathrm{Re}(\theta)>0$,

$$A^\theta T(\rho)y=\frac{1}{2\pi\imath}\int\limits_{\Gamma_\mu}v^\theta\mathrm{e}^{-\rho v}R(v;A)y\,\mathrm{d}v\quad\textit{for all } y\in Y,$$

and hence there exists a constant $C'=C'(\gamma,\theta)>0$ *such that*

$$\left\|A^\theta T(\rho)\right\|_{L(Y)}\leqslant C'\rho^{-\gamma-\mathrm{Re}(\theta)-1}\quad\textit{for all } \rho>0;$$

(iv) *If* $\Sigma_T=\{y\in Y\colon \lim_{\rho\to0^+}T(\rho)y=y\}$, *then* $D(A^\theta)\subset\Sigma_T$ *if* $\theta>1+\vartheta$;
(v) $(vI-A)^{-1}=\int_0^\infty\mathrm{e}^{-v\nu}T(\nu)\,\mathrm{d}\nu$, $v\in\mathbb{C}$ *and* $\mathrm{Re}(v)>0$.

Definition 2. (See [32].) The left-side Riemann–Liouville fractional integral of order λ with the lower limit c for the function $\mathcal{G}:[c,\infty)\to\mathbb{R}$ is given by

$$I^\lambda_{c+}\mathcal{G}(\nu)=\frac{1}{\Gamma(\lambda)}\int\limits_c^\rho\frac{\mathcal{G}(\nu)}{(\rho-\nu)^{1-\lambda}}\,\mathrm{d}\nu,\quad \rho>0,\ \lambda>0,$$

if the right side is point-wise determined on $[c,+\infty)$, $\Gamma(\cdot)$ is the gamma function.

Definition 3. (See [32].) The left-sided Riemann–Liouville fractional derivative of order $\lambda>0$, $m-1\leqslant\lambda<m$, $m\in\mathbb{N}$, for a function $\mathcal{G}:[c,+\infty)\to\mathbb{R}$, is given by

$${}^LD^\lambda_{c+}\mathcal{G}(\nu)=\frac{1}{\Gamma(m-\lambda)}\frac{\mathrm{d}^m}{\mathrm{d}\rho^m}\int\limits_c^\rho\frac{\mathcal{G}(\nu)}{(\rho-\nu)^{\lambda+1-m}}\,\mathrm{d}\nu,\quad \rho>c,$$

provided the right-hand side is defined a.e. on $(c,\infty]$.

Definition 4. (See [32].) The left-sided Caputo derivative of type of order $\lambda>0$, $m-1\leqslant\lambda<m$, $m\in\mathbb{N}$, for a function $\mathcal{G}:[c,+\infty)\to\mathbb{R}$, is given by

$${}^CD^\lambda_{c+}\mathcal{G}(\nu)=\frac{1}{\Gamma(m-\lambda)}\int\limits_c^\rho\frac{\mathcal{G}^m(\nu)}{(\rho-\nu)^{\lambda+1-m}}\,\mathrm{d}\nu=I^{m-\rho}_{y+}\mathcal{G}^m(\nu),\quad \rho>c,$$

provided the right-hand side is defined a.e. on $[c,+\infty)$.

Definition 5. (See [14].) The left-sided HFD of order $0<\lambda<1$ and type $\mu\in[0,1]$, of function $\mathcal{G}:[c,+\infty)\to\mathbb{R}$, is given by

$${}^HD^{\lambda,\mu}_{c^+}\mathcal{G}(\nu)=\left[I^{(1-\lambda)\mu}_{c^+}D\left(I^{(1-\lambda)(1-\mu)}_{c^+}\mathcal{G}\right)\right](\nu).$$

Remark 1. (See [14].)

(i) If $\mu = 0$, $0 < \lambda < 1$, and $c = 0$, then the HFD is equivalent to the Riemann–Liouville fractional derivative:

$$^{H}D_{0+}^{\lambda,0}\mathcal{G}(\nu) = \frac{\mathrm{d}}{\mathrm{d}\nu} I_{0+}^{1-\lambda}\mathcal{G}(\rho) = {}^{L}D_{0+}^{\lambda}\mathcal{G}(\nu).$$

(ii) If $\mu = 1$, $0 < \lambda < 1$, and $c = 0$, then the HFD is equivalent to the Caputo fractional derivative:

$$^{H}D_{0+}^{\lambda,1}\mathcal{G}(\nu) = I_{0+}^{1-\lambda}\frac{\mathrm{d}}{\mathrm{d}\nu}\mathcal{G}(\rho) = {}^{C}D_{0+}^{\lambda}\mathcal{G}(\nu).$$

Definition 6. (See [31].) Define the Wright function $W_\lambda(\beta)$ by

$$W_\lambda(\beta) = \sum_{n\in\mathbb{N}} \frac{(-\beta)^{n-1}}{\Gamma(1-\lambda n)(n-1)!}, \quad \beta \in \mathbb{C},$$

with the following property:

$$\int_0^\infty \theta^\iota W_\lambda(\theta)\,\mathrm{d}\theta = \frac{\Gamma(1+\iota)}{\Gamma(1+\lambda\iota)}, \quad \text{for } \iota \geqslant 0.$$

Definition 7. (See [29].) $\mathcal{G}$ is said to be upper semicontinuous (u.s.c.) on Y if for any $y_0 \in Y$, the set $\mathcal{G}(y_0)$ is a nonempty, closed subset of Y, and if for each open set $\mathcal{V}$ of Y containing $\mathcal{G}(y_0)$, there exists an open neighbourhood $\mathcal{V}$ of y_0 such that $\mathcal{G}(\mathcal{V}) \subseteq \mathcal{V}$.

If a multivalued map $\mathcal{G}$ is completely continuous with nonempty compact values, then $\mathcal{G}$ is upper semicontinuous iff $\mathcal{G}$ has a closed graph i.e., $y_m\mathcal{G}y_0$, $z_m \to z_0$, $z_n \in \mathcal{G}(y_n)$ imply $z_0 \in \mathcal{G}(y_0)$.

Lemma 1. (*See* [10].) *The differential system* (1)–(2) *is equivalent to an integral inclusions presented as*

$$y(\rho) \in \frac{y_0}{\Gamma(\mu(1-\lambda)+\lambda)}\rho^{(1-\lambda)(\mu-1)} + \frac{1}{\Gamma(\lambda)}\int_0^\rho (\rho-\nu)^{\lambda-1}\big[Ay(\nu)\,\mathrm{d}\nu + g(\nu)\,\mathrm{d}W(\nu)\big].$$

Lemma 2. (*See* [10].) *Let $y(\rho)$ be a solution of the integral inclusions given in Lemma* 1*, then $y(\rho)$ satisfies*

$$y(\rho) = \mathcal{S}_{\lambda,\mu}(\rho)y_0 + \int_0^\rho \mathcal{K}_\lambda(\rho-\nu)g(\nu)\,\mathrm{d}W(\nu), \quad \rho \in \mathcal{V}',$$

where

$$\mathcal{S}_{\lambda,\mu}(\rho) = I_0^{\mu(1-\lambda)}\mathcal{K}_\lambda(\rho), \qquad \mathcal{K}_\lambda(\rho) = \rho^{\lambda-1}\mathcal{Q}_\lambda(\rho),$$

and

$$\mathcal{Q}(\rho) = \int_0^\infty \lambda\theta W(\theta)T\big(\rho^\lambda\theta\big)\,\mathrm{d}\theta.$$

Definition 8. An $\mathscr{E}_\rho$-adpapted stochastic process $y(\rho) \in C(\mathcal{V}', Y)$ is called a mild solution of the Cauchy problem (1)–(2) if $I_0^{(1-\lambda)(1-\mu)} y(0) = y_0$, $y_0 \in L_2^0(\Omega, Y)$, there exists $g \in L^2(\Omega, Y)$ such that $g(\rho) \in \mathcal{G}(\rho, y(\rho))$ on $\rho \in \mathcal{V}'$, and

$$y(\rho) = \mathcal{S}_{\lambda,\mu}(\rho) y_0 + \int_0^\rho \mathcal{K}_\lambda(\rho - \nu) g(\nu) \, \mathrm{d}W(\nu), \quad \rho \in \mathcal{V}',$$

where

$$(Fy)(\nu) = \int_0^\nu f\big(\nu, w, y(w)\big) \, \mathrm{d}w, \qquad (Hy)(\nu) = \int_0^c h\big(\nu, w, y(w)\big) \, \mathrm{d}w.$$

