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Controllability of multi-agent systems with input and communication delays*

Yuanchao Si[a,b], **Deyuan Meng**[c], **JinRong Wang**[a,b,1]

[a]Department of Mathematics, Guizhou University,
Guiyang 550025, China
siyuanchao@126.com; jrwang@gzu.edu.cn

[b]Gui'an Supercomputing Center,
Kechuang Industrial Development Company Limited,
Gui'an New Area, Guiyang, Guizhou 550025, China

[c]The Seventh Research Division,
Beihang University (BUAA),
Beijing 100191, China
dymeng@buaa.edu.cn

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Abstract. Distributed cooperative control of multi-agent systems is broadly applied in artificial intelligence in which time delay is of great concern because of its ubiquitous. This paper considers the controllability of leader-follower multi-agent systems with input and communication delays. For the first-order systems with input delay, neighbor-based protocol is adopted to realize the interactions among agents, yielding a system with delay existed in state and control input. New notions of interval controllability and interval structural controllability for the system are defined. Algebraic criterion is established for interval controllability, and graph-theoretic interpretation is put forward for the interval structural controllability. Results imply that input delay of the multi-agent systems has significant influence on the interval controllability and interval structural controllability. Corresponding conclusions are generalized to the first-order systems and the high-order ones with communication delays, respectively. Example is attached to illustrate the work.

Keywords: multi-agent systems, interval structural controllability, time delay, linear parameterization, maximum matching.

1 Introduction

The multi-agent systems are in fact a number of individual agents with independent dynamics acting together to achieve a goal by using their neighbors to realize information

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[1]Corresponding author.

interactions. In recent decades, the distributed cooperative control of multi-agent systems has been a hot topic [2, 18] because of the extensive applications in military, industrial and society, etc.

Controllability of multi-agent systems, which concerns the ability of leaders educating followers and is put forward by Tanner [26], is a fundamental issue in this field. Abundant literature is committed to the research of this topic (see [7, 19, 28] etc.), and numerous conclusions are deduced from the perspectives of the graph theory [7,19], the eigenvalues of matrix [28], the invariant subspace [10], etc.

The weights in an interaction topology are of paramount importance for the controllability of multi-agent systems. For instance, an unweighted complete graph in [26] is proved to be uncontrollable with a leader selected; however, a group of weights is specified to the weighted complete graph, rendering such a graph to be controllable for an assigned leader (see [31]). Thus, for the controllability of multi-agent systems, it usually assumed that the interaction topology is directed/undirected, and the link edges among agents are prespecified some weights (see more in [18, 25]).

However, it is no picnic to obtain the precise measurement of weights because of the measurement error and other uncertainties in real world, except for some values identically equaling to zeros, which are caused by no edge existence between some two agents. Thus, it is a significant work to consider the structural controllability of multi-agent systems. For structural controllability, Lin [9] makes a pathbreaking work and deals with the case of single-input from the graph theoretic perspective. Shields et al. [22] extend Lin's work from single-input systems to multi-input ones with a pure algebraic approach utilized. Subsequently, the graph theoretic interpretations about Shields' result are analysed in [13]. Other conclusions about structural controllability can be found in [3] etc.

Structural controllability of the multi-agent systems with first-order integrator is considered by Zamani [31]. Partovi [17] extends above results to the high-order multi-agent systems, respectively. However, relative protocol is widely adopted to realize interactions among agents (see [4, 12, 17, 31]), rendering the independence among the elements of structured matrices fails to hold (the structured matrices of the structured system are the block matrices of Laplacian matrix). In addition, symmetric network is a typical instance with absence of independent elements of structured matrix (see [15]). Thus, the conclusions of Lin [9] and Shields [22] might not be matched to deal with such parameter-dependent problems. To avoid the parameter dependence of structured matrix, Guan et al. [5] propose the absolute protocol to establish the interaction network among agents.

In the information transmitting of multi-agent systems, there exist communication delays because the bandwidth constraints of the communication channels or burn-in of sensors. In addition, connecting and processing the data packets from the neighbors may case input delay for agents [27]. Thus, copious literature is paid close attention to the multi-agent systems with delays (see [20, 23, 27]). In fact, information transmission has timeliness, such as earthquake early warning network system [16], it is apparently crucial for the time cost of information transmission. For delay multi-agent network systems, what we concern is not only whether the predetermined goal be achieved, but also whether the delay contributes to the time cost when we arrived at the goal, such as finite-time synchronization [32], and consensus on finite [20].

Whereas, almost the existed literature concerning structural controllability of the structured systems ignores the time delay although it is a very common phenomenon in mechanical engineering, information technology, economics, etc. (see more in [21, 24]). In fact, from [23, 30] we know that for a system with delay, controllability on some time interval does not imply that on any time interval. It consequently renders that for a structured system with delay, structural controllability on some time interval is not equivalent to that on any interval [8, 9]. Thus, how time delay influencing structural controllability of a delay structured system is an worthwhile topic [21], especially, for delay multi-agent systems.

In this paper, we consider the (structural) controllability of multi-agent systems with input and communication delays. The notions of interval controllability and interval structural controllability are defined. Criteria on interval controllability and interval structural controllability are established, respectively, and the delay influence on controllability is illustrated. Contributions of this paper are below. Firstly, we establish algebraic criteria for interval controllability of the systems with delay existed in states and control inputs in which we point out how time delay influences on the interval controllability. Secondly, we generalize structural controllability of the nondelay structured system to the delay one, and we propose a new notion of interval structural controllability. Thirdly, we reveal the relationship between interval structural controllability and the topology structure of multi-agent systems, and we figure out how time delay influences on the interval structural controllability. Fourthly, a new protocol is designed for the interactions among agents of higher-order multi-agent systems.

This paper is organized as follows. In Section 2, the knowledge of some basic graph theory and structured theory are presented. Interval controllability and interval structural controllability of the first-order multi-agent systems with input delay are considered in Section 3. In Section 4, we consider the interval controllability and interval structural controllability of first-order systems with communication delay. In Section 5, corresponding conclusions are generalized to the higher-order systems with communication delay.

2 Preliminaries

2.1 Elements of graph theory

Denote by $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ a directed graph with $\mathcal{V} = \{v_1, \dots, v_n\}$ the vertex set and $\mathcal{E} = \{e_{ij} = (v_j, v_i),\ v_i, v_j \in \mathcal{V}\} \subset \mathcal{V} \times \mathcal{V}$ the edge set. For an edge $e_{ij} = (v_j, v_i) \in \mathcal{E}$ with v_j called tail and v_i called head, it implies that there exists an information flow from vertex v_j to v_i, but might not inversely. For a special case $e_{ii} = (v_i, v_i)$, it implies that there exists a self-loop edge from v_i to itself. We denote the neighbor set of vertex v_i by $\mathcal{N}_i = \{v_j \in \mathcal{V}\colon (v_j, v_i) \in \mathcal{E},\ i \neq j\}$.

Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be two nonempty subsets of vertex set $\mathcal{V}$. We say there exists a path from $\mathcal{V}_1$ to $\mathcal{V}_2$ if there is a sequence of finite vertices $v_1, v_2, v_3, \dots, v_{k-1}, v_k$, where $v_1 \in \mathcal{V}_1$ and $v_k \in \mathcal{V}_2$ with corresponding edges $(v_1, v_2), (v_2, v_3), \dots, (v_{k-1}, v_k)$ in $\mathcal{E}$. We call v_1 the beginning vertex and v_k the ending vertex of the path, respectively. Further, if every vertex on the path occurs only once, then we call the path a simple one. Two paths from $\mathcal{V}_1$ to $\mathcal{V}_2$ are said to be disjoint if they share no common vertices. The set of mutually disjoint paths with beginning vertices in a vertex set U is called a U-rooted path family. A simple path with beginning and ending vertices sharing a common one is called a cycle. Similarly, two cycles without common vertices are said to be disjoint, and the set of disjoint cycles are called a cycle family. A union graph of a set of subgraphs is the unions of vertex sets and edge sets of each subgraph, respectively (see more in [3]).

A matching of a graph is a set of edges in which no two edges share a common tail or head. A vertex is matched if it is a head of an edge in the matching, otherwise, it is unmatched. A matching of maximum cardinality is called a maximum matching of the graph.

2.2 Structured system

With referring to [9, 13], a structured matrix is a matrix with elements either fixed zeros in some certain locations or indeterminate entries in the remaining locations, which are assumed to be independent of one another. A system

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$

where $x \in \mathbb{R}^n$, is called structured system and denoted by (A, B) if A and B are structured matrices. A structured system (A, B) is called structurally controllable if there exists an admissible numerical realization $(\tilde{A}, \tilde{B})$ by fixing all indeterminate entries at some particular values such that it is controllable in the usual sense. There are two main methods dealing with the structural controllability of structured system (A, B) (see [9,13] for the graph-theoretic approach and [22] for the algebraic one).

For a structured system (A, B) with $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times l}$, we define the corresponding flow structure $\mathcal{G}_s(A, B) = (\mathcal{V}_s, \mathcal{E}_s)$ with $n+l$ vertices, where the ijth entry of structured matrix $[A, B]$ corresponds to the edge (j, i). If the ijth entry is nonzero, then there exists an edge from the jth vertex to the ith one; otherwise, there is no edge between the two vertices. The vertex corresponding to the elements of A is called state vertex, and the set of state vertices is denoted by in $\mathcal{V}_A = \{v_1, \dots, v_n\}$. Similarly, the vertex corresponding to the elements of B is called input vertices, and the set of input vertices is denoted by $\mathcal{V}_B = \{v_{n+1}, \dots, v_{n+l}\}$. If for each state vertex, there is a path beginning with one of the input vertex and ending with it, then we say the flow structure is input reachable. A state vertex $v_i \in \mathcal{V}_A$ is said to be unaccessible if there exists no path beginning with the vertex in $\mathcal{V}_B$ and ending with it. For $l = 1$, we say the flow structure $\mathcal{G}_s(A, B)$ has a spanning tree if $\mathcal{G}_s(A, B)$ is input reachable; we call the input vertex v_{n+1} the root of the tree. For $l > 1$, the flow structure $\mathcal{G}_s(A, B)$ has a spanning forest if it contains a spanning subgraph consisted of multiple disjoint trees with roots at $\mathcal{V}_B$. Denote by $|E|$ the number of vertices in $E \subset \mathcal{V}_s$ and $T(E)$ the set of vertices v_js with the property that there is an edge $(j, i) \in \mathcal{E}_s$ for any $v_i \in E$. We say the flow structure $\mathcal{G}_s(A, B)$ contains a dilation if there exists a subset $E \subset \mathcal{V}_A$ such that $|T(E)| < |E|$.

2.3 Notation

In what follows, we denote by $\overline{n}$ the set of integers $\{1, \dots, n\}$, $\mathbb{L}^2([0, t_f]; \mathbb{R}^l)$ the space consisted of all the functions, which are square integrable on $[0, t_f]$ and take value in $\mathbb{R}^l$, θ the zero vector, I the identity matrix, and Θ the zero matrix of appropriate dimensions.

3 Controllability of the first-order systems with input delay

In this section, we consider the first-order multi-agent systems consisting of $n + l$ agents with each agent i suffering from input delay. The interaction topology of the multi-agent systems is assumed to be modeled by a simple weighted digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$, where $\mathcal{V}$ is the set of $n + l$ vertices, $\mathcal{E}$ is the set of edges, and $\mathcal{W}$ is the associated weighted matrix. The vertices in $\mathcal{V}$ are indexed from v_1 to v_{n+l}. Suppose the last l vertices are assigned as leaders, and the others are followers. Dynamics of each agent obeys the following rule:

$$\dot{x}_i(t) = u_i(t - \tau), \quad i \in \overline{n + l}, \tag{2}$$

where $x_i \in \mathbb{R}$ is the state variable, $u_i \in \mathbb{R}$ is control input, and $\tau > 0$ is time delay.

With referring to [20], introduce the following neighbor-based protocol to realize the interactions among agents:

$$u_i(t) = \sum_{v_j \in \mathcal{N}_i} w_{ij}\bigl(x_j(t) - x_i(t)\bigr), \quad i \in \overline{n}, \tag{3}$$

where $\mathcal{N}_i$ represents the neighbor set of v_i. If v_j is a neighbor of v_i, then w_{ij} is a nonzero indeterminate parameter. Otherwise, w_{ij} is zero.

Denote the Laplacian matrix by $\mathcal{L} = [l_{ij}]$, where

$$l_{ij} = \begin{cases} \sum_{v_r \in \mathcal{N}_i} w_{ir}, & i = j, \\ -w_{ij}, & i \neq j,\ v_j \in \mathcal{N}_i, \end{cases}$$

and assume that it can be partitioned as

$$\mathcal{L} = \begin{bmatrix} L_{ff} & L_{fl} \\ L_{lf} & L_{ll} \end{bmatrix},$$

where $L_{ff} \in \mathbb{R}^{n \times n}$, $L_{fl} \in \mathbb{R}^{n \times l}$, $L_{lf} \in \mathbb{R}^{l \times n}$, and $L_{ll} \in \mathbb{R}^{l \times l}$. Under (3), the multi-agent systems modeled by (2) become

$$\dot{x}(t) = Ax(t - \tau) + Bu(t - \tau), \tag{4}$$

where $x = [x_1, \dots, x_n]^{\mathrm{T}}$, $u = [x_{n+1}, \dots, x_{n+l}]^{\mathrm{T}}$, $A = -L_{ff}$, and $B = -L_{fl}$.

System (4) has two characteristics. Firstly, A and B are submatrices of the Laplacian matrix $\mathcal{L}$. Thus, the elements of A and B are in fact linear dependent. Secondly, delay exists in state and control input, rendering that both initial function and initial control input have impact on the state of (4).

If (4) is controllable, then we say the multi-agent systems modeled by (2) with protocol (3) is controllable. Next, we establish the controllability criterion of system (4).

3.1 Interval controllability with input delay

With referring to [30], the notion of interval controllability is defined below.

Definition 1. System (4) is called controllable on $[0, t_f]$ if for any terminal state x_f, differentiable initial function $\varphi(t)$, and initial control input $u_0(t)$ on $[-\tau, 0]$, there exists a bounded measurable control function $u^*(\cdot) \in \mathbb{L}^2([0, t_f]; \mathbb{R}^l)$ such that corresponding response of (4) satisfies $x^*(t_f) = x_f$ and $x^*(t) \equiv \varphi(t)$ for $t \in [-\tau, 0]$.

