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Existence of a solution for a nonlinear integral equation by nonlinear contractions involving simulation function in partially ordered metric space*

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Abstract. In a recent paper, Khojasteh et al. presented a new collection of simulation functions, said Z-contraction. This form of contraction generalizes the Banach contraction and makes different types of nonlinear contractions. In this article, we discuss a pair of nonlinear operators that applies to a nonlinear contraction including a simulation function in a partially ordered metric space. For this pair of operators with and without continuity, we derive some results about the coincidence and unique common fixed point. In the following, many known and dependent consequences in fixed point theory in a partially ordered metric space are deduced. As well, we furnish two interesting examples to explain our main consequences, so that one of them does not apply to the principle of Banach contraction. Finally, we use our consequences to create a solution for a particular type of nonlinear integral equation.

Keywords: simulation functions, coincidence point, compatible, partially ordered metric space, integral equation.

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1 Introduction

Fixed point theory is a clear subject, which affords beneficial techniques and senses for dealing with diverse problems. Particularly, we mention the being of solutions of mathematical questions diminishable to equivalent fixed point problems. Thus, we remember that the Banach contraction principle [5] is based on this theory. Nevertheless, the fixed point theory has been able to attract many researchers. In 2018, Vetro [31] proved the existence and uniqueness of a fixed point in the setting of ordered metric spaces by introducing the notion of ordered S-G-contraction. Hoc and his colleagues [13] provided some new fixed point theorems in compact metric space. In 2022, Kim [18] studied the existence of a coupled fixed point in Hilbert space. Also, Gautam et al. [11] introduced the notion of interpolative Matkowski-type contraction, and they obtained the solution for the nonlinear matrix equations. Therefore, there are many achievements for enthusiasts, look, for example, [6, 8–10, 19–22, 28, 29, 33].

Recently, lots of conclusions became apparent linked to fixed point theorems in an ordered metric space. Run and Reurings [27] expressed the first conclusion in this orientation, where they expanded the Banach contraction principle in metric space equipped with a partial order. Subsequently, Nieto and Rodríguez-López [24] generalized the previous results and used them to find a unique solution for a specific type of ordinary differential equation. More progress in the above-argued results is detected in [2, 3, 12, 23, 25, 30].

Recently, the concept of simulation function was introduced and studied by Khojasteh et al. [17]. By using the simulation functions Vetro [32] investigated the existence of a common fixed point and coincidence point in both metric space and partial metric space. In this article, we presume a pair of nonlinear operators satisfying in nonlinear contractions including a simulation function in a metric space with a partial order. We generalize some results Khojaste et al. [17] to obtain coincidence and common fixed point results for this pair of operators with and without continuity. Also, we process two interesting examples to explain our main results, so that one of them does not apply to the principle of Banach contraction. Then we exploit our achievements to create a solution for a particular type of nonlinear integral equation.

2 Preliminaries

The following definition was given by Argoubi et al. [4].

Definition 1. Let (X, d) be a metric space, and let $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ satisfies the following conditions:

(ζ_1) $\zeta(p, q) < q - p$ for all $p, q > 0$;
(ζ_2) If $\{p_n\}$ and $\{q_n\}$ are sequences in $(0, \infty)$ such that $\lim_{n\to\infty} p_n = \lim_{n\to\infty} q_n = l > 0$, then

$$\limsup_{n\to\infty} \zeta(p_n, q_n) < 0.$$

Then ζ is a simulation function.

Remark 1. Initially, Khojasteh et al. [17] defined the simulation function as a mapping $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ satisfying $\zeta(0,0) = 0$ and conditions (ζ_1) and (ζ_2) of Definition 1. In the following, we will use the modified definition by Argoubi et al. [4].

Before starting the main results of this research, we render many examples that highlight their possible applicability to the field of fixed point theory.

Example 1. (See [17].) Let $\zeta_i : [0,\infty) \times [0,\infty) \to \mathbb{R}$, $i = 1, 2, \dots, 6$, be defined by

(i) $\zeta_1(p,q) = \psi(q) - \phi(p)$ for all $p, q \in [0,\infty)$, where $\psi, \phi : [0,\infty) \to [0,\infty)$ are two continuous functions such that $\psi(t) = \phi(t) = 0$ if only if $t = 0$ and $\psi(t) < t \leqslant \phi(t)$ for all $t > 0$.

(ii) $\zeta_2(p,q) = \alpha q - p$ for all $p, q \in [0,\infty)$ is a particular case of ζ_1 with $\phi(t) = t$ and $\psi(t) = \alpha t$ for all $t \geqslant 0$ and $\alpha \in [0,1)$.

(iii) $\zeta_3(p,q) = q - \varphi(q) - p$ for all $p, q \in [0,\infty)$, where $\varphi : [0,\infty) \to [0,\infty)$ is a lower semicontinuous function such that $\phi^{-1}(0) = \{0\}$.

(iv) $\zeta_4(p,q) = q\varphi(q) - p$ for all $p, q \in [0,\infty)$, where $\varphi : [0,\infty) \to [0,1)$ is a function such that $\limsup_{t\to r^+} \varphi(t) < 1$ for all $r > 0$.

(v) $\zeta_5(p,q) = q - (f(p,q)/g(p,q))p$ for all $p, q \in [0,\infty)$, where $f, g : [0,\infty) \to (0,\infty)$ are two continuous functions with respect to each variable such that $f(p,q) > g(p,q)$ for all $p, q > 0$.