Lemma 3. (*See* [31].) *If $T(\rho)_{\rho>0}$ is a compact operator, then $\mathcal{S}_{\lambda,\mu}(\rho)_{\rho>0}$ and $\mathcal{Q}_\lambda(\rho)_{\rho>0}$ are also compact operators.*

Lemma 4. (*See* [31].) *Let $\theta > 1 - \vartheta$. For all $y \in D(A^\theta)$, we have*

$$\lim_{\rho \to 0^+} \mathcal{Q}(\rho) y = \frac{y}{\Gamma(\lambda)}.$$

Lemma 5. (*See* [31].) *Assume that $T(\rho)_{\rho>0}$ is a compact operator. Then $T(\rho)_{\rho>0}$ is equicontinuous.*

Lemma 6. (*See* [31].) *For each fixed $\rho > 0$, $\mathcal{Q}_\lambda(\rho)$, $\mathcal{K}_\lambda(\rho)$, and $\mathcal{S}_{\lambda,\mu}(\rho)$ are linear operators, and for any $y \in Y$,*

$$\big\|\mathcal{Q}_\lambda(\rho) y\big\| \leqslant \kappa_p \rho^{\lambda(\vartheta - 1)} \|y\|, \qquad \big\|\mathcal{K}_\lambda(\rho) y\big\| \leqslant \kappa_p \rho^{\lambda\vartheta - 1} \|y\|,$$

and

$$\big\|\mathcal{S}_{\lambda,\mu}(\rho) y\big\| \leqslant \kappa_s \rho^{-1+\mu-\lambda\mu+\lambda\vartheta} \|y\|,$$

where

$$\kappa_p = \frac{\kappa_0 \Gamma(\vartheta)}{\Gamma(\lambda\vartheta)}, \quad \kappa_s = \frac{\kappa_0 \Gamma(\vartheta)}{\Gamma(\mu(1-\lambda) + \lambda\vartheta)}.$$

Lemma 7. (*See* [31].) *Assume that $\{T(\rho)\}_{\rho>0}$ is equicontinuous. Then $\{\mathcal{Q}_\lambda(\rho)\}_{\rho>0}$, $\{\mathcal{K}_\lambda(\rho)\}_{\rho>0}$, and $\{\mathcal{S}_{\lambda,\mu}\}_{\rho>0}$ are strongly continuous, that is, for any $y \in Y$ and $\rho'' > \rho' > 0$,*

$$\big\|\mathcal{Q}_\lambda(\rho') y - \mathcal{Q}_\lambda(\rho'') y\big\| \to 0, \quad \big\|\mathcal{K}_\lambda(\rho') y - \mathcal{K}_\lambda(\rho'') y\big\| \to 0,$$
$$\big\|\mathcal{S}_{\lambda,\mu}(\rho') y - \mathcal{S}_{\lambda,\mu}(\rho'') y\big\| \to 0 \quad \text{as } \rho'' \to \rho'.$$

Theorem 2. (*See* [32].) *$\mathcal{S}_\lambda(\rho)$ and $\mathcal{Q}_\lambda(\rho)$ are continuous in the uniform operator topology, for $\rho > 0$, for any $c > 0$, the continuity is uniform on $[c, \infty)$.*

Lemma 8. (*See* [18].) *Suppose $BCC(Y)$ be the set of all nonempty, bounded, closed, and convex subset of Y, $\mathcal{V}$ be a compact real interval. Let $\mathcal{G}$ be the L^2-Caratheodory* *multivalued map, and $\mathcal{G} : \mathcal{V} \times Y \times Y \times Y \to BCC(Y)$ be measurable to ρ for any fixed $y \in Y$, u.s.c. to y for every $\rho \in \mathcal{V}$, and for all $y \in \mathcal{C}$, the set*

$$S_{\mathcal{G},y} = \{g \in L^2(\mathcal{V}, L(U,Y))\colon\ g(\rho) \in \mathcal{G}(\rho, y(\rho), (Fy)(\rho), (Hy)(\rho)),\ \rho \in \mathcal{V}\}$$

be nonempty. Let Γ be the linear continuous function from $L^2(\mathcal{V}, Y)$ to $\mathcal{C}$, then

$$\Gamma \circ S_{\mathcal{G}} : \mathcal{C} \to BCC(\mathcal{C}), \qquad y \to (\Gamma \circ S_{\mathcal{G}})(y) = \Gamma(S_{\mathcal{G},y}),$$

is closed graph operator in $\mathcal{C} \times \mathcal{C}$.

Lemma 9 [Bohnenblust-Karlins fixed point theorem]. (*See* [5].) *Suppose that Y be a closed, bounded, and convex subset Y of y. Assume that $\mathcal{D} : Y \to 2^Y \setminus \{\emptyset\}$ is upper semicontinuous with closed, convex values such that $\mathcal{D}(Y) \subset Y$ and $\mathcal{D}(Y)$ is compact. Then $\mathcal{D}$ has a fixed point.*

3 Existence of mild solution

We require the following hypotheses:

(H1) The operator $\{T(\rho), \rho \geqslant 0\}$ is compact.

(H2) The map $\mathcal{G} : \mathcal{V} \times Y \times Y \times Y \to BCC(Y)$ is measurable to ρ for any fixed $y \in Y$, u.s.c. to y for all $\rho \in \mathcal{V}$ and for any $y \in \mathcal{C}$, the set

$$S_{\mathcal{G},y} = \{g \in L^1(\mathcal{V}, Y)\colon\ g(\rho) \in \mathcal{G}(\rho, y(\rho), (Fy)(\rho), (Hy)(\rho)),\ \rho \in \mathcal{V}\}$$

is nonempty.

(H3) For $\rho \in \mathcal{V}$, $\mathcal{G}(\rho, \cdot, \cdot, \cdot) : Y \times Y \times Y \to Y$, $f(\rho, \nu, \cdot)$, $h(\rho, \nu, \cdot) : Y \to Y$ are continuous functions, and for each $y \in \mathcal{C}$, $\mathcal{G}(\cdot, y, (Fy), (Hy)) : \mathcal{V} \to \mathcal{V}$ and $f(\cdot, \cdot, y), h(\cdot, \cdot, y) : \mathcal{V} \times \mathcal{V} \to Y$ are strongly measurable.

(H4) For $r > 0$, $y \in \mathcal{C}$ along with $\|y\|_{\mathcal{C}} \leqslant r$ and $\mathcal{L}_{\mathcal{G},r}(\rho) \in L^1(\mathcal{V}, \mathbb{R}^+)$ satisfying $\lim_{\rho\to 0^+} \rho^{1-\mu+\lambda\mu-\lambda\vartheta} I_{0+}^{\lambda\vartheta} \mathcal{L}_{\mathcal{G},r}(\rho) = 0$,

$$\sup\{\mathbf{E}\|g\|^2\colon\ g(\rho) \in \mathcal{G}(\rho, y(\rho), (Fy)(\rho), (Hy)(\rho))\} \leqslant \mathcal{L}_{\mathcal{G},r}(\rho)$$

for a.e. $\rho \in \mathcal{V}$.

(H5) The function $\nu \to (\rho - \nu)^{2(\lambda\vartheta-1)} \mathcal{L}_{\mathcal{G},r}(\nu) \in L^1(\mathcal{V}, \mathbb{R}^+)$, and there exists a constant $\gamma > 0$ such that

$$\lim_{r\to\infty} \inf \frac{\int_0^\rho (\rho-\nu)^{2(\lambda\vartheta-1)} \mathcal{L}_{\mathcal{G},r}(\nu)\,\mathrm{d}\nu}{r} = \gamma < \infty.$$

Theorem 3. *Assume that hypotheses* (H1)–(H5) *hold. Then the Hilfer fractional stochastic system* (1)–(2) *has a mild solution on $\mathcal{V}$, provided*

$$2\,\mathrm{Tr}(Q)\kappa_p^2 c^{2(1-\mu+\lambda\mu-\lambda\vartheta)}\gamma < 1$$

and $y_0 \in D(A^\theta)$ with $\theta > 1 + \vartheta$.

Proof. We define the multivalued operator $\Sigma : \mathcal{C} \to 2^{\mathcal{C}}$ by

$$\Sigma\big(y(\rho)\big) = \Bigg\{ y \in \mathcal{C}\colon\ z(\rho) = \rho^{1-\mu+\lambda\mu-\lambda\vartheta}\Bigg[\mathcal{S}_{\lambda,\mu}(\rho)y_0 + \int_0^\rho (\rho-\nu)^{\lambda-1}\mathcal{Q}_\lambda(\rho-\nu)g(\nu)\,\mathrm{d}W(\nu)\Bigg],\ \rho \in (0,c]\Bigg\},$$

and we prove that Σ has a fixed point.