Introduce the delayed matrix function [8] for some positive integer k as follows:

$$e_\tau^{At} = \begin{cases} \Theta, & -\infty < t < -\tau, \\ I, & -\tau \leqslant t < 0, \\ I + At + \cdots + A^k \frac{(t-(k-1)\tau)^k}{k!}, & (k-1)\tau \leqslant t < k\tau, \end{cases} \tag{5}$$

where A is the matrix in (4). Solution of (4) with any differentiable initial function $\varphi(t)$ and initial control input $u_0(t)$, $t \in [-\tau, 0]$, gives

$$\begin{aligned} x(t) = {}& e_\tau^{At}\varphi(-\tau) + \int_{-\tau}^{0} e_\tau^{A(t-s-\tau)}\varphi'(s)\,\mathrm{d}s \\ & + \int_{-\tau}^{0} e_\tau^{A(t-s-2\tau)}Bu_0(s)\,\mathrm{d}s + \int_{0}^{t-\tau} e_\tau^{A(t-s-2\tau)}Bu(s)\,\mathrm{d}s. \end{aligned} \tag{6}$$

For the interval controllability of (4), we have below theorem.

Theorem 1. *System* (4) *is controllable on* $[0, t_f]$ *if and only if* $t_f > n\tau$ *and* $\operatorname{rank} Q = n$, *where* Q *is the controllability matrix defined by*

$$Q \triangleq \left[B, AB, \ldots, A^{n-1}B\right].$$

Proof. Necessity. Assume that (4) is controllable on $[0, t_f]$. Then for arbitrary terminal state x_f, differentiable initial function $\varphi(t)$, and initial control input $u_0(t)$, $t \in [-\tau, 0]$, there exists a bounded measurable control function $u^*(\cdot) \in \mathbb{L}^2([0, t_f]; \mathbb{R}^l)$ such that corresponding response of (4) satisfies $x^*(t_f) = x_f$. From (6) we have

$$\begin{aligned} x^*(t_f) &= e_\tau^{At_f}\varphi(-\tau) + \int_{-\tau}^{0} e_\tau^{A(t_f-s-\tau)}\varphi'(s)\,\mathrm{d}s \\ &\quad + \int_{-\tau}^{0} e_\tau^{A(t_f-s-2\tau)}Bu_0(s)\,\mathrm{d}s + \int_{0}^{t_f-\tau} e_\tau^{A(t_f-s-2\tau)}Bu^*(s)\,\mathrm{d}s \\ &= x_f. \end{aligned}$$

Assume there is a positive integer $k \in \mathbb{N}$ such that $(k-1)\tau \leqslant t_f < k\tau$. Then from (5) and [8] we arrive at

$$\begin{aligned}
&\int_0^{t_f-\tau} e_\tau^{A(t_f-s-2\tau)} Bu^*(s)\,\mathrm{d}s \\
&\quad = B\int_{-\tau}^{t_f-2\tau} u^*(t_f-s-2\tau)\,\mathrm{d}s + AB\int_0^{t_f-2\tau} su^*(t_f-s-2\tau)\,\mathrm{d}s + \cdots \\
&\qquad + A^{k-2}B\int_{(k-3)\tau}^{t_f-2\tau} \frac{(s-(k-3)\tau)^{k-2}}{(k-2)!} u^*(t_f-s-2\tau)\,\mathrm{d}s.
\end{aligned}$$

Denote that $u^* = [\tilde{u}_1^*, \dots, \tilde{u}_l^*]^{\mathrm{T}}$,

$$\eta = x_f - e_\tau^{At_f}\varphi(-\tau) - \int_{-\tau}^{0} e_\tau^{A(t_f-s-\tau)}\varphi'(s)\,\mathrm{d}s - \int_{-\tau}^{0} e_\tau^{A(t_f-s-2\tau)}Bu_0(s)\,\mathrm{d}s,$$

and $\Phi_j(t_f) = [\Phi_{j,1}(t_f), \dots, \Phi_{j,l}(t_f)]^{\mathrm{T}}$, where

$$\Phi_{j,r}(t_f) = \int_{(j-2)\tau}^{t_f-2\tau} \frac{(s-(j-2)\tau)^{j-1}}{(j-1)!}\tilde{u}_r^*(t_f-s-2\tau)\,\mathrm{d}s$$

with $j \in \overline{k-1}$, $r \in \bar{l}$. Thus, we obtain

$$\eta = \sum_{j=0}^{k-2} A^j B\Phi_{j+1}(t_f) = \sum_{j=0}^{k-2} A^j B\big[\Phi_{j+1,1}(t_f), \dots, \Phi_{j+1,l}(t_f)\big]^{\mathrm{T}}. \tag{7}$$

Rewrite (7) to yield

$$\big[B, AB, \dots, A^{k-2}B\big]\Phi(t_f) = \eta, \tag{8}$$

where

$$\Phi(t_f) = \big[\Phi_{1,1}(t_f), \dots, \Phi_{1,l}(t_f), \dots, \Phi_{k-1,1}(t_f), \dots, \Phi_{k-1,l}(t_f)\big]^{\mathrm{T}}.$$

Namely, (8) is n equations with $(k-1)l$ variables. The controllability of (4) is equivalent to that (8) always has a solution for any η. Thus, $[B, \dots, A^{k-2}B]$ must be row full rank. If $k-1 < n$, $[B, \dots, A^{k-2}B]$ is not row full rank, then (8) might have no solution for some η. Thus, it must be $k-1 \geqslant n$. If $t_f = (k-1)\tau = n\tau$, it follows from $\Phi_{k-1}(t_f) = \theta$ that $[B, \dots, A^{k-3}B]$ is not row full rank. Thus, we need $t_f > (k-1)\tau \geqslant n\tau$. Based on Cayley–Hamilton theorem, (8) can be further simplified as

$$B\hat{\Phi}_1(t_f) + AB\hat{\Phi}_2(t_f) + \cdots + A^{n-1}B\hat{\Phi}_n(t_f) = \eta,$$

where $\hat{\Phi}_j(t_f)$, $j \in \overline{n}$, is some function obtained by transforming the terms with A^i, $i > n-1$, in (8) to the linear combination of the ones with $I, A, \dots, A^{n-1}$. Thus, we obtain that $\operatorname{rank} Q = n$.

The sufficiency is analogous to [8], thus we omit it. □

Remark 1. The system in [8] is the one with delay in state variable and with scale control input. The result in [8] implies that system is controllable on $[0, t_f]$ if and only if the controllability matrix is full rank and terminal time satisfies $t_f \geqslant (n-1)\tau$. (In fact, the terminal time should satisfy $t_f > (n-1)\tau$. If $t_f = (n-1)\tau$, then the controllability matrix is rank deficient for that $\Psi_{k-1}(t_1) = 0$ in (18) of [8].)

System (4) generalizes [8] to the case with multiple inputs and with delay in both state and control input. From the proof of Theorem 1 we know that if $t_f \leqslant n\tau$, there could exist a terminal state that cannot be reached accurately at t_f for any control input. Namely, the multi-agent systems modeled by (2) with relative protocol (3) are not controllable on $[0, t_f]$ for the existence of input delay.