(vi) $\zeta_6(p,q) = q - \int_0^p \phi(u)\,\mathrm{d}u$ for all $p, q \in [0,\infty)$, where $\phi : [0,\infty) \to [0,\infty)$ is a function such that $\int_0^\epsilon \phi(u)\,\mathrm{d}u$ exists, and $\int_0^\epsilon \phi(u)\,\mathrm{d}u > \epsilon$ for each $\epsilon > 0$.

Definition 2. Let (X,d) be a metric space and $S, T : X \to X$. If $v = Su = Tu$ for some u in X, then u is called a coincidence point of S and T.

Definition 3. (See [15].) Let (X,d) be a metric space. The mappings $S, T : X \to X$ are compatible if and only if for any sequence $\{u_n\}$ in X such that $\lim_{n\to\infty} Su_n = \lim_{n\to\infty} Tu_n$, $\lim_{n\to\infty} d(STu_n, TSu_n) = 0$.

Definition 4. (See [16].) Let (X,d) be a metric space. The mappings $S, T : X \to X$ are weakly compatible if and only if $Su = Tu$ for some $u \in X$ implies that $STu = TSu$ or S and T commute at their coincidence points.

If S and T are compatible, then S and T are weakly compatible.

Definition 5. (See [7].) Let $(X, \preccurlyeq)$ is a partially ordered set and $S, T : X \to X$. S is said to be T-nondecreasing if for $u, v \in X$,

$$Tu \preccurlyeq Tv \quad \Longrightarrow \quad Su \preccurlyeq Sv.$$

3 Main result

The following main theorem is a generalized coincidence point theorem for maps that are not necessarily continuous.

Theorem 1. *Let $(X, \preccurlyeq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is complete metric space. Suppose that there exist a simulation function ζ and $S, T : X \to X$ such that*

$$\zeta\bigl(d(Sx, Sy), d(Tx, Ty)\bigr) \geqslant 0 \quad \forall x, y \in X\colon Tx \preccurlyeq Ty, \tag{1}$$

and suppose the following hypotheses:

(i) *$SX \subseteq TX$ and TX is closed*;
(ii) *S is T-nondecreasing*;
(iii) *If $\{Tx_n\} \subset X$ is a nondecreasing sequence, which converges to Tu in TX, then $Tx_n \preccurlyeq Tu$ for all $n \geqslant 0$.*

If there exists $x_0 \in X$ such that $Tx_0 \preccurlyeq Sx_0$, then S and T have a coincidence point, that is, there exists $v \in X$ such that $Sv = Tv$.

Proof. Using the theorem condition, we have $x_0 \in X$ such that $Tx_0 \preccurlyeq Sx_0$. Since $SX \subseteq TX$, then there exists $x_1 \in X$ such that $Tx_1 = Sx_0$ and $Tx_0 \preccurlyeq Sx_0 = Tx_1$. Since S is T-nondecreasing, we have $Sx_0 \preccurlyeq Sx_1$. Continuing this process, we construct the sequence $\{x_n\}$ with the following conditions:

$$Sx_n = Tx_{n+1} \quad \forall n \geqslant 0, \tag{2}$$

and

$$\begin{aligned} Tx_0 \preccurlyeq Sx_0 = Tx_1 \preccurlyeq Sx_1 = Tx_2 \preccurlyeq Sx_2 \preccurlyeq \cdots \\ \preccurlyeq Sx_{n-1} = Tx_n \preccurlyeq Sx_n = Tx_{n+1} \preccurlyeq \cdots. \end{aligned} \tag{3}$$

If two consecutive members of the sequences $\{Sx_n\}$ or $\{Tx_n\}$ are equal, then the conclusion of the theorem follows. So we have

$$d(Sx_n, Sx_{n+1}) \neq 0, \quad d(Tx_n, Tx_{n+1}) \neq 0 \quad \forall n \geqslant 0. \tag{4}$$

If for some $n \in \mathbb{N}$, we assume that $d(Tx_{n-1}, Tx_n) < d(Tx_n, Tx_{n+1})$, then by property (ζ_1) of simulation function and (2)–(4) we have

$$\begin{aligned} 0 &\leqslant \zeta\bigl(d(Sx_{n-1}, Sx_n), d(Tx_{n-1}, Tx_n)\bigr) \\ &= \zeta\bigl(d(Tx_n, Tx_{n+1}), d(Tx_{n-1}, Tx_n)\bigr) \\ &< d(Tx_{n-1}, Tx_n) - d(Tx_n, Tx_{n+1}) < 0. \end{aligned}$$

This contradiction shows that

$$d(Tx_n, Tx_{n+1}) \leqslant d(Tx_{n-1}, Tx_n).$$

This implies that the sequence $\{d(Tx_{n-1}, Tx_n)\}$ is a monotone decreasing sequence of nonnegative real numbers, and consequently, there exists $r \geqslant 0$ such that the sequence $\{d(Tx_{n-1}, Tx_n)\}$ converges to r.

