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Bounded solutions and Hyers–Ulam stability of quasilinear dynamic equations on time scales

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Abstract. We consider the quasilinear dynamic equation in a Banach space on unbounded above and below time scales $\mathbb{T}$ with rd-continuous, regressive right-hand side. We define the corresponding Green-type map. Using the integral functional technique, we find a new simpler, but at the same time, more general sufficient condition for the existence of a bounded solution on the time scales expressed in terms of integrals of the Green-type map. We construct previously unknown linear scalar differential equation, which does not possess exponentially dichotomy, but for which the integral of the corresponding Green-type map is uniformly bounded. The existence of such example allows, on the one hand, to obtain the new sufficient condition for the existence of bounded solution and, on the other hand, to prove Hyers–Ulam stability for a much broader class of linear dynamic equations even in the classical case.

Keywords: dynamic equations on time scales, Green-type map, bounded solution, periodic solution, Hyers–Ulam stability.

1 Introduction

The following result is known in the theory of dynamical systems as Bohl–Perron-type theorem; see, for example, [4, 10, 11, 23].

In his dissertation [4] (see also [6, p. 147] and the translation of the dissertation [5, p. 74]), Bohl studied the problem of the existence of a bounded solution $x \in \mathbb{R}^n$ for the quasilinear differential equation

$$\dot{x} = Ax + f(t, x),$$

where real parts of matrix A eigenvalues are not equal to zero. Bohl proved the theorem for the cases $n = 1$ and $n = 2$. Later, Demidovich proved the theorem in the general case [12] (see also [13, p. 300]), and he called it the Bohl theorem.

In our research, we generalize these results, even for $\mathbb{R}^n$, by relaxing conditions on the linear part A and strengthening conditions on the nonlinear part f. We use Green-type map and integral functional equation technique [27–29] to substantially simplify the proof. Furthermore, for more general point of view, we consider dynamic equations on time scales in arbitrary Banach space. To highlight our improvement comparing to previous results, we use an example where the linear part of the differential equation even does not possess a nonuniform exponential dichotomy.

The field of dynamic equations on time scales created by Hilger in 1990 [16, 17] is an emerging area with great potential. This new and compelling area of mathematics is more general and versatile than the traditional theories of differential and difference equations. The field of dynamic equations on time scales contains and extends the classical theory of differential, difference, integral and summation equations as special cases. To understand the notation in this article, some basic definitions are needed (for details, see [7, 8, 20]).

Many applications lead to the task of finding the conditions under which various types of quasilinear dynamical systems have a bounded global solution. In fact, the study has become an important area of research due to the fact that such equations arise in a variety of real world problems such as in the study of p-Laplace equations, non-Newtonian fluid theory and the turbulent flow of a polytrophic gas in a porous medium and so [9, 21].

Let us note that the Keller–Segel models arising in mathematical biology idealize chemotaxis phenomena, and many related variants have been finding interest in the mathematical community. In this regard, in recent articles [14, 15, 22] and the references therein, the interested reader can find an extensive and rigorous theory on existence and properties of global, uniformly bounded or blow-up solutions.

In 1940, at the Mathematics Club of the University of Wisconsin, Ulam [31] raised the question when a solution of an equation, differing slightly from a given one, must be somehow near to the exact solution of the given equation. In the following year, Hyers [18] gave an affirmative answer to the question of Ulam for Cauchy additive functional equation in a Banach space. So the stability concept proposed by Ulam and Hyers, was named as *Hyers–Ulam stability*. Afterwards, Rassias [26] introduced new ideas of Hyers–Ulam stability using unbounded right-hand side in the involved inequalities, depending on certain functions, introducing therefore the so-called *Hyers–Ulam–Rassias stability*. However, we will use only the term Hyers–Ulam stability in this article.

Hyers–Ulam stability of some linear and nonlinear dynamic equations on time scales has been studied; see [1, 30] and the references therein. The connections between Hyers–Ulam stability and uniform exponential stability was examined by [2,35]. We derive a new sufficient condition for the Hyers–Ulam stability of a linear dynamic equation in the case when the integral of Green-type map is uniformly bounded.