Step 1. $\Sigma(y)$ is convex for all $y \in \mathcal{C}$.

Suppose $z_1, z_2 \in \mathcal{C}$ and $g_1, g_2 \in S_{\mathcal{G},y}$, $\rho \in \mathcal{V}$. We know

$$z_i(\rho) = \rho^{1-\mu+\lambda\mu-\lambda\vartheta}\Bigg[\mathcal{S}_{\lambda,\mu}(\rho)y_0 + \int_0^\rho (\rho-\nu)^{\lambda-1}\mathcal{Q}_\lambda(\rho-\nu)g_i(\nu)\,\mathrm{d}W(\nu)\Bigg], \quad i = 1, 2.$$

Let $0 \leqslant \chi \leqslant 1$, then for each of $\rho \in \mathcal{V}$, we obtain

$$\begin{aligned}
&\chi z_1 + (1-\chi)z_2(\rho) \\
&\quad = \rho^{1-\mu+\lambda\mu-\lambda\vartheta}\mathcal{S}_{\lambda,\mu}(\rho)y_0 \\
&\qquad + \rho^{1-\mu+\lambda\mu-\lambda\vartheta}\int_0^\rho (\rho-\nu)^{\lambda-1}\mathcal{Q}_\lambda(\rho-\nu)\big[\chi g_1(\nu) + (1-\chi)g_2(\nu)\big]\,\mathrm{d}W(\nu).
\end{aligned}$$

Since $S_{\mathcal{G},y}$ is convex, $\chi g_1 + (1-\chi)g_2 \in S_{\mathcal{G},y}$.

Therefore,

$$\chi z_1 + (1-\chi)z_2 \in \Sigma y(\rho),$$

hence Σ is convex.

Step 2. On the space $\mathcal{C}$, consider $B_r = \{y \in \mathcal{C}\colon\ \|y\|_{\mathcal{C}}^2 \leqslant r\}$ for $r > 0$. Clearly, B_r is bounded, closed, and convex set of $\mathcal{C}$. Now we prove that there exists $r > 0$ such that $\Sigma(B_r) \subseteq B_r$.

If not, then for all $r > 0$, there exists $y^r \in B_r$, but $\Sigma(y^r) \notin B_r$, i.e.,

$$\big\|\Sigma\big(y^r\big)\big\|_{\mathcal{C}} \equiv \sup\big\{\big\|z^r\big\|_{\mathcal{C}}\colon\ z^r \in \big(\Sigma y^r\big)\big\} > r,$$

and

$$z^r(\rho) = \rho^{1-\mu+\lambda\mu-\lambda\vartheta}\Bigg[\mathcal{S}_{\lambda,\mu}(\rho)y_0 + \int_0^\rho (\rho-\nu)^{\lambda-1}\mathcal{Q}_\lambda(\rho-\nu)g^r(\nu)\,\mathrm{d}W(\nu)\Bigg]$$

for some $g^r \in S_{\mathcal{G},y^r}$,

$$\begin{aligned}
r &\leqslant \mathbf{E}\big\|\big(\Sigma y^r\big)(\rho)\big\|^2 \\
&\leqslant \mathbf{E}\left\|\rho^{1-\mu+\lambda\mu-\lambda\vartheta}\left[\mathcal{S}_{\lambda,\mu}(\rho)y_0 + \int_0^\rho (\rho-\nu)^{\lambda-1}\mathcal{Q}_\lambda(\rho-\nu)g^r(\nu)\,\mathrm{d}W(\nu)\right]\right\|^2 \\
&\leqslant 2\mathbf{E}\big\|\rho^{1-\mu+\lambda\mu-\lambda\vartheta}\mathcal{S}_{\lambda,\mu}(\rho)y_0\big\|^2 \\
&\quad + 2\mathbf{E}\left\|\rho^{1-\mu+\lambda\mu-\lambda\vartheta}\int_0^\rho (\rho-\nu)^{\lambda-1}\mathcal{Q}_\lambda(\rho-\nu)g^r(\nu)\,\mathrm{d}W(\nu)\right\|^2 \\
&\leqslant 2\sup_{\rho\in\mathcal{V}}\rho^{2(1-\mu+\lambda\mu-\lambda\vartheta)}\big[\kappa_s^2\rho^{-2+2\mu-2\lambda\mu+2\lambda\vartheta}\|y_0\|^2\big] \\
&\quad + 2\,\mathrm{Tr}(Q)\sup_{\rho\in\mathcal{V}}\rho^{2(1-\mu+\lambda\mu-\lambda\vartheta)}\int_0^\rho \kappa_p^2(\rho-\nu)^{2(\lambda\vartheta-1)}\mathbf{E}\big\|g^r(\nu)\big\|^2\,\mathrm{d}\nu \\
&\leqslant 2\sup_{\rho\in\mathcal{V}}\rho^{2(1-\mu+\lambda\mu-\lambda\vartheta)}\kappa_s^2\rho^{-2+2\mu-2\lambda\mu+2\lambda\vartheta}\|y_0\|^2 \\
&\quad + 2\,\mathrm{Tr}(Q)\sup_{\rho\in\mathcal{V}}\rho^{2(1-\mu+\lambda\mu-\lambda\vartheta)}\kappa_p^2\int_0^\rho (\rho-\nu)^{2(\lambda\vartheta-1)}\mathcal{L}_{\mathcal{G},r}(\nu)\,\mathrm{d}\nu.
\end{aligned}$$

Dividing both sides by r and taking $r\to\infty$, we obtain that

$$2\,\mathrm{Tr}(Q)\kappa_p^2c^{2(1-\mu+\lambda\mu-\lambda\vartheta)}\gamma \geqslant 1,$$

which is a contradiction to our assumption. Thus for $\delta>0$, there exists $r>0$ and some $g\in S_{\mathcal{G},y},\ \Sigma(B_r)\subset B_r$.

Step 3. Σ maps bounded sets into equicontinuous sets of $\mathcal{C}$.
For any $z\in\Sigma(y)$ and $y\in B_r$, there exists $\mathcal{G}\in S_{\mathcal{G},y}$. We define

$$z(\rho) = \rho^{1-\mu+\lambda\mu-\lambda\vartheta}\left[\mathcal{S}_{\lambda,\mu}(\rho)y_0 + \int_0^\rho (\rho-\nu)^{\lambda-1}\mathcal{Q}_\lambda(\rho-\nu)g(\nu)\,\mathrm{d}W(\nu)\right].$$

Let $0<\rho_1<\rho_2\leqslant c$.

$$\begin{aligned}
&\mathbf{E}\big\|z(\rho_2)-z(\rho_1)\big\|^2 \\
&\quad\leqslant \mathbf{E}\Bigg\|\rho_2^{1-\mu+\lambda\mu-\lambda\vartheta}\left(\mathcal{S}_{\lambda,\mu}(\rho_2)y_0 + \int_0^{\rho_2}(\rho_2-\nu)^{\lambda-1}\mathcal{Q}_\lambda(\rho_2-\nu)g(\nu)\,\mathrm{d}W(\nu)\right) \\
&\qquad - \rho_1^{1-\mu+\lambda\mu-\lambda\vartheta}\left(\mathcal{S}_{\lambda,\mu}(\rho_1)y_0 + \int_0^{\rho_1}(\rho_1-\nu)^{\lambda-1}\mathcal{Q}_\lambda(\rho_1-\nu)g(\nu)\,\mathrm{d}W(\nu)\right)\Bigg\|^2
\end{aligned}$$