Suppose $r > 0$. By (3) we know that the elements Tx_n and Tx_{n+1} are comparable, so using property (ζ_2) of a simulation function with $p_n = d(Sx_n, Sx_{n+1})$ and $q_n = d(Sx_{n-1}, Sx_n)$, we have

$$\begin{aligned} 0 &\leqslant \limsup_{n\to\infty} \zeta\big(d(Sx_{n-1}, Sx_n), d(Tx_{n-1}, Tx_n)\big) \\ &= \limsup_{n\to\infty} \zeta\big(d(Tx_n, Tx_{n+1}), d(Tx_{n-1}, Tx_n)\big) < 0, \end{aligned}$$

which is a contradiction, and hence,

$$\lim_{n\to\infty} d(Tx_{n-1}, Tx_n) = 0.$$

The next step is to show that the sequence $\{Tx_n\}$ is Cauchy. By contradiction and by Lemma 2.1 of [14] there exist an $\epsilon > 0$ and $\{Tx_{m(k)}\}, \{Tx_{n(k)}\} \subset \{Tx_n\}$ with $n(k) > m(k) \geqslant k$ for all $k \in \mathbb{N}$ such that

$$\lim_{k\to\infty} d(Tx_{m(k)}, Tx_{n(k)}) = \lim_{k\to\infty} d(Tx_{m(k)+1}, Tx_{n(k)+1}) = \epsilon, \tag{5}$$

$$d(Tx_{m(k)}, Tx_{n(k)}) \geqslant \epsilon. \tag{6}$$

Then we can assume that

$$d(Tx_{m(k)+1}, Tx_{n(k)+1}) > 0 \quad \forall k \in \mathbb{N}. \tag{7}$$

Again, by (3) we know that the elements $Tx_{m(k)}$ and $Tx_{n(k)}$ are comparable, so using (5)–(7) and property (ζ_2) of a simulation function with $p_n = d(Tx_{m(k)+1}, Tx_{n(k)+1})$ and $q_n = d(Tx_{m(k)}, Tx_{n(k)})$, we have

$$\begin{aligned} 0 &\leqslant \limsup_{k\to\infty} \zeta\big(d(Sx_{m(k)}, Sx_{n(k)}), d(Tx_{m(k)}, Tx_{n(k)})\big) \\ &= \limsup_{k\to\infty} \zeta\big(d(Tx_{m(k)+1}, Tx_{n(k)+1}), d(Tx_{m(k)}, Tx_{n(k)})\big) < 0, \end{aligned}$$

which is a contradiction. We conclude that the sequence $\{Tx_n\}$ is a Cauchy sequence, and hence, $\{Tx_n\}$ is convergent in the complete metric space (X, d). TX is closed, therefore, by (2) there exists $u \in X$ such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = Tu. \tag{8}$$

From (3) and (8) we know that $\{Tx_n\}$ is a nondecreasing sequence in TX such that $Tx_n \to Tu$, then by condition (iii) and (4) we have

$$Tx_n \prec Tu. \tag{9}$$

Again, by (3), (4) and since S is T-nondecreasing, we have

$$Sx_n \prec Su. \tag{10}$$

Using property (ζ_1) of a simulation function, (9) and (10), we have

$$0 \leqslant \zeta\big(d(Sx_n, Su), d(Tx_n, Tu)\big) < d(Tx_n, Tu) - d(Sx_n, Su) \quad \forall n \in \mathbb{N}.$$

Taking $n \to \infty$ in the above inequality, we have $\lim_{n\to\infty} Sx_n = Su$. Then

$$Su = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = Tu. \tag{11}$$

This completes the proof. □

Now, we will prove the existence and uniqueness theorem of a common fixed point.

Theorem 2. *If in Theorem 1, it is additionally assumed that S and T are weakly compatible and $Tu \preccurlyeq TTu$, where u is a coincidence point of S and T, then S and T have a common fixed point in X. Moreover, if a set of fixed points of T is totally ordered, then S and T have a unique common fixed point.*

Proof. We prove $v = Su = Tu$. Since S and T are weakly compatible, by (11) we have $STu = TSu$. Then

$$Tv = TTu = TSu = STu = SSu = Sv. \tag{12}$$

If $Tv = v$ or $Sv = v$, then v is a common fixed point. Otherwise, i.e., if $Tv \neq v$ and $Sv \neq v$, by property (ζ_1) of a simulation function with $Tu \preccurlyeq TTu$

$$\begin{aligned} 0 &\leqslant \zeta\big(d(v, Sv), d(v, Tv)\big) = \zeta\big(d(Su, SSu), d(Tu, TTu)\big) \\ &< d(Tu, TTu) - d(Su, SSu). \end{aligned}$$

Using (11) and (12) in the above inequality, we have

$$d(Su, SSu) < d(Tu, TTu) = d(Su, SSu),$$

which is a contradiction. Therefore, $Tv = v$ or $Sv = v$, and we conclude that $v = Sv = Tv$.

Now, suppose that the set of fixed points of T is totally ordered. Assume on the contrary that $v = Sv = Tv$ and $v' = Sv' = Tv'$ but $v \neq v'$. Since v and v' contain a set of fixed points of T, without loss of generality, we assume that $Tv \preccurlyeq Tv'$. If $Sv = Sv'$ or $Tv = Tv'$, then $v = v'$, which is a contradiction. Otherwise, i.e., if $Sv \neq Sv'$ and $Tv \neq Tv'$, by property (ζ_1) of a simulation function we have

$$\begin{aligned} 0 &\leqslant \zeta\big(d(Sv, Sv'), d(Tv, Tv')\big) = \zeta\big(d(v, v'), d(v, v')\big) \\ &< d(v, v') - d(v, v') = 0, \end{aligned}$$

which is a contradiction. Therefore, S and T have a unique common fixed point. □

In the next theorem, we will omit condition (iii) of Theorem 1, and we will assume that $S, T : X \to X$ are continuous and compatible.