3.2 Interval structural controllability with input delay

To explore the relationship between interval controllability and the topology structure of multi-agent systems, in what follows, we consider the structural controllability of the multi-agent systems with input delay. For structural controllability of the system with commensurate delay in state and input, Sename [21] transforms corresponding system into a compact form $(A(\nabla), B(\nabla))$ leaning on the delay operator ∇. Whereas, Sename does not consider the case that the parameters of system matrices are dependent on one another.

Systems (4) is with delay in both state and control input in which the matrices A and B are inherited from the graph Laplacian matrix $\mathcal{L}$, rendering parameters of A and B are dependent on one another. For (4), the notion of interval structural controllability is defined below.

Definition 2. System (4) is called structurally controllable on $[0, t_f]$ if there is a numerical realization such that (4) is controllable on $[0, t_f]$.

For the interval structural controllability of (4), we will establish the graph-theoretic criterion and figure out the influence of delay on it. The parameterization technique is adopted to tackle the interval structural controllability for the parameters of A and B dependent on one another. We assume that A and B in (4) have p nonzero independent parameters, which are mapped to $\xi_1, \dots, \xi_p$ by a bijective mapping. Also, the following linear parameterizations are assumed to hold (see more in [11]):

$$A = \sum_{j=1}^{p} \rho_j \xi_j \phi_j, \qquad B = \sum_{j=1}^{p} \rho_j \xi_j \varphi_j,$$

where $\rho_j \in \mathbb{R}^n$, $\phi_j \in \mathbb{R}^{1\times n}$ and $\varphi_j \in \mathbb{R}^{1\times l}$ are the parameterized vectors, $j \in \overline{p}$.

Denote by $\mathcal{G}_F$ the flow structure of matrix pair (A, B), A and B the matrices in (4), with node set $\mathcal{V}_F = \{f_1, \dots, f_{n+l}\}$ and edge set $\mathcal{E}_F$, where there is an edge from node f_j to f_i if and only if the ijth entry of matrix $[A, B]$ is nonzero.

Lemma 1. (*See* [11].) *The pair (A, B) is irreducible if and only if the associated flow structure $\mathcal{G}_F$ has a spanning forest rooted at $f_{n+1}, \dots, f_{n+l}$.*

Denote the transfer matrix by $\Omega = [\eta_{ij}]$, where

$$\eta_{ij} = \begin{cases} \phi_i \rho_j, & i, j \in \overline{p}, \\ \varphi_i, & i \in \overline{p},\ j = p+1. \end{cases}$$

Further, define the associated transfer graph, an unweighted digraph, which is denoted by $\mathcal{G}_\Omega$, with node set $\mathcal{V}_\Omega = \{\omega_1, \dots, \omega_{p+1}\}$ and edge set $\mathcal{E}_\Omega$, where there is an edge from node ω_j to ω_i if and only if η_{ij} is nonzero.

Next, we establish an equivalence relation between the irreducibility of (A, B) and the transfer graph. In [11], it requires that the linearly parameterized pair (A, B) satisfies the binary assumption, namely, just 0 and 1 are allowed to appear in the parameterized vectors ρ_j, ϕ_j, and φ_j. Mehrabadi et al. [14] extend this result to the case that A is a symmetric one.

In our work, for the interaction topology of the multi-agent systems, a weighted directed graph symmetric property of the associated Laplacian matrix $\mathcal{L}$ is no longer maintained, rendering A no longer maintains symmetry, and each nonzero parameter of A appears in two different places of $[A, B]$ simultaneously. We will generalize the result of [14] to the case when A is a asymmetric one.

Lemma 2. *If the pair (A, B) is irreducible, then the associated transfer graph $\mathcal{G}_\Omega$ has a spanning tree rooted at ω_{p+1}.*

Proof. Denote the line graph of the flow structure $\mathcal{G}_F$ by $\mathcal{G}_L$ with node set $\mathcal{V}_L$ and edge set $\mathcal{E}_L$. If the flow structure $\mathcal{G}_F$ has an edge (f_i, f_j) with weighted parameter ξ_k, then there exists a corresponding node, denoted by ijk, in the line graph $\mathcal{G}_L$. If there exist an edge (f_i, f_j) with weighted parameter ξ_r and an edge (f_j, f_k) with weighted parameter ξ_s in $\mathcal{G}_F$, simultaneously, then there is an edge from node ijr to node jks in $\mathcal{G}_L$. Next, we introduce a partition $\mathfrak{D}$ about the nodes in $\mathcal{G}_L$ as follows. If the edges in $\mathcal{G}_F$ share a common weighted parameter, then the associated nodes in $\mathcal{G}_L$ are collected into a cell of the partition. Introduce the quotient graph $\mathcal{G}_L/\mathfrak{D}$ induced by the partition $\mathfrak{D}$ with node set $\mathcal{V}_D = \{d_1, \dots, d_p\}$ and edge set $\mathcal{E}_D$. The edge of the quotient graph is formed as follows. Denote that $\mathfrak{D} = \{D_1, \dots, D_p\}$. Assume that $d_i \in \mathcal{V}_D$ corresponds to the cell D_i, and $d_j \in \mathcal{V}_D$ corresponds to D_j. If there exists at least one edge from a node in D_i to a node in D_j in the line graph $\mathcal{G}_L$, then there is an edge from d_i to d_j in the quotient graph $\mathcal{G}_L/\mathfrak{D}$. This implies that there exists a node f_k such that an edge with weight ξ_i arrives at f_k, and an edge with weight ξ_j leaves f_k in the flow structure $\mathcal{G}_F$.

Based on the structure of A and B, we have that

$$\rho_j^{(r)} = 1, \qquad \phi_j^{(j)} = 1, \qquad \phi_j^{(s)} = -1,$$

where $\rho_j^{(r)} = 1$ implies that there is an edge with weight ξ_j arriving at f_r, $\phi_j^{(s)} = -1$ implies that there is an edge with weight ξ_j leaving f_s in the flow structure $\mathcal{G}_F$, and $\phi_j^{(j)} = 1$ implies that the same parameter ξ_j appears in the main diagonal of A. Thus, if there is an edge from ω_i to ω_j in the transfer graph $\mathcal{G}_\Omega$, we have

$$\eta_{ji} = \phi_j \rho_i = \sum_{k=1}^{n} \phi_j^{(k)} \rho_i^{(k)} \neq 0.$$

Thus, there must exist a $k \in \overline{n}$ such that there are an edge with weight ξ_i arriving at f_k and an edge with weight ξ_j leaving f_k in the flow structure $\mathcal{G}_F$. Denote by $\tilde{\mathcal{G}}_\Omega$ the induced subgraph by $\{\omega_1, \dots, \omega_p\}$ in $\mathcal{G}_\Omega$. Thus, $\tilde{\mathcal{G}}_\Omega$ is isomorphic with the quotient graph $\mathcal{G}_L/\mathfrak{D}$ because there exists a bijective between d_i and ω_i, $i \in \overline{p}$.