$$
\begin{aligned}
&\leqslant 2\mathbf{E}\big\|\big[\rho_2^{1-\mu+\lambda\mu-\lambda\vartheta}\mathcal{S}_{\lambda,\mu}(\rho_2)-\rho_1^{1-\mu+\lambda\mu-\lambda\vartheta}\mathcal{S}_{\lambda,\mu}(\rho_1)\big]y_0\big\|^2\\
&\quad+2\mathbf{E}\Bigg\|\rho_2^{1-\mu+\lambda\mu-\lambda\vartheta}\int\limits_0^{\rho_1}(\rho_2-\nu)^{\lambda-1}\mathcal{Q}_\lambda(\rho_2-\nu)g(\nu)\,\mathrm{d}W(\nu)\\
&\quad+\rho_2^{1-\mu+\lambda\mu-\lambda\vartheta}\int\limits_{\rho_1}^{\rho_2}(\rho_2-\nu)^{\lambda-1}\mathcal{Q}_\lambda(\rho_2-\nu)g(\nu)\,\mathrm{d}W(\nu)\\
&\quad-\rho_1^{1-\mu+\lambda\mu-\lambda\vartheta}\int\limits_0^{\rho_1}(\rho_1-\nu)^{\lambda-1}\mathcal{Q}_\lambda(\rho_1-\nu)g(\nu)\,\mathrm{d}W(\nu)\Bigg\|^2\\
&\leqslant 2\mathbf{E}\big\|\big[\rho_2^{1-\mu+\lambda\mu-\lambda\vartheta}\mathcal{S}_{\lambda,\mu}(\rho_2)-\rho_1^{1-\mu+\lambda\mu-\lambda\vartheta}\mathcal{S}_{\lambda,\mu}(\rho_1)\big]y_0\big\|^2\\
&\quad+6\mathbf{E}\Bigg\|\rho_2^{1-\mu+\lambda\mu-\lambda\vartheta}\int\limits_{\rho_1}^{\rho_2}(\rho_2-\nu)^{\lambda-1}\mathcal{Q}_\lambda(\rho_2-\nu)g(\nu)\,\mathrm{d}W(\nu)\Bigg\|^2\\
&\quad+6\mathbf{E}\Bigg\|\rho_2^{1-\mu+\lambda\mu-\lambda\vartheta}\int\limits_0^{\rho_1}(\rho_2-\nu)^{\lambda-1}\mathcal{Q}_\lambda(\rho_2-\nu)g(\nu)\,\mathrm{d}W(\nu)\\
&\quad-\rho_1^{1-\mu+\lambda\mu-\lambda\vartheta}\int\limits_0^{\rho_1}(\rho_1-\nu)^{\lambda-1}\mathcal{Q}_\lambda(\rho_2-\nu)g(\nu)\,\mathrm{d}W(\nu)\Bigg\|^2\\
&\quad+6\mathbf{E}\Bigg\|\rho_1^{1-\mu+\lambda\mu-\lambda\vartheta}\int\limits_0^{\rho_1}(\rho_1-\nu)^{\lambda-1}\mathcal{Q}_\lambda(\rho_2-\nu)g(\nu)\,\mathrm{d}W(\nu)\\
&\quad-\rho_1^{1-\mu+\lambda\mu-\lambda\vartheta}\int\limits_0^{\rho_1}(\rho_1-\nu)^{\lambda-1}\mathcal{Q}_\lambda(\rho_1-\nu)g(\nu)\,\mathrm{d}W(\nu)\Bigg\|^2\\
&=I_1+I_2+I_3+I_4.
\end{aligned}
$$

By the strong continuity of $\mathcal{S}_{\lambda,\mu}(\rho)y_0$ we get $I_1\to 0$ as $\rho_2\to\rho_1$. Also,

$$
\begin{aligned}
I_2&=6\mathbf{E}\Bigg\|\rho_2^{1-\mu+\lambda\mu-\lambda\vartheta}\int\limits_{\rho_1}^{\rho_2}(\rho_2-\nu)^{\lambda-1}\mathcal{Q}_\lambda(\rho_2-\nu)g(\nu)\,\mathrm{d}W(\nu)\Bigg\|^2\\
&\leqslant 6\,\mathrm{Tr}(Q)\rho_2^{2(1-\mu+\lambda\mu-\lambda\vartheta)}\int\limits_{\rho_1}^{\rho_2}(\rho_2-\nu)^{-2\lambda-2}\big\|\mathcal{Q}_\lambda(\rho_2-\nu)\big\|^2\mathbf{E}\big\|g(\nu)\big\|^2\,\mathrm{d}\nu\\
&\leqslant 6\,\mathrm{Tr}(Q)\rho_2^{2(1-\mu+\lambda\mu-\lambda\vartheta)}\int\limits_{\rho_1}^{\rho_2}(\rho_2-\nu)^{2\lambda-2}\kappa_p^2(\rho_2-\nu)^{-2\lambda(1-\vartheta)}\mathcal{L}_{\mathcal{G},r}(\nu)\,\mathrm{d}\nu
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant 6\,\mathrm{Tr}(Q)\rho_2^{2(1-\mu+\lambda\mu-\lambda\vartheta)}\kappa_p^2\int_{\rho_1}^{\rho_2}(\rho_2-\nu)^{2(\lambda\vartheta-1)}\mathcal{L}_{\mathcal{G},r}(\nu)\,\mathrm{d}\nu\\
&\leqslant 6\,\mathrm{Tr}(Q)\kappa_p^2\Biggl[\rho_2^{2(1-\mu+\lambda\mu-\lambda\vartheta)}\int_0^{\rho_2}(\rho_2-\nu)^{2(\lambda\vartheta-1)}\mathcal{L}_{\mathcal{G},r}(\nu)\,\mathrm{d}\nu\\
&\quad-\rho_1^{2(1+\lambda\vartheta)(1-\mu)}\int_0^{\rho_1}(\rho_1-\nu)^{2(\lambda\vartheta-1)}\mathcal{L}_{\mathcal{G},r}(\nu)\,\mathrm{d}\nu\Biggr]\\
&\quad+6\,\mathrm{Tr}(Q)\kappa_p^2\int_0^{\rho_1}\bigl[\rho_1^{2(1-\mu+\lambda\mu-\lambda\vartheta)}(\rho_1-\nu)^{2(\lambda\vartheta-1)}-\rho_2^{2(1-\mu+\lambda\mu-\lambda\vartheta)}(\rho_2-\nu)^{2(\lambda\vartheta-1)}\bigr]\\
&\qquad\times\mathcal{L}_{\mathcal{G},r}(\nu)\,\mathrm{d}\nu.
\end{aligned}
$$

We get $I_2\to 0$ as $\rho_2\to\rho_1$ by using (H4) and the Lebesgue dominated convergent theorem.

$$
\begin{aligned}
I_3&=6\mathbf{E}\Biggl\|\rho_2^{1-\mu+\lambda\mu-\lambda\vartheta}\int_0^{\rho_1}(\rho_2-\nu)^{\lambda-1}\mathcal{Q}_\lambda(\rho_2-\nu)g(\nu)\,\mathrm{d}W(\nu)\\
&\qquad-\rho_1^{1-\mu+\lambda\mu-\lambda\vartheta}\int_0^{\rho_1}(\rho_1-\nu)^{\lambda-1}\mathcal{Q}_\lambda(\rho_2-\nu)g(\nu)\,\mathrm{d}W(\nu)\Biggr\|^2\\
&\leqslant 6\mathbf{E}\Biggl\|\int_0^{\rho_1}\bigl[\rho_2^{1-\mu+\lambda\mu-\lambda\vartheta}(\rho_2-\nu)^{\lambda-1}-\rho_1^{1-\mu+\lambda\mu-\lambda\vartheta}(\rho_1-\nu)^{\lambda-1}\bigr]\\
&\qquad\times\mathcal{Q}_\lambda(\rho_2-\nu)g(\nu)\,\mathrm{d}W(\nu)\Biggr\|^2\\
&\leqslant 6\,\mathrm{Tr}(Q)\int_0^{\rho_1}\mathbf{E}\bigl\|\rho_2^{1-\mu+\lambda\mu-\lambda\vartheta}(\rho_2-\nu)^{\lambda-1}-\rho_1^{1-\mu+\lambda\mu-\lambda\vartheta}(\rho_1-\nu)^{\lambda-1}\bigr\|^2\\
&\qquad\times\bigl\|\mathcal{Q}_\lambda(\rho_2-\nu)\bigr\|^2\mathbf{E}\bigl\|g(\nu)\bigr\|^2\,\mathrm{d}\nu\\
&\leqslant 6\,\mathrm{Tr}(Q)\kappa_p^2\int_0^{\rho_1}(\rho_2-\nu)^{-2\lambda(1-\vartheta)}\mathbf{E}\bigl\|\rho_2^{1-\mu+\lambda\mu-\lambda\vartheta}(\rho_2-\nu)^{(\lambda-1)}\\
&\qquad-\rho_1^{1-\mu+\lambda\mu-\lambda\vartheta}(\rho_1-\nu)^{(\lambda-1)}\bigr\|^2\mathcal{L}_{\mathcal{G},r}(\nu)\,\mathrm{d}\nu.
\end{aligned}
$$