Theorem 3. *Let $(X, \preccurlyeq)$ be a partially ordered set, and let there exists a metric d on X such that (X, d) is complete metric space. Suppose that there exist a simulation function ζ and $S, T : X \to X$ such that*

$$\zeta\bigl(d(Sx, Sy), d(Tx, Ty)\bigr) \geqslant 0 \quad \forall x, y \in X\colon\ Tx \preccurlyeq Ty.$$

We suppose the following hypotheses:

(i) *$SX \subseteq TX$*;
(ii) *S is T-nondecreasing*;
(iii) *S and T are continuous*;
(iv) *The pair $\{S, T\}$ is compatible.*

If there exists $x_0 \in X$ such that $Tx_0 \preccurlyeq Sx_0$, then S and T have a coincidence point, that is, there exists $u \in X$ such that $Su = Tu$. Further, if $Tu \preccurlyeq TTu$ and the set of fixed points of T is totally ordered, then S and T have a unique common fixed point.

Proof. Following the proof of Theorem 1, we have that $\{Tx_n\}$ is a Cauchy sequence in the complete metric space (X, d). Then there exists $u \in X$ such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = u. \tag{13}$$

Since S and T are compatible, this implies that

$$\lim_{n\to\infty} \bigl(S(Tx_n), T(Sx_n)\bigr) = 0. \tag{14}$$

From (13) and the continuity of S and T we have

$$\lim_{n\to\infty} T(Tx_n) = Tu, \qquad \lim_{n\to\infty} S(Tx_n) = Su. \tag{15}$$

By the triangular inequality we have

$$d(Su, Tu) \leqslant \bigl(Su, S(Tx_n)\bigr) + d\bigl(S(Tx_n), T(Sx_n)\bigr) + d\bigl(T(Tx_{n+1}), Tu\bigr).$$

By (14) and (15) and letting $n \to \infty$, we obtain:

$$d(Su, Tu) \leqslant 0,$$

therefore, $Su = Tu$, that is, u is the coincidence point of S and T.

Finally, because S and T are compatible (therefore, they are weakly compatible) and, on the other hand, $Tu \preccurlyeq TTu$ and set of fixed points of T is totally ordered, then by Theorem 2, S and T have a unique common fixed point. □

If $T : X \to X$ is the identity mapping, we can deduce easily the following fixed point results. It is an immediate consequence of Theorem 1.

Theorem 4. *Let $(X, \preccurlyeq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is complete metric space. Suppose that there exist a simulation function ζ and $S : X \to X$ such that*

$$\zeta\big(d(Sx, Sy), d(x, y)\big) \geqslant 0 \quad \forall x, y \in X\colon\ x \preccurlyeq y.$$

We suppose the following hypotheses:

(i) *S is a nondecreasing function*;
(ii) *If $\{u_n\}$ is a nondecreasing sequence, which converges to u in X, then $u_n \preccurlyeq u$ for all $n \geqslant 0$.*

If there exists $x_0 \in X$ such that $x_0 \preccurlyeq Sx_0$, then S has a fixed point.

The following result is an immediate consequence of Theorem 3.

Theorem 5. *Let $(X, \preccurlyeq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is complete metric space. Suppose that there exist a simulation function ζ and $S : X \to X$ such that*

$$\zeta\big(d(Sx, Sy), d(x, y)\big) \geqslant 0 \quad \forall x, y \in X\colon\ x \preccurlyeq y.$$

We suppose the following hypotheses:

(i) *S is a nondecreasing function*;
(ii) *S is continuous.*

If there exists $x_0 \in X$ such that $x_0 \preccurlyeq Sx_0$, then S has a fixed point.

4 Consequences

In this section, as applications, we obtain some results of Theorem 1 in fixed point theory in partially ordered metric space via specific choices of simulation functions.

Let $(X, \preccurlyeq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space.

Corollary 1. *Let $S, T : X \to X$ be mappings such that there exist two continuous functions $\phi, \psi : [0, \infty) \to [0, \infty)$ verifying $\psi(t) = \phi(t) = 0$ if and only if $t = 0$, $\psi(t) < t \leqslant \phi(t)$ for all $t > 0$, and*

$$\phi\big(d(Sx, Sy)\big) \leqslant \psi\big(d(Tx, Ty)\big) \quad \forall x, y \in X\colon\ Tx \preccurlyeq Ty.$$

We suppose the following hypotheses:

(i) *$SX \subseteq TX$ and TX is closed*;
(ii) *S is T-nondecreasing*;
(iii) *If $\{Tx_n\} \subset X$ is a nondecreasing sequence converges to Tu in TX, then $Tx_n \preccurlyeq Tu$ for all $n \geqslant 0$.*

If there exists $x_0 \in X$ such that $Tx_0 \preccurlyeq Sx_0$, then S and T have a coincidence point, that is, there exists $u \in X$ such that $Su = Tu$. Further, if S and T are weakly compatible, $Tu \preccurlyeq TTu$, and the set of fixed points of T is totally ordered, then S and T have a unique common fixed point.