If the linearly parameterized pair (A, B) is irreducible, from Lemma 1 we know that the flow structure $\mathcal{G}_F$ has a spanning forest rooted at $f_{n+1}, \dots, f_{n+l}$. Thus, the quotient graph $\mathcal{G}_L/\mathfrak{D}$ has a spanning forest rooted at $\{d_j, \varphi_j \neq 0\}$. Further, we obtain that $\tilde{\mathcal{G}}_\Omega$ has a spanning forest rooted at ω_js because of the isomorphism between $\tilde{\mathcal{G}}_\Omega$ and $\mathcal{G}_L/\mathfrak{D}$, where js are the indices of the roots of the spanning forest in $\mathcal{G}_L/\mathfrak{D}$. For $\varphi_j \neq 0$, there exist edges from ω_{p+1} to ω_j. Thus, there is spanning tree rooted at ω_{p+1}. This ends the proof. □

For positive integers r and $j_1, \dots, j_r$, we assume that $j_1 < j_2 < \cdots < j_r$, and we define the matrices as follows:

$$\Lambda_r = [\rho_{j_1}, \dots, \rho_{j_r}], \qquad \Xi_r = \begin{bmatrix} \phi_{j_1}^{\mathrm{T}}, \phi_{j_2}^{\mathrm{T}}, \cdots, \phi_{j_r}^{\mathrm{T}} \\ \varphi_{j_1}^{\mathrm{T}}, \varphi_{j_2}^{\mathrm{T}}, \cdots, \varphi_{j_r}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}.$$

For the structural controllability of (A, B), we have the following lemma.

Lemma 3. (*See* [11].) *A linearly parameterized pair* (A, B) *is structurally controllable if and only if*

$$\min_{\overline{r} \subset \overline{p}} (\operatorname{rank} \Lambda_r + \operatorname{rank} \Xi_{p-r}) = n$$

and the transfer graph $\mathcal{G}_\Omega$ *has a spanning tree rooted at* ω_{p+1}.

For the interval structural controllability of (4), we have the following theorem.

Theorem 2. *System* (4) *is structurally controllable on* $[0, t_f]$ *if and only if* $t_f > n\tau$ *and* $\mathcal{G}_F$ *is input reachable.*

Proof. Sufficiency. Assume that $\mathcal{G}_F$ is input reachable, then there exist at least n edges consisting multiple paths such that for every state vertex, there is a corresponding path beginning with one of $v_{n+1}, \dots, v_{n+l}$ and ending with it. Assume that there exist exactly n edges. Then from [14] we have that ρ_js, $j \in \overline{p}$, are independent on one another because each state vertex has one and only one ingoing edge. Denote that $\chi_j = [\phi_j, \varphi_j]$, then

$\chi_j^{(k)} = -1$ implies that there is an edge with weighted parameter ξ_j leaving vertex f_k in $\mathcal{G}_F$, and $\chi_j^{(j)} = 1$ implies that the same parameter ξ_j appearing in the main diagonal of A. Thus, χ_js, $j \in \overline{p}$, are independent on one another. If we assume that $\operatorname{rank} \Lambda_r = q$, then $\operatorname{rank} \Xi_{p-r} = n - q$, where q is the cardinality of the set $\overline{r}$. Namely, $\min_{\overline{r} \subset \overline{p}}(\operatorname{rank} \Lambda_r + \operatorname{rank} \Xi_{p-r}) = n$. If there are more than n edges, then we can find a subgraph, which is consisted of exactly n edges and keeps input reachability.

On the other hand, the input reachability implies that the pair (A, B) is irreducible. Then from Lemma 1 we know the transfer graph $\mathcal{G}_\Omega$ has a spanning tree rooted at ω_{p+1}. It finally follows from Lemma 3 that system (A, B) is structurally controllable. Thus, there is a numerical realization $(\tilde{A}, \tilde{B})$ such that

$$\operatorname{rank}\bigl[\tilde{B}, \tilde{A}\tilde{B}, \dots, \tilde{A}^{n-1}\tilde{B}\bigr] = n.$$

From Theorem 1 we obtain that if terminal time satisfies $t_f > n\tau$, (4) is controllable on $[0, t_f]$. Namely, system (4) is structurally controllable on $[0, t_f]$.

Necessity. Suppose that (4) is structurally controllable on $[0, t_f]$, whereas $\mathcal{G}_F$ is input unreachable. Then it follows from [9] that (A, B) is reducible. Thus, there is a permutation P such that

$$PAP^{-1} = \begin{bmatrix} A_{11} & \Theta \\ A_{21} & A_{22} \end{bmatrix}, \qquad PB = \begin{bmatrix} \Theta \\ B_{22} \end{bmatrix}, \tag{9}$$

where $A_{11} \in \mathbb{R}^{s \times s}$, $A_{21} \in \mathbb{R}^{(n-s) \times s}$, $A_{22} \in \mathbb{R}^{(n-s) \times (n-s)}$, and $B_{22} \in \mathbb{R}^{(n-s) \times l}$. Thus, we have

$$\begin{aligned} &\operatorname{rank}\bigl[B, AB, \dots, A^{n-1}B\bigr] \\ &\quad = \operatorname{rank}\bigl[PB, PAB, \dots, (PAP^{-1})^{n-1}PB\bigr] \\ &\quad = \operatorname{rank} \begin{bmatrix} \Theta & \Theta & \Theta & \cdots & \Theta \\ B_{22} & A_{22}B_{22} & * & \cdots & * \end{bmatrix} \\ &\quad < n, \end{aligned}$$

where $*$ represents zero or nonzero block matrix. From Theorem 1 we know that system (4) cannot be structurally controllable on $[0, t_f]$. This contradicts with the assumption, and we complete the proof. □

In [14], for structural controllability of the multi-agent systems without delay, it requires that the interaction topology modeled by an undirected graph is connected. In our work, the interaction topology is modeled by a directed graph and suffers from time delay. Thus, for the structural controllability of (4) on $[0, t_f]$, it requires that the leaders have directed paths arriving at all followers, and the terminal time satisfies $t_f > n\tau$.

Example 1. Consider the multi-agent systems consisted of 4 agents with interaction topology in Fig. 1 (without cycles), where each node is indexed from v_1 to v_4. The agent v_4 is selected as leader, which is manipulated by some external input, and the others are followers. Assume that the delay $\tau = 1$ exists in the control input of agent, and the dynamics of each agent is (2).

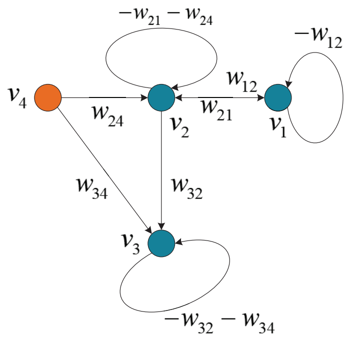

Figure 1. The flow structure $\mathcal{G}_F$ of (10).

Under relative protocol (3), we obtain

$$\begin{aligned}\dot{x}_1(t) &= w_{12}\big(x_2(t-1) - x_1(t-1)\big),\\ \dot{x}_2(t) &= w_{21}\big(x_1(t-1) - x_2(t-1)\big) + w_{24}\big(x_4(t-1) - x_2(t-1)\big),\\ \dot{x}_3(t) &= w_{32}\big(x_2(t-1) - x_3(t-1)\big) + w_{34}\big(x_4(t-1) - x_3(t-1)\big).\end{aligned} \tag{10}$$

As shown in Fig. 1, $\mathcal{G}_F$ is input reachable. From Theorem 2, if the terminal time satisfies $t_f > 3\tau$, then (10) is structurally controllable on $[0, t_f]$. The flow structure of system (10) is shown in Fig. 1. In what follows, we illustrate the difference between interval structural controllability and structural controllability.