Consider

$$
\begin{aligned}
&(\rho_2-\nu)^{-2\lambda(1-\vartheta)}\\
&\quad\times\mathbf{E}\bigl\|\rho_2^{1-\mu+\lambda\mu-\lambda\vartheta}(\rho_2-\nu)^{(\lambda-1)}-\rho_1^{1-\mu+\lambda\mu-\lambda\vartheta}(\rho_1-\nu)^{(\lambda-1)}\bigr\|^2\mathcal{L}_{\mathcal{G},r}(\nu)
\end{aligned}
$$

$$\begin{aligned}
&\leqslant \big[2\rho_2^{2(1-\mu+\lambda\mu-\lambda\vartheta)}(\rho_2-\nu)^{2(\lambda\vartheta-1)} \\
&\quad + 2\rho_1^{2(1-\mu+\lambda\mu-\lambda\vartheta)}(\rho_1-\nu)^{2(\lambda-1)}(\rho_2-\nu)^{-2\lambda(1-\vartheta)}\big]\mathcal{L}_{\mathcal{G},r}(\nu) \\
&\leqslant \big[2\rho_2^{2(1-\mu+\lambda\mu-\lambda\vartheta)}(\rho_2-\nu)^{2(\lambda\vartheta-1)} + 2\rho_1^{2(1-\mu+\lambda\mu-\lambda\vartheta)}(\rho_1-\nu)^{2(\lambda\vartheta-1)}\big]\mathcal{L}_{\mathcal{G},r}(\nu) \\
&\leqslant 4\rho_1^{2(1-\mu+\lambda\mu-\lambda\vartheta)}(\rho_1-\nu)^{2(\lambda\vartheta-1)}\mathcal{L}_{\mathcal{G},r}(\nu)
\end{aligned}$$

and

$$\int_0^{\rho_1} 4\rho_1^{2(1-\mu+\lambda\mu-\lambda\vartheta)}(\rho_1-\nu)^{2(\lambda\vartheta-1)}\mathcal{L}_{\mathcal{G},r}(\nu)\,\mathrm{d}\nu$$

existing for $\nu \in (0,\rho_1]$. Then by Lebesgue's dominated convergence theorem we obtain

$$\begin{aligned}
&\int_0^{\rho_1}(\rho_2-\nu)^{-2\lambda(1-\vartheta)}\mathbf{E}\big\|\rho_2^{1-\mu+\lambda\mu-\lambda\vartheta}(\rho_2-\nu)^{\lambda-1} - \rho_1^{1-\mu+\lambda\mu-\lambda\vartheta}(\rho_1-\nu)^{\lambda-1}\big\|^2 \\
&\quad \times \mathcal{L}_{\mathcal{G},r}(\nu)\,\mathrm{d}\nu \to 0 \quad \text{as } \rho_2 \to \rho_1,
\end{aligned}$$

so, we conclude $\lim_{\rho_2\to\rho_1} I_3 = 0$. For any $\epsilon > 0$, we have

$$\begin{aligned}
I_4 &\leqslant 6\mathbf{E}\left\|\int_0^{\rho_1}\rho_1^{1-\mu+\lambda\mu-\lambda\vartheta}\big[\mathcal{Q}_\lambda(\rho_2-\nu)-\mathcal{Q}_\lambda(\rho_1-\nu)\big](\rho_1-\nu)^{\lambda-1}g(\nu)\,\mathrm{d}W(\nu)\right\| \\
&\leqslant 6\,\mathrm{Tr}(Q)\int_0^{\rho_1}\rho_1^{2(1-\mu+\lambda\mu-\lambda\vartheta)}\big\|\mathcal{Q}_\lambda(\rho_2-\nu)-\mathcal{Q}_\lambda(\rho_1-\nu)\big\|^2(\rho_1-\nu)^{2\lambda-2} \\
&\qquad \times \mathbf{E}\big\|g(\nu)\big\|^2\,\mathrm{d}\nu \\
&\leqslant 6\,\mathrm{Tr}(Q)\int_0^{\rho_1}\rho_1^{2(1-\mu+\lambda\mu-\lambda\vartheta)}\big\|\mathcal{Q}_\lambda(\rho_2-\nu)-\mathcal{Q}_\lambda(\rho_1-\nu)\big\|^2(\rho_1-\nu)^{2\lambda-2} \\
&\qquad \mathcal{L}_{\mathcal{G},r}(\nu)\,\mathrm{d}\nu \\
&\leqslant 6\,\mathrm{Tr}(Q)\left\{\int_0^{\rho_1-\epsilon}\rho_1^{2(1-\mu+\lambda\mu-\lambda\vartheta)}\big\|\mathcal{Q}_\lambda(\rho_2-\nu)-\mathcal{Q}_\lambda(\rho_1-\nu)\big\|^2(\rho_1-\nu)^{2(\lambda-1)}\right. \\
&\qquad \times \mathcal{L}_{\mathcal{G},r}(\nu)\,\mathrm{d}\nu \\
&\quad \left. + \int_{\rho_1-\epsilon}^{\rho_1}\rho_1^{2(1-\mu+\lambda\mu-\lambda\vartheta)}\big\|\mathcal{Q}_\lambda(\rho_2-\nu)-\mathcal{Q}_\lambda(\rho_1-\nu)\big\|^2(\rho_1-\nu)^{2(\lambda-1)}\mathcal{L}_{\mathcal{G},r}(\nu)\,\mathrm{d}\nu\right\} \\
&\leqslant 6\,\mathrm{Tr}(Q)\left\{\rho_1^{2(1-\mu+\lambda\mu-\lambda\vartheta)}\int_0^{\rho_1-\epsilon}(\rho_1-\nu)^{2(\lambda-1)}\mathcal{L}_{\mathcal{G},r}(\nu)\,\mathrm{d}\nu\right. \\
&\qquad \times \sup_{\nu\in[0,\rho_1-\epsilon]}\big\|\mathcal{Q}_\lambda(\rho_2-\nu)-\mathcal{Q}_\lambda(\rho_1-\nu)\big\|^2
\end{aligned}$$

$$
\begin{aligned}
&+ \kappa_p^2 \int\limits_{\rho_1-\epsilon}^{\rho_1} \rho_1^{2(1-\mu+\lambda\mu-\lambda\vartheta)} \big[(\rho_2-\nu)^{-2\lambda(1-\vartheta)} + (\rho_1-\nu)^{-2\lambda(1-\vartheta)}\big] \\
&\qquad \times (\rho_1-\nu)^{2(\lambda-1)} \mathcal{L}_{\mathcal{G},r}(\nu)\,\mathrm{d}\nu \Bigg\} \\
&\leqslant 6\,\mathrm{Tr}(Q) \Bigg\{ \rho_1^{2(1-\mu+\lambda\mu-\lambda\vartheta)+2\lambda(1-\vartheta)} \int\limits_0^{\rho_1} (\rho_1-\nu)^{2(\lambda\vartheta-1)} \mathcal{L}_{\mathcal{G},r}(\nu)\,\mathrm{d}\nu \\
&\qquad \times \sup_{\nu\in[0,\rho_1-\epsilon]} \big\| \mathcal{Q}_\lambda(\rho_2-\nu) - \mathcal{Q}_\lambda(\rho_1-\nu) \big\|^2 \\
&+ 4\kappa_p^2 \int\limits_{\rho_1-\epsilon}^{\rho_1} \rho_1^{2(1-\mu+\lambda\mu-\lambda\vartheta)} (\rho_1-\nu)^{2(\lambda\vartheta-1)} \mathcal{L}_{\mathcal{G},r}(\nu)\,\mathrm{d}\nu.
\end{aligned}
$$

From Theorem 2 and $\lim_{\rho_2\to\rho_1} I_2 = 0$ we get $I_4 \to 0$ independently of $y \in \mathcal{C}$ as $\rho_2 \to \rho_1$, $\epsilon \to 0$. Hence $\|z(\rho_2) - z(\rho_1)\| \to 0$ independently of $y \in \mathcal{C}$ as $\rho_2 \to \rho_1$. This implies that $\{\Sigma y(\rho)\colon\ y \in \mathcal{C}\}$ is equicontinuous on $\mathcal{V}$.