Proof. The result follows from Theorems 1 and 2 by taking as simulation function

$$\zeta_1(p,q) = \psi(q) - \phi(p) \quad \forall p,q \geqslant 0,$$

which was introduced in Example 2. □

Corollary 2 [Banach type]. *Let $S,T : X \to X$ be mappings such that there exists $\alpha \in [0,1)$ verifying*

$$d(Sx,Sy) \leqslant \alpha d(Tx,Ty) \quad \forall x,y \in X\colon\ Tx \preccurlyeq Ty.$$

We suppose the following hypotheses:

(i) *$SX \subseteq TX$ and TX is closed*;
(ii) *S is T-nondecreasing*;
(iii) *If $\{Tx_n\} \subset X$ is a nondecreasing sequence converges to Tu in TX, then $Tx_n \preccurlyeq Tu$ for all $n \geqslant 0$.*

If there exists $x_0 \in X$ such that $Tx_0 \preccurlyeq Sx_0$, then S and T have a coincidence point, that is, there exists $u \in X$ such that $Su = Tu$. Further, if S and T are weakly compatible, $Tu \preccurlyeq TTu$, and the set of fixed points of T is totally ordered, then S and T have a unique common fixed point.

Proof. The result follows from Theorems 1 and 2 by taking as simulation function

$$\zeta_2(p,q) = \alpha q - p \quad \forall p,q \geqslant 0,$$

which was introduced in Example 2. □

Corollary 3. *Let $S,T : X \to X$ be mappings such that there exists a lower semicontinuous function $\varphi : [0,\infty) \to [0,\infty)$ verifying $\varphi^{-1}(\{0\}) = \{0\}$ and*

$$d(Sx,Sy) \leqslant d(Tx,Ty) - \varphi\bigl(d(Tx,Ty)\bigr) \quad \forall x,y \in X\colon\ Tx \preccurlyeq Ty.$$

We suppose the following hypotheses:

(i) *$SX \subseteq TX$ and TX is closed*;
(ii) *S is T-nondecreasing*;
(iii) *If $\{Tx_n\} \subset X$ is a nondecreasing sequence, which converges to Tu in TX, then $Tx_n \preccurlyeq Tu$ for all $n \geqslant 0$.*

If there exists $x_0 \in X$ such that $Tx_0 \preccurlyeq Sx_0$, then S and T have a coincidence point, that is, there exists $u \in X$ such that $Su = Tu$. Further, if S and T are weakly compatible, $Tu \preccurlyeq TTu$, and the set of fixed points of T is totally ordered, then S and T have a unique common fixed point.

Proof. The result follows from Theorems 1 and 2 using simulation function

$$\zeta_3(p,q) = q - \varphi(q) - p \quad \forall p,q \geqslant 0,$$

which was introduced in Example 2. □

Corollary 4. *Let $S,T : X \to X$ be mappings such that there exists a function $\varphi : [0,\infty) \to [0,1)$ with $\limsup_{t\to r^+} \varphi(t) < 1$ for all $r > 0$, and*

$$d(Sx,Sy) \leqslant d(Tx,Ty)\varphi\bigl(d(Tx,Ty)\bigr) \quad \forall x,y \in X\colon Tx \preccurlyeq Ty,$$

We suppose the following hypotheses:

(i) *$SX \subseteq TX$ and TX is closed*;
(ii) *S is T-nondecreasing*;
(iii) *If $\{Tx_n\} \subset X$ is a nondecreasing sequence converges to Tu in TX, then $Tx_n \preccurlyeq Tu$ for all $n \geqslant 0$.*

If there exists $x_0 \in X$ such that $Tx_0 \preccurlyeq Sx_0$, then S and T have a coincidence point, that is, there exists $u \in X$ such that $Su = Tu$. Further, if S and T are weakly compatible, $Tu \preccurlyeq TTu$, and the set of fixed points of T is totally ordered, then S and T have a unique common fixed point.

Proof. The result follows from Theorems 1 and 2 by taking as simulation function

$$\zeta_4(p,q) = q\varphi(q) - p \quad \forall p,q \geqslant 0,$$

which was introduced in Example 2. □

Corollary 5. *Let $S,T : X \to X$ be mappings and $f,g : [0,\infty) \to (0,\infty)$ be two continuous functions with respect to each variable such that $f(p,q) > g(p,q)$ for all $p,q > 0$, and*

$$\frac{f(d(Sx,Sy),d(Tx,Ty))}{g(d(Sx,Sy),d(Tx,Ty))} d(Sx,Sy) \leqslant d(Tx,Ty) \quad \forall x,y \in X\colon Tx \preccurlyeq Ty.$$

We suppose the following hypotheses:

(i) *$SX \subseteq TX$ and TX is closed*;
(ii) *S is T-nondecreasing*;
(iii) *If $\{Tx_n\} \subset X$ is a nondecreasing sequence converges to Tu in TX, then $Tx_n \preccurlyeq Tu$ for all $n \geqslant 0$.*

If there exists $x_0 \in X$ such that $Tx_0 \preccurlyeq Sx_0$, then S and T have a coincidence point, that is, there exists $u \in X$ such that $Su = Tu$. Further, if S and T are weakly compatible, $Tu \preccurlyeq TTu$, and the set of fixed points of T is totally ordered, then S and T have a unique common fixed point.