Denote by the initial function $\varphi(t) = [\varphi_1(t), \varphi_2(t), \varphi_3(t)]^{\mathrm{T}}$ and terminal state $x_f = [x_{f_1}, x_{f_2}, x_{f_3}]^{\mathrm{T}}$, where $\varphi_j(t)$ is a scale function, and $x_{f_j} \in \mathbb{R}$, $j = 1, 2, 3$. Choose that $w_{12} = 2$, $w_{21} = 5$, $w_{24} = 3$, $w_{32} = 1$, $w_{34} = 4$, then we have

$$A = \begin{bmatrix} -2 & 2 & 0 \\ 5 & -8 & 0 \\ 0 & 1 & -5 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 3 \\ 4, \end{bmatrix}, \tag{11}$$

and

$$\operatorname{rank}\big[B, AB, A^2B\big] = 3. \tag{12}$$

If we select the terminal time $t_f \leqslant 3\tau$, for instance, $t_f = 2\tau$, then from (6) we can choose x_f such that

$$\theta \neq \xi = [\tilde{\xi}_1, \tilde{\xi}_2, \tilde{\xi}_3]^{\mathrm{T}},$$

where

$$\begin{aligned}\tilde{\xi}_1 = {} & 6\varphi_2(-1) - 4\varphi_1(-1) - \int_{-1}^{0} \big(-1 + 2s + 7s^2\big)\varphi_1'(s)\,\mathrm{d}s \\ & + x_{f_1} - \int_{-1}^{0} \big(2 - s - 10s^2\big)\varphi_2'(s)\,\mathrm{d}s + 6\int_{-1}^{0} s u_0(s)\,\mathrm{d}s,\end{aligned}$$

$$\tilde{\xi}_2 = x_{f_2} + 15\varphi_1(-1) - \int_{-1}^{0} \big(5 - 5s - 25s^2\big)\varphi_1'(s)\,\mathrm{d}s$$
$$- 22\varphi_2(-1) - \int_{-1}^{0} \big(-7 + 8s + 37s^2\big)\varphi_2'(s)\,\mathrm{d}s - \int_{-1}^{0} (3 + 24s)u_0(s)\,\mathrm{d}s,$$

and

$$\tilde{\xi}_3 = x_{f_3} - 2.5\varphi_1(-1) + 4.5\varphi_2(-1) - 2.5\int_{-1}^{0} s^2\varphi_1'(s)\,\mathrm{d}s$$
$$- 3.5\varphi_3(-1) - \int_{-1}^{0} \big(1 - s - 6.5s^2\big)\varphi_2'(s)\,\mathrm{d}s$$
$$- \int_{-1}^{0} \big(-4 + 5s + 12.5s^2\big)\varphi_3'(s)\,\mathrm{d}s - \int_{-1}^{0} (4 + 17s)u_0(s)\,\mathrm{d}s,$$

whereas

$$\int_{0}^{t_f-\tau} e_\tau^{A(t_f-s-2\tau)} Bu(s)\,\mathrm{d}s = \int_{0}^{1} e_\tau^{-As} Bu(s)\,\mathrm{d}s = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} \int_{0}^{1} u(s)\,\mathrm{d}s.$$

Thus, whatever the control input $u(t)$ is selected, the first component $\tilde{\xi}_1$ of ξ cannot be controlled. Namely, system (10) is not structurally controllable on $[0, t_f]$ with $t_f = 2\tau$. However, it follows from (12) that (A, B) is structurally controllable on $[0, t_f]$, thus, it is structurally controllable on any time interval.

If fact, if we select $t_f = 4\tau$, then from

$$G(t_f) = \int_{0}^{t_f-\tau} e_\tau^{A(t_f-s-2\tau)} BB^{\mathrm{T}} e_\tau^{A^{\mathrm{T}}(t_f-s-2\tau)}\,\mathrm{d}s$$
$$= \int_{-1}^{0} BB^{\mathrm{T}}\,\mathrm{d}s + \int_{0}^{1} (I + As)BB^{\mathrm{T}}(I + As)^{\mathrm{T}}\,\mathrm{d}s$$
$$+ \int_{1}^{2} \big(I + As + 0.5A^2(s-\tau)^2\big)BB^{\mathrm{T}}\big(I + As + 0.5A^2(s-\tau)^2\big)^{\mathrm{T}}\,\mathrm{d}s$$
$$= \begin{bmatrix} 66 & -471/2 & -171/4 \\ -471/2 & 4266/5 & 3157/20 \\ -171/4 & 3157/20 & 3137/15 \end{bmatrix}$$

we have $\operatorname{rank}(G(t_f)) = 3$. Construct the control input as

$$u(t) = B^{\mathrm{T}} e_{\tau}^{A^{\mathrm{T}}(t_f - s - 2\tau)} \big(G(t_f)\big)^{-1} \xi,$$

Thus, for the system (10) with values taken in (11), any given terminal state x_f can be arrived at $t_f = 4\tau$. Namely, system (10) is structurally controllable on $[0, t_f]$ with $t_f = 4\tau$.

4 Controllability of the first-order systems with communication delay

Next, we suppose that the dynamics of each agent follows

$$\dot{x}_i(t) = u_i(t), \quad i \in \overline{n + l}, \tag{13}$$

and the communications among agents suffer from time delay $\tau > 0$. Communication network among agents is established by the following neighbor-based protocol:

$$u_i(t) = \gamma_i x_i(t) + \sum_{v_j \in \mathcal{N}_i} w_{ij} x_j(t - \tau), \quad i \in \overline{n}, \tag{14}$$

where γ_i is an indeterminate constant, and w_{ij} is the same definition with the one in Section 3.1. Denote that $A = [a_{ij}] \in \mathbb{R}^{n \times n}$, where $a_{ij} = w_{ij}$ for $i \in \overline{n}$, $j \in \overline{n}$, and $v_j \in \mathcal{N}_i$ or zero otherwise. Similarly, $B = [b_{ij}] \in \mathbb{R}^{n \times l}$ with $b_{ij} = w_{i,\, j+n}$ for $i \in \overline{n}$, $j \in \overline{l}$, and $v_{j+n} \in \mathcal{N}_i$ or zero otherwise. The matrices A and B are inherited from the adjacent matrix of digraph $\mathcal{G}$, where A captures the information flows among followers, and B captures information flows from leaders to followers, respectively.

Under (14), the multi-agent systems modeled by (13) are converted into

$$\dot{x}(t) = Dx(t) + Ax(t - \tau) + Bu(t - \tau), \tag{15}$$

where $D = \operatorname{diag}\{\gamma_1, \ldots, \gamma_n\}$.

We denote (15) by (D, A, B). If (15) is controllable, we say the multi-agent systems modeled by (13) with protocol (14) are controllable.

4.1 Interval controllability with communication delay

In what follows, we present the interval controllability of (15) in the sense of Definition 1. For the solution of (15), we have the following lemma.