Step 4. $\Pi(\rho) = \{z(\rho) : z \in \Sigma(B_r)\}$ is relatively compact for $\rho \in \mathcal{V}$.
For $\varepsilon \in (0,\rho)$ and $\eta > 0$, consider the operator $z'(\rho)$ on B_r by

$$
\begin{aligned}
z'_{\varepsilon,\eta}(\rho) &= \rho^{1-\mu+\lambda\mu-\lambda\vartheta} \mathcal{S}_{\lambda,\mu}(\rho) y_0 \\
&\quad + \rho^{1-\mu+\lambda\mu-\lambda\vartheta} \int\limits_0^{\rho-\varepsilon} (\rho-\nu)^{\lambda-1} \mathcal{Q}_\lambda(\rho-\nu) g(\nu)\,\mathrm{d}W(\nu) \\
&= \rho^{1-\mu+\lambda\mu-\lambda\vartheta} \mathcal{S}_{\lambda,\mu}(\rho) y_0 \\
&\quad + \lambda\rho^{1-\mu+\lambda\mu-\lambda\vartheta} \int\limits_0^{\rho-\varepsilon} \int\limits_\eta^\infty \theta M_\lambda(\theta)(\rho-\nu)^{\lambda-1} T\big((\rho-\nu)^\lambda\theta\big) g(\nu)\,\mathrm{d}\theta\,\mathrm{d}W(\nu) \\
&= \rho^{1-\mu+\lambda\mu-\lambda\vartheta} \mathcal{S}_{\lambda,\mu}(\rho) y_0 + \lambda\rho^{1-\mu+\lambda\mu-\lambda\vartheta} T\big(\varepsilon^\lambda\eta\big) \\
&\quad \times \int\limits_0^{\rho-\varepsilon} \int\limits_\eta^\infty \theta M_\lambda(\theta)(\rho-\nu)^{\lambda-1} T\big((\rho-\nu)^\lambda\theta - \varepsilon^\lambda\eta\big) g(\nu)\,\mathrm{d}\theta\,\mathrm{d}W(\nu).
\end{aligned}
$$

Hence $\Pi_{\varepsilon,\eta}(\rho) = \{z'_{\varepsilon,\eta}(\rho)\colon\ y \in B_r\}$ is precompact in Y for all $0 < \varepsilon < \rho$ and $\eta > 0$ due to the compactness of $T(\varepsilon^\lambda q)$. For any $y \in B_r$, we get

$$
\begin{aligned}
&\mathbf{E}\big\|z(\rho) - z'_{\varepsilon,\eta}(\rho)\big\|^2 \\
&\quad \leqslant \mathbf{E}\Bigg\| \rho^{1-\mu+\lambda\mu-\lambda\vartheta} \mathcal{S}_{\lambda,\mu} y_0 + \rho^{1-\mu+\lambda\mu-\lambda\vartheta} \int\limits_0^\rho (\rho-\nu)^{\lambda-1} \mathcal{Q}(\rho-\nu) g(\nu)\,\mathrm{d}W(\nu)
\end{aligned}
$$

$$
\begin{aligned}
&- \Bigg[\rho^{1-\mu+\lambda\mu-\lambda\vartheta}\mathcal{S}_{\lambda,\mu}y_0 + \lambda\rho^{1-\mu+\lambda\mu-\lambda\vartheta}T\big(\varepsilon^\lambda\eta\big) \\
&\times \int_0^{\rho-\varepsilon}\int_\eta^\infty \theta M_\lambda(\theta)(\rho-\nu)^{\lambda-1}T\big((\rho-\nu)^\lambda\theta - \varepsilon^\lambda\eta\big)g(\nu)\,\mathrm{d}\theta\,\mathrm{d}W(\nu)\Bigg]\Bigg\|^2 \\
\leqslant{}& 2\mathbf{E}\Bigg\|\lambda\rho^{1-\mu+\lambda\mu-\lambda\vartheta}\int_0^\rho\int_0^\eta \theta M_\lambda(\theta)(\rho-\nu)^{\lambda-1}T\big((\rho-\nu)^\lambda\theta\big)g(\nu)\,\mathrm{d}\theta\,\mathrm{d}W(\nu)\Bigg\|^2 \\
&+ 2\mathbf{E}\Bigg\|\lambda\rho^{1-\mu+\lambda\mu-\lambda\vartheta}\int_{\rho-\varepsilon}^\rho\int_\eta^\infty (\rho-\nu)^{\lambda-1}\theta M_\lambda(\theta)T\big((\rho-\nu)^\lambda\theta\big)g(\nu)\,\mathrm{d}\theta\,\mathrm{d}W(\nu)\Bigg\|^2 \\
\leqslant{}& 2\,\mathrm{Tr}(Q)\lambda\rho^{2(1-\mu+\lambda\mu-\lambda\vartheta)} \\
&\times \int_0^\rho\int_0^\eta \theta^2 M_\lambda^2(\theta)(\rho-\nu)^{2(\lambda-1)}\big\|T\big((\rho-\nu)^\lambda\theta\big)\big\|^2\mathbf{E}\big\|g(\nu)\big\|^2\,\mathrm{d}\theta\,\mathrm{d}\nu \\
&+ 2\,\mathrm{Tr}(Q)\lambda\rho^{2(1-\mu+\lambda\mu-\lambda\vartheta)} \\
&\times \int_{\rho-\varepsilon}^\rho\int_\eta^\infty (\rho-\nu)^{2(\lambda-1)}\theta^2 M_\lambda^2(\theta)\big\|T\big((\rho-\nu)^\lambda\theta\big)\big\|^2\mathbf{E}\big\|g(\nu)\big\|^2\,\mathrm{d}\theta\,\mathrm{d}\nu \\
\leqslant{}& 2\,\mathrm{Tr}(Q)\lambda\kappa_0\rho^{2(1+\lambda\vartheta)(1-\mu)} \\
&\times \int_0^\rho\int_0^\eta \theta^2 M_\lambda^2(\theta)(\rho-\nu)^{2(\lambda-1)}(\rho-\nu)^{-2\lambda(1-\vartheta)}\theta^{2\vartheta-2}\mathcal{L}_{\mathcal{G},r}(\nu)\,\mathrm{d}\theta\,\mathrm{d}\nu \\
&+ 2\,\mathrm{Tr}(Q)\lambda\kappa_0\rho^{2(1-\mu+\lambda\mu-\lambda\vartheta)} \\
&\times \int_{\rho-\varepsilon}^\rho\int_\eta^\infty (\rho-\nu)^{2(\lambda-1)}\theta^2 M_\lambda^2(\theta)(\rho-\nu)^{-2\lambda(1-\vartheta)}\theta^{2\vartheta-2}\mathcal{L}_{\mathcal{G},r}(\nu)\,\mathrm{d}\nu \\
\leqslant{}& 2\,\mathrm{Tr}(Q)\lambda\kappa_0\rho^{2(1-\mu+\lambda\mu-\lambda\vartheta)}\int_0^\rho (\rho-\nu)^{2(\lambda\vartheta-1)}\mathcal{L}_{\mathcal{G},r}(\nu)\,\mathrm{d}\nu\int_0^\eta \theta^{2\vartheta}M_\lambda(\theta)\,\mathrm{d}\theta \\
&+ 2\,\mathrm{Tr}(Q)\lambda\kappa_0\rho^{2(1-\mu+\lambda\mu-\lambda\vartheta)}\int_{\rho-\varepsilon}^\rho (\rho-\nu)^{2(\lambda\vartheta-1)}\mathcal{L}_{\mathcal{G},r}(\nu)\,\mathrm{d}\nu\int_0^\infty \theta^{2\vartheta}M_\lambda^2(\theta)\,\mathrm{d}\theta \\
\leqslant{}& 2\,\mathrm{Tr}(Q)\lambda\kappa_0\rho^{2(1-\mu+\lambda\mu-\lambda\vartheta)}\int_0^\rho (\rho-\nu)^{-2(1+\lambda\vartheta)}\mathcal{L}_{\mathcal{G},r}(\nu)\,\mathrm{d}\nu\int_0^\eta \theta^{2\vartheta}M_\lambda^2(\theta)\,\mathrm{d}\theta
\end{aligned}
$$

$$+ 2\,\mathrm{Tr}(Q)\lambda\kappa_0\frac{\Gamma(1+2\vartheta)}{\Gamma(1+2\lambda\vartheta)}\rho^{2(1-\mu+\lambda\mu-\lambda\vartheta)}\int_{\rho-\varepsilon}^{\rho}(\rho-\nu)^{2(\lambda\vartheta-1)}\mathcal{L}_{\mathcal{G},r}(\nu)\,\mathrm{d}\nu$$

$$\to 0 \quad \text{as}\ \varepsilon\to 0,\ \eta\to 0.$$

So, $\Pi_{\varepsilon,\eta}(\rho) = \{z'_{\varepsilon,\eta}(\rho),\ \rho\in\mathcal{V}\}$ are arbitrary closed to $\Pi(\rho) = \{z(\rho),\ \rho\in\mathcal{V}\}$. As a result of the Arzela–Ascoli theorem, $\{z(\rho),\ \rho\in\mathcal{V}\}$ is relatively compact. As a result, $z(\rho)$ is a completely continuous operator due to the continuity of $z(\rho)$ and relatively compactness of $\{z(\rho),\ \rho\in\mathcal{V}\}$.

Step 5. Σ has a closed graph.