Proof. The result follows from Theorems 1 and 2 by taking as simulation function

$$\zeta_5(p,q) = q - \frac{f(p,q)}{g(p,q)}p \quad \forall p,q \geqslant 0,$$

which was introduced in Example 2. □

Corollary 6. *Let $S, T : X \to X$ be mappings and there exists a function $\varphi : [0,\infty) \to [0,\infty)$ such that $\int_0^\epsilon \phi(u)\,\mathrm{d}u$ exists, $\int_0^\epsilon \phi(u)\,\mathrm{d}u > \epsilon$ for each $\epsilon > 0$, and*

$$\int_0^{d(Sx,Sy)} \phi(u)\,\mathrm{d}u \leqslant d(Tx,Ty) \quad \forall x,y \in X\colon Tx \preccurlyeq Ty.$$

We suppose the following hypotheses:

(i) *$SX \subseteq TX$ and TX is closed*;
(ii) *S is T-nondecreasing*;
(iii) *If $\{Tx_n\} \subset X$ is a nondecreasing sequence converges to Tu in TX, then $Tx_n \preccurlyeq Tu$ for all $n \geqslant 0$.*

If there exists $x_0 \in X$ such that $Tx_0 \preccurlyeq Sx_0$, then S and T have a coincidence point, that is, there exists $u \in X$ such that $Su = Tu$. Further, if S and T are weakly compatible, $Tu \preccurlyeq TTu$, and the set of fixed points of T is totally ordered, then S and T have a unique common fixed point.

Proof. The result follows from Theorems 1 and 2 using the simulation function

$$\zeta_6(p,q) = q - \int_0^p \phi(u)\,\mathrm{d}u \quad \forall p,q \geqslant 0,$$

which was introduced in Example 2. □

5 Example

In this section, we suppose that $\zeta(p,q) : [0,\infty)\times[0,\infty) \to \mathbb{R}$ with $\zeta(p,q) = q-((p+2)/(p+1))p$. Clearly, ζ is a simulation function.

Example 2. Let $X = \{0,1,2,3,4,\dots\}$ be endowed with the metric $d : X \times X \to \mathbb{R}$ given by

$$d(x,y) = \begin{cases} x+y & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

We define a partial order $\preccurlyeq$ in X as $x \preccurlyeq y$ if and only if $x \geqslant y$, $(y-x)$ is divisible by 2 for all $x,y \in \{2,3,4,\dots\}$, and $1 \preccurlyeq 0$, $2 \preccurlyeq 1$. Clearly, $(X,d,\preccurlyeq)$ is a complete partially ordered metric space.

Let $S, T : X \to X$ be defined as

$$Sx = \begin{cases} x-2 & \text{if } x \geqslant 3, \\ 0 & \text{if } x = 0, 1, 2, \end{cases} \qquad Tx = \begin{cases} x-1 & \text{if } x > 1 \\ 0 & \text{if } x = 0, 1. \end{cases}$$

Without loss of generality, assume that $x > y$ and verify inequality (1). Then the following cases are possible.

(C1) If $x = 1$, then $y = 0$ and $Sx = Sy = Tx = Ty = 0$. Then (1) is satisfied.

(C2) If $x = 2$, then $y = 0$ or 1 and $Sx = Sy = 0$. Then (1) is satisfied.

(C3) If $x \in \{3, 4, 5, \dots\}$, then we have the following subcases:

(a) If $y = 0$ or $y = 1$, then $Sx = x - 2$, $Sy = 0$, $Tx = x - 1$, and $Ty = 0$. Therefore,

$$\zeta\bigl(d(Sx, Sy), d(Tx, Ty)\bigr) = x - 1 - \frac{x-2+2}{x-2+1}(x-2) = \frac{1}{x-1} \geqslant 0.$$

(b) If $y = 2$, then $Sx = x - 2$, $Sy = 0$, $Tx = x - 1$, and $Ty = 1$. Therefore,

$$\zeta\bigl(d(Sx, Sy), d(Tx, Ty)\bigr) = x - \frac{x-2+2}{x-2+1}(x-2) = \frac{x}{x-1} \geqslant 0.$$

(c) If $y \in \{3, 4, 5, \dots\}$, then $Sx = x - 2$, $Sy = y - 2$, $Tx = x - 1$, and $Ty = y - 1$. Then

$$\begin{aligned} \zeta\bigl(d(Sx, Sy), d(Tx, Ty)\bigr) &= x + y - 2 - \frac{x+y-4+2}{x+y-4+1}(x+y-4) \\ &= \frac{x+y-2}{x+y-3} \geqslant 0. \end{aligned}$$

Thus, (1) is verified.

On the other hand, if $\{Tx_n\}$ is a nondecreasing sequence in TX with respect to $\preccurlyeq$ such that $Tx_n \to Tu$. By the definition of the metric d, there exists $m \in \mathbb{N}$ such that $Tx_n = Tu$ for all $n \geqslant m$, so condition (iii) of Theorem 1 is satisfied.

Thus, S and T satisfy all other hypotheses of Theorem 1. Therefore, S and T have a coincidence point. Moreover, since S and T satisfy all the hypotheses of Theorem 2, we obtain S and T have a unique common fixed point in 0.

Remark 2. We know by using the Archimedean property for all $\alpha \in (0, 1)$ there exists $x \in \{3, 4, \dots\}$ such that

$$(1-\alpha)x > 2,$$

therefore,

$$(\alpha - 1)x < -2 \quad \Longrightarrow \quad \alpha x - (x-2) < 0.$$

If $\zeta(p, q) = \alpha q - p$ and $y = 2$ in the previous example, then we have

$$\zeta\bigl(d(Sx, Sy), d(Tx, Ty)\bigr) = \alpha x - (x-2) < 0.$$

Then the previous example does not apply to Banach contraction.