Lemma 4. *For any initial function $\varphi(t)$ and initial control input $u_0(t)$, $t \in [-\tau, 0]$, solution of* (15) *has the following form*:

$$x(t) = \phi(t, 0, \varphi, u_0) + \int_{-\tau}^{0} R(s + \tau,\, t) B u_0(s)\, \mathrm{d}s + \int_{0}^{t-\tau} R(s + \tau,\, t) B u(s)\, \mathrm{d}s,$$

where $\phi(t,0,\varphi,u_0)$ *is the solution of force-free equation corresponding to* (15) *with initial function* $\varphi(t)$ *on* $[-\tau,0]$, *and* $R(s,t)$ *is the solution of the following matrix equation*:

$$\begin{aligned} \frac{\partial R(s,t)}{\partial s} &= \begin{cases} -R(s,t)D - R(s+\tau,t)A, & 0 \leqslant s \leqslant t-\tau, \\ -R(s,t)D, & t-\tau \leqslant s \leqslant t, \end{cases} \\ R(s,t) &= \begin{cases} I, & s=t, \\ \Theta, & s>t. \end{cases} \end{aligned} \tag{16}$$

Proof. The proof is similar to that of [1, Thm. 10.3], thus we omit it. □

For the interval controllability of (15), we have the following lemma.

Lemma 5. *System* (15) *is controllable on* $[0,t_f]$ *if and only if* $\operatorname{rank}\tilde{Q}=n$, *where*

$$\tilde{Q} = \int_0^{t_f-\tau} R(s+\tau,\, t_f)BB^{\mathrm{T}}R^{\mathrm{T}}(s+\tau,\, t_f)\,\mathrm{d}s.$$

Proof. From Lemma 4 and an analogous process of [29] we obtain the result. □

With referring to [6], introduce the following matrix sequence:

$$\begin{aligned} Q_1^1 &= B, \\ Q_j^{k+1} &= DQ_j^k + AQ_{j-1}^k, \quad j \in \overline{k},\ k \in \overline{n}, \end{aligned} \tag{17}$$

and $Q_j^k = \Theta$ for $j=0$ or $j>k$.

From (16) we know that for $t-\tau \leqslant s \leqslant t$, it holds

$$\frac{\partial^{q-1}R(s,t)}{\partial s^{q-1}}B = (-1)^{q-1}R(s,t)Q_1^q,$$

and for $0 \leqslant s \leqslant t-\tau$, it has

$$\frac{\partial^{q-1}R(s,t)}{\partial s^{q-1}}B = (-1)^{q-1}\sum_{j=1}^{q}R(s+(j-1)\tau,t)Q_j^q,$$

where $q \in \mathbb{N}^+$. From [6] introduce the following matrix:

$$\hat{Q} = \left[Q_1^1,\dots,Q_1^n,Q_2^2,\dots,Q_2^n,\dots,Q_n^n\right].$$

For the interval controllability of (15), we have the following lemma.

Theorem 3. *If* $t_f > n\tau$ *and* $\operatorname{rank}\hat{Q}=n$, *then* (15) *is controllable on* $[0,t_f]$.

Proof. The proof is analogous to [29]. Thus we omit it. □

Remark 2. From [30] we know that for system (1) with control input taking value in $\mathbb{R}^l$, controllability on some time interval is equivalent to that on any time interval. Whereas, from Theorems 1 and 3 we know that for the system with delay in both state and control input, it is different for the controllability on some time interval and on any one.

4.2 Interval structural controllability with communication delay

Next, we present the interval structural controllability of (15) in the sense of Definition 2. Denote the flow structure of matrix $D + A$ by $\mathcal{G}(D + A) = (\mathcal{V}_s, \mathcal{E}_s)$, where $\mathcal{V}_s = \{s_1, \dots, s_n\}$ is the state vertex set, and $\mathcal{E}_s$ is the edge set with an edge $e_{ji} = (s_i, s_j)$ in $\mathcal{E}_s$ if and only if the jith entry of matrix $D + A$ being nonzero. Further, denote flow structure of the pair $(D + A, B)$ by $\mathcal{G}_A = \mathcal{G}(D + A,\ B) = (\mathcal{V}_c, \mathcal{E}_c)$, where $\mathcal{V}_c = \mathcal{V}_s \cup \mathcal{V}_t$ with $\mathcal{V}_t = \{c_1, \dots, c_l\}$ the input vertex set, and $\mathcal{E}_c = \mathcal{E}_s \cup \mathcal{E}_t$ with $\mathcal{E}_t = \{e_{ij} = (c_j, s_i)\colon c_j \in \mathcal{V}_t,\ s_i \in \mathcal{V}_s,\ b_{ij} \neq 0\}$ the edge set.

For the interval structural controllability of (15), we have the following theorem.

Theorem 4. *System* (15) *is structurally controllable on* $[0, t_f]$ *if and only if* $t_f > n\tau$ *and* $\mathcal{G}_A$ *is input reachable.*

Proof. Sufficiency. Suppose $\mathcal{G}_A$ is input reachable. From [13] we know that $(D + A,\ B)$ is irreducible. From the definition of $\mathcal{G}(D + A)$ we know that $\mathcal{G}(D + A)$ is in fact the union of $\mathcal{G}(D)$ and $\mathcal{G}(A)$. Similarly, $\mathcal{G}_A$ is the union of $\mathcal{G}(D)$ and $\mathcal{G}(A, B)$. Thus, there is a maximum matching consisted of a disjoint cycle family in the digraph $\mathcal{G}_A$, which covers all the vertices in $\mathcal{G}_A$. Thus, the cardinality of the maximum matching is n, and the matrix $[D + A,\ B]$ has generally full rank. From [22] we obtain that $(D + A,\ B)$ is structurally controllable. Namely, there exists a numerical realization $(\tilde{D} + \tilde{A}, \tilde{B})$ such that

$$\operatorname{rank}\bigl[\tilde{B}, (\tilde{D} + \tilde{A})\tilde{B}, \dots, (\tilde{D} + \tilde{A})^{n-1}\tilde{B}\bigr] = n. \tag{18}$$

From [6] we obtain that

$$\begin{aligned}
&\operatorname{rank}\bigl[Q_1^1, Q_1^2, \dots, Q_1^n, Q_2^2, \dots, Q_2^n, Q_3^3, \dots, Q_3^n, \dots, Q_n^n\bigr] \\
&\quad = \operatorname{rank}\bigl[Q_1^1, Q_1^2 + Q_2^2, \dots, Q_1^n + Q_2^n + \dots + Q_n^n, \\
&\qquad\qquad Q_2^2, \dots, Q_2^n, Q_3^3, \dots, Q_3^n, \dots, Q_n^n\bigr] \\
&\quad = \operatorname{rank}\bigl[\tilde{B}, (\tilde{D} + \tilde{A})\tilde{B}, \dots, (\tilde{D} + \tilde{A})^{n-1}\tilde{B}, \\
&\qquad\qquad Q_2^2, \dots, Q_2^n, Q_3^3, \dots, Q_3^n, \dots, Q_n^n\bigr] \\
&\quad = n.
\end{aligned}$$

It finally follows from $t_f > n\tau$ and Theorem 3 that system (15) is controllable on $[0, t_f]$. Thus, system (15) is structurally controllable on $[0, t_f]$.