2 Notations and preliminaries

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of real numbers $\mathbb{R}$. We assume throughout that a time scale $\mathbb{T}$ has the topology that it inherits from the standard topology on the real numbers $\mathbb{R}$. Since a time scale may or may not be connected, the concept of jump operator is useful for describing the structure of the time scale under consideration and is also used in defining the delta derivative. The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by the equality

$$\sigma(t) = \inf\{s \in \mathbb{T} \mid s > t\},$$

while the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by the equality

$$\rho(t) = \sup\{s \in \mathbb{T} \mid s < t\}.$$

In these definitions, we put $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$, where $\emptyset$ denotes the empty set. The graininess function $\mu : \mathbb{T} \to [0, +\infty)$ is defined by

$$\mu(t) = \sigma(t) - t.$$

The jump operators allow the classification of points in a time scale $\mathbb{T}$. If $\sigma(t) > t$, then the point $t \in \mathbb{T}$ is called right-scattered, while if $\rho(t) < t$, then the point $t \in \mathbb{T}$ is called left-scattered. If $\sigma(t) = t$, then $t \in \mathbb{T}$ is called right-dense, while if $\rho(t) = t$, then $t \in \mathbb{T}$ is called left-dense. If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^\kappa = \mathbb{T} \setminus \{m\}$, otherwise, set $\mathbb{T}^\kappa = \mathbb{T}$.

Let $\mathbf{X}$ be a Banach space. Assume that $g : \mathbb{T} \to \mathbf{X}$ is a map and fix $t \in \mathbb{T}^\kappa$. The delta derivative (also, Hilger derivative) $g^\Delta(t) \in \mathbf{X}$ exists if for every $\varepsilon > 0$, there exists a neighbourhood $U = (t - \delta,\, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$ such that

$$\begin{gathered} \big|\big(g\big(\sigma(t)\big) - g(s)\big) - g^\Delta(t)\big(\sigma(t) - s\big)\big| \\ \leqslant \varepsilon\big|\sigma(t) - s\big| \quad \text{for all } s \in U. \end{gathered}$$

Take $\mathbb{T} = \mathbb{R}$ and g is differentiable in the ordinary sense at $t \in \mathbb{T}$. Then $g^\Delta(t) = g'(t)$ is the derivative used in standard calculus. Take $\mathbb{T} = \mathbb{Z}$. Then $g^\Delta(t) = \Delta g(t)$ is the forward difference operator used in difference equation.

If $F^\Delta(t) = g(t)$, then define the (Cauchy) delta integral by

$$\int\limits_r^s g(t)\,\Delta t = F(s) - F(r) \quad \text{for all } r, s \in \mathbb{T}.$$

If $\mathbb{T} = \mathbb{R}$, then

$$\int\limits_r^s g(t)\,\Delta t = \int\limits_r^s g(t)\,\mathrm{d}t,$$

while if $\mathbb{T} = \mathbb{Z}$, then

$$\int\limits_r^s g(t)\,\Delta t = \sum_{t=r}^{s-1} g(t) \quad \text{for } r, s \in \mathbb{T} \text{ and } r < s.$$

A map $g : \mathbb{T} \to \mathbf{X}$ is called rd-continuous, provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist at left-dense points in $\mathbb{T}$. A map $h : \mathbb{T} \times \mathbf{X} \to \mathbf{X}$ is called rd-continuous if g defined by $g(t) = h(t, x(t))$ is rd-continuous for any continuous map $x : \mathbb{T} \to \mathbf{X}$. A map h is called regressive at $t \in \mathbb{T}^\kappa$ if the map

$$I + \mu(t)h(t, \cdot) : \mathbf{X} \to \mathbf{X} \quad \text{is invertible}$$

(where I is the identity map), and h is called regressive on $\mathbb{T}^\kappa$ if h is regressive at each $t \in \mathbb{T}^\kappa$.