Example 3. Let $X = [0, \infty)$ be endowed with the metric $d : X \times X \to \mathbb{R}$ given by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ \max\{x, y\} & \text{if } x \neq y. \end{cases}$$

Now consider the usual order of real numbers and define the mappings $S, T : X \to X$ by $Sx = x$ and $Tx = 2x$ for all $x \in X$. Then we have

$$\begin{aligned} &\zeta\bigl(d(Sx, Sy), d(Tx, Ty)\bigr) \\ &\quad = 2y - \frac{y+2}{y+1}y = \frac{2y(y+1) - y(y+2)}{y+1} = \frac{y^2}{y+1} \geqslant 0 \end{aligned}$$

for all $x, y \in X$ with $x \leqslant y$. Therefore, inequality (1) is satisfied. Thus, S and T satisfy all the hypotheses of Theorem 3. Here $v = 0$ is a coincidence point as well as a unique common fixed point of S and T.

6 Existence of solution for a nonlinear integral equation

Consider the integral equation

$$\begin{aligned} x(t) = f_1(t) - f_2(t) + \epsilon \int_0^t n_1(t, s) k_1\bigl(s, x(s)\bigr)\, \mathrm{d}s \\ + \delta \int_0^T n_2(t, s) k_2\bigl(s, x(s)\bigr)\, \mathrm{d}s, \quad t \in I, \end{aligned} \tag{16}$$

where $I = [0, T]$, $T > 0$.

The purpose of this section is to give an existence theorem for a solution of (16) using Theorem 1. This application was inspired by [1, 26].

Previously, we considered the space

$$C(I) := \{x\colon\, I \to \mathbb{R} \mid x \text{ is continuous on I}\}.$$

Obviously, $C(I)$ with the metric given by

$$d(x, y) = \sup_{t \in I} \bigl|x(t) - y(t)\bigr| \quad \forall x, y \in C(I),$$

is a complete metric space. $C(I)$ can also be equipped with the partial order $\preccurlyeq$ given by

$$x, y \in C(I), \quad x \preccurlyeq y \quad \Longleftrightarrow \quad x(t) \leqslant y(t) \quad \forall t \in I.$$

Moreover, in [24], it was proved that if a nondecreasing sequence $\{u_n\} \subseteq C(I)$ converges to u in $C(I)$, then $u_n \preccurlyeq u$ for all $n \geqslant 0$.

Now, we will prove the following result.

Theorem 6. *For each $x \in C(I)$, define the operators*

$$Sx(t) = -f_2(t) + \epsilon \int_0^t n_1(t,s) k_1\bigl(s, x(s)\bigr)\,\mathrm{d}s$$

and

$$Tx(t) = x(t) - f_1(t) - \delta \int_0^T n_2(t,s) k_2\bigl(s, x(s)\bigr)\,\mathrm{d}s,$$

where $t \in I$, $\epsilon, \delta \in \mathbb{R}$, $f_1, f_2 \in C(I)$ with $f_1(t) \geqslant f_2(t)$, and $n_1(t,s)$, $n_2(t,s)$, $k_1(s,x(s))$, $k_2(s,x(s))$ are continuous real-valued functions in $I \times \mathbb{R}$.

Suppose that the following hypotheses hold:

(i) $\int_0^T \sup_{t\in I} |n_i(t,s)|\,\mathrm{d}s = N_i < \infty$, $i \in \{1,2\}$;

(ii) *For each $s \in I$ and for all $x, y \in C(I)$ with $Tx \preccurlyeq Ty$, there is $P_i \geqslant 0$ such that*

$$\bigl|K_i\bigl(s, x(s)\bigr) - k_i\bigl(s, y(s)\bigr)\bigr| \leqslant P_i\bigl|x(s) - y(s)\bigr|, \quad i \in \{1,2\};$$

(iii)
$$\delta \int_0^T n_2(t,s) k_2\left(s,\ \epsilon \int_0^s n_1(s,v) k_1\bigl(v, x(v)\bigr)\,\mathrm{d}v + f_1(s) - f_2(s)\right) \mathrm{d}s = 0;$$

(iv) *$Tx \preccurlyeq Ty$ implies $Sx \preccurlyeq Sy$ for all $x, y \in C(I)$*;

(v) *There exists $x_0 \in C(I)$ such that*

$$\begin{aligned} x_0 &\leqslant f_1(t) - f_2(t) \\ &\quad + \epsilon \int_0^t n_1(t,s) k_1\bigl(s, x_0(s)\bigr)\,\mathrm{d}s + \delta \int_0^T n_2(t,s) k_2\bigl(s, x_0(s)\bigr)\,\mathrm{d}s. \end{aligned}$$

If

$$\frac{|\epsilon| P_1 N_1}{1 - |\delta| P_2 N_2} < 1 \quad \textit{and} \quad |\delta| < \frac{1}{P_2 N_2},$$

then the integral equation (16) *has a solution.*

Proof. Note that the integral equation (16) has a solution if and only if the operators S and T have a coincidence point. Clearly, S and T are self-operators on $C(I)$. Now, for all $x, y \in C(I)$ with $Tx \preccurlyeq Ty$, by assumptions (i) and (ii) we have

$$\begin{aligned} \bigl|Sx(t) - Sy(t)\bigr| &\leqslant |\epsilon| \int_0^t \bigl|n_1(t,s)\bigr| \bigl|k_1\bigl(s, x(s)\bigr) - k_1\bigl(s, y(s)\bigr)\bigr|\,\mathrm{d}s \\ &\leqslant |\epsilon| \int_0^t \sup_{t\in I} \bigl|n_1(t,s)\bigr| \bigl|k_1\bigl(s, x(s)\bigr) - k_1\bigl(s, y(s)\bigr)\bigr|\,\mathrm{d}s \end{aligned}$$

$$
\begin{aligned}
&\leqslant |\epsilon| \int_0^t \sup_{t\in I} \bigl|n_1(t,s)\bigr| P_1 \bigl|x(s)-y(s)\bigr| \,\mathrm{d}s \\
&\leqslant |\epsilon| P_1 \|x-y\| \int_0^t \sup_{t\in I} \bigl|n_1(t,s)\bigr| \,\mathrm{d}s \\
&\leqslant |\epsilon| P_1 N_1 \|x-y\|.
\end{aligned}
$$