Necessity. Suppose that (15) is structurally controllable on $[0, t_f]$, whereas $\mathcal{G}_A$ is input unreachable, then there is a permutation P such that (9) and the following equation hold:

$$PDP^{-1} = \begin{bmatrix} D_{11} & \Theta \\ \Theta & D_{22} \end{bmatrix},$$

where $D_{11} \in \mathbb{R}^{s \times s}$ and $D_{22} \in \mathbb{R}^{(n-s)\times(n-s)}$. From (17) we have

$$PQ_1^1 = \begin{bmatrix} \Theta \\ B_{22} \end{bmatrix}, \quad PQ_1^2 = \begin{bmatrix} \Theta \\ D_{22}B_{22} \end{bmatrix}, \quad \dots, \quad PQ_1^n = \begin{bmatrix} \Theta \\ D_{22}^{n-1}B_{22} \end{bmatrix}$$

and

$$PQ_2^2 = \begin{bmatrix} \Theta \\ A_{22}B_{22} \end{bmatrix}, \quad PQ_2^3 = \begin{bmatrix} \Theta \\ (D_{22}A_{22} + A_{22}D_{22})B_{22} \end{bmatrix}, \quad \dots,$$
$$PQ_2^n = \begin{bmatrix} \Theta \\ (D_{22}^{n-2}A_{22} + A_{22}D_2^{n-2})B_{22} \end{bmatrix},$$

where $A_{11} \in \mathbb{R}^{s\times s}$, $A_{22} \in \mathbb{R}^{(n-s)\times(n-s)}$, and $B_{22} \in \mathbb{R}^{(n-s)\times l}$. Following the analogous process, we can obtain that

$$PQ_3^3 = \begin{bmatrix} \Theta \\ A_{22}^2 B_{22} \end{bmatrix}, \quad \dots, \quad PQ_n^n = \begin{bmatrix} \Theta \\ A_{22}^{n-1} B_{22} \end{bmatrix}.$$

Thus, we obtain that

$$\begin{aligned} &\operatorname{rank}\left[Q_1^1, \dots, Q_1^n, Q_2^2, \dots, Q_2^n, \dots, Q_n^n\right] \\ &\quad = \operatorname{rank}\left[PQ_1^1, \dots, PQ_1^n, PQ_2^2, \dots, PQ_2^n, \dots, PQ_n^n\right] \\ &\quad = \operatorname{rank}\begin{bmatrix} \Theta & \cdots & \Theta & \Theta & \cdots & \Theta \\ B_{22} & \cdots & D_{22}^{n-1}B_{22} & A_{22}B_{22} & \cdots & A_{22}^{n-1}B_{22} \end{bmatrix} \\ &\quad < n. \end{aligned}$$

From Theorem 3 we know that system (15) cannot be structurally controllable on $[0, t_f]$. This contradicts with the assumption, and we complete the proof. □

Remark 3. For system (1), if it is structurally controllable on some time interval, it must be structurally controllable on any time interval. However, from Theorems 2 and 4 we know that for the systems with delay in both state and control input, structural controllability on some time interval does not imply that on any time interval.

From Theorems 2 and 4 we know that the criterion for interval structural controllability with input delay and the one with communication delay are the same. Next, we generalize the result to high-order multi-agent systems.

5 Controllability of high-order systems with communication delay

For high-order multi-agent systems without delay, Alireza [17] and Guan [4] deal with the structural controllability with graph-theoretic approach, respectively. More information can be found in [5]. In what follows, we will present the interval structural controllability of high-order multi-agent systems with delay in the interaction topology. Assume that the interaction topology, the indices of agents and the selection of leaders are the same with the ones in Section 3. Dynamics of each agent obeys the following high-order rule:

$$\dot{x}_i^{(1)}(t) = x_i^{(2)}(t), \quad \dot{x}_i^{(2)}(t) = x_i^{(3)}(t), \quad \dots, \quad \dot{x}_i^{(m)}(t) = u_i(t), \tag{19}$$

where $x_i^{(1)} \in \mathbb{R}$ is the state of the ith agent, and $x_i^{(j)} \in \mathbb{R}$ is the $(j-1)$th derivative of $x_i^{(1)}$, $i \in \overline{n+l}$, $j \in \overline{m} - \{1\}$. Construct the following neighbor-based protocol:

$$u_i(t) = \sum_{k=0}^{m-1} \beta_k \biggl(\gamma_i x_i^{k+1}(t) + \sum_{v_j \in \mathcal{N}_i} w_{ij} x_j^{(k+1)}(t-\tau) \biggr), \tag{20}$$

where β_k and γ_i are indeterminate parameters, and w_{ij} is the same definition with the one in (3), $i \in \overline{n}$, $k \in \{0\} \bigcup \overline{m-1}$. Under (20), system (19) is converted into the form

$$\dot{x}(t) = \hat{D}x(t) + \hat{A}x(t-\tau) + \hat{B}u(t-\tau), \tag{21}$$

where $x = [x^{[1]}, \dots, x^{[m]}]$ with $x^{[k]} = [x_1^{(k)}, \dots, x_n^{(k)}]$, $k \in \overline{m}$, $u = [u^{[1]}, \dots, u^{[m]}]$ with $u^{[r]} = [x_{n+1}^{(r)}, \dots, x_{n+l}^{(r)}]$, $r \in \overline{m}$,

$$\hat{D} = \begin{bmatrix} \theta_{(m-1)n \times n}, I_{m-1} \otimes I_n \\ \beta^{\mathrm{T}} \otimes D \end{bmatrix}$$

with $\beta = [\beta_0, \dots, \beta_{m-1}]$, and

$$\hat{A} = \begin{bmatrix} \Theta_{(m-1) \times m} \\ \beta^{\mathrm{T}} \end{bmatrix} \otimes A, \hat{B} = \begin{bmatrix} \Theta_{(m-1) \times m} \\ \beta^{\mathrm{T}} \end{bmatrix} \otimes B,$$

D, A and B the same definition with the ones in (15).

For the interval controllability of (21), we have the following lemma.

Lemma 6. *If $t_f > mn\tau$ and $(D + A, B)$ is controllable on $[0, t_f]$, then (21) is controllable on $[0, t_f]$.*

Proof. The proof is analogous to [5]. Thus we omit it. □

Next, we present the interval structural controllability of (21).

Theorem 5. *System (21) is structurally controllable on $[0, t_f]$ if and only if $t_f > mn\tau$ and $\mathcal{G}_A$ is input reachable.*

Proof. Sufficiency. Suppose that $\mathcal{G}_A$ is input reachable, then it follows from the proof of Theorem 4 that system $(D + A, B)$ is structurally controllable. Namely, there exists a numerical realization $(\tilde{D} + \tilde{A}, \tilde{B})$ of $(D + A, B)$ such that (18) holds. Thus, if $t_f > mn\tau$, from Lemma 6 we know that system (21) is structurally controllable on $[0, t_f]$.

The proof of necessity is analogous to that of Theorem 4, thus we omit it. □

6 Conclusion

Controllability is considered for the multi-agent systems with input and communication delays. For the first-order systems, interval controllability and structural controllability are considered, respectively. Results imply that both interval controllability and structural controllability of the delay systems have significant different with the ones of nondelay systems. Corresponding conclusions are generalized to the high-order multi-agent systems.

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