The set of all regressive and rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}^\kappa, \mathbb{R})$. The set of functions being both regressive and rd-continuous on time scales $\mathbb{T}$ forms an Abelian group under the addition $\oplus$ defined by

$$(p \oplus q)(t) = p(t) + q(t) + \mu(t)p(t)q(t) \quad \text{for all } t \in \mathbb{T}^\kappa,$$

where $p, q \in \mathcal{R}(\mathbb{T}^\kappa, \mathbb{R})$, and additive inverse in this group is given by

$$(\ominus p)(t) = -\frac{p(t)}{1 + \mu(t)p(t)} \quad \text{for all } t \in \mathbb{T}^\kappa.$$

3 Bounded solution

Hereinafter, $\mathbb{T}$ will be unbounded above and below time scales. Let $\mathfrak{L}(\mathbf{X})$ be the Banach space of linear bounded endomorphisms. Let us look closely to the following quasilinear regressive dynamic equation:

$$x^\Delta = A(t)x + f(t, x), \tag{1}$$

where:

(i) the map $A : \mathbb{T} \to \mathfrak{L}(\mathbf{X})$ is rd-continuous, and the corresponding linear dynamic equation

$$x^\Delta = A(t)x \tag{2}$$

is regressive;

(ii) the map $f : \mathbb{T} \times \mathbf{X} \to \mathbf{X}$ is rd-continuous with respect to t for fixed x, and it satisfies the Lipschitz conditions

$$\left|f(t, x) - f(t, x')\right| \leqslant \varepsilon(t)|x - x'|,$$

and in addition, it satisfies the following inequality:

$$\left|f(t, 0)\right| \leqslant N(t) < +\infty,$$

where $N : \mathbb{T} \to \mathbb{R}_+$ and $\varepsilon : \mathbb{T} \to \mathbb{R}_+$ are integrable scalar functions.

From conditions (i) and (ii) we obtained that the solutions of (1) are continuable in the negative direction. Moreover, the solution for initial value problem defined on $\mathbb{T}$ is unique because of Lipschitz condition with respect to x of the right hand.

The solution of dynamic equation (1) is denoted by $x(\cdot, s, x) : \mathbb{T} \to \mathbf{X}$ with initial condition $x(s) = x$. So $x(s, s, x) = x$, and because of uniqueness of solutions, for

$t, \tau, s \in \mathbb{T}$, we have

$$x(t,s,x) = x\bigl(t,\tau,x(\tau,s,x)\bigr).$$

For short, we will use the notation $x(t) = x(t,s,x)$.

Local results, which hold under more realistic assumptions on the nonlinearity, can be deduced using standard bump function techniques.

The exponential operator (transition operator) $e_A(\cdot,s) : \mathbb{T} \to \mathfrak{L}(\mathbf{X})$ is solution of the corresponding operator-valued initial value problem

$$X^{\Delta} = A(t)X, \quad X(s) = I,$$

where $t, s \in \mathbb{T}$, $e_A(s,s) = I$, and $I \in \mathfrak{L}(\mathbf{X})$ is identity operator. Let us note that linear cocycle property holds for $\tau, s, t \in \mathbb{T}$:

$$e_A(t,\tau)e_A(\tau,s) = e_A(t,s).$$

The solution of (1) can be represented in the form

$$x(t,s,x) = e_A(t,s)x + \int_s^t e_A\bigl(t,\sigma(\tau)\bigr) f\bigl(\tau, x(\tau,s,x)\bigr)\,\Delta\tau.$$

Green-type map can be represented in the form

$$G(t,s) = \begin{cases} e_A(t,s)P(s) & \text{if } s \leqslant t, \\ e_A(t,s)(P(s) - I & \text{if } t < s, \end{cases}$$

where $P(s) \in \mathfrak{L}(\mathbf{X})$ is rd-continuous with respect to $s \in \mathbb{T}$. Note that the linear dynamic equation (2) has infinitely many Green-type maps. But if $\mathbb{T} = \mathbb{R}$ and the linear regressive dynamic equation (2) has uniform exponential dichotomy, then moreover there exists a unique Green-type map, which satisfies the inequality

$$\bigl|G(t,s)\bigr| \leqslant K \exp\bigl(-\lambda|t-s|\bigr), \quad K \geqslant 1,\ \lambda > 0.$$

Let us formulate sufficient conditions for the existence of bounded solution to the quasilinear dynamic equation (1). Note that the weak condition of the nonlinear member and the weak condition of the nonlinear member Lipschitz coefficient play an important role in the formulation of the theorem.

Theorem 1. *Suppose that the linear dynamic equation* (2) *has an rd-continuous Green-type map* $G(s,\tau) \in \mathfrak{L}(\mathbf{X})$ *such that*

$$\sup_{t\in\mathbb{T}} \int_{-\infty}^{+\infty} \bigl|G(t,\sigma(\tau))\bigr| N(\tau)\,\Delta\tau < +\infty, \tag{3}$$

$$\sup_{t\in\mathbb{T}} \int_{-\infty}^{+\infty} \bigl|G(t,\sigma(\tau))\bigr| \varepsilon(\tau)\,\Delta\tau = q < 1. \tag{4}$$

Then the quasilinear equation (1) *has a bounded solution.*

Proof. Let $\mathcal{C}_{\mathrm{rd}}(\mathbb{T}, X)$ be a set of rd-continuous maps $\eta : \mathbb{T} \to X$. The set

$$\mathfrak{B} = \Bigl\{ \eta \in \mathcal{C}_{\mathrm{rd}}(\mathbb{T}, X) \Bigm| \sup_{t \in \mathbb{T}} \bigl|\eta(t)\bigr| < +\infty \Bigr\}$$

is Banach space with the supremum norm $\|\eta\| = \sup_{t\in\mathbb{T}} |\eta(t)|$. Let us consider the integro-functional equation on time scales

$$\eta(t) = \int_{-\infty}^{+\infty} G\bigl(t, \sigma(\tau)\bigr) f\bigl(\tau, \eta(\tau)\bigr)\, \Delta\tau \tag{5}$$

and the map $\eta \mapsto \mathfrak{T}\eta$, $\eta \in \mathfrak{B}$, defined by the equality

$$\mathfrak{T}\eta(t) = \int_{-\infty}^{+\infty} G\bigl(t, \sigma(\tau)\bigr) f\bigl(\tau, \eta(\tau)\bigr)\, \Delta\tau.$$

We have

$$\begin{aligned}
\bigl|\mathfrak{T}\eta(t)\bigr| &\leqslant \int_{-\infty}^{+\infty} \bigl|G\bigl(t, \sigma(\tau)\bigr)\bigr| \bigl|f\bigl(\tau, \eta(\tau)\bigr)\bigr|\, \Delta\tau \\
&\leqslant \int_{-\infty}^{+\infty} \bigl|G\bigl(t, \sigma(\tau)\bigr)\bigr| \varepsilon(\tau) \|\eta\|\, \Delta\tau + \int_{-\infty}^{+\infty} \bigl|G\bigl(t, \sigma(\tau)\bigr)\bigr| \bigl|N(\tau)\bigr|\, \Delta\tau \\
&\leqslant q\|\eta\| + \int_{-\infty}^{+\infty} \bigl|G\bigl(t, \sigma(\tau)\bigr)\bigr| \bigl|N(\tau)\bigr|\, \Delta\tau.
\end{aligned}$$

It follows that $\mathfrak{T}\eta \in \mathfrak{B}$. Next, we get

$$\begin{aligned}
\bigl|\mathfrak{T}\eta(t) - \mathfrak{T}\eta'(t)\bigr| &= \left| \int_{-\infty}^{+\infty} G\bigl(s, \sigma(\tau)\bigr) \bigl(f\bigl(\tau, \eta(\tau)\bigr) - f\bigl(\tau, \eta'(\tau)\bigr)\bigr)\, \Delta\tau \right| \\
&\leqslant \int_{-\infty}^{+\infty} \bigl|G\bigl(s, \sigma(\tau)\bigr)\bigr| \varepsilon(\tau) \bigl|\eta(\tau) - \eta'(\tau)\bigr|\, \Delta\tau \\
&\leqslant \sup_t \int_{-\infty}^{+\infty} \bigl|G\bigl(s, \sigma(\tau)\bigr)\bigr| \varepsilon(\tau)\, \Delta\tau \|\eta - \eta'\| \\
&= q\|\eta - \eta'\|,
\end{aligned}$$

where $q < 1$. Thus the map $\mathfrak{T}$ is a contraction, and consequently, the integro-functional equation (5) has a unique solution in $\mathfrak{B}$.

Next, we will prove that the bounded solution of integro-functional equation (5) is also the solution of (1).

First, consider the case when $s < t$. Then

$$\begin{aligned}
\eta(t) &= \int_{-\infty}^{+\infty} G\big(t, \sigma(\tau)\big) f\big(\tau, \eta(\tau)\big)\, \Delta\tau \\
&= \int_{-\infty}^{t} G\big(t, \sigma(\tau)\big) f\big(\tau, \eta(\tau)\big)\, \Delta\tau + \int_{t}^{+\infty} G\big(t, \sigma(\tau)\big) f\big(\tau, \eta(\tau)\big)\, \Delta\tau \\
&= \int_{-\infty}^{s} e_A(t,s) G\big(s, \sigma(\tau)\big) f\big(\tau, \eta(\tau)\big)\, \Delta\tau + \int_{s}^{t} e_A\big(t, \sigma(\tau)\big) P\big(\sigma(\tau)\big) f\big(\tau, \eta(\tau)\big)\, \Delta\tau \\
&\quad + \int_{s}^{+\infty} e_A(t,s) G\big(s, \sigma(\tau)\big) f\big(\tau, \eta(\tau)\big)\, \Delta\tau \\
&\quad + \int_{t}^{s} e_A\big(t, \sigma(\tau)\big)\big(P\big(\sigma(\tau)\big) - I\big) f\big(\tau, \eta(\tau)\big)\, \Delta\tau \\
&= e_A(t,s) \int_{-\infty}^{+\infty} G\big(s, \sigma(\tau)\big) f\big(\tau, \eta(\tau)\big)\, \Delta\tau + \int_{s}^{t} e_A\big(t, \sigma(\tau)\big) f\big(\tau, \eta(\tau)\big)\, \Delta\tau \\
&= e_A(t,s)\eta(s) + \int_{s}^{t} e_A\big(t, \sigma(\tau)\big) f\big(\tau, \eta(\tau)\big)\, \Delta\tau.
\end{aligned}$$

Analogously, we consider the case when $s > t$.

The bounded solution η is unique in $\mathfrak{B}$, and $\|\eta\| = \sup_{t\in\mathbb{T}} |\eta(t)| < +\infty$. The theorem is proved. $\square$

Remark 1. Note that conditions (3) and (4) can be simplified if the improper integral of the Green-type map is uniformly bounded, that is, if

$$\sup_{t\in\mathbb{T}} \int_{-\infty}^{+\infty} \big|G\big(t, \sigma(\tau)\big)\big|\, \Delta\tau < +\infty. \tag{6}$$

Condition (6) is satisfied if the linear dynamic equation (2) admits a uniform exponential dichotomy [10, 11]. In the case $\mathbb{T} = \mathbb{R}$, it can be concluded that there exists a projector $P : \mathbb{R} \to \mathfrak{L}(\mathbf{X})$ such that $P^2(s) = P(s)$ and

$$e_A(t,s)P(s) = P(t)e_A(t,s) \tag{7}$$

holds, and there exist constants $K \geqslant 1$ and $\lambda > 0$ such that

$$\begin{aligned} \bigl|e_A(t,s)P(s)\bigr| &\leqslant K\mathrm{e}^{-\lambda(t-s)} \quad \text{if } s \leqslant t, \\ \bigl|e_A(t,s)\bigl(I - P(s)\bigr)\bigr| &\leqslant K\mathrm{e}^{\lambda(t-s)} \quad \text{if } t \leqslant s. \end{aligned}$$

Linear dynamic equation (2) is said to have an exponential dichotomy [24, 25, 32, 33, 37, 38, 40] on $\mathbb{T}$ if there exists a projector $P : \mathbb{T} \to \mathfrak{L}(\mathbf{X})$ such that $P^2(s) = P(s)$ and (7) holds, and there exist positive constants K_i and λ_i, $i = 1, 2$, such that

$$\begin{aligned} \bigl|e_A(t,s)P(s)\bigr| &\leqslant K_1 e_{\ominus\lambda_1}(t,s) \quad \text{for } s \leqslant t,\ s,t \in \mathbb{T}, \\ \bigl|e_A(t,s)\bigl(I - P(s)\bigr)\bigr| &\leqslant K_2 e_{\ominus\lambda_2}(t,s) \quad \text{for } t \leqslant s,\ s,t \in \mathbb{T}. \end{aligned}$$

Example 1. We construct the linear scalar differential equation, the solution of which on the one hand has infinite Lyapunov exponent [10, 11], but on the other hand, integral of corresponding Green-type map is uniformly bounded. Consider

$$\dot{x} = -\biggl(\frac{a'(t)}{a(t)} + a(t)\biggr)x, \quad a(t) = \alpha + \kappa(t),\ \alpha > 0,\ \kappa(t) \geqslant 0, \tag{8}$$

where the function $\kappa : \mathbb{R} \to \mathbb{R}$ is sawtooth piecewise linear and satisfies the estimates

$$\int_{-\infty}^{+\infty} \kappa(t)\,\mathrm{d}t < +\infty, \qquad \limsup_{t\to+\infty} \frac{\ln(\kappa(t))}{t} = +\infty.$$

Then the Cauchy initial value problem $x(s) = 1$ of equation (8) has solution

$$x(t,s) = \exp\biggl(-\int_s^t a(\tau)\,\mathrm{d}\tau\biggr)\frac{a(s)}{a(t)} > 0$$

with property

$$\begin{aligned} \limsup_{t\to+\infty} \frac{\ln(x(t,s))}{t} &= -\alpha - \limsup_{t\to+\infty} \frac{\ln(a(t))}{t} = -\alpha - \limsup_{t\to+\infty} \frac{\ln(\kappa(t))}{t} \\ &= -\infty. \end{aligned}$$

For example, the function $\kappa : \mathbb{R} \to \mathbb{R}$ can be taken as follows: κ is piecewise linear continuous scalar function, $\kappa(-t) = \kappa(t)$, and for $n \in \mathbb{N}$,

$$\kappa(t) = \begin{cases} 0 & \text{if } t \in [0, 1/2), \\ 0 & \text{if } t \in [n - 1/2,\ n - 2^{-1}\mathrm{e}^{-n^2}n^{-2}], \\ \text{linear} & \text{if } t \in (n - 2^{-1}\mathrm{e}^{-n^2}n^{-2},\ n), \\ \mathrm{e}^{n^2} & \text{if } t = n, \\ \text{linear} & \text{if } t \in (n,\ n + 2^{-1}\mathrm{e}^{-n^2}n^{-2}), \\ 0 & \text{if } t \in [n + 2^{-1}\mathrm{e}^{-n^2}n^{-2},\ n + 1/2). \end{cases}$$

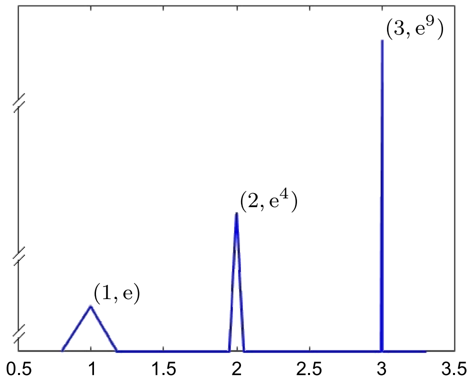

Figure 1. Sketch of the sawtooth function $\kappa : \mathbb{R} \to \mathbb{R}$.

Then

$$\int\limits_{-\infty}^{+\infty} \kappa(t)\,\mathrm{d}t = 2\sum_{n=1}^{+\infty} \frac{1}{2n^2} = \frac{\pi^2}{6} < +\infty$$

and

$$\limsup_{t\to+\infty} \frac{\ln(\kappa(t))}{t} = \lim_{n\to+\infty} \frac{\ln(\kappa(n))}{n} = \lim_{n\to+\infty} n = +\infty.$$

We choose Green-type map

$$G(t,s) = \begin{cases} x(t,s) & \text{if } s \leqslant t, \\ 0 & \text{if } t < s. \end{cases}$$

Let us note that

$$\int\limits_{-\infty}^{+\infty} \bigl|G(t,s)\bigr|\mathrm{d}s = \int\limits_{-\infty}^{t} x(t,s)\,\mathrm{d}s = \frac{1}{a(t)} \leqslant \frac{1}{\alpha}.$$

If $\sup_{t,x\in\mathbb{R}} |f(t,x)| < +\infty$ and $\sup_{t\in\mathbb{R}} \varepsilon(t) < \alpha$, then corresponding quasilinear equation (1) has bounded solution although the solutions of the corresponding linear differential equation has infinite Lyapunov exponent.

We say that the linear differential equation

$$\dot{x} = A(t)x, \quad t \in [\alpha, +\infty) \subset \mathbb{R}$$

satisfies the *Perron conjecture* [13, 23] if for every nonhomogeneous differential equation

$$\dot{x} = A(t)x + f(t),$$

where $f : [\alpha, +\infty) \to \mathbb{R}^n$ is a continuous and bounded map, there exists a bounded solution $x : [\alpha, +\infty) \to \mathbb{R}^n$, $x(\alpha) = 0$, if and only if the corresponding exponential operator satisfies the inequality

$$\left|e_A(t,s)\right| \leqslant K\exp\bigl(-\lambda(t-s)\bigr), \quad K \geqslant 1,\ \lambda > 0,\ \alpha \leqslant s \leqslant t.$$

Example 1 shows that the Peron's conjecture can be incorrect in the case where the matrix A is unbounded with respect to $t \in [\alpha, +\infty)$.

Illustrative Example 2. Let uss look at the case where $\mathbb{T} = \mathbb{R}$. Suppose that linear dynamic equation (2) admits a nonuniform exponential dichotomy [3, 34, 36, 39]. Then there exists a projector $P : \mathbb{T} \to \mathfrak{L}(\mathbf{X})$, (7) holds, and there exist constants $K \geqslant 1$, $\lambda > 0$ and $\delta > 0$ such that

$$\begin{aligned}\left|e_A(t,s)P(s)\right| &\leqslant K\mathrm{e}^{-\lambda(t-s)}\mathrm{e}^{\delta|s|} \quad \text{if } s \leqslant t,\\ \left|e_A(t,s)\bigl(I-P(s)\bigr)\right| &\leqslant K\mathrm{e}^{\lambda(t-s)}\mathrm{e}^{\delta|s|} \quad \text{if } t \leqslant s.\end{aligned}$$

When $\delta \equiv 0$, we obtain the classical notion of uniform exponential dichotomy. If $\delta \equiv 0$ and $\lambda \equiv 0$, we obtain a uniform dichotomy.

If $N(t) \leqslant \mu\mathrm{e}^{-\delta|t|}$, $\varepsilon(t) \leqslant r\mathrm{e}^{-\delta|t|}$ and $2Kr < \lambda$, then dynamic equation (1) in the case of nonuniform exponential dichotomy has bounded solution [34].

If $\int_{-\infty}^{+\infty} N(t) < +\infty$ and

$$\sup_{t,s\in\mathbb{R}}\left|G(t,s)\right|\int\limits_{-\infty}^{+\infty}\varepsilon(t)\,\mathrm{d}t = q < 1,$$

then dynamic equation (1) in the case of uniform dichotomy has bounded solution [19].

Illustrative Example 3. Let us illustrate Theorem 1. For simplicity, let us consider nonhomogeneous linear differential equation on $\mathbb{R}$

$$\dot{x} = A(t)x + f(t) = -\frac{2t}{1+t^2}x - \frac{2t\mathrm{e}^{-t^2}}{1+t^2}$$

with nonuniform dichotomy. In our case the exponential map is

$$e_A(t,s) = \frac{1+s^2}{1+t^2},$$

and choose the Green-type map in the following form:

$$G(t,s) = \begin{cases}\frac{1+s^2}{1+t^2} & \text{if } s \leqslant t,\\ 0 & \text{if } t < s.\end{cases}$$

It follows that bounded solution is

$$\eta(t) = -\int\limits_{-\infty}^{t}\frac{1+s^2}{1+t^2}\frac{2s\mathrm{e}^{-s^2}}{1+s^2}\,\mathrm{d}s = \frac{\mathrm{e}^{-t^2}}{1+t^2}.$$

Remark 2. Let the time scale be periodic, that is, if $t \in \mathbb{T}$, then $t + \omega \in \mathbb{T}$, where $\omega > 0$ is the period of the time scale $\mathbb{T}$. Then the graininess function is also periodic $\mu(t) = \mu(t + \omega)$.

Let us consider the quasilinear dynamic equation (1) where the right-hand side is periodic with period $\omega > 0$, consequently, $A(t+\omega) = A(t)$ and $f(t+\omega, x) = f(t, x)$. We obtain that exponential map satisfies equality $e_A(t+\omega,\, s+\omega) = e_A(t, s)$. If $P(s+\omega) = P(s)$, it follows that $G(t + \omega, s + w) = G(t, s)$.

It is important to note that

$$\begin{aligned} \eta(t+\omega) &= \int\limits_{-\infty}^{+\infty} G\bigl(t+\omega,\, \sigma(\tau)\bigr) f\bigl(\tau, \eta(\tau)\bigr)\, \Delta\tau \\ &= \int\limits_{-\infty}^{+\infty} G\bigl(t+\omega,\, \sigma(r+\omega)\bigr) f\bigl(r+\omega,\, \eta(r+\omega)\bigr)\, \Delta r \\ &= \int\limits_{-\infty}^{+\infty} G\bigl(t, \sigma(r)\bigr) f\bigl(r,\, \eta(r+\omega)\bigr)\, \Delta r. \end{aligned}$$

Because the integro-functional equation (5) has a unique bounded solution, we get that unique bounded solution is periodic

$$\eta(t+w) = \eta(t).$$

4 Hyers–Ulam stability

Definition 1. We say that linear dynamic equation (2) is Hyers–Ulam stable if there exists a positive constant $C > 0$ such that for each real number $\varepsilon > 0$ and for each solution x of the inequality

$$\bigl|x^\Delta - A(t)x\bigr| \leqslant \varepsilon,$$

there exists a solution x_0 of the linear dynamic equation (2) with the property

$$\sup_{t\in\mathbb{T}} \bigl|x(t) - x_0(t)\bigr| \leqslant C\varepsilon.$$

We will give a new sufficient condition for the Hyers–Ulam stability of a linear dynamic equation (2) in the case when the integral of Green-type map is uniformly bounded. Example 1 shows that there are cases where, on the one hand, the solution of a linear dynamical system has an infinite Lyapunov exponent, but on the other hand, the integral from the Green-type map is uniformly bounded.

Theorem 2. *Suppose that the linear dynamic equation* (2) *has a rd-continuous Green-type map* $G(s,\tau) \in \mathfrak{L}(\mathbf{X})$ *such that*

$$\sup_{t\in\mathbb{T}} \int_{-\infty}^{+\infty} \bigl|G\bigl(t,\sigma(\tau)\bigr)\bigr| \,\Delta\tau < +\infty.$$

Then the linear dynamic equation (2) *is Hyers–Ulam stable.*

Proof. Let

$$\eta(t) = x^{\Delta} - A(t)x.$$

Then $|\eta(t)| \leqslant \varepsilon$, and general solution of

$$x^{\Delta} = A(t)x + \eta(t)$$

is

$$x(t) = e_A(t,s)C_1 + \int_{-\infty}^{+\infty} G\bigl(t,\sigma(\tau)\bigr)\eta(\tau)\,\Delta\tau.$$

Let us take

$$x_0(t) = e_A(t,s)C_1,$$

where $C_1 \in \mathbf{X}$. Then

$$\begin{aligned}\bigl|x(t) - x_0(t)\bigr| &\leqslant \left|\int_{-\infty}^{+\infty} G\bigl(t,\sigma(\tau)\bigr)\eta(\tau)\,\Delta\tau\right| \leqslant \varepsilon \int_{-\infty}^{+\infty} \bigl|G\bigl(t,\sigma(\tau)\bigr)\bigr|\,\Delta\tau \\ &\leqslant \varepsilon \sup_{t\in\mathbb{T}} \int_{-\infty}^{+\infty} \bigl|G\bigl(t,\sigma(\tau)\bigr)\bigr|\,\Delta\tau,\end{aligned}$$

where

$$C = \sup_{t\in\mathbb{T}} \int_{-\infty}^{+\infty} \bigl|G\bigl(t,\sigma(\tau)\bigr)\bigr|\,\Delta\tau.$$

The theorem is proved. □

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