This implies that

$$
\|Sx-Sy\| = \sup_{t\in I} \bigl|Sx(t)-Sy(t)\bigr| \leqslant |\epsilon| P_1 N_1 \|x-y\|. \tag{17}
$$

On the other hand, we have

$$
\begin{aligned}
&\left| \delta \int_0^T n_2(t,s) k_2\bigl(s,x(s)\bigr) \,\mathrm{d}s - \delta \int_0^T n_2(t,s) k_2\bigl(s,y(s)\bigr) \,\mathrm{d}s \right| \\
&\quad \leqslant |\delta| \int_0^T \bigl|n_2(t,s)\bigr| \bigl|k_2\bigl(s,x(s)\bigr) - k_2\bigl(s,y(s)\bigr)\bigr| \,\mathrm{d}s \\
&\quad \leqslant |\delta| \int_0^T \sup_{t\in I} \bigl|n_2(t,s)\bigr| \bigl|k_2\bigl(s,x(s)\bigr) - k_2\bigl(s,y(s)\bigr)\bigr| \,\mathrm{d}s \\
&\quad \leqslant |\delta| \int_0^T \sup_{t\in I} \bigl|n_2(t,s)\bigr| P_2 \bigl|x(s)-y(s)\bigr| \,\mathrm{d}s \\
&\quad \leqslant |\delta| P_2 N_2 \|x-y\|,
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\sup_{t\in I} \left| \delta \int_0^T n_2(t,s) k_2\bigl(s,x(s)\bigr) \,\mathrm{d}s - \delta \int_0^T n_2(t,s) k_2\bigl(s,y(s)\bigr) \,\mathrm{d}s \right| \\
&\quad \leqslant |\delta| P_2 N_2 \|x-y\|.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
&\|Tx-Ty\| \\
&\quad \geqslant \|x-y\| - \sup_{t\in I} \left| \delta \int_0^T n_2(t,s) k_2\bigl(s,x(s)\bigr) \,\mathrm{d}s - \delta \int_0^T n_2(t,s) k_2\bigl(s,y(s)\bigr) \,\mathrm{d}s \right| \\
&\quad \geqslant \bigl(1 - |\delta| P_2 N_2\bigr) \|x-y\|,
\end{aligned}
$$

which implies

$$\|x - y\| \leqslant \frac{1}{1 - |\delta| P_2 N_2} \|Tx - Ty\|. \tag{18}$$

From (17) and (18) we get

$$\|Sx - Sy\| \leqslant \frac{|\epsilon| P_1 N_1}{1 - |\delta| P_2 N_2} \|Tx - Ty\|,$$

and, since $\alpha = |\epsilon| P_1 N_1 / (1 - |\delta| P_2 N_2) < 1$, if we define $\zeta(p, q) = \alpha q - p$ for all $p, q \in [0, \infty)$, then we have

$$\zeta\bigl(d(Sx, Sy), d(Tx, Ty)\bigr) \geqslant 0$$

for all $x, y \in C(I)$ with $Tx \preccurlyeq Ty$. Thus, condition (1) is trivially satisfied. Next, we can show that $S(C(I)) \subseteq T(C(I))$. Indeed, by (iii) for $x(t) \in C(I)$ we have

$$\begin{aligned}
&T\bigl(Sx(t) + f_1(t)\bigr) \\
&\quad = Sx(t) + f_1(t) - f_1(t) - \delta \int_0^T n_2(t, s) k_2\bigl(s,\ Sx(s) + f_1(s)\bigr)\, \mathrm{d}s \\
&\quad = Sx(t) - \delta \int_0^T n_2(t, s) k_2\left(s,\ \epsilon \int_0^s n_1(s, v) k_1\bigl(v, x(v)\bigr)\, \mathrm{d}v + f_1(s) - f_2(s)\right) \mathrm{d}s \\
&\quad = Sx(t).
\end{aligned}$$

Clearly, hypothesis (iv) means that S is T-nondecreasing. Next, by (v) we get

$$x_0 - f_1(t) - \delta \int_0^T n_2(t, s) k_2\bigl(s, x_0(s)\bigr)\, \mathrm{d}s \leqslant -f_2(t) + \epsilon \int_0^t n_1(t, s) k_1\bigl(s, x_0(s)\bigr)\, \mathrm{d}s,$$

that is, $Tx_0 \preccurlyeq Sx_0$. Thus, all the cases of Theorem 1 are satisfied, and hence, its result holds, that is, S and T have at last a coincidence point. Consequently, the integral equation (16) has a solution in $C(I)$. $\square$

7 Conclusion

In this work, we consider a pair of nonlinear operators satisfying a nonlinear contraction involving a simulation function in a metric space endowed with a partial order. For this pair of operators with and without continuity, we establish coincidence and unique common fixed point results. Moreover, an application of our results obtained to prove the existence of a solution to an integral equation is presented.

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