Let $y_n\to y_0$ as $n\to\infty$, $z_n(\rho)\in\Sigma(y_n)$ and $z_n\to z_0$ as $n\to\infty$. We need to prove that $z_0\in\Sigma(y_0)$. Since $z_n\in\Sigma(y_n)$, then there exists a function $g_n\in S_{\mathcal{G},y_n}$ such that

$$z_n(\rho) = \rho^{1-\mu+\lambda\mu-\lambda\vartheta}\left[\mathcal{S}_{\lambda,\mu}(\rho)y_0 + \int_0^{\rho}(\rho-\nu)^{\lambda-1}\mathcal{Q}_\lambda(\rho-\nu)g_n(\nu)\,\mathrm{d}W(\nu)\right].$$

We have to show that there exists $g_0\in S_{\mathcal{G},y_0}$ such that

$$z_0(\rho) = \rho^{1-\mu+\lambda\mu-\lambda\vartheta}\left[\mathcal{S}_{\lambda,\mu}(\rho)y_0 + \int_0^{\rho}(\rho-\nu)^{\lambda-1}\mathcal{Q}_\lambda(\rho-\nu)g_0(\nu)\,\mathrm{d}W(\nu)\right].$$

Clearly, as $n\to\infty$,

$$\left\|\left(z_n(\rho)-\rho^{1-\mu+\lambda\mu-\lambda\vartheta}\mathcal{S}_{\lambda,\mu}(\rho)y_0\right) - \left(z_0(\rho)-\rho^{1-\mu+\lambda\mu-\lambda\vartheta}\mathcal{S}_{\lambda,\mu}(\rho)y_0\right)\right\|\to 0.$$

Now, we consider an operator $\Gamma : L^2(\mathcal{V},Y)\to\mathcal{C}(\mathcal{V},Y)$,

$$\Gamma(g)(\rho) = \int_0^{\rho}(\rho-\nu)^{\lambda-1}\mathcal{Q}_\lambda(\rho-\nu)g(\nu)\,\mathrm{d}W(\nu).$$

By Lemma 8 $\Gamma\circ S_{\mathcal{G},y}$ is a closed graph operator. So, by comparing with Γ we have

$$\left(z_n(\rho)-\rho^{1-\mu+\lambda\mu-\lambda\vartheta}\mathcal{S}_{\lambda,\mu}(\rho)y_0\right)\in\Gamma(S_{\mathcal{G},y_n}).$$

Since $g_n\to g_0$, it follows from Lemma 8 that

$$\left(z_0(\rho)-\rho^{1-\mu+\lambda\mu-\lambda\vartheta}\mathcal{S}_{\lambda,\mu}(\rho)y_0\right)\in\Gamma(S_{\mathcal{G},y_0}).$$

Hence, Σ is a closed graph.

As a result of applying Arzela–Ascoli theorem on Steps 1–5, Σ is a u.s.c. multivalued mapping because it is a completely continuous multivalued mapping with compact value. As a result of Lemma 9, Σ has a fixed point $z(\cdot)$ on $B_r(\cdot)$, and $z(\cdot)$ is the mild solution of (1)–(2). □

4 Example

As an example of how our findings can be put to use, think about the following: HF stochastic Volterra–Fredholm integro-differential inclusions

$$
\begin{aligned}
&D_{0^+}^{4/7,\mu} w(\rho,\tau) \in w_{\rho\rho}(\rho,\tau) + \bar{\mathcal{G}}\big(\rho, w(\rho,\tau), (Fw)(\rho,\tau), (Hw)(\rho,\tau)\big) \\
&\qquad\qquad \times \frac{\mathrm{d}W(\rho)}{\mathrm{d}\rho}, \quad \rho \in (0,c],\ \tau \in [0,\pi], \\
&w(\rho,0) = w(\rho,\pi) = 0, \quad \rho \in [0,c], \\
&I^{(1-4/7)(1-\mu)} w(0,\tau) = w_0(\tau), \quad \tau \in [0,\pi],
\end{aligned} \tag{3}
$$

where $D_{0^+}^{4/7,\mu}$ is the HFD of order $4/7$ and type μ, $I^{(1-4/7)(1-\mu)}$ is the RL integral of order $(3/7)(1-\mu)$, $\bar{\mathcal{G}}(\rho, w(\rho,\tau), (Ew)(\rho,\tau), (Hw)(\rho,\tau))$, $(Fw)(\rho,\tau)$, and $(Hw)(\rho,\tau)$ are the given functions.

Let $W(\rho)$ be a one-dimensional standard Brownian motion in Y defined on the filtered probability space $(\Omega, \mathscr{E}, \mathbf{P})$, and $Y = L^2[0,\pi]$ with the norm $\|\cdot\|_Y$ to write system (3) in the abstract form of (1)–(2). Define the operator A, $D(A) \subset Y \times Y$, by $Aw = w_{\rho\rho}$ with the domain

$$
D(A) = \big\{ w \in Y\colon\ w_\rho, w_{\rho\rho} \in Y,\ w(\rho,0) = w(\rho,\pi) = 0 \big\}.
$$

Then

$$
Aw = \sum_{n=1}^{\infty} n^2 \langle w, e_n \rangle e_n, \quad w \in D(A),
$$

where $e_n(w) = \sqrt{2/\pi}\sin(n\tau)$, $n = 1, 2, \dots$, is the orthogonal set of eigenvectors of A. It can be easily shown that A is an almost sectorial operators of an analytic semigroup $T(\rho)$, $\rho > 0$, in Y and is presented as

$$
T(\rho)w = \sum_{n=1}^{\infty} \mathrm{e}^{-n^2\rho} \langle w, e_n \rangle e_n, \quad w \in Y,
$$

and satisfies hypothesis (H1). There are constants $\delta, \epsilon > 0$ as a result of the work in [22] such that $(A+\delta) \in \Theta_{\pi/2-\epsilon}^{\pi/2-1}(Y)$. The compactness of the semigroup $T(\rho)$ follows from [32, Lemma 4.66].

$y(\rho)(\tau) = w(\rho,\tau)$, $\rho \in \mathcal{V} = [0,c]$, $\tau \in [0,\pi]$. Now for any $y \in Y = L^2[0,\pi]$, $\tau \in [0,\pi]$, we define the function $\mathcal{G} : \mathcal{V} \times Y \times Y \times Y \to Y$,

$$
\begin{aligned}
&\mathcal{G}\big(\rho, y(\rho), (Fy)(\rho), (Hy)(\rho)\big) \\
&\quad = \bar{\mathcal{G}}\big(\rho, w(\rho,\tau), (Fw)(\rho,\tau), (Hw)(\rho,\tau)\big) \\
&\quad = \frac{\mathrm{e}^{-\rho}}{1+\mathrm{e}^{-\rho}} \sin\left(w(\rho,\tau) + \int_0^\rho \sin(\rho s) w(s,\tau)\,\mathrm{d}s + \int_0^c \cos(\rho s) w(s,\tau)\,\mathrm{d}s \right),
\end{aligned}
$$

where

$$(Fy)(\rho)(\tau) = \int_0^{\rho} f\big(\rho, s, w(s,\tau)\big)\,\mathrm{d}s = \int_0^{\rho} \sin(\rho s) w(s,\tau)\,\mathrm{d}s,$$

$$(Hy)(\rho)(\tau) = \int_0^{c} h\big(\rho, s, w(s,\tau)\big)\,\mathrm{d}s = \int_0^{c} \cos(\rho s) w(s,\tau)\,\mathrm{d}s.$$

As a result, the nonlocal Cauchy problem (1)–(2) can be restated as the fractional system (3). Obviously, $\bar{\mathcal{G}}(\rho, w(\rho,\tau), (Fw)(\rho,\tau), (Hw)(\rho,\tau))$ is uniformly bounded. Then hypotheses (H1)–(H5) are fulfilled. The problem has a mild solution on $\mathcal{V}$ according to Theorem 3.

Remark. Now, to discuss the existence of Hilfer fractional stochastic Volterra–Fredholm integro-differential inclusions via almost sectorial operators, we have to use the fractional calculus, stochastic analysis theory, multivalued maps, Brownian motions, and Bohnenblust–Karlin's fixed point theorem. Then we extended the proposed systems with the impulsive effects and nonlocal conditions. In the future, we will use the fixed point theorem to examine the approximate controllability and Hyers–Ulam stability of Hilfer fractional stochastic evolution Volterra–Fredholm integro-differential systems with impulses.

Conclusion. The existence of Hilfer fractional stochastic Volterra–Fredholm integro-differential inclusions via almost sectorial operators was the focus of our article. The major conclusions are established by applying the concepts and ideas from fractional calculus, stochastic analysis theory, and the fixed point approach. We first established the existence of mild solutions to the fractional system under consideration. Finally, we exemplify the theory using an example.

Acknowledgment. The authors are grateful to the reviewers of this article who gave insightful comments and advice that allowed us to revise and improve the content of the paper.

References

1. B. Ahmad, A. Alsaedi, S.K. Ntouyas, J. Tariboon, *Hadamard-Type Fractional Differential Equations, Inclusions and Inequalities*, Springer, Cham, 2017.
2. B. Ahmad, D. Garout, S.K. Ntouyas, A. Alsaedi, Caputo fractional differential inclusions of arbitrary order with non-local integro-multipoint boundary conditions, *Miskolc Math. Notes*, **20**(2):683–699, 2019, 10.18514/MMN.2019.2241.
3. P. Balasubramaniam, P. Tamilalagan, Approximate controllability of a class of fractional neutral stochastic integro-differential inclusions with infinite delay by using Mainardi's function, *Appl. Math. Comput.*, **256**:232–246, 2015, https://doi.org/10.1016/j.amc.2015.01.035.

4. P. Bedi, A. Kumar, T. Abdeljawad, Z.A. Khan, A. Khan, Existence and approximate controllability of Hilfer fractional evolution equations with almost sectorial operators, *Adv. Contin. Discrete Models*, **2020**(615):615, 2020, https://doi.org/10.1186/s13662-020-03074-1.
5. Y.-K. Chang, D.N. Chalishajar, Controllability of mixed Volterra–Fredholm-type integro-differential inclusions in Banach spaces, *J. Franklin Inst.*, **345**(5):499–507, 2012, https://doi.org/10.1016/j.jfranklin.2008.02.002.
6. C. Dineshkumar, K.S. Nisar, R. Udhayakumar, V. Vijayakumar, A discussion on approximate controllability of Sobolev-type Hilfer neutral fractional stochastic differential inclusions, *Asian J. Control*, **24**(5):2378–2394, 2022, https://doi.org/10.1002/asjc.2650.
7. X.-L. Ding, B. Ahmad, Analytical solutions to fractional evolution equations with almost sectorial operators, *Adv. Difference Equ.*, **2016**(203):1–25, 2016, https://doi.org/10.1186/s13662-016-0927-y.
8. J. Du, Wei Jiang, A.U. Khan Niazi, Approximate controllability of impulsive Hilfer fractional differential inclusions, *J. Nonlinear Sci. Appl.*, **10**(2):595–611, 2017, https://doi.org/10.22436/jnsa.010.02.23.
9. K.M. Furati, M.D. Kassim, N.E. Tatar, Existence and uniqueness for a problem involving Hilfer fractional derivative, *Comput. Math. Appl.*, **64**(6):1616–1626, 2012, https://doi.org/10.1016/j.camwa.2012.01.009.
10. H. Gu, J.J. Trujillo, Existence of mild solution for evolution equation with Hilfer fractional derivative, *Appl. Math. Comput.*, **257**:344–354, 2015, https://doi.org/10.1016/j.amc.2014.10.083.
11. Y. Guo, M. Chen, X.-B. Shu, F. Xu, The existence and Hyers-Ulam stability of solution for almost periodical fractional stochastic differential equation with fBm, *Stochastic Anal. Appl.*, **39**(4):643–666, 2020, https://doi.org/10.1080/07362994.2020.1824677.
12. Y. Guo, X.-B. Shu, Y. Li, F. Xu, The existence and Hyers–Ulam stability of solution for an impulsive Riemann–Liouville fractional neutral functional stochastic differential equation with infinite delay of order $1 < \beta < 2$, *Bound. Value Probl.*, **2019**(59):1–18, 2019, https://doi.org/10.1186/s13661-019-1172-6.
13. A. Harrat, J.J. Nieto, A. Debbouche, Solvability and optimal controls of impulsive Hilfer fractional delay evolution inclusions with Clarke sub-differential, *J. Comput. Appl. Math.*, **344**: 725–737, 2018, https://doi.org/10.1016/j.cam.2018.05.031.
14. R. Hilfer, *Application of Fractional Calculus in Physics*, World Scientific, Singapore, 2000, https://doi.org/10.1142/3779.
15. A. Jaiswal, D. Bahuguna, Hilfer fractional differential equations with almost sectorial operators, *Differ. Equ. Dyn. Syst.*, 2020, https://doi.org/10.1007/s12591-020-00514-y.
16. K. Karthikeyan, A. Debbouche, D.F.M. Torres, Analysis of Hilfer fractional integro-differential equations with almost sectorial operators, *Fractal Fract.*, **5**(1):22, 2021, https://doi.org/10.3390/fractalfract5010022.
17. K. Kavitha, V. Vijayakumar, R. Udhayakumar, N. Sakthivel, K.S. Nissar, A note on approaximate controllability of the Hilfer fractional neutral differential inclusions with infinite delay, *Math. Methods Appl. Sci.*, **44**(6):4428–4447, 2021, https://doi.org/10.1002/mma.7040.

18. A. Lasota, Z. Opial, An application of the Kakutani-Ky-Fan theorem in the theory of ordinary differential equations or noncompact acyclic-valued map, *Bull. Acad. Pol. Sci., Sér. Sci. Math. Astron. Phys.*, **13**:781–786, 1965.

19. F. Li, Mild solutions for abstract differential equations with almost sectorial operators and infinite delay, *Adv. Difference Equ.*, **2013**:327, 2013, `https://doi.org/10.1186/1687-1847-2013-327`.

20. X. Ma, X.-B. Shu, J. Mao, Existence of almost periodic solutions for fractional impulsive neutral stochastic differential equations with infinite delay, *Stoch. Dyn.*, **20**(1):2050003, 2020, `https://doi.org/10.1142/S0219493720500033`.

21. M. Mohan Raja, V. Vijayakumar, R. Udhayakumar, A new approach on the approximate controllability of fractional evolution inclusion of order $1 < r < 2$ with infinite delay, *Chaos Solitons Fractals*, **141**:110343, 2020, `https://doi.org/10.1016/j.chaos.2020.110343`.

22. F. Periago, B. Straub, A functional calculus for almost sectorial operators and applications to abstract evolution equations, *J. Evol. Equ.*, **2**:41–68, 2002, `https://doi.org/10.1007/s00028-002-8079-9`.

23. I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, CA, 1999.

24. R. Sakthivel, Y. Ren, A. Debbouche, N.I. Mahmudov, Approximate controllability of fractional stochastic differential inclusions with nonlocal conditions, *Appl. Anal.*, **95**(11):2361–2382, 2016, `https://doi.org/10.1080/00036811.2015.1090562`.

25. S. Sivasankar, R. Udhayakumar, New outcomes regarding the existence of Hilfer fractional stochastic differential systems via almost sectorial operators, *Fractal Fract.*, **6**:522, 2022, `https://doi.org/10.3390/fractalfract6090522`.

26. C.S. Varun Bose, R. Udhayakumar, Existence of mild solutions for Hilfer fractional neutral integro-differential inclusions via almost sectorial operators, *Fractal Fract.*, **6**:532, 2022, `https://doi.org/10.3390/fractalfract6090532`.

27. J. Wang, Y. Zhou, Existence and controllability results for fractional semilinear differential inclusions, *Nonlinear Anal., Real World Appl.*, **12**(6):3642–3653, 2011, `https://doi.org/10.1016/j.nonrwa.2011.06.021`.

28. R.-N. Wang, D.-H. Chen, T.-J. Xiao, Abstract fractional Cauchy problems with almost sectorial operators, *J. Differ. Equations*, **252**(1):202–235, 2012, `https://doi.org/10.1016/j.jde.2011.08.048`.

29. M. Yang, Q.-R. Wang, Approximate controllability of Hilfer fractional differential inclusions with nonlocal conditions, *Math. Methods Appl. Sci.*, **40**(4):1126–1138, 2017, `https://doi.org/10.1002/mma.4040`.

30. L. Zhang, Y. Zhou, Fractional Cauchy problems with almost sectorial operators, *Appl. Math. Comput.*, **257**:145–157, 2014, `https://doi.org/10.1016/j.amc.2014.07.024`.

31. M. Zhou, C. Li, Y. Zhou, Existence of mild solutions for Hilfer fractional evolution equations with almost sectorial operators, *Axioms*, **11**(4):144, 2022, `https://doi.org/10.3390/axioms11040144`.

32. Y. Zhou, *Basic Theory of Fractional Differential Equations*, World Scientific, Singapore, 2014, `https://doi.org/10.1142/9